Simulating Statistical Power in Latent Growth Curve Modeling: A Strategy for Evaluating Age-Based Changes in Cognitive Resources

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1 Introduction

Variability across and within individuals is a fundamental property of adult age changes in behavior [20, 21, 24]. Some people seem young for their age, others seem old; shining examples of older individuals who maintained high levels of intellectual functioning well into very old age, such as Johann Wolfgang von Goethe or Sophocles, stand in contrast to individuals whose cognitive resources are depleted by the time they reach later adulthood. A similar contrast exists between different intellectual abilities. For example, if one looks at the speed needed to identify and discriminate between different percepts, one is likely to find monotonic decline after late adolescence and early adulthood. But if one looks at verbal knowledge, one will find age stability or positive change into very old age [36]. As a general rule, tasks that assess individual differences in speed, reliability, and coordination of elementary processing operations show greater decline, whereas tasks that assess individual differences in acquired knowledge show less decline.

The simultaneous presence of resource growth, maintenance, and decline, both across individuals and across abilities, calls for statistical methods that are able to efficiently capture both the commonalities and the differences of age-based changes in levels of functioning across the lifespan [5]. In this context, a family of methods known as latent growth curve models (LGCMs), multi-level models, random-coefficient modeling has gained prominence in recent years [15]. Despite their widespread and increasing application, central statistical properties of these models have not yet been explored or formally analyzed. In this chapter, we introduce a general strategy for evaluating the suitability of LGCM for charting lifespan changes in behavior, with a specific emphasis on statistical power.

LGCMs [26] are a particular set of structural equation models (SEMs) aimed at describing the general, average trend in change as well as the individual differences around the group trend. Extensions allow for including predictors of interindividual

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differences in change parameters obtained from the time series analyzed. Because longitudinal data are composed of repeated measures, the non-zero within-person correlation violates the common statistical assumption of independence in ordinary least squares regression analysis. A family of analyses for correlated data (e.g., multi-level, hierarchical linear, mixed effects, and random effects models) thus provides an appealing analytical strategy for longitudinal data [18]. The LGCM and the correlated data approach to repeated measures are statistically equivalent [30, 33]. For simplicity, we will here subsume both statistical approaches under the heading of LGCM.

LGCMs have become the favorite analytical tool of many psychological researchers for theoretical investigations about development and change phenomena. Several long-standing goals of longitudinal analyses may be achieved by implementing proper LGCMs under specific assumptions [4]. Because of their popularity, many software packages allow for easily reproducible LGCM analyses, either within the more general SEM framework or within the analogous correlated data approach (software implementation may highlight and optimize different statistical aspects; see [13, 22]). However, not all scientific inquiries nor all longitudinal data are amenable to LGCM analyses, and some warnings have been raised as to the limits of LGCMs [25, 34]. In particular, LGCMs have been utilized to reach substantive conclusions about concomitant interindividual differences in a multivariate space, especially in cognitive aging research. Yet, in samples of aging individuals it is often very hard to detect interindividual differences in change and subsequently, covariances among change components. Given the need to more fully understand statistical properties of LGCMs simulation work to this end has appeared in the literature (e.g., [11, 14, 16]).

LGCMs are most commonly computed with SEM software by applying a maximum likelihood estimation procedure to a moment matrix containing covariance and mean information about the repeated measures. For the common case of incomplete data, the Full Information Maximum Likelihood (FIML) variant allows analyzing raw data of all observations, without excluding observations with an incomplete data vector (cf. [1, 19, 27]). The FIML algorithm is now the choice of incomplete data treatment in many SEM software packages, including Mx [32], Lisrel [17], AMOS [2], EQS [6], and MPlus [31]. The mathematical formulation of the FIML algorithm can be found in the original source by Lange et al. (1976) or in some SEM manuals cited above. However, the general implementation of Lange et al.'s formulation within each SEM software package and the remaining elements of the general computation procedure used in the parameter estimation process are not easily documented, hence generally not available to SEM users.

In this chapter, we aim at (a) presenting a general simulation procedure for testing specific statistical properties of LGCMs and (b) describing the mathematical formulation of the estimation procedure adopted within our data-generation-plus-analysis engine. The simulation tool can be found at http://www.mpib-berlin.mpg.de/en/forschung/computerscience/.

In Sect. 2, we discuss the general LGCM and its assumptions. In Sect. 3, we describe two fitting functions, the Least Squares and the Minus 2 Log Likelihood,

and their implementation in our engine. The issues of starting values and non-admissible estimation areas is discussed. Section 4 describes the general simulation procedure. Data are generated in accordance with the LGCM and a set of known parameter values (we shall call "population values"), then selected following specific considerations about time sampling, and finally analyzed. Comparisons of slightly different LGCMs are presented to allow for inferential conclusions about single parameters of interest. Section 5 presents an illustration of the engine to investigate a particular set of parameters within the LGCM. A more detailed analysis of LGCM parameters under a wide variety of empirically plausible conditions is presented in [14, 16]. The present simulation serves illustrative purposes. Finally, in Sect. 6 we discuss our conclusions.

2 The Latent Growth Curve Model

Consider N units (e.g., persons) with K data points, corresponding to V variables measured at T time points (i.e., $K = VT$). The data points are obtained by applying a continuous function f_v defining the relations among P parameters, R Gaussian distributed random numbers, and T time points to each variable $v = 0, \ldots, V - 1$. We call such a continuous function a *model*. Let C denote this function space. Then $\Sigma \in C^{K \times K}$ denotes the covariance matrix of the data points with respect to the parameters. Likewise, $\mu \in C^K$ denotes the vector of means with respect to the parameters. We denote the vector of parameters by $\overline{p}$.

We define here a particular linear model with equal interval measurement, where for each variable v and for each unit $i = 0, \ldots, N - 1$ we consider a level $l_{v,i}$ and a slope $s_{v,i}$. A data point for a unit i is defined by

$$f_{t,v,i} = l_{v,i} + \frac{t}{T} \cdot s_{v,i} + err_{t,v,i} \tag{1}$$

where t is a time point (i.e., $t = 0, \ldots, T - 1$) and $err_{t,v,i}$ is a normally distributed error term. While $err_{t,v,i}$ contains a time subscript t, indicating that its value changes across a unit's time series, both $l_{v,i}$ and $s_{v,i}$ are time invariant. All three terms are dependent on the unit i and variable v.

The means, variances, and covariances of $l_{v,i}$, $s_{v,i}$, and $err_{t,v,i}$ represent the parameters of the linear LGCM for each variable v (e.g., [26]). Assuming that the error components have mean zero, do not covary with any other parameter, and have a time-invariant variance, the parameters of the model are $2V$ means for all levels and slopes, $3V$ variances for all levels, slopes, and errors, and $\frac{2V(2V-1)}{2}$ covariances among all levels and slopes. The total number of parameters is $3V + \frac{2V(2V+1)}{2} = 2v^2 + 4v$.

