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ALGEBRAIC DECOMPOSITION OF INDIVIDUAL CHOICE BEHAVIOR

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Abstract

In this study a general framework for algebraic decomposition models of individual decision behavior is proposed. A specific decomposition that is based on the sequence of intransitive choices is investigated in classical domains of binary decision tasks.

In the theoretical part a general algebraic decomposition is applied to individual choice behavior. It is based on directed cycles of length k, the graph-theoretical analogue of intransitive choice. Two major results are obtained by applying techniques from graph theory and algebra to pair comparisons.

First, internal consistency of choice behavior can be characterized by a polynomial expression that provides detailed information about the number and size of strong components together with the number of k-dicycles in each component. The attempt, however, to identify a minimal set of critical choices which can explain all intransitive choices leads to a well known optimization problem.

Second, a specific decomposition model, the ear decomposition by sequence, is suggested which is based on the sequence of intransitive choices in a pair comparison. The ear decomposition determines a unique and minimal collection of dicycles that can generate all intransitive choices. In an additional technique, that makes slightly stronger assumptions, intransitive subchains are defined on the sequence of choices. It is shown that under an appropriate closure operation these subchains form a lattice structure.

In the experimental part it is investigated whether or not the algebraic decomposition of choice behavior provides a suitable alternative to traditional algebraic and probabilistic models. In three domains inconsistent choice behavior is studied in block designs with differently optimized arrangements of choice-trials. Experiments in the domain of riskless and risky choice show that subjects interact with the sequence of choice-trials in different block designs. This supports the ear decomposition by sequence rather than the general assumption of independent choices that underlies classical decision models. Experiments on the visual discrimination of contrast do not provide such evidence and inconsistency of choice varies unsystematically.

Algebraic decompositions of choice behavior are considered as an important step toward a qualitative theory of error on ordinal scale and further experimental and theoretical developments are encouraged.

Zusammenfassung

In der vorliegenden Arbeit wird ein Rahmenkonzept für die algebraische Zerlegung individuellen Entscheidungsverhaltens vorgestellt. Eine spezifische Zerlegung, die auf der Reihenfolge von intransitiven Wahlen beruht, wird in klassischen Alternativenbereichen binärer Entscheidungsaufgaben untersucht.

Im theoretischen Teil wird eine generelle algebraische Zerlegung auf individuelles Wahlverhalten angewandt. Sie beruht auf gerichteten Zyklen der Länge k, dem graphentheoretischen Analog intransitiven Wählens. Die Anwendung von Techniken aus der Graphentheorie und Algebra auf Paarvergleiche hat zwei grundlegende Ergebnisse.

Zum einen kann die interene Konsistenz des Wahlverhaltens durch einen polynomischen Ausdruck charakterisiert werden, der detailierte Information über die Anzahl und Größe streng zusammenhängender Komponenten, sowie die Anzahl gerichteter Zyklen in jeder Komponente bereithält. Allerdings führt der Versuch, eine minimale Menge von kritischen Wahlen zu identifizieren, die sämtliche intransitiven Wahlen erklären kann, zu einem bekannten Optimierungsproblem.

Zum anderen wird ein spezifisches Zerlegungsmodell, die Ear-Zerlegung nach Reihenfolge, vorgestellt, die auf der Reihenfolge von intransitiven Wahlen im Paarvergleich beruht. Mit Hilfe der Ear-Zerlegung kann eine eindeutig bestimmte minimale Menge gerichteter Zyklen ermittelt werden, die sämtliche intransitiven Wahlen generieren. In einer weiteren Technik, die auf etwas strengeren Annahmen beruht, werden intransitive Teilketten auf der Reihenfolge der Wahlen definiert. Es wird gezeigt, daß diese Teilketten unter geeigneter Abschlußoperation eine Verbandsstruktur bilden.

Im experimentellen Teil wird überprüft, ob die algebraische Zerlegung von Entscheidungsverhalten eine Alternative zu klassischen algebraischen und probabilistischen Modellen darstellt. In drei Alternativenbereichen wird anhand von optimierten Anordnungen der Wahldurchgänge in unterschiedlichen Blockdesigns inkonsistentes Entscheidungsverhalten untersucht. Experimente zu Entscheidungen mit und ohne Risiko zeigen, daß die Versuchspersonen mit der Anordnung der Wahldurchgänge in unterschiedlichen Blockdesigns interagieren. Ein Befund, der für die Ear-Zerlegung nach Reihenfolge spricht und die für klassische Entscheidungsmodelle grundlegende Annahme von unabhängigen Wahldurchgängen widerlegt. Experimente mit Diskriminierungsaufgaben zur visuellen Kontrastwahrnehmung zeigen keinen vergleichbaren Befund und inkonsistentes Entscheidungsverhalten variiert unsystematisch.

Die algebraische Zerlegung von Entscheidungsverhalten wird als ein wichtiger Schritt in Richtung einer qualitativen Fehlertheorie auf ordinalem Meßniveau betrachtet und mögliche experimentelle und theoretische Entwicklungen werden angeregt.

Chapter 1

Introduction

In this study a general framework for algebraic decomposition models of individual choice behavior is developed. It is argued that algebraic decompositions provide an alternative to classical deterministic and probabilistic models of choice, especially when the sample is small and choice-trials are not repeated.

The thesis consists of four chapters and three appendices which are organized as follows. In the first chapter rationality and probability of choice are discussed. Both concepts have been essential to models of human decision-making and provide a normative framework. Classical models of choice behavior are reviewed that are exposed to violations of rational choice principles. No attempt is made to provide a complete survey of decision-making theories. Only a few models from different approaches are highlighted with a preference for axiomatized models. Finally, algebraic decomposition techniques are outlined which have been concerned with substructures in binary data. In the second chapter theoretical considerations lead to a characterization of choice behavior in terms of intransitive choices, that is directed cycles in a directed graph. This characterization includes a general algebraic decomposition of a preference matrix into irreducible components. In the following a more specific model is established. The ear decomposition by sequence employs the sequence of choice-trials in a pair comparison which leads to a unique algebraic decomposition of choice behavior. In the third chapter it is experimentally investigated whether choice behavior in pair comparisons can be modeled by algebraic decompositions such as the ear decomposition by sequence. In six experiments inconsistency of choice behavior is tested under different block designs and in three different domains: decision-making under certainty, decision-making under uncertainty or risk, and decisions in psychophysical discrimination tasks. In the fourth chapter conclusions are drawn from the theoretical and experimental results and possibilities for further research is discussed.

At this point a remark on the presentation of the first two chapters appears necessary. The survey of decision models in the first chapter is far from being complete. It was attempted to present concepts and models which have been influential on decision making research or which can be linked to algebraic decompositions. In both chapters it was tried to provide a comprehensible account of theoretical ideas whilst avoiding a mixture of intuitive ideas with mathematical terms. Therefore, both forms of presentation were only given whenever this seemed advisable and straightforward. Progressing in this way has the disadvantage that formal definitions and expressions may disturb the flow of reading and that Appendix A needs to be addressed which provides short accounts of the mathematical background.

On the other hand, all theoretical ideas are given precisely. In addition, redundancy is avoided because each theoretical idea is defined only once and can be referred to throughout the text. A subject and author index is provided at the end. To some extent such a presentation of ideas might even improve the comprehensibility of this dissertation, especially when more technical sections are skipped at first reading.

1.1 Rationality and Probability of Choice

Describing human decision behavior in terms of deterministic models has a longer tradition than psychology as an empirical science and dates back to the eighteenth century. For example, the mathematician Daniel Bernoulli (1738) laid down the foundations of expected utility, the majority rule traces back to Condorcet (1785), who applied this rule in the context of social choice, and the weighted sets of differences rule was already used by Benjamin Franklin (1772) in the context of individual choice. Since then researchers in disciplines such as mathematics, economics, philosophy, and psychology have been involved in the development of decision models.

In general, models of individual choice can be divided into deterministic and probabilistic models depending on their particular use of mathematical concepts.[1] A further classification which is based on the alternatives was proposed by Edwards (1954c). He distinguished between risky choice in which

[1] A deterministic model may be regarded as a probabilistic one by assigning a probability of 0 to non-preference, $\frac{1}{2}$ to indifference, and 1 to preference but this does not change the qualitative nature of the deterministic model.

decisions are made under risk or uncertainty about the outcomes, and riskless choice in which decisions are made under certainty.[2] Additionally, one may distinguish between models for single- and multiattribute alternatives and between models for single-stage and multi-stage decision processes. Models of the latter type are omitted here (see however Keeney & Raiffa, 1976; 1993).

Before a brief survey of models from some of these categories is given, two important aspects of decision theory are introduced: rationality and probability. A rather simplistic view of rationality and probability is adopted here for there is no universal definition of rationality or probability, especially not in the context of human decision-making. Nevertheless, both concepts play an important role in deterministic and probabilistic models of choice behavior.