μ and Σ are created with the matrix $\Lambda \in \mathbb{Q}^{K \times 2V}$, defined as

$$\Lambda_{ij} = \begin{cases} 1 & \lfloor i/T \rfloor = j \\ \frac{i-(j-V)T}{t} & \lfloor i/t \rfloor = j + V \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Λ will hence look like

$$\Lambda = \begin{pmatrix}
1 & 0 & & 0 & \frac{0}{T} & 0 & & 0 \\
1 & 0 & & 0 & \frac{1}{T} & 0 & & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & & 0 & \frac{T-1}{T} & 0 & & 0 \\
0 & 1 & & 0 & 0 & \frac{0}{T} & & 0 \\
0 & 1 & & 0 & 0 & \frac{1}{T} & & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 1 & & 0 & 0 & \frac{T-1}{T} & & 0 \\
 & & & & \vdots & & & \\
0 & & & 1 & 0 & & & \frac{0}{T} \\
0 & & & 1 & 0 & & & \frac{1}{T} \\
\vdots & & \cdots & \vdots & \vdots & & \cdots & \vdots \\
0 & & & 1 & 0 & & & \frac{T-1}{T}
\end{pmatrix} \tag{3}$$

To obtain the vector μ, we multiply Λ by the vector of the means of all levels and slopes:

$$\mu = \Lambda \left(\mu_{l_1}, \ldots, \mu_{l_{V-1}}, \mu_{s_1}, \ldots, \mu_{s_{V-1}} \right)^T \tag{4}$$

Let M denote the covariance matrix of levels and slopes in the same order. Σ is then obtainable by

$$\Sigma = \Lambda M \Lambda^T \tag{5}$$

Note that in the linear LGCM, all entries in μ and Σ are linear. Furthermore, the parameters in μ are only means of levels and slopes, while the parameters in Σ are covariances of levels, slopes, and error.

In sum, then, the usual application of LGCM in psychological research consists in analyzing N time series, one for each unit i of analysis, spanning over T time points, for a total of V variables. Researchers then wish to obtain information about the level $l_{v,i}$, the slope $s_{v,i}$, and the error $err_{t,v,i}$ for each variable v. The overall level means, slope means, level variances, slope variances, covariances among all levels and slopes, and error variances are the elements of $\overline{p}$.

3 Least Squares and Minus Two Log Likelihood Fitting Functions

To obtain the optimal parameter values by applying the LGCM to an $N \cdot K$ data matrix, *indices* are defined that mathematically define the distance between the observed data points and the expectations of the LGCM contingent upon the parameter values. Indices are norm functions on the parameters and the data, which are minimal iff the most suitable parameter values are estimated. We consider two indices, the *Euclidean distance* and the *Deviance*. We then discuss two associated fitting functions, which minimize these distances given the observed data and estimated parameters in $\bar{p}$. For the Deviance, we use an iterative procedure that needs starting values. Inadmissible estimation areas of the fitting functions are defined and finally we discuss how our engine handles them.

The Euclidean distance fit index defines the distance between the covariance matrix and mean vector of the data and the covariance matrix $\Sigma(\bar{p})$ and mean vector $\mu(\bar{p})$ predicted by the model given $\bar{p}$. Let S be the covariance matrix of the observed data and m be the mean vector of the observed data. Then, the Euclidean distance ls is defined as

$$ ls = \sqrt{\sum_{i=0}^{K-1}(\mu(\bar{p})_i - m_i)^2 + \sum_{i=0}^{K-1}\sum_{j=0}^{K-1}\left(\Sigma(\bar{p})_{i.j} - S_{i.j}\right)^2} \tag{6}$$

So, the Euclidean distance is the Frobenius norm on the difference of covariance matrices plus the absolute value of the difference of the mean vectors. If ls is minimal, the Euclidean distance (in the $K^2 + K$ dimensional space) between (Σ, μ) and (S, m) is minimal. We call the point of global minimum the least square estimate.

The square root can be omitted for computing the least square estimate, and if the parameters are distinguished between those associated to the means (i.e., parameters appearing in μ, but not in Σ) and those associated to the variances–covariances (i.e., parameters appearing in Σ, but not in μ), both can be estimated separately. In linear models, the ls index is a polynomial of degree two, and hence its extremes are uniquely determined. ls can be obtained by computing the first two derivatives with respect to $\bar{p}$. The first derivative with respect to one parameter θ is

$$\frac{\partial ls^2}{\partial\theta} = \sum_{i=0}^{K}2\left(\frac{\partial\mu_i}{\partial\theta} + m_i\right) + \sum_{i=0}^{K}\sum_{j=0}^{K}2\left(\frac{\partial\Sigma_{i.j}}{\partial\theta} + S_{i.j}\right) \tag{7}$$

The second derivative with respect to θ_1 and θ_2 is

$$\frac{\partial^2 ls^2}{\partial\theta_1\partial\theta_2} = \sum_{i=0}^{K}2\frac{\partial^2\mu_i}{\partial\theta_1\partial\theta_2} + \sum_{i=0}^{K}\sum_{j=0}^{K}2\frac{\partial^2\Sigma_{i.j}}{\partial\theta_1\partial\theta_2} \tag{8}$$

If the model is linear, $\dfrac{\partial^2 ls^2}{\partial\theta_1\partial\theta_2}$ is zero for $\theta_1 \neq \theta_2$ and constant otherwise, so a single step in Newton's Method (with any starting value, take $\overline{0}$ for simplicity) obtains the least square estimates.

The second fit index we consider is the *Deviance*, also commonly called the *Minus Two Log Likelihood* or $-2LL$. Lange [19] defined the *Deviance* to easily accommodate incomplete data patterns (for instance in pedigree analysis for behavioral genetics research). The *Deviance* is defined by

$$F(\overline{p}) = N \cdot K \cdot \ln 2\pi + \sum_{i=1}^{N} \ln|\Sigma(\overline{p})| + (x^{(i)} - \mu(\overline{p}))^T \Sigma(\overline{p})^{-1}(x^{(i)} - \mu(\overline{p})) \quad (9)$$

where $x^{(i)}$ is the data vector of the ith person. Since $\Sigma(\overline{p})$ is a covariance matrix, it follows that it must be positive definite and consequently has a positive determinant; $F(\overline{p})$ is considered undefined otherwise. Hence, the image of F is in $\mathbb{R}$.

The above definition of the *Minus Two Log Likelihood* easily allows handling of incomplete data by deleting, or filtering, the rows and columns in Σ and μ corresponding to missing data points (cf. [1, 27]). Formally, let $\mathcal{M}_i \subseteq \{1,\ldots,K\}$ denote the incomplete, or missing, values in $x^{(i)}$. Let $\Sigma_{\mathcal{M}_i}$ denote the matrix Σ with all columns and rows in $\mathcal{M}_i$ deleted, and $\mu_{\mathcal{M}_i}$ denote the vector of means with all elements in $\mathcal{M}_i$ deleted. Then,

$$F(\overline{p}) = N \cdot K \cdot \ln 2\pi + \sum_{i=1}^{N} \ln|\Sigma_{\mathcal{M}_i}(\overline{p})| + (x^{(i)} - \mu_{\mathcal{M}_i}(\overline{p}))^T \Sigma_{\mathcal{M}_i}(\overline{p})^{-1}(x^{(i)} - \mu_{\mathcal{M}_i}(\overline{p}))$$
$$(10)$$

Analogous to the ls fit index, we wish to obtain the minimal value of $F(\overline{p})$. We will call the $\overline{p}$ parameter values globally minimizing $F(\overline{p})$ the $-2LL$ estimates.

3.1 Minimization of the Fitting Functions

A convenient minimization method for the $-2LL$ index is Newton's Method applied to the first derivatives of $F(\overline{p})$, finding an extremum of $F(\overline{p})$ by searching for a common zero of the first derivatives. The following propositions are used:

$$\frac{\partial \ln|\Sigma|}{\partial\theta} = \text{trace}\left(\Sigma^{-1}\frac{\partial\Sigma}{\partial\theta}\right)$$

$$\frac{\partial\text{trace}(\Sigma)}{\partial\theta} = \text{trace}\left(\frac{\partial\Sigma}{\partial\theta}\right) \quad (11)$$

$$\frac{\partial\Sigma^{-1}}{\partial\theta} = -\Sigma^{-1}\frac{\partial\Sigma}{\partial\theta}\Sigma^{-1}$$

We then obtain (see also [19])

$$\frac{\partial F}{\partial \theta} = \sum_{i=1}^{N} \text{trace}\left(\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta}\right) - 2\left(\frac{\partial \mu}{\partial \theta}\right)^{T}\Sigma^{-1}(x^{(i)} - \mu)$$

$$-(x^{(i)} - \mu)^{T}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta}\Sigma^{-1}(x^{(i)} - \mu) \tag{12}$$

The second derivatives can be computed in full generality by

$$\frac{\partial^{2} F}{\partial \theta_{1}\partial \theta_{2}} = \sum_{i=1}^{N} \text{trace}\left(-\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{2}}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{1}} + \Sigma^{-1}\frac{\partial^{2}\Sigma}{\partial \theta_{1}\partial \theta_{2}}\right)$$

$$-2\left(\frac{\partial^{2}\mu}{\partial \theta_{1}\partial \theta_{2}}\right)^{T}\Sigma^{-1}(x^{(i)} - \mu) + 2\left(\frac{\partial \mu}{\partial \theta_{1}}\right)^{T}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{2}}\Sigma^{-1}(x^{(i)} - \mu)$$

$$+2\left(\frac{\partial \mu}{\partial \theta_{1}}\right)^{T}\Sigma^{-1}\left(\frac{\partial \mu}{\partial \theta_{2}}\right) + 2\left(\frac{\partial \mu}{\partial \theta_{2}}\right)^{T}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{1}}\Sigma^{-1}(x^{(i)} - \mu)$$

$$+(x^{(i)} - \mu)^{T}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{2}}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{1}}\Sigma^{-1}(x^{(i)} - \mu) \tag{13}$$

$$-(x^{(i)} - \mu)^{T}\Sigma^{-1}\frac{\partial^{2}\Sigma}{\partial \theta_{1}\partial \theta_{2}}\Sigma^{-1}(x^{(i)} - \mu)$$

$$+(x^{(i)} - \mu)^{T}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{1}}\Sigma^{-1}\frac{\partial \Sigma}{\partial \theta_{2}}\Sigma^{-1}(x^{(i)} - \mu)$$

Given that the above formulae are too complex for quick computation, we make use of the following simplifications:

$$\sum_{i=1}^{N} x^{(i)^{T}}\Sigma x^{(i)} = \sum_{i=1}^{N}\sum_{j,k=1}^{K} x_{j}^{(i)}x_{k}^{(i)}\sigma_{j,k} \tag{14}$$

$$= \sum_{j,k=1}^{K}\sigma_{j,k}\left(\sum_{i=1}^{N} x_{j}^{(i)}x_{k}^{(i)}\right)$$

and similarly

$$\sum_{i=1}^{N} x^{(i)^{T}}\Sigma \mu = \sum_{i=1}^{N}\sum_{j,k=1}^{K} x_{j}^{(i)}\mu_{k}\sigma_{j,k} \tag{15}$$

$$= \sum_{j,k=1}^{K}\sigma_{j,k}\mu_{k}\sum_{i=1}^{N} x_{j}^{(i)}$$

By computing the vector of all sums $X = \sum_{i=1}^{N} x^{(i)}$ and all moments $M_{jk} = \sum_{i=1}^{N} x_{j}^{(i)}X_{k}$ in advance, we get rid of the outer sums in all formulae.

In some models such as the LGCM, there are parameters associated to the means and others associated to the variances–covariances. That is, the set of parameters

$\theta_1, \ldots, \theta_P$ decomposes into two disjunct subsets $\theta_1, \ldots, \theta_l$ and $\theta_{l+1}, \ldots, \theta_P$, such that $\mu \in \mathbb{R}[\theta_1, \ldots, \theta_l]^K$ and $\Sigma \in \mathbb{R}[\theta_{l+1}, \ldots, \theta_P]^{K \times K}$. Thus, $\dfrac{\partial \Sigma}{\partial \theta_i}$ is zero if $i \leq l$ and $\dfrac{\partial \mu}{\partial \theta_i}$ is zero if $i > l$.

If we consider linear models, all entries in Σ and μ are linear, and therefore $\dfrac{\partial \Sigma}{\partial \theta_i \partial \theta_j} = \dfrac{\partial \mu}{\partial \theta_i \partial \theta_j} = 0$, regardless of i and j.

Consequently, the terms above simplify to the following five terms:

If $i \leq l$, that is, θ_i is a parameter associated to the means, then

$$\frac{\partial F}{\partial \theta_i} = 2 \left(\frac{\partial \mu}{\partial \theta_i} \right)^T \Sigma (X - K \cdot \mu) \tag{16}$$

for $i > l$

$$A := \Sigma^{-1} \frac{\partial \Sigma}{\partial \theta} \Sigma^{-1}$$

$$\frac{\partial F}{\partial \theta_i} = n \cdot \text{trace} \left(\Sigma^{-1} \frac{\partial \Sigma}{\partial \theta_i} \right) - M \odot A + 2\mu^T A X - n\mu^T A \mu \tag{17}$$

where M denotes the moment matrix above and $\odot$ the sum over component-wise products.