In general, choice may be described as the specification of a nonempty subset X of objects from a set S, with finite or infinite cardinality, where every possible subset of the set denotes a possible choice. Consequently, the complete choice behavior of an individual decision-maker is only captured if all chosen subsets of alternatives from all possible subsets of S are collected. This is clearly impracticable and a number of suitable restrictions are normally imposed. Then of course, choice behavior is reduced to a subcollection of choices. Choosing a single object out of a set of objects is more common than selecting two or more alternatives which might be considered as repeated choice of single alternatives from the same set.[3] In fact, binary choice or choosing a single object out of two in a forced choice pair comparison is considered as a valid way of assessing individual choice behavior. In the usual incomplete forced choice pair comparison procedure a person is forced to make a choice between *unordered* pairs of objects. Clearly, under this paradigm choice behavior satisfies *connectedness* and *asymmetry*, for arbitrary alternatives $a, b \in S$ satisfy either $a \succ b$ or $b \succ a$, where $\succ$ denotes the preference relation (see Definition A.2.1). The preference relation is often said to be a subset of the cartesian product of the set of alternatives, written as $\succ \subseteq S \times S$. Hence, in the case of a complete pair comparison the preference relation is a connected subset of the cartesian product whereas in the case of an incomplete forced choice pair comparison binary preferences are restricted to an asymmetric and connected subset of the cartesian

[2] In the following risky choice means choice with known probabilities whereas choice under uncertainty also includes ambiguous or unknown probabilities.

[3] From this point of view ranking of alternatives is another special case because it is equivalent to choosing single objects from a given set of objects without replacement.

product. This is easily overlooked because it is convenient to assume that choice behavior is at least asymmetric and therefore unaffected by this standard paradigm. The restriction to asymmetric and connected relations in an incomplete forced choice pair comparison gives a trivial example of how the experimental paradigm affects choice behavior since preference reversals and missing preferences are simply excluded. A non-trivial interaction between the pair comparison paradigm and choice behavior is investigated in Chapter 3.

Despite their presupposed properties, incomplete forced choice pair comparisons[4] have been used in numerous empirical investigations and are closely related to the axiomatization of preference by binary relations. By imposing reasonable constraints or axioms on a preference relation an ordered structure can be defined as described in the next section and Appendix A.2. When defining an ordered structure it is important whether the set of alternatives S is considered as a set of finite or infinite cardinality, and whether the objects in the set are represented as single- or as multiattribute alternatives.

Rational choice behavior has been interpreted as optimal choice, emphasizing the economic origin of decision research. Optimal choice maximizes the outcome of one or more choices. This interpretation requires an interval scale where different outcomes can be compared quantitatively. Such a scale is the utility scale that was originally defined in terms of money but became a more abstract construct since the days of Bernoulli (1738). The interval scale requires stronger assumptions than the more fundamental ordinal scale which only preserves the order of objects (Stevens, 1946; 1951). Consequently, the assumptions for the representation on ordinal scale are of primary interest and rational choice behavior is discussed in Chapter 2 and 3 exclusively in terms of ordinal preference relations.

Transitivity of choice is a necessary condition for the existence of an ordinal utility scale. Together with connectedness, transitivity is also a sufficient condition for the existence of such a scale, provided the number of alternatives is finite or countable. Since transitivity is so fundamental for the ordinal representation of choice behavior it is considered as a cornerstone of normative and descriptive decision theories (Edwards, 1954c; Luce, 1990). Indeed, persons behave irrationally if they violate transitivity in their choice behavior and the following example is commonly cited: Assume a person chooses intransitively by preferring alternative a over alternative b, b over c, and c over a. Then, it can be argued that this person is willing to pay a certain

[4] In the following 'pair comparison' always refers to an incomplete forced choice comparison.

amount of money to exchange a for b, another amount to exchange c for a, and a third amount to exchange b for c. Thereby, the person loses money but ends up with the same alternative. Obviously, the circular argument $a \succ b$, $b \succ c$, and $c \succ a$ can be used recursively and may be extended to more than three alternatives.

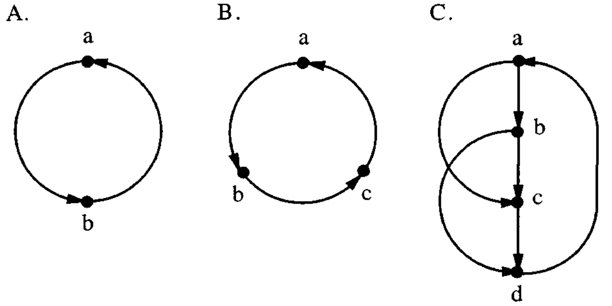

Figure 1.1: Examples of inconsistent choices: A. Preference reversal; B. Intransitive triple; C. Intransitive preferences between three and four objects.

On the other hand, if we think of choices in terms of preference and indifference, written as $a \succeq b$, $b \succeq c$, and $c \succeq a$, the phenomenon of intransitive choices may be interpreted as indifference between alternatives. This can be modeled by an equivalence relation between indifferent alternatives, written as $a \sim b$, $b \sim c$, and $c \sim a$ (Definition A.2.3). However, it has been shown that indifference is not necessarily transitive, and therefore not an equivalence relation. The economist Armstrong (1939) was one of the first to argue that indifference is not necessarily transitive. He discussed the following example: Suppose a boy is indifferent about receiving as a gift a pony or a bicycle. He will undoubtedly prefer the bicycle with a bell attached to the bicycle without a bell. But he is still likely to be indifferent about the bicycle with bell and the pony. Intransitive indifference in the form of limited discriminatory ability of the decision-maker has been incorporated into deterministic and probabilistic models.

Another axiom that has been associated with rational choice and that has implications for the representation on ordinal scale is asymmetry. If somebody prefers a over b, he or she should not prefer b over a. Again, the phenomenon of preference reversals might be considered as transitive or intransitive indifference, but only a few preference reversals between alternatives are believed to be non-systematic caused by indifference or limited discriminatory ability between multiattribute alternatives (e.g., Tversky, 1969). As mentioned before, in an incomplete forced choice pair comparison preference reversals are ruled out by the experimental design unless the pair comparison itself is repeated.

Preference reversals and intransitivity of preference may be considered as violations of rational choice behavior especially when indifference between alternatives can be excluded as a possible explanation. The phenomena are related and have a common feature in the language of graph theory. Preference reversals and intransitivity can be described as directed cycles connecting two, three, or more objects in a digraph (see Chapter 2 and Appendix A.2). It is suggested here that irrational choice is conveniently described as directed k-cycles, including directed 2-cycles, rather than a violation of stronger principles that can be derived from axiom systems of expected utility theory and related theories. The graph-theoretical definition of intransitive preference is exploited in the next two chapters. Intransitive preferences which form directed cycles between more than two alternatives are investigated theoretically in Chapter 2 and empirically in Chapter 3. In the following the violation of asymmetry and transitivity in terms of preference reversals and intransitive preferences is considered as irrational choice behavior. To acknowledge this simplistic view of rationality the general term *inconsistent choice* or *inconsistency* instead of irrationality is used throughout the text and refers to preference cycles of any length. Figure 1.1 gives a graphical representation of inconsistencies. Graph A on the left-hand side shows a preference reversal (2-dicycle), an intransitive triple (3-dicycle) is depicted in Graph B and a preference cycle of length 4 (4-dicycle) appears in Graph C on the right-hand side. It is argued that longer preference cycles are less convincingly interpreted in terms of intransitive indifference because all alternatives that belong to a directed cycle have to be indifferent. This means as soon as a single preference connects the least with the most preferred alternative all previously ordered alternatives become indifferent. To explain $d \succeq a$ in Graph C, for example, would require that $a \sim b \sim c \sim d$. On the contrary, it seems more plausible to assume that the preference order between a, b, c, and d is valid and that the choice $d \succ a$ marks a new decision strategy. The problem of critical arcs is addressed Section 2.2.2.

The concept of probability plays a crucial role in modeling riskless as well as risky or uncertain choice behavior. It offers a possibility to model violations of rational principles such as asymmetry and transitivity. It also provides theoretical tools for modeling risky or uncertain outcomes. When stating a probabilistic choice model, it is essential how and where the concept of probability is incorporated. In models of riskless choice, for example, the relative frequency of repeated choices between the same pair of alternatives is modeled by functions of constant or random utilities, whereas in models of risky choice the (subjective) probabilities of the outcomes are taken into account to model a single choice between alternatives with uncertain outcomes.