For $i \leq l$ and $j \leq l$, the second derivatives are

$$\frac{\partial^2 F}{\partial \theta_i \partial \theta_j} = 2K \left(\frac{\partial \mu}{\partial \theta_i} \right)^T \Sigma^{-1} \left(\frac{\partial \mu}{\partial \theta_j} \right) \tag{18}$$

for $i \leq l$ and $j > l$

$$\frac{\partial^2 F}{\partial \theta_i \partial \theta_j} = 2 \left(\frac{\partial \mu}{\partial \theta_i} \right)^T \Sigma^{-1} \frac{\partial \Sigma}{\partial \theta_j} \Sigma^{-1} (X - K\mu) \tag{19}$$

and finally for $i > l$ and $j > l$

$$A := \Sigma^{-1} \frac{\partial \Sigma}{\partial \theta_j} \Sigma^{-1}$$

$$B := A \frac{\partial \Sigma}{\partial \theta_i} \Sigma^{-1} + \Sigma^{-1} \frac{\partial \Sigma}{\partial \theta_i} A$$

$$\frac{\partial^2 F}{\partial \theta_i \partial \theta_j} = K \cdot \text{trace} \left(A \frac{\partial \Sigma}{\partial \theta_i} \right) + M \odot B - 2\mu^T B X - n\mu^T B \mu \tag{20}$$

These computations allow us to find the extremum of $F(\overline{p})$ by applying Newton's Method. In a simulation work, as is ours, data are generated by population values given a priori. Thus, it would be possible to take the population values as starting

values for the iteration process. In empirical research situations, however, the population values are not known, and an alternative way of providing starting values must be applied.

One common method is to use the least square estimates as starting values. In this line, the simulation engine first computes the least square estimates and uses them as starting values for the iterative process.

3.2 Inadmissable Estimation Areas

The $-2LL$ fit index is not defined on those parameter vectors p for which $\Sigma_{\bar{p}}$ is not positive definite. We call these areas *inadmissible estimation areas*. A strategy to avoid the estimation algorithm from falling into inadmissible estimation areas is to apply a *penalty function* (cf. [32]), which artificially increases the fit index adopted when the algorithm is approaching inadmissible areas. This will force the estimation algorithm away from inadmissible estimation areas.

Let Σ_k denote the upper $k \times k$ submatrix of Σ. We define the following penalty function *pen* which is added to the $F(\bar{p})$:

$$pen(\Sigma) = \sum_{k=1}^{K} p(\Sigma_k) \ , \quad p(\Sigma_k) = \begin{cases} 0 & |\Sigma_k| \geq 0 \\ e^{-c|\Sigma_k|^2} - 1 & |\Sigma_k| \leq 0 \end{cases} \tag{21}$$

Thus,

$$\frac{\partial pen}{\partial \theta}(\Sigma) = \sum_{k=1}^{N} \frac{\partial p}{\partial \theta}(\Sigma_k) \tag{22}$$

and

$$\frac{\partial p}{\partial \theta}(\Sigma_k) = \begin{cases} 0 & |\Sigma_k| \geq 0 \\ -2c \frac{\partial|\Sigma_k|}{\partial\theta} |\Sigma_k| e^{-c|\Sigma_k|^2} & |\Sigma_k| \leq 0 \end{cases} \tag{23}$$

where

$$\frac{\partial|\Sigma_k|}{\partial \theta} = |\Sigma_k| \text{trace}\left(\Sigma_k^{-1} \frac{\partial \Sigma_k}{\partial \theta} \right) \tag{24}$$

We can hence simplify

$$-2c \frac{\partial|\Sigma_k|}{\partial\theta} |\Sigma_k| e^{-c|\Sigma_k|^2} = -2c|\Sigma_k|^2 \text{trace}\left(\Sigma_k^{-1} \frac{\partial \Sigma_k}{\partial \theta} \right) e^{-c|\Sigma_k|^2} \tag{25}$$

The derivative of *pen* is continuous. The second derivative for non-positive $|\Sigma_k|$ is

$$\frac{\partial^2 p}{\partial\theta_1\partial\theta_2}(\Sigma_k) = |\Sigma_k|\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_2}\right)(-4c)|\Sigma_k|\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_1}\right)e^{-c|\Sigma_k|^2}$$

$$+(-2c)|\Sigma_k|^2\text{trace}\left(-\Sigma_k\frac{\partial\Sigma_k}{\partial\theta_2}\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_1}+\Sigma_k^{-1}\frac{\partial^2\Sigma_k}{\partial\theta_1\partial\theta_2}\right)e^{-c|\Sigma_k|^2}$$

$$+(-2c)|\Sigma_k|^2\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_1}\right)(-2c)|\Sigma_k|^2\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_2}\right)e^{-c|\Sigma_k|^2}$$

$$= (-2c)|\Sigma_k|^2 e^{-c|\Sigma_k|^2}\left((2-2c|\Sigma_k|^2)\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_2}\right)\right.\tag{26}$$

$$\left.\text{trace}\left(\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_1}\right)+\text{trace}\left(-\Sigma_k\frac{\partial\Sigma_k}{\partial\theta_2}\Sigma_k^{-1}\frac{\partial\Sigma_k}{\partial\theta_1}+\Sigma_k^{-1}\frac{\partial^2\Sigma_k}{\partial\theta_1\partial\theta_2}\right)\right)$$

For non-negative $|\Sigma_k|$, the derivative is constant zero. For $|\Sigma_k| = 0$, the derivatives are zero by both definitions, so *pen* is twice differentiable. In the linear model, the first derivative of *pen* is zero for the mean parameters, because *pen* only depends on parameters associated to variances and covariances. Moreover, in the linear model, the second derivative of Sigma is zero for all parameters.

Computationally, the determinants of all Σ_k are computed first. If all of them are above zero, the penalty function and its derivatives are zero. The (relatively complex) computations for the derivatives of the penalty function are only applied otherwise.

However, when the matrix is not positive definite and relatively far away from the boundary of positive definiteness, the penalty function results in high values, which are likely to produce computational difficulties. We therefore do not want to rely solely on a penalty function. Moreover, Newton's Method on the first derivatives of $F(\overline{p})$ only reveals an extreme point, but not necessarily a minimum. To circumvent both problems, we introduce a modification of Newton's Method by extension of the idea of damping (cf. [10]). This modification has so far not been used elsewhere for parameter estimation. We choose a damping factor dependent on the two-dimensional situation along the gradient. We quickly summarize the idea.