Both types of models point to different underlying concepts of probability which have been labeled as the frequentist and Bayesian approach. According to the frequentist approach probability of an event means that if an experiment is repeated over and over again the Laplace probability[5] tends to be the probability. According to the Bayesian approach, probability of an event is a quantification of the speaker's uncertainty about the outcome of the experiment and thus has a personal or subjective notion; the probability for a certain event may be different for different speakers, depending on their experience and knowledge of the situation. Ramsey (1931) and de Finetti (1937) laid the groundwork for a subjective theory of probability. There has been vigorous debates among proponents of various versions of these points of view (for a collection of papers on this important topic see Kyburg & Smokler, 1980) Without entering this controversy we give the following general definition of a probability measure on a countably additive probability space which is widely accepted. It was first correctly stated by Kolmogorow (1933).

Definition 1.1.1 (Probability Measure) *A* probability measure *on the probability space* $\Omega \neq \emptyset$ *is a function* P *from a* σ*-algebra* $\mathcal{A}$ *of subsets of* Ω *into* $[0,1]$ *that satisfies the following axioms:*

(i) $P(\Omega) = 1$.

(ii) *If* $A \subset \mathcal{A}$, *then* $P(A) \geq 0$.

(iii) *If* $A_1, A_2, \ldots \in \mathcal{A}$ *are mutually disjoint, then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The first two axioms are quite obvious. Since the sample space Ω consists of all possible outcomes, its probability equals $P(\Omega) = 1$. The second axiom

[5]The Laplace probability of event A is defined as the number of possible ways A can occur divided by the number of all possible outcomes in a suitable sample space Ω.

simply states that a probability is non-negative. In the finite case, the third axiom requires that if A and B are disjoint then $P(A \cup B) = P(A) + P(B)$, and in the infinite case it is assumed that this property extends to limits.[6] The importance of the definition for the infinite case lies in the fact that it holds for discrete and continuous probability measures and allows to distinguish elementary probabilities from any probabilistic or stochastic mechanism operating upon this structure.

The probability measure on the sample space determines the probabilities of a random variable X; in the discrete case by a frequency function p, and in the continuous case by a density function f. The expectancy $E(X)$ of a discrete or continuous random variable X will be of importance in the context of expected utility and is defined as follows.

Definition 1.1.2 (Expectancy) *Let X be a discrete random variable with frequency function $p(x)$, then*

$$E(X) = \sum_i x_i p(x_i) \tag{1.1}$$

is the expectancy *provided that $\sum_i |x_i| p(x_i) < \infty$. If X is a continuous variable with density $f(x)$, then*

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{1.2}$$

provided that $\int |x| f(x)\, dx < \infty$.

The definition explicitly states that the expectancy is undefined if the sum or integral does not converge.

Probabilistic mechanisms and their distributional assumptions rather than the probability measure itself are usually at the center of interest when studying choice behavior. But the qualitative aspects of probability have also been employed to model choice behavior (e.g., Luce & Suppes, 1965; Narens, 1991). A few traditional probabilistic choice models are discussed in Section 1.3 but it is already pointed out here that properties of these models are derived from limit theorems in which underlying mechanisms are studied in the infinite case; a powerful assumption which demands reasonably large samples of individual choice behavior levelling out any systematic changes.

Having specified rationality and probability as the main concepts for deterministic and probabilistic investigations of choice behavior a brief survey of deterministic and probabilistic decision models is provided in the next sections.

[6] In the more unusual case of a finitely additive probability space, $\mathcal{A}$ is reduced to an algebra of subsets of Ω.

1.2 Deterministic Models

The following order structures have been suggested to model preference as well as indifference in binary choice without using probabilities. They are defined by a set of axioms or conditions which are imposed on a collection of binary choices. They all require or imply transitivity or negative transitivity (see Definition A.2.1) but some of them are too weak for a representation into the numbers, thus not even an ordinal utility function is available. The definitions combine preference and transitive indifference if the ordered structures are reflexive but indifference is intransitive or excluded if they are irreflexive. In the following the reflexive relation $\succeq$ always denotes preference and indifference and the irreflexive relation $\succ$ denotes strict preference.

1.2.1 Order Structures

The weak order allows for a unique ordinal representation of alternatives into the real numbers $\mathbf{R}$ if the set S of alternatives is finite. In the infinite case the underlying set S has to be order-dense.

Definition 1.2.1 (Weak Order) *Let S be a non-empty set and $\succeq$ a binary relation on S, with $\succeq \subset S \times S$. $\succeq$ is called a* weak order *on S, if and only if for all $a, b, c \in S$ holds:*

(i) *if $a \succeq b$ and $b \succeq c$, then $a \succeq c$* (transitive)

(ii) *$a \succeq b$ or $b \succeq a$* (connected)

The axiomatization of the weak order has been generalized in two ways to accommodate infinite and multiattribute sets of alternatives as described in Section 1.4.1. Only the simple case of a weak order on a finite set of single-attribute alternatives is considered here. The axioms of the weak order guarantee the existence of an ordinal utility function $u : S \rightarrow \mathbf{R}$ so that the representation

$$a \succeq b \Leftrightarrow u(a) \geq u(b) \tag{1.3}$$

exists where u is unique up to monotone increasing transformations, thus defining an ordinal scale.

The corresponding representation theorem is due to Cantor (1895) and has been extended to infinite sets by Milgram (1939), Birkhoff (1967), and Krantz, Luce, Suppes and Tversky (1971). As mentioned before, empirical results suggest that the indifference relation is not necessarily transitive, and therefore not necessarily an equivalence relation.

Definition 1.2.2 (Semiorder) *Let S be a non-empty set and $\succ$ a binary relation on S, with $\succ \subset S \times S$. $\succ$ is called a* semiorder *on S, if and only if for all $a, b, c, d \in S$ holds:*

(i) *not $a \succ a$* (irreflexive)

(ii) *if $a \succ b$ and $c \succ d$, then $a \succ d$ or $c \succ b$*

(iii) *if $a \succ b$ and $b \succ c$, then $a \succ d$ or $d \succ c$*

The semiorder allows for intransitive indifference whilst the preference relation remains transitive. It has a representation into the real numbers

$$a \succ b \Leftrightarrow u(a) > u(b) + \delta \tag{1.4}$$

where u is unique up to monotone transformations and δ is a positive number. It denotes a certain threshold or just-noticeable difference which must be exceeded in order to establish a preference relation (Luce, 1956). Therefore, the threshold δ, which may be set to 1 by scaling, represents the limited ability to discriminate between alternatives. If only axiom (i) and (ii) of Definition 1.2.2 are required the structure is called an interval order with a representation given by ordered intervals rather than real numbers (Fishburn, 1970).

$$a \succ b \Leftrightarrow u(a) > u(b) + g(b) \tag{1.5}$$

This representation generalizes Eq. 1.4. Clearly, every interval order is a semiorder.

Definition 1.2.3 (Strict Partial Order) *Let S be a non-empty set and $\succ$ a binary relation on S, with $\succ \subset S \times S$. $\succ$ is called a* strict partial order *on S, if and only if for all $a, b, c \in S$ holds:*

(i) *not $a \succ a$* (irreflexive)

(ii) *if $a \succ b$ and $b \succ c$, then $a \succ c$* (transitive)

Every semiorder is a strict partial order because transitivity follows from (i) by taking $d = c$ in (iii) of Definition 1.2.2. The strict partial order as well as the biorder are too weak for a representation into the real numbers but they do appear in algebraic decomposition models.

Definition 1.2.4 (Biorder) *Let S, and D be non-empty sets and $\succ$ a binary relation with $\succ \subset S \times D$. $\succ$ is called a* biorder, *if and only if for all $a, b \in S$, and for all $d, e \in D$*

(i) *$a \succ d$ and $b \succ e$ imply $a \succ e$ or $b \succ d$.*

If the underlying sets S and D are the same then an irreflexive biorder on S is connected and transitive and equivalent to an interval order. A reflexive biorder on a set S is asymmetric and negatively transitive (Definition A.2.1) and corresponds to a partial order (Doignon, Ducamp & Falmagne, 1984). For a thorough introduction into measurement theory and the role of order in measurement theory consult Krantz, Luce, et al. (1971) and Narens (1985).[7]

The weak order and the semiorder form the backbone of deterministic choice models. Together with the strict partial order and the biorder, which are both beyond a representation on ordinal scale, they share a common property: They obey transitivity or, in the case of a reflexive biorder, negative transitivity. The next section deals with classical deterministic models in which alternatives are described by the probabilities of outcomes.

1.2.2 Expected Utility Models

Gabriel Cramer (1728) and Daniel Bernoulli (1738) stated hypotheses which marked a change in the understanding of risky choices. Until then, it was assumed that individual decision-making is best modeled by the evaluation of alternative monetary gambles on the basis of their expected values, so that a lottery offering the payoffs $(x_1, \ldots, x_n)$ with respective probabilities $(p_1, \ldots, p_n)$ would yield as much satisfaction as a sure payment equal to its expected value $\sum x_i p_i$. Such an approach was justified by appealing to the law of large numbers, which states that if a gamble is indefinitely and independently repeated, its long-run average payoff will necessarily converge to its expected value. Instead of applying plain expectancies the psychologically significant concept of utility was introduced which may be expressed as a square-root or logarithmic function.