In the original Newton's Method, to find a common zero of $f_1, \ldots, f_K : \mathbb{R}^K \to \mathbb{R}$, in each step the *Jacobian* J of $f_1, \ldots, f_K$ defined by

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_K}{\partial p_1} \\ \cdots & \ddots & \cdots \\ \frac{\partial f_1}{\partial p_K} & \cdots & \frac{\partial f_K}{\partial p_K} \end{pmatrix}\tag{27}$$

is computed. If $p^{(i)}$ denotes the position of the ith step, then

$$p^{(i+1)} = p^{(i)} - J(p^{(i)})^{-1}f(p^{(i)})\tag{28}$$

To i mprove the quality of $p^{(i+1)}$, replace this line by

$$p^{(i+1)} = p^{(i)} + \lambda J(p^{(i)})^{-1} f(p^{(i)})$$

where $\lambda \in \mathbb{R}$ is a parameter that can be freely chosen. The original Newton's Method is achieved by setting $\lambda = -1$. We make a successive search for the best λ by the following method:

Let $p_\lambda := p^{(i)} + \lambda J(p^{(i)})^{-1} f(p^{(i)})$ and $f_\lambda = f(p_\lambda)$. Consider three values $\lambda_0 < \lambda_1 < \lambda_2$ initially set to $-1, 0, 1$. By decreasing λ_0, respectively increasing λ_2, we change the three values until f_{λ_1} is the minimum of the three values. Because the direction of the gradient $J(p^{(i)})^{-1} f(p^{(i)})$ points toward an extremum, we expect to find three values with the requested condition quickly. Otherwise, we take the preceding best value of λ corresponding to the lowest f_λ.

When $\lambda_0 < \lambda_1 < \lambda_2$ with $f_{\lambda_1} < f_{\lambda_0}$ and $f_{\lambda_1} < f_{\lambda_2}$ are found, we check whether $f_{\lambda_0} > f_{\lambda_2}$. Let λ_i correspond to the higher of the two, and let $\lambda_4 := \dfrac{\lambda_i + \lambda_1}{2}$ be the mean of λ_i and λ_1. We then check whether f_{λ_4} or f_{λ_1} is smaller and repeat the process with the corresponding λ and its two neighboring λ. These three λs again respect the condition that the middle λ corresponds to the lowest value. We repeat this process until λ_0, λ_1, and λ_3 only differ by a small a priori distance and continue with Newton's Method on p_{λ_1}.

4 General Simulation Procedure

To test certain statistical properties of the LGCM, we will create data points with an LGCM and a given set of known parameters (similarly to [14, 16]). Here we explain how the completed data points were created, how the data points were selected to match substantive questions of interest, and the evaluation criteria we computed to appraise the quality of the statistical methods (in particular how the parameter estimates themselves can be compared to the initial population parameter values).

4.1 Data Generation

Consistent with the LGCM, we specify a priori a variance–covariance matrix $M \in \mathbb{R}^{2V \times 2V}$ of the levels and slopes, vector $\mu \in \mathbb{R}^{2V}$ of the means of the levels and slopes, and an error variance θ uncorrelated with any other variable. From these initial parameters, we generate general level $l_{v,i}$ and slope $s_{v,i}$ scores, for variable v and unit of analysis i, such that the first and second moments of $l_{v,i}$ and $s_{v,i}$ correspond to μ and M, respectively.

To this end, we perform a Cholesky decomposition of the covariance matrix M of the parameters, i.e., we find a lower left triangular matrix C, such that $CC^T = M$. Since M is a covariance matrix, it is positive definite, and thus the Cholesky Decomposition exists. The matrix C can be computed recursively by

$$c_{i,j} = \begin{cases} \dfrac{m_{i,j} - \sum_{k=1}^{i-1} c_{k,i} c_{k,j}}{c_{i,i}} & i < j \\[2ex] \sqrt{m_{i,j} - \sum_{k=1}^{i-1} c_{k,i} c_{k,j}} & i = j \\[2ex] 0 & i > j \end{cases} \tag{29}$$

Let $r \in \mathbb{R}^N$ be N Gaussian-distributed random variables. The levels and slopes can then be obtained by taking

$$(l_{0,0}, \ldots, l_{V-1,N-1}, s_{0,0}, \ldots, s_{V-1,N-1}) = Cr + \mu$$

$l_{v,i}$ and $s_{v,i}$ are then normally distributed with means μ and variance–covariance matrix M, as can be checked

$$\int_r Cr(Cr)^T \omega(r) = \int_r Crr^T C^T \omega(r)$$
$$= C \left(\int_r rr^T \omega(r) \right) C^T$$
$$= CIC^T \tag{30}$$
$$= M$$

where

$$\int_r f(r)\omega(r) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (2\pi)^{\frac{-n}{2}} e^{\frac{-x_1^2}{2} - \cdots - \frac{-x_n^2}{2}} f(r) dr_1 \cdots dr_n$$

as a short notation for integrals over some Gaussian distribution.

The error term $err_{t,v,i}$ can be computed by multiplying a Gaussian-distributed variable by θ, independently for all t, v, and i.

Hence, in the end we generate N level scores $l_{v,i}$, slope scores $s_{v,i}$, and error scores $err_{t,v,i}$, which all correspond to the initial population LGCM parameters M, μ, and θ. These values are finally combined according to the LGCM equation (1) to obtain K final data points for each unit of observation.

4.2 Data Selection

Let $x^{(i)}$ be the data points created as described above given the population variance–covariance matrix Σ, the population mean vector μ, and the population error variance–covariance matrix θ. At this point we may apply the estimation procedures explained above to obtain the LGCM parameter values from the observed data $x^{(i)}$. Alternatively, if we are interested in specific statistical properties of the LGCM, we

may *select* some of the data from $x^{(i)}$. For instance, we may test the robustness of LGCM to incomplete data under certain conditions (e.g., [28, 29, 12]).

One dimension of interest in our simulation is the time interval spanned by the data points T. Large scale longitudinal studies, or panels, are very expensive and laborious, and consequently typically last less than a decade (although notable exceptions exist). Of interest to many applied researchers is the necessary duration of a longitudinal study in order to detect reliable variance in change (cf. [15]). In the context of LGCM, this question translates into the number of longitudinal data points necessary to detect reliable variance in the slope scores $s_{v,k}$. Other LGCM parameters defined in Σ, μ, or θ can of course be examined.

To select the data to be examined, we perform a selection operation on the complete data set $x^{(i)}$ of each unit of observation i. Let A be the full data set of all data points for each unit of observation. For each selection condition, we select (randomly or according to some predefined criterion, such as the number of longitudinal measurements) a subset $J \subseteq A$ of the observed data points. We then restrict the data vector $x^{(i)}$ of each unit of observation, the matrix Σ, the vector μ, and the matrix θ to the rows and columns corresponding to the indices in J (for each variable v).

This selection operation allows testing the robustness of LGCM to incomplete data. The LGCM is then applied to the resulting subset of data J and the overall fit of the model (*ls* or *Deviance*) and the resulting parameter estimates $\overline{p}$ are evaluated in light of prespecified criteria.