With this utility function it was tried to explain the so called 'St. Petersburg Paradox', first presented in 1728 by Nicholas Bernoulli, a cousin of Daniel Bernoulli: Why does nearly everybody prefer the option of receiving about \$25 for sure instead of winning $\$2^n$ where n is the number of 'tails' until the first toss of 'head' when flipping a fair coin?[8] If the two options are compared in terms of their expectancy values to reach an optimal decision

[7] Further developments in measurement theory are summarized in Suppes, Krantz, Luce, and Tversky (1989) and Luce, Krantz, Suppes, and Tversky (1990).

[8] Note that the problem is stated here in dollars rather than in ducats as in the original problem. It has been shown that different currencies can have an effect on the utility function (Roskam, 1987).

the following problem arises. According to Definition 1.1.2, the expectancy of the gamble is simply not defined when the sum or integral is infinite. Since the probability that k 'tails' followed by one 'head' is $2^{-(k+1)}$, which is $P(X = 2^k) = \frac{1}{2^{k+1}}$, it follows that $E(X) = \sum_{n=0}^{\infty} np(n) = \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} = \infty$. Consequently and contrary to the original claim, there is no paradox to be solved for the infinite or unlimited gamble. However, if the number of throws is finite, people still tend to choose in favor of a lower expectancy value based on certainty rather than a higher expectancy value based on risky or uncertain outcomes. Bernoulli suggested a logarithmic utility function u for the subjective value of money which gives reasonable small expected utilities in this gamble.

There are several arguments which cast doubt on the solution of this problem in terms of logarithmic utility.[9] For instance, Savage (1954) suggested a modified version of this gamble where the first toss of 'head' leads to a win of $\$2^{2^n}$. Even if a logarithmic utility function is assumed the expected utility quickly becomes very large but most people are still willing to sell the option for a considerably lower but certain amount of money.

More than two centuries passed by until von Neumann and Morgenstern (1944) formally axiomatized the idea of expected utility in the second an third edition of their classic book *Theory of Games and Economic Behavior* (1947; 1953). They established a system of axioms which is based on a binary preference relation on a set of alternatives which includes probability mixtures. A probability mixture denoted by (x, p, y) can be understood as a gamble where outcome x occurs with probability p and outcome y with probability $1 - p$.

Definition 1.2.5 (Expected Utility) *Let $\succeq$ be a relation on a set of outcomes S which includes all probability mixtures (x, p, y). For all $x, y, z \in S$ and $p, q \neq 0, 1$*

(i) *$\succeq$ is a weak order on S, $\succ$ strict preference, and $\sim$ indifference.*

(ii) $[(x,p,y),q,y] \sim (x,pq,y)$.

(iii) *If $x \sim y$, then $(x,p,z) \sim (y,p,z)$.*

(iv) *If $x \succ y$, then $x \succ (x,p,y) \succ y$.*

[9] For example, it can be questioned: What is the probability of winning more than about \$25 in this gamble? This turns out to be $P(X > 2^m) = 1 - P(X \leq 2^m) = 1 - \sum_{k=1}^{m} \frac{1}{2^k}$. For $k = 5$ the probability $P(X > 25) \approx 0.03$ is already quite low; see Vlek and Wagenaar (1979), Lopes (1981), and Treisman (1983) for different heuristics.

(v) *If $x \succ y \succ z$, then there exists a probability p such that $y \sim (x, p, z)$. This structure is known as the* expected utility *model.*

The axioms permit a representation of the alternatives into the reals. The use of a utility function implies a stronger notion of rationality than ordinal preference because probability mixtures are represented by their expectancy and choice behavior is therefore required to maximize utility.

$$\begin{aligned} x \succeq y \Leftrightarrow u(x) &\geq u(y) \\ u(x,p,y) &= pu(x) + (1-p)u(y) \end{aligned} \tag{1.6}$$

Although this axiomatization of expected utility is very important because it laid down the foundation of game theory and provided a framework to test conditions underlying expected utility, its empirical limitations for modeling choice behavior were soon discovered (for a review on expected utility theory and its empirical violations see Camerer, 1989).

Modified conditions for the representation of linear expected utility have been suggested. Jensen (1967), for example, formulated the following set of axioms. For all $x, y, z \in S$ and for all $\lambda \in (0,1)$

(i) Order: $\succ$ is asymmetric; $\succ$ and $\sim$ are transitive

(ii) Independence: If $x \succ y$ and $0 < \lambda < 1$ then $\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z)$

(iii) Continuity: If $x \succ y \succ z$ then $\alpha x + (1-\alpha)z \succ y$ and $y \succ \beta x + (1-\beta)z$ for some $\alpha, \beta \in (0,1)$

Fishburn (1982) explored a system of axioms which yields a representation on ratio scale by replacing the assumption of independence and transitivity by a symmetry condition.

The expected utility model uses utility as a source of subjectivity but entertains an undesirable numerical representation of probability in form of probability mixtures, thereby representing numerical objects into numbers. The first theory with a subjective concept of probability was sketched by Ramsey (1931) in his treatise on the philosophy of beliefs. Savage (1954) proposed a system of axioms which employs *subjective* probability and utility. In his work on the foundations of statistical inference Savage stated conditions on preference relations which are sufficient for the derivation of both a subjective probability measure and a utility function.[10] By means of

[10]Scott (1964) formulated necessary and sufficient conditions for the finite case.

several axioms, Savage proved the existence of a subjective probability function s, which obeys the axioms of a probability measure (Definition 1.1.1), and a utility function u on interval scale such that

$$\begin{aligned} x \succeq y &\Leftrightarrow u(x) \geq u(y) \\ u(x, A, y) &= s(A)u(x) + [1 - s(A)]\, u(y) \end{aligned} \tag{1.7}$$

where (x, A, y) denotes the gamble where outcome x is obtained if event A occurs and outcome y, otherwise.

Generalized versions of utility and subjective probability are frequently used in economics and psychology, but systematic violations of some of the axioms are well documented (Schoemaker, 1982; Lopes, 1990; Birnbaum, 1992). For example Ellsberg (1961) created a problem[11] which can be described as follows: Imagine an urn that contains 90 balls. Thirty of the balls are red; the remaining 60 are black and yellow in unknown proportion. One ball is to be drawn at random from the urn. Consider the two choice situations in A and B and their payoffs in Table 1.1.

Table 1.1: *The 'Ellsberg Paradox'*

Number of Balls	30	60	
Situation A	Red	Black	Yellow
1. Bet on red	$100	$0	$0
2. Bet on black	$0	$100	$0
Situation B	Red	Black	Yellow
3. Bet on red or yellow	$100	$0	$100
4. Bet on black or yellow	$0	$100	$100

If a person bets on red balls in Situation A he or she will win $100 if a red ball is drawn and nothing if a black or yellow ball is drawn. If the person bets on black or yellow balls in Situation B he or she will win $100 if a black or yellow ball is drawn and nothing if a red ball is drawn.

In this problem, the extended sure-thing principle which immediately follows from the independence axiom as well as the axioms of subjective expected utility is violated. It implies that a decision maker who maximizes utility must choose either Option 1 and 3 or Options 2 and 4 in the two situ-

[11] The earlier 'Allais Paradox' (Allais, 1953) is similar but has the disadvantage that it incorporates a probability of 1 and a payoff of 0 as well as large sums of money. The problem was reformulated and tested by Slovic and Tversky (1974) but still includes extreme values which make the interpretation of the empirical result less convincing.

ations. The extended sure-thing principle which is also related to the dominance principle asserts that if two alternatives have a common outcome under a particular state of nature, then the ordering of the alternatives should be independent of the value of that common outcome. This is illustrated by the fact that the options of the 'Ellsberg Paradox' become equivalent if the last column in Table 1.1 is ignored. However, most people select Option 1 and 4, thus violating the principle. Presumably, they prefer to bet on payoffs whose probabilities are known precisely rather than on payoffs with ambiguous or uncertain probabilities (Grether & Plott, 1979). This phenomenon is important because it does not only violate expected utility but also generalized versions such as Fishburn's nontransitive expected utility, and non-expected utility models which are discussed next.

1.2.3 Non-Expected Utility Models

The systematic violations of expected utility led to generalizations of this model introducing nonlinear functions for expected values in form of non-expected utility models. Many of these models are flexible enough to maintain desirable properties of expected utility in a non-expected utility framework and have proven to be both theoretically and empirically useful. As for expected utility, preferences can be empirically assessed and used to predict individual choice behavior in other situations.