4.3 Evaluation Criteria

We specified several evaluation criteria based on the statistical distribution of the χ^2 statistic and its degrees of freedom. For each LGCM computed on every generated data set, we obtain the parameters and evaluation criteria summarized in Table 1.

Table 1 Parameters and evaluation criteria

1	Least square estimates
2	$-2LL$ estimates
1	$-2LL$ value for the estimates
2	$-2LL$ value for the saturated model
3	$-2LL$ value for the population parameters
4	$-2LL$ value for the independent model with free means
5	$-2LL$ value for the independent model with averaged means
6	$-2LL$ value for the parameters of the free model
7	SRMR for the variance–covariance matrix
8	SRMR for the mean vector

Here, we have (a) the following estimates:

1. The estimated parameters for the minimal Least Square index. This includes all elements of Σ and μ estimated with the *ls* procedure specified above.

2. The estimated parameters for the minimal $-2LL$ index. This includes all elements of Σ and μ estimated with the $-2LL$ procedure specified above.

and (b) the following fit indices:

1. The $-2LL$ value of the $-2LL$ estimates. This is the actual value of the $-2LL$ fit index corresponding to the parameter estimates obtained with the $-2LL$ procedure (cf. the estimates in point 2 above). In this application, the model is a LGCM with all parameters freely estimated.
2. The $-2LL$ value for the saturated model. This is the $-2LL$ value for the model estimating one parameter for each unknown, that is, one parameter for each element in Σ and in μ. This model is the least parsimonious and yields the best fit to the data. Indeed, to obtain this $-2LL$ we substituted Σ with S and μ with m in the equation for $-2LL$.
3. The $-2LL$ value for population parameters. This corresponds to the $-2LL$ fit index when the parameters are not estimated, but fixed at the known population values, with which the data were generated in the first place. In this application, the model is a LGCM with all parameters fixed (hence not estimated) to the initial population values.
4. The $-2LL$ value for the independent model with free means. This is a common baseline comparison model in the structural equation modeling literature (e.g., [7]). This $-2LL$ value is the value obtained when the independence model, rather than the LGCM, is compared to the observed data. In the independence model, all longitudinal measures are posited independent of each other, but with possibly different variance values. The model expectation variance–covariance matrix S is hence a diagonal matrix, where all covariances are equal to zero. To separate the effects due to the variance–covariance matrix Σ from those of the mean vector μ, this first independence model estimates all mean values separately, so that m counts $(V - 1) \cdot (T - 1)$ parameters.
5. The $-2LL$ value for the independent model with averaged means. This fit index is equivalent to the previous, except that it is also restrictive on the mean structure. Here only $(V - 1)$ parameters are estimated for the means, that is, only one mean value for each variable v. Hence this model posits independence among the longitudinal measurements and no average longitudinal change.
6. The $-2LL$ value of the free model. To reach statistical conclusions about a specific LGCM parameter (e.g., the correlation between two variables' slope scores), the initial LGCM model, whose $-2LL$ fit is (1), will be constrained with a value of 0 for that specific parameter. The statistical inference about that parameter can be based on the $-2LL$ contribution due to that parameter, which is the difference between the $-2LL$ in (1) and the $-2LL$ of the free model, in which the parameter is freely estimated rather than constrained at 0.
7. The Standardized Root Mean Residual (SRMR) for the variance–covariance matrix. This fit index is similar to the squared Euclidean distance between the standardized data variance–covariance matrix S and the model expected

variance–covariance matrix Σ with the parameter values $\overline{p}$. This SRMR is defined by

$$SRMR_{\text{cov}}(\overline{p}) = \sum_{i=0}^{K} \sum_{j=0}^{K} \left(\frac{\Sigma(\overline{p})_{ij}}{\sqrt{\Sigma(\overline{p})_{ii} \, \Sigma(\overline{p})_{jj}}} - \frac{S_{ij}}{\sqrt{S_{ii} S_{jj}}} \right)^2 \tag{31}$$

8. The Standardized Root Mean Residual (SRMR) for the mean vector. This SRMR ignores all variance–covariance information and assesses the squared Euclidean distance between the mean vectors μ and m:

$$SRMR_{\text{mean}}(\overline{p}) = \sum_{i=0}^{K} (\mu(\overline{p})_i - m_i)^2 \tag{32}$$

All models described above are statistically nested in the free model, meaning that the parameters estimated by each model are a subset of the parameters of the free model. The free model will always obtain a Deviance, which is smaller or equal to the any other model, because it describes the overall data structure better or as equally well as any other model. The $-2LL$ difference between any other model and the free model is distributed as a χ^2 statistic with as many degrees of freedom as the difference between the number of parameters estimated by the two models, because they are statistically nested. Differences of $-2LL$ statistics can be re-expressed as Comparative Root Mean Square Error of Approximation (cf. [8]), defined as

$$CRMSEA = \sqrt{\frac{\max[(\frac{\Delta\chi^2 - \Delta df}{N-1}), 0]}{\Delta df}} \tag{33}$$

where $\Delta\chi^2$ corresponds to the difference in χ^2 values and Δdf to the difference in degrees of freedom (df) between the two nested models.

4.4 Summarizing the Simulation Procedure

The total simulation procedure is illustrated in Fig. 1. First, the population values and the model for data creation are used to generate the data set as described in Sect. 4.1. Possibly, specific data points are selected as described in Sect. 4.2, and the model is restricted by constraining specific parameters of focus. The restricted model is then applied to the selected data set to minimize the least squares or the $-2LL$ index and to obtain the estimated parameters. The estimated parameters may then be compared to the original parameters (i.e., the population values) or by certain evaluation criteria to finally ascertain their quality, as described in Sect. 4.3.
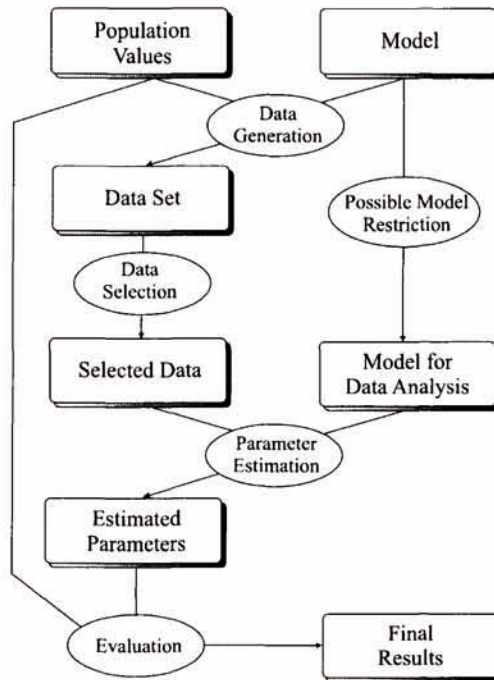
Fig. 1 Representation of simulation strategy

5 An Illustration

In this section, we present an illustration of our engine for data generation plus least squares and full information maximum likelihood estimation to test statistical properties of LGCMs. We will limit the analyses to two LGCM parameters. A more extensive analysis of LGCM statistical properties is provided in [14, 16].