Luce and Narens (1985) investigated how general the representation of utility could be and still retain the property of interval scalability. These models are centered around evidence that utility of risky or uncertain outcomes and probability are not independent (Birnbaum & Stegner, 1979; Lopes, 1984; 1987; Birnbaum, Coffey, Mellers & Weiss, 1992).

The rank-dependent utility model was developed for uncertain choice alternatives and for the restricted case of two-outcome gambles (Luce, 1992). The rank-dependent utility of a gamble (x, A, y), where outcome x occurs when event A happens and outcome y occurs otherwise, depends on whether $x \succ y$ or $y \succ x$ and is defined as:

$$\begin{aligned} x \succeq y \;&\Leftrightarrow\; u(x) \geq u(y) \\ u(x, A, y) \;&=\; s(A)u(x) + [1 - s(A)]\, u(y) + r(A)|u(x) - u(y)| \end{aligned} \tag{1.8}$$

In comparison to Equation 1.7 the expression on the right-hand side is extended by the weighted difference of utilities where r is the rank-dependent weighting function which can have different forms (Weber, 1994; Luce &

von Winterfeldt, 1994). The rank-dependent model was generalized by Luce (1988; 1991) and by Luce and Fishburn (1991) to a rank- and sign-dependent linear utility model resulting in a representation for multiattribute alternatives.

Although these models permit modeling and analysis of preferences which are more general than those allowed by expected utility, they have two limitations. First, each model requires a different set of conditions on its component functions to model desirable properties of choice behavior so that expected utility theorems linking properties of utility to such aspects of choice behavior will typically not extend to the corresponding properties of the component functions. Second, non-expected utility models replace the independence axiom by some other more general restriction on preferences, possibly subject to similar types of systematic empirical violations already observed.

An alternative approach to the study of non-expected utility considers nonlinear functions in general and uses calculus to extend results from expected utility in the same manner as it is used to extend results involving linear functions (Machina, 1983). More specifically, linear approximations in form of the first order Taylor expansion of a smooth differentiable preference function are investigated. In the case of a preference function over cumulative distribution functions, this expansion includes the classical multivariate derivative of the function plus an remainder in terms of the standard L^1 norm.

1.2.4 Miscellaneous Models

The failures of expected utility as revealed by the 'Ellsberg Paradox' and related phenomena inspired researchers to find alternative explanations of choice behavior which are not directly linked to expected utility theory. Edwards (1953, 1954a, 1954b) was the first to study probability and variance preferences. Coombs (1969; 1975) proposed portfolio theory which highlights risk preference as a determinant of choices among gambles. One important characteristic of portfolio theory is the single peakedness of the curves of preference indifference which has been investigated theoretically and empirically by Coombs and Avrunin (1977a, 1977b) and Aschenbrenner (1981; 1984).

Kahneman and Tversky (1979) developed an deterministic framework, called prospect theory, to overcome some of the failures of expected utility theory. Three phenomena, the certainty effect (overestimating certain outcomes), the reflection effect (risk aversion for positive expectancies vs risk proneness for negative expectancies), and the isolation effect (simplifying gambles) were identified and incorporated in their model. In a later article Tversky and Kahneman (1992) extended the model to cumulative prospect theory which corresponds to a rank- and sign-dependent expected utility model for multiattribute alternatives.

Bell (1982), Loomes and Sugden, (1982) and Sage and White (1983) developed sophisticated alternative theories of decision-making under risk and uncertainty. In their models, the preference $a \succ b$ means that choosing a and rejecting b is preferable to choosing b and rejecting a. Since every choice depends on the regret involved in the particular pair of options being considered, transitivity of choice is not ensured in models of regret theory.

In many cases these models originated from a list of side conditions which were derived from experimental results and have been extended to a full theory at a later stage. Some of the models received a mathematical axiomatization which revealed their multiattribute nature. For example, an axiomatization was developed which casts portfolio theory in the framework of conjoint measurement with weak orderings in respect to expected value, risk, and preference (van Santen, 1978; Suck & Getta, 1994).

1.3 Probabilistic Models

Traditional probabilistic choice models are outlined. Some of their properties have an deterministic counterpart. To admit preference reversals and intransitive preferences in individual decision behavior axioms of rational choice are replaced by corresponding probabilistic assumptions. So transitivity is replaced by weak, strong and other probabilistic versions of transitivity. Although the distinction between constant and random utility models is not entirely adequate, probabilistic models are introduced under these categories to keep this account brief and simple.

1.3.1 Constant Utility Models

Constant utility models are models of riskless choice. They are closely related to probabilistic versions of transitivity. Their interrelationship is summarized at the end of this section. The description here is due to Roberts (1979, Chapter 6) and Falmagne (1985, Chapter 5) but can be found elsewhere (e.g., Luce & Suppes, 1965).

In the following a system (S, p) is called a system of pair comparison probabilities if for all distinct alternatives $a \neq b$ in a set S holds: $p_{a,b}+p_{b,a}=1$ with

$$p_{a,b} = P(a \succ b) \tag{1.9}$$

the empirical probability that alternative a is preferred to alternative b. For convenience assume that the probability of choosing between two identical alternatives equals $p_{a,a} = \frac{1}{2}$.

First, the representation for the weak utility model is introduced. If for all $a, b \in S$

$$p_{a,b} > p_{b,a} \Leftrightarrow u(a) > u(b) \tag{1.10}$$

then this model is equivalent to a probabilistic version of transitivity, called weak (stochastic) transitivity[12] which is defined as follows:

$$p_{a,b} \geq \tfrac{1}{2} \quad \text{and} \quad p_{b,c} \geq \tfrac{1}{2} \Rightarrow p_{a,c} \geq \tfrac{1}{2} \tag{1.11}$$

Two stronger probabilistic versions of transitivity, moderate and strong (stochastic) transitivity, are defined below:

$$p_{a,b} \geq \tfrac{1}{2} \quad \text{and} \quad p_{b,c} \geq \tfrac{1}{2} \quad \Rightarrow \quad p_{a,c} \geq \min(p_{a,b}, p_{b,c}) \tag{1.12}$$

$$p_{a,b} \geq \tfrac{1}{2} \quad \text{and} \quad p_{b,c} \geq \tfrac{1}{2} \quad \Rightarrow \quad p_{a,c} \geq \max(p_{a,b}, p_{b,c}) \tag{1.13}$$

A collection of different probabilistic versions of transitivity can be found in Fishburn (1979). The strong utility model fulfills

$$p_{a,b} > p_{c,d} \Leftrightarrow u(a) - u(b) > u(c) - u(d) \tag{1.14}$$

and implies the strong version of probabilistic transitivity.

A general variant of the strong utility model is the Fechnerian utility model. A forced choice pair comparison system (S, p) satisfies this model if there is a real-valued function u on S and a (strictly) monotone increasing function $\phi : \mathbf{R} \rightarrow \mathbf{R}$ so that for all $a, b \in S$

$$p_{a,b} = \phi\left[u(a) - u(b)\right] \tag{1.15}$$

This model also appears in psychophysics where it is used to relate a physical intensity to a psychological sensation. The model implies that $p_{a,b} = p_{c,d} \Rightarrow u(a) - u(b) = u(c) - u(d)$. Thus, pairs of stimuli which are equally often confused are equally far apart. This property was a critical assumption in Fechner's derivation of the psychophysical function as a logarithmic one (Fechner, 1860).

[12]Probabilistic forms of transitivity are usually called stochastic transitivity but as Luce and Suppes (1965) have pointed out the term stochastic should only refer to some process over time.

Luce's choice axiom (1959), reformulated by Luce and Suppes (1965), states that choices from any subset are independent of what else may have been available.

Definition 1.3.1 (Luce's Choice Axiom) *A set of preference probabilities defined for all subsets of a finite set S satisfy* Luce's choice axiom *provided that for all x, Y and X such that $x \in Y \subseteq X \subseteq S$ whenever the conditional probability exists,*

(i) $p_Y(x) = p_X(x \mid Y)$.

More specifically, the axiom requires that the probability of choosing x from Y when only Y was presented, written as $p_Y(x)$, equals the probability of those occasions where x is chosen from Y even though X may have been presented, written as $p_X(x \mid Y)$. For binary choice the axiom leads to a simple requirement for the utility function u also known as the Bradley-Terry-Luce model (BTL model).

$$p_{a,b} = \frac{u(a)}{u(a)+u(b)}. \tag{1.16}$$

However, for all $a, b \in S$ the additional assumption that $p_{a,b} \neq 0, 1$ has to be fulfilled. This model was also used by Zermelo (1929) to measure the playing power of a chess player. Adams and Messick (1957) pointed out that if (S, p) meets this assumption, the BTL model implies the Fechnerian model as it is shown here. Define $u' = \ln u$ for all $u(a) > 0$, $a \in S$. Let ϕ be the logistic distribution function $\phi(\lambda) = \frac{1}{(1+e^{-\lambda})}$. Then

$$\begin{aligned}
p_{a,b} &= \frac{u(a)}{u(a)+u(b)} \\
&= \frac{1}{1+u(b)/u(a)} \\
&= \frac{1}{1+\exp\left(-\left[u'(a)-u'(b)\right]\right)} \\
&= \phi\left[u'(a)-u'(b)\right]
\end{aligned}$$

so u' and ϕ satisfy the Fechnerian utility scale (see Equation 1.15). Consequently, if the set S is finite, the BTL model implies the Fechnerian utility model. The Fechnerian implies the strong utility model for

$$\begin{aligned}
p_{a,b} > p_{c,d} &\Leftrightarrow \phi\left[u(a)-u(b)\right] > \phi\left[u(c)-u(d)\right] \\
&\Leftrightarrow u(a)-u(b) > u(c)-u(d).
\end{aligned}$$

Moreover, when S is finite, they are equivalent (cf., Roberts, 1979, Chap. 6). The strong utility model implies the three probabilistic transitivity conditions, where weak transitivity is equivalent to the weak utility model. Note that if S is infinite neither of the probabilistic versions of transitivity implies the weak utility model. Hence, weak (stochastic) transitivity is as fundamental to constant utility models as transitivity to deterministic models.

The most thoroughly examined ideas are the three probabilistic versions of transitivity. Experimental findings are not too conclusive because these assumptions are difficult to test (e.g., Davidson & Marschak, 1959). Nevertheless, Tversky (1969) demonstrated systematic violations of weak transitivity under a carefully constructed situation where lotteries varied in probability and payoff. Although evidence against transitivity is less convincing in the context of money, other domains with multiattribute outcomes appear to be good candidates for intransitive choices (May, 1954).

1.3.2 Random Utility Models

In a related approach, one can assign a random variable U_a to each $a \in S$ so that

$$p_{a,b} = P\left(U_a \geq U_b\right) \tag{1.17}$$

This model is known as the random utility model (Block & Marschak, 1960). The interpretation is that the utilities are no longer assumed to stay fixed, but are determined by some probabilistic process. Indeed, the constant utility models may be considered as special cases of the random utility model with random variables remaining constant.

The following models originated from applications within psychophysics but have also been considered as utility models. They can be looked at as random utility models or as special versions of the Fechnerian model introduced in the previous section.

In variation of the Fechnerian model, if the function ϕ is required to be a cumulative distribution function, then $\phi\left[u(a) - u(b)\right]$ is the probability that $u(a)$ is larger than $u(b)$. This idea goes back to Thurstone (1927a, 1927b) and is known as 'Thurstone's Law of Comparative Judgment'. Scale values $u(a)$ are sought so that for all $a, b \in S$

$$p_{a,b} = \int_{-\infty}^{u(a)-u(b)} \mathcal{N}(x)\, dx \tag{1.18}$$

where $\mathcal{N}(x)$ denotes the standard normal distribution. Obviously, this is a special case of the random utility model introduced above. The cases most frequently used arise when the random variables are pairwise independent (Thurstone's Case III) and have equal variances (Case V).

As mentioned earlier, another variation of the Fechnerian model uses ϕ as the logistic distribution $\phi(x) = \frac{1}{1+e^{-x}}$ and seeks scale values $u(a)$ so that

$$p_{a,b} = \frac{1}{1 + e^{-[u(a)-u(b)]}} \tag{1.19}$$

The logistic model is due to Guilford (1954) and Luce (1959). If the random variables are double exponential distributions then their difference is the logistic distribution and therefore equivalent to Luce's Choice Axiom and the BTL model (Yellott, 1977).

Models of random utility with different distributional assumptions have been used and tested against each other extensively. Underlying assumptions or axioms of random utility, however, have been scarcely tested, simply because no set of necessary and sufficient conditions for the representation of binary choice probabilities have been found. Despite some progress in this area this problem remains a major theoretical and empirical obstacle (Koppen, 1995).

It seems straightforward to combine expected utility with probabilistic models leading to further generalizations. However, most of these models are not very different in their predictions from subjective expected utility models and similar violations have been reported (Becker, DeGroot, & Marschak, 1963a, 1963b, 1963c). The development of random utility models on the basis of non-expected utility is in some respect similar to rank- and sign-dependent utility models but addresses more abstract issues within measure theory and probability theory (Gilboa, 1987; Schmeidler, 1989; Wakker, 1989). This generalization of expected utility can accommodate preference reversals as observed in the Ellsberg Paradox because individuals' probabilistic beliefs are represented by non-additive probability measures which do not satisfy Definition 1.1.1. However, some axioms of these models are impossible to test. Therefore, they have limited psychological interpretation and relevance. As with non-expected utility models the success of these models depends on the extent to which they can usefully address issues in the theory of individual and group choice under uncertainty.

1.4 Multiattribute Models

Another generalization is of interest here. It originates from economics and the representation of choice alternatives as commodity bundles. A multiattribute alternative a can be expressed as a vector[13] of attributes $a = (a_1, a_2, \ldots, a_n)$ where the attributes belong to sets in a cartesian product $S_1 \times \ldots \times S_n$. Such product structures are mostly represented by additive conjoint measurement.

Most of the deterministic and probabilistic models in the previous sections can be generalized to multiattribute alternatives. However, a number of additional assumptions are needed to establish a representation into the numbers (Krantz, Luce, Suppes, & Tversky 1971).

1.4.1 Deterministic Multiattribute Models

A simple conjoint structure of multiple attributes is given by this representation in conjunctive form

$$a \succeq b \Leftrightarrow u_1(a_1) \geq u_1(b_1) \wedge \ldots \wedge u_n(a_n) \geq u_n(b_n) \tag{1.20}$$

where a preference holds if and only if the utilities of all attributes in one alternative exceed the utilities of all attributes in another alternative. In other words reversed utilities on a single dimension would prevent a preference relation from being established. The right-hand side of Equation 1.20 is also known as the dominance principle. The representation in disjunctive form is defined dually

$$a \succeq b \Leftrightarrow u_1(a_1) \geq u_1(b_1) \vee \ldots \vee u_n(a_n) \geq u_n(b_n) \tag{1.21}$$

where a difference on a single dimension suffices to create a preference. Coombs (1964) and Dawes (1964) defined a conjunctive and disjunctive decision rule in terms of thresholds. The conjunctive rule states that the attribute values for the chosen alternative must exceed a threshold specific on each attribute and that for at least one attribute, the other alternative falls below this threshold. The disjunctive rule states that the evaluation on at least one attribute for the chosen alternative must exceed a threshold specific to that attribute and that all aspects of the other alternative fall below a critical value specific to each attribute.

[13]To keep the notation simple no distinction is made between single-attribute and multiattribute alternatives which are sometimes written as vectors in bold-faced letters.

The weak order (Definition 1.2.1) has to fulfill the dominance and a continuity condition to yield the following representation of multiattribute alternatives with a single utility function.

$$a \succeq b \Leftrightarrow u(a_1, \ldots, a_n) \geq u(b_1, \ldots, b_n) \tag{1.22}$$

In contrast to conjoint structures the multiattribute alternatives are represented by a single utility function.

The most prominent multiattribute representation is additive conjoint measurement where different attributes are being measured conjointly (Debreu, 1959; Luce & Tukey, 1964).

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} u_i(a_i) \geq \sum_{i=1}^{n} u_i(b_i) \tag{1.23}$$

Several axioms are needed to formulate a representation for the finite as well as the infinite case (Luce & Tukey, 1964; Krantz, Luce, et al. 1971). Polynomial conjoint measurement is an example for a non-additive representation and has also been studied by Krantz, Luce, et al. (1971). In general, multiattribute utility models decompose the utility function u into single-attribute u_i, and then aggregate according to the appropriate model.

It has been argued that gambles or lotteries as introduced in Section 1.2.2 are in fact multiattribute alternatives which can be characterized by their probability of winning, amount to win, probability of losing, and amount to lose (Slovic & Lichtenstein, 1968). In the same line of argument rank- and sign-dependent utility models have been extended to multiple outcomes and can accommodate empirical findings such as dependence between outcomes and probability as well as asymmetrical loss and gain functions (Luce & Fishburn, 1991; Tversky & Kahneman, 1992). The axioms of utility and probability are generalized while preserving some desirable properties of expected utility, namely the dominance principle.

Additional examples of deterministic multiattribute models are the majority and lexicographic rule. The majority rule

$$a \succ b \Leftrightarrow \sum_{i=1}^{n} v\left[u_i(a_i)\right] > \sum_{i=1}^{n} v\left[u_i(b_i)\right] \tag{1.24}$$

where v is defined as

$$v = \begin{cases} 1 : u_i(a_i) > u_i(b_i) \\ 0 : \text{otherwise} \end{cases}$$

can produce intransitive choice behavior as demonstrated by May (1954). The rule counts the number of dominating attributes for each multiattribute alternative in order to establish a preference relation between the two alternatives. If, for example, certain attributes are missing then intransitivity can occur.