In this application, we were particularly interested in testing the power of LGCMs in estimating variances and covariances of the slope components. Much recent work in our main research field, cognitive aging, has focused on interindividual differences in change, or differential change, and relationships of change over time. General salient questions in adult cognitive development concern whether aging individuals change similarly or display subgroups according to their developmental patterns (e.g., the successful aging paradigm proposed by [35, 3]) and whether changes in one domain, such as cognitive abilities, are related to changes in other domains, such as sensory functions (e.g., [9]).

5.1 Population Parameters

Based on existing research examples (cf. [14, 16]), we generated data on $V = 2$ variables for $N = 200$ and $N = 500$ units of analysis over $T = 20$ time points. We

chose the following variance–covariance matrix M and mean vector μ for levels and slopes (in the order level of variable one, level of variable two, slope of variable one, and slope of variable two):

$$M = \begin{pmatrix} 100 & & & \\ 50 & 100 & & \\ X & 0 & 50 & \\ 0 & X & 25 & 50 \end{pmatrix} \quad \mu = \begin{pmatrix} 50 \\ 50 \\ -20 \\ -20 \end{pmatrix} \quad (34)$$

In this example, we were interested in the effect of the within-variable level-slope covariance X on the power to detect the covariance between the two slopes. To this end, we varied X from -21.21 to 0.00 to 21.21, which corresponds to varying the within-variable level-slope correlation from -0.3 to 0.0 and 0.3, respectively. The error variance of both variables was set to 10, which is equivalent to an initial reliability of 0.91 for both variables. This corresponds to the reliability of good cognitive tests (e.g., [23]).

5.2 Data Selection

Two hundred replicate data sets per combination of population parameters are generated and successively altered and analyzed. The data selection concerned the time dimension, or length of the individual time series, only. The full data sets generated in the first step simulate cognitive aging studies lasting $T = 20$ epochs (e.g., weeks, months, or, relevant to the choice of the population parameters, years). In empirical terms, it is very laborious and problematic for several reasons to repeatedly measure a sample for 20 years. One practical question for researchers is then: For how many years, or on how many occasions, do I need to measure my participants to detect reliable variance in change and reliable covariance in change with LGCM analyses (as a function of sample size, real variance and covariance in change, and variables' reliabilities)?

Hence we selected data according to the time dimension, by retaining 3 ($t = 0, 2, 4$), 4 ($t = 0, 2, 4, 6$), 5 ($t = 0, 2, 4, 6, 8$), 6 ($t = 0, 2, 4, 6, 8, 10$), 10 ($t = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18$), and all 20 (from 0 to 19, by increments of 1) repeated measures. This condition will allow comparing the power to detect reliable variance in change and covariance in change as a function of study length, sample size, and the effect size of within-variable level-slope correlations (X in (34)), accounting for variables' reliabilities and the effect sizes of the variances and means of levels and slopes.

5.3 Parameters of Focus

In this illustration, we focus on the two variances of slope scores $Variance(s_{1,k})$ and $Variance(s_{2,k})$ and the covariance between them, $Covariance(s_{1,k}; s_{2,k})$. The three parameters are boldfaced in the matrix representation of M in (34). During the parameter estimation procedure, we will hence solve three models: (M1) the LGCM with all parameters freely estimated; (M2) an LGCM with the covariance of slope scores fixed at 0, hence not estimated; and (M3) an LGCM with both variances of slope scores and their dependent covariances fixed at 0. Model (M1) estimates the total number of parameters of a bivariate LGCM, that is 16, model (M2) estimates 15 parameters, and model (M3) estimates 9 parameters.

Besides the evaluation of the three models by means of the fit indices described in 4.3, relative model comparisons are possible. Models (M2) and (M3) are statistically nested within (M1) and model (M3) is statistically nested in (M2), so that statistical pairwise model comparisons are justified.

5.4 Definition of Power

The main dependent variable of our illustration concerns the power of LGCMs to correctly reject the null hypothesis that the parameters of focus are equal to 0 (when their analogous population parameters are different from 0). In this illustration, power of the parameters of focus is defined for all combinations of population parameters, because $Variance(s_{1,k}) = Variance(s_{2,k}) = 50$ and $Covariance(s_{1,k}; s_{2,k}) = 25$ in all combinations of population parameters.

To define power in our simulation, we computed for all 200 replicates of each combination of population parameters two statistical comparisons: We compared models (M1) and (M2) to calculate the loss in $-2LL$ fit due to not estimating the $Covariance(s_{1,k}; s_{2,k})$ and models (M1) and (M3) for the loss in fit attributable to the $Variance(s_{1,k})$ and $Variance(s_{2,k})$ and all dependent covariances (which are not defined when their relative variances are zero). The two differences in fit are distributed as a χ^2 with 1 and 7 degree(s) of freedom, respectively. A significant model comparison is obtained when the comparison χ^2 value is greater, at an alpha level of 0.05, than 3.84 and 14.07, respectively. Power is then defined as the ratio of significant model comparisons out of the total 200 for each combination of population parameters.

5.5 Results

The power estimates for detecting the $Variance(s_{1,k})$ and $Variance(s_{2,k})$ are plotted in Fig. 2, and those for $Covariance(s_{1,k}; s_{2,k})$ in Fig. 3.