The lexicographic rule selects the alternative which dominates on the most important dimension. If alternatives have equal utility on this attribute the next important is considered and so forth. Hence,

$$\begin{aligned} a \succ b \;\Leftrightarrow\; & \text{for some } (1 \leq j \leq n)\; u(a_j) > u(b_j) \\ & \text{and for all } (i < j)\; u(a_i) = u(b_i) \end{aligned} \tag{1.25}$$

As long as the importance of the dimensions remains unchanged and u is monotonically increasing the lexicographic rule is transitive. Again, if certain attributes are indifferent or missing then intransitive choice can result (Luce, 1956).

Since the information processing approach was launched in cognitive psychology it has been at the center of interest in psychological decision research. Mainly initiated by Simon (1959) and his notion of bounded rationality the cognitive limitations of human information processing were investigated in decision-making. A number of simple decision rules, heuristics and strategies which are based on multiattribute alternatives emerged from this approach (e.g., Gigerenzer & Goldstein, 1996). Another example that has also been applied in statistical decision theory is the maximin and the maximax rule (Dahlstrand & Montgomery, 1984). According to the first rule, the chosen alternative has the highest lowest attribute value. The second rule implies that the chosen alternative has the highest highest evaluation.

It is not possible to review all the models here as it is difficult to evaluate and categorize the different algorithms and heuristics, especially when they are lacking a precise mathematical description. For various attempts to provide a framework for different choice heuristics and strategies consult for example Payne (1976; 1982), Beach and Mitchell (1978), Billings and Marcus (1983), and Ford, Schmitt, Schlechtman, Hults, and Doherty (1989).

1.4.2 Probabilistic Multiattribute Models

Restle (1961) proposed a specific BTL model where each alternative is understood as a set A of aspects $A = \{a_1, a_2, \ldots, a_n\}$. He assumed that one examines only certain relevant aspects of each stimulus before making a choice.

He also assumed that any characteristic contained by both alternatives becomes irrelevant to the decision process, and that the same is true for aspects contained in neither alternative. In a comparison of alternative A and B, the utility for alternative A depends on the utilities of all the aspects contained in A minus those contained in the intersection, written as $A \cap B$. Hence, the utility of A paired with B could be represented by $u(A) - u(A \cap B)$ and similarly for B by $u(B) - u(A \cap B)$. In terms of the BTL model the following representation results

$$p_{A,B} = \frac{u(A) - u(A \cap B)}{u(A) + u(B) - 2u(A \cap B)} \tag{1.26}$$

If the sets A and B are disjoint, then Restle's model and the BTL model make equivalent predictions about choice probabilities. An example of the superiority of Restle's choice model over the BTL model has been demonstrated by Edgell, Geisler, and Zinnes (1973), who based their analysis on data collected by Rumelhart and Greeno (1971).

Tversky's elimination by aspects rule (Tversky 1972a, 1972b) views each alternative as a collection of measurable attributes called aspects and describes choice as a covert process of successive elimination. At each stage in the process, an attribute of the alternatives is selected with a probability proportional to its value. Any alternative that does not include the chosen attribute is eliminated and the process continues until a single alternative remains. In the case of binary choice, elimination by aspects coincides with Restle's model. Moreover, elimination by aspects can be considered as a probabilistic version of the lexicographic rule.

In addition to the empirical violations in the context of expected utility, there are a number of empirical phenomena which are known to occur among multiattribute alternatives and which have a potential to create inconsistent choice. For instance, choice behavior is affected if an alternative is added to the set of multiattribute alternatives. It was recognized that the frequency of choosing alternatives can be either reduced or increased (Kahneman & Tversky, 1979; Huber, Payne, & Puto, 1982; Wedell, 1991) depending on certain attributes alternatives share with the added alternative. The systematic selection of a set of multiattribute alternatives due to similarity (Rumelhart & Greeno, 1971) or difference in attractiveness can also have a considerable effect on individual decision behavior (Albert, Aschenbrenner & Schmalhofer, 1989).

As the number of empirically verified effects increase, the requirements for decision models become more and more demanding. It seems reasonable to assume that most alternatives are multiattribute and that decision making processes vary even when the experimental context of choices does not. So far choice heuristics are the only candidates which try to explain inconsistent preferences in terms of adaptive individual choice behavior (e.g., Payne, 1976; Payne, Bettman, & Johnson, 1988; 1990; Schmalhofer, Albert, Aschenbrenner & Gertzen, 1986). Unfortunately, these models and in fact all multiattribute models, require a lot of domain-specific information before they can be applied.

In the next chapter a non-probabilistic approach for modeling adaptive choice is developed which has been neglected in decision research. In this approach it is tried to assess and explain inconsistent choices in a decomposition model without imposing strong assumptions on domain-specific attributes or underlying information processing. Inconsistencies are viewed as signals of adapting individual choice behavior. This view suggests the decomposition of a preference structure into substructures which is based on inconsistencies. In the next section models and techniques are discussed which can be seen as predecessors of an algebraic decomposition approach.

1.5 Algebraic Decomposition

Some non-probabilistic multidimensional scaling techniques can be related to the decomposition of a preference structure into substructures. Usually substructures refer to attributes or dimensions of alternatives underlying the decision process. In general, substructures meet stronger assumptions than the preference structure itself. Therefore, these models are related to deterministic multiattribute models.

In a theoretical paper Doignon, Ducamp, and Falmagne (1984) showed that an arbitrary relation R, defined on the cartesian product of two finite sets $R \subseteq S \times D$, can be expressed by the intersection of $|\overline{R}|$ biorders B_i.[14] Koppen (1987) investigated reductions and algorithms that compute the minimal number of biorders for data matrices of moderate size using tools of graph theory. He used the fact that the minimal number of biorders or bidimension of a relation is identical with the chromatic number of an associated hypergraph. This idea also applies to square preference matrices resulting in specific partial orders instead of biorders. In a more general approach Chubb

[14] $|\overline{R}|$ denotes the cardinal number of the complement of relation R.

(1986) reported an algebraic technique that can reduce preference matrices to a $\cap$-core of relations to find the minimal number of dimensions.

A more explicit idea for the decomposition of a pair comparison was proposed by Barthélemy (1990). He suggested that substructures of the decomposition should refer to different points of view of the decision maker. The decomposition follows from a lexicographic decision rule that includes in its generalized form both the conjunctive and disjunctive rule as well as decision rules with thresholds. The main goal is a minimization of the number of substructures. Thereby, the lexicographic sum of the minimal number of partial orders can explain all kinds of preferences, including intransitive choices.

In order to give an example of this decomposition the preference relation R, defined on the cartesian product of a set of four elements $S = \{a, b, c, d\}$, is mapped onto a (0,1)-matrix $A = [a_{ij}]$ of size 4×4 by the indicator function $\iota : S \times S \to \{0, 1\}$.

$$\iota(a_{ij}) = \begin{cases} 1 : (a, b) \in R \\ 0 : (a, b) \notin R \end{cases}$$

Let $R = \{(a, b), (a, c), (a, d), (b, c), (c, d), (d, b)\}$ be a set of preferences on S then the (0,1)-matrix[15] is written as

$$A = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note that the pairs $\{(b, c), (c, d), (d, b)\} \subset R$ describe an intransitive triple. Due to the generalized lexicographic rule, matrix A may be decomposed into two transitive partial orders O_1, and O_2. At this point it should be emphasized that Barthélemy did not suggest an explicit decomposition technique which can determine the underlying partial orders. He derived upper bounds for the minimal number of partial orders. Nevertheless, the lexicographic decomposition is illustrated in the example below to give an idea of how such an algebraic decomposition might work.

$$\begin{array}{ccccc} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \oplus & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ O_1 & \oplus & O_2 & = & A \end{array}$$

[15]The indices of a_{ij} run over $i, j = (1, \ldots, 4)$; 1's denote preference of alternative in row i over alternative in column j, 0's denote no preference.

The operation '$\oplus$' denotes the lexicographic sum which for the case of a combination of only two partial orders is defined as

$$(a,b) \in O_1 \oplus O_2 \quad \Leftrightarrow \quad (a,b) \in O_1 \text{ or } ((a,b) \in O_2 \text{ and } (b,a) \notin O_1) \tag{1.27}$$

The two partial orders O_1 and O_2 can be interpreted as different points of view held by the same person.[16] The lexicographic decomposition can explain intransitive choices in A. According to the lexicographic rule, it is not possible to compensate the preference on a more important dimension by less important ones.