In general, the within-variable level-slope correlation seems to affect power to detect the variance of slopes only when three occasions are retained with $N = 200$
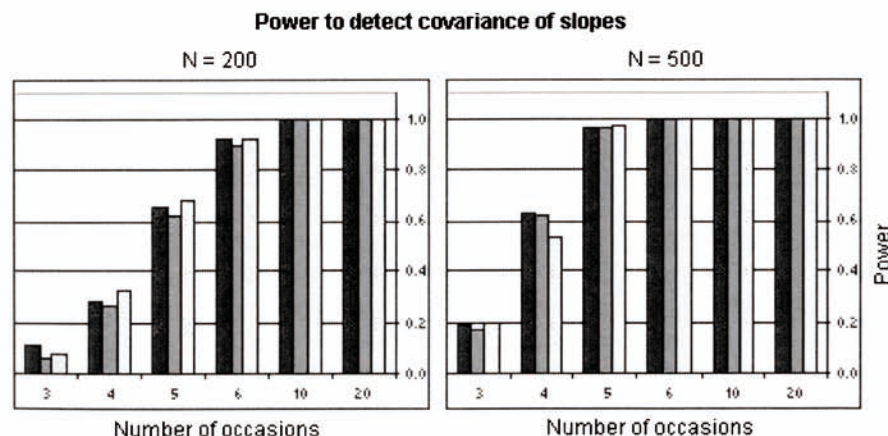
Power to detect variance of slopes
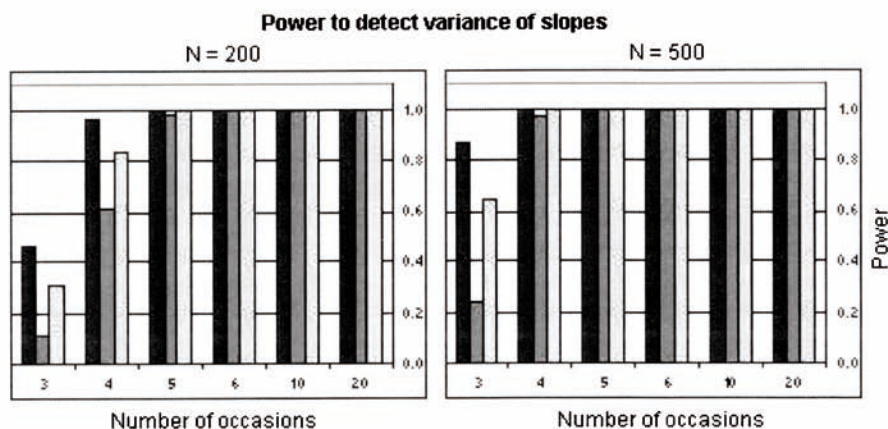
Fig. 2 Power to detect variances of slopes as a function of within-variable level-slope correlation (*left, dark bar r* $= -0.30$, *middle, grey bar r* $= 0$, *right, white bar r* $= 0.30$) and occasions retained

or $N = 500$ and with four occasions with $N = 200$. When five or more occasions are retained, power is very high with both sample sizes across the three values of the within-variable level-slope correlation.

Power to detect the covariance of slopes is acceptable with six occasions or more when $N = 200$ and five or more occasions with $N = 500$, independently of the within-variable level-slope correlation.

To test formally the effects observed with the barplots, we tested the differences between the analogous power columns of both sample sizes for significance. The event to successfully reject the null hypothesis is binary distributed with the power

Power to detect covariance of slopes

Fig. 3 Power to detect covariance of slopes as a function of within-variable level-slope correlation (*left, dark bar r* $= -0.30$, *middle, grey bar r* $= 0$, *right, white bar r* $= 0.30$) and occasions retained

as probability for rejection. Thus, the number of rejections in 200 replicates is bino-
mially distributed with the power equal to the probability. Two cells are therefore
significantly different to an α level if the probability of being drawn from the same
binomial distribution is less than α.

This analysis confirmed our assumptions that occasions strongly affect the power
to detect both variables' slope variance and the across-variable slope covariance. All
comparisons between two analogous cells of the same parameter but differing with
respect to occasions were significantly different ($p < 0.001$), unless the power was
maximum (i.e., 100%). Likewise, the effect of sample size was equally significant
($p < 0.001$) for all analogous cells differing only with respect to sample size, unless
power was 100%.

The within-variable level-slope covariance had no significant effects on the
power to detect the across-variable covariance of slopes at an $\alpha = 0.01$ level.
Yet, the within-variable level-slope covariance had a significant effect on the power
to detect the two slope variances. For the power to detect variance in change, all
cells different only in level-slope covariance were highly significantly different
($\alpha < 0.001$, unless power was 100%).

In sum, the simulation showed that detection power of variance is higher with a
positive within-variable correlation between level and slope, and even higher when
this correlation is negative. The power of across-variable covariance of change, on
the other hand, does not appear to be significantly affected by the within-variable
level-slope correlations.

6 Discussion and Outlook

In this chapter, we presented a simulation procedure for testing statistical properties
of Latent Growth Curve Models (LGCM). In the application, we applied the pro-
cedure to study the power of LGCMs to detect variances and covariances of linear
change. The simulation engine (http://www.mpib-berlin.mpg.de/en/
forschung/computerscience/.) produced data according to a linear LGCM
with different parameters, then selected those data sets, and finally analyzed them
under different parameter restrictions to compute nested model comparisons focused
around parameters of interest (variances in and covariance of change).

To estimate the LGCM parameters for each generated data set, we provided some
technically equivalent transformations of the derivatives of the Minus Two Log
Likelihood index, which allowed us quickly finding minimal points by a variant
of Newton's Method. In this manner, we were able to avoid areas of the parameter
space that are inadmissible for covariance matrices and to separate minima from
maxima.

In the illustration, we showed that the power to detect variances of change in a
LGCM is dependent on the within-variable level-slope covariance, while the power
to detect across-variable covariance of changes in a LGCM apparently is not. A
possible explanation of this effect can be found in detail in [16].

In short, there is more than one possible statistical test for the variance in change. One may nest a model with both variances in change constrained to be zero within an unrestricted model. This will lead to a 2-degree-of-freedom chi square comparison. Alternatively, one may also compare a model with both variances in change fixed to zero as well as all related covariances, which corresponds to a 7-degree-of-freedom chi square comparison (2 variances and 7 covariances). In [16], we showed that while the latter method is superior when the real covariances associated to the change factors in the population are zero, the former is superior when those covariances are different from zero. Because in the present illustration we applied the former method, the resulting power to detect variances in change increased with higher within-variable level-slope covariance.

In future work, we intend to elaborate our research of simulation methods and expand the simulation engine to cope with the incongruence due to creating data with one LGCM specification and subsequently analyzing those data with a different LGCM specification. Also, the effects of more complex data selection strategies will be addressed.

LGCMs have become a prominent method of data analysis in much psychological researchs. These models are appealing because they (a) allow disentangling level from change in information; (b) allow specifying a wide variety of pre-defined change patterns (e.g., polynomial, exponential, and Gompertz) or estimating the change pattern empirically from the data analyzed; (c) allow analyzing all data available, even in the presence of incomplete data, as long as the missing at random assumption is met; and most importantly (d) have contributed significantly to the advancement of theoretical knowledge about the cognitive aging literature.

The study of statistical behaviors of LGCMs is however still a very active research field. Although much cognitive aging literature focuses on change parameters, especially variance and covariance, the field as a whole still does not know the limits of LGCMs. We showed that even under ideal and empirically unrealistic assumptions about the data (e.g., group homogeneity with respect to the change phenomenon examined, nonexistent longitudinal dropout, and correct a priori specification of the change function) certain LGCM parameters of chief substantive importance are estimated with low to very low power.

Simulation studies such as this allow us furthering our knowledge about the limits and tenability of LGCMs under given research situations. We believe that much more research is needed to persuade LGCM users not to rest on substantive findings, which might be invalid because of inherent LGCM lack of power under specific conditions, most of which still in need of being discovered.

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