In a theoretical note Lages (1989) suggested a simple decomposition which is based on the embedding of a partial order into the cartesian product of two linear orders. Suck (1994) established solvability conditions for the general case of an embedding into product structures. In another approach Doignon (1995) and Doignon and Falmagne (1997) investigated the family of all possible semiorders on a given set. If a family of semiorders or similar order relations is well graded it can be described by a permutahydron where the addition or removal of a single relation generates semiorders. These combinatorial results can be used to model a stochastic learning process which operates on this lattice structure.

The following critical remarks apply to the decompositions mentioned so far. None of the decompositions has been tested empirically as a model of choice behavior. The reasons for the lack of empirical data are quickly explained. These models are new to decision research and their application to choice behavior was not always considered. Moreover, no explicit techniques for assessing substructures have been proposed. This problem arises because algorithms for decompositions are inefficient and solutions are not necessarily unique. Depending on the observed preference structure, there can be several sets of substructures, each of which is computationally hard to determine but explains a given preference structure. As a consequence, if techniques for the previously described decompositions were available, they would be difficult to test. This is true because such techniques have at least one trivial but adequate solution and observe preferences as the only variable. So far no other variables have been considered or incorporated in a decomposition model.

For the reason that decomposition models are difficult to falsify they

[16] For example, when choosing between motorbikes, O_1 may reflect preference due to price, and O_2 preference due to fuel consumption with the two points of view decreasing in importance, respectively.

should be based on weak and plausible assumptions. A variable which is easy to observe is the sequence of choice-trials. In decision research the sequence of choices has been treated as a nuisance variable of the experimental procedure or has been ignored altogether. It is proposed here that the sequence of choice-trials can be used as a variable of choice behavior which offers weak and plausible assumptions for an algebraic decomposition.

In the next chapter an explicit decomposition model, the ear decomposition by sequence, is proposed which is efficient and solves the uniqueness problem by incorporating the sequence of inconsistent choices.

It follows from the survey and discussion of decision models in this chapter that the violation of asymmetry and transitivity has been crucial for the development of deterministic and probabilistic decision models. What is missing is a general approach in which individual choice behavior can be characterized in terms of inconsistencies and which provides a psychologically plausible algebraic decomposition. This decomposition should have a minimally and uniquely determined solution. In Chapter 2, we will pursue this goal by employing methods of graph theory and algebra. A graph-theoretical definition of intransitivity is suggested which leads to the algebraic decomposition of preference matrices into irreducible components, first established by Frobenius (1912). In contrast to other decompositions it naturally follows from intransitive choice behavior. This leads on to a more specific decomposition, the ear decomposition of so-called irreducible components, which has a unique solution if it is based on the sequence of intransitive choices. The solution is a directed ear basis, a minimal collection of choices which covers all intransitivities within an irreducible component. In Chapter 3 quantitative measures of inconsistency are tested in order to determine if choice behavior is adaptive so that the sequence of choice-trials affects inconsistency as postulated by the ear decomposition. In six experiments the ear decomposition by sequence is applied to three different domains of alternatives and various quantitative measures of inconsistency are compared.

1.6 Summary

This chapter has been concerned with basic assumptions of rational choice behavior. Asymmetry and transitivity were identified as fundamental assumptions for the ordinal representation of rational choice. Both assumptions appear in almost all deterministic choice models such as order structures, subjective expected and expected utility and related models. The violation of transitivity and related rational principles such as the extended sure-thing principle might favor a representation in form of probabilistic choice models satisfying probabilistic versions of transitivity. But probabilistic models such as constant utility and random utility require a lot more empirical data. These models have been successful when random noise enters the decision process as in psychophysical decision tasks, but violations of weak probabilistic versions of transitivity have been demonstrated in a number of preference data. It was argued that all traditional single- and multiattribute deterministic and probabilistic models face the problem of explaining systematic or adapting inconsistencies.

In Chapter 3 we will provide evidence that different arrangements of choice-trials in a pair comparison can affect the consistency of choice behavior; a finding that cannot be explained by classical deterministic or probabilistic models and makes a strong case for adaptive choice behavior and the application of decomposition models as introduced in Chapter 2.

Chapter 2

Theory

A decomposition model is developed which assumes that inconsistent choice originates from adaptive choice behavior. Consequently, inconsistent choices do not occur randomly but in response to changing choice situations. Intransitivities are therefore no longer regarded as random errors; they are considered as indicators of adapting choice behavior signaling a shift of strategy or a change in the information process of the decision maker. In a slightly stronger model it is assumed that subjects maintain their strategy when choosing in a pair comparison until they are confronted with a critical choice pair that suggests a different way to reach a decision. This is most likely due to an attribute which was ignored or forgotten in preceding choices but becomes important in current choices. The reasons for inconsistencies caused by altered information processing such as selective information processing are believed to be highly subjective and domain-specific eluding efforts to be described in the framework of a normative model such as expected utility, prospect theory or related models.

Consider, for example, a person who chooses between magazines in a pair comparison. In the first few choices he or she might develop a strong preference for magazines with a large section on cooking recipes. When confronted with a magazine which has a detailed TV preview he or she realizes that this is another attribute of importance. Taking this attribute into account would have reversed an earlier choice and leads now to inconsistencies in terms of intransitive preferences.

The difficulty is to identify critical choices that caused a shift in perspective without assessing domain-specific information by costly methods as in verbal protocols (Ericsson & Simon, 1984), without introducing strong assumptions as in multiattribute decision models, and without running into computational problems as in techniques of non-probabilistic multidimensional scaling and related decompositions.

In the previous chapter we have encountered some decision rules which can create intransitive choices. The majority rule and the lexicographic rule, for example, are heuristics that can cause preference cycles. However, it can be argued that such heuristics are not adaptive and should produce invariant amounts of inconsistency unless selective or altered information processing is assumed.

As outlined in Chapter 1 the assumption of transitivity is predominant in both algebraic and probabilistic choice theories. Perhaps due to the focus on transitivity, elaborate measures of inconsistency have not yet been developed. The methods almost invariably use Kendall's consistency index ζ and τ (Kendall, 1970). Slater (1961) proposed a measure of consistency by computing the minimal number of comparisons in which the choice should be reversed to obtain a linear order. A linear order resulting from reversing a suitable set of choices is called the nearest adjoining order and the number is known as Slater's i. Again, determining such a minimal set leads to an optimization problem which is related to the 'minimal acyclic subgraph problem' mentioned in Section 2.2.1 and 2.2.2. Bezembinder (1981) suggested a more sophisticated measure of inconsistency. This measure is related to the size of strong components, a characteristic of pair comparisons which is discussed in the following sections.

In the following preferences from a pair comparison are represented by sets, graphs, and matrices. First, intransitive choice in a pair comparison is defined in terms of directed cycles in a tournament. A theorem which goes back to Frobenius (1912) is applied and leads to a decomposition of any tournament into strong components. Related to the decomposition is the partition of a polynomial expression which can be used as a quantitative measure to characterize intransitive choice behavior in a tournament (Lages, 1995). Simple matrix operations on the adjacency matrices of tournaments are discussed that were designed to identify critical arcs of intransitive choices. As for the characterization of intransitive choice a fundamental optimization problem is encountered.

Second, a more specific decomposition is outlined which is based on the sequence of intransitive choices in a pair comparison. The ear decomposition by sequence has the advantage that it determines a small and uniquely defined subset of directed cycles that generate all intransitivities within a feasible algebraic structure. Moreover, the underlying algorithm is known to be efficient.

Finally, in a digression on the sequence of choices the completion by cuts is applied to a family of intransitive choices. This technique does not result in a unique set of critical choices but it offers a refined interpretation for solutions of algebraic decompositions.

As mentioned before, theoretical ideas in this chapter are presented in mathematical terms. They are summarized in an example at the end of each section. Although Appendix A provides some of the mathematical background, it is beyond the scope of this thesis to give a self-contained account of the mathematics involved. Standard textbooks on order theory, graph theory and algebra are cited in Appendix A.

2.1 Directed Cycles and Tournaments

There are three equivalent ways to represent a collection of relations between distinct objects: sets, graphs, and matrices. Each representation refers to a different theoretical approach: Sets of ordered pairs are mainly used in order theory, diagrams in graph theory, and matrices in algebra. The basic concepts of each approach which are of interest here are provided in Appendix A2, A3, and A4, respectively. In the following, diagrams are used to exemplify structures, whereas sets and matrices are mainly used to achieve theoretical results.

A definition of intransitive relations is given by *directed cycles*, abbreviated to *dicycles* (see also Appendix A.3).[1]

Definition 2.1.1 (Directed Cycle) *Let D be a digraph. A closed directed walk of length k is of the form*

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{k-1}, a_k)$$

where $a_k = a_0$ are identical vertices. If a_k, and a_0 are the only identical vertices then the closed directed walk is a directed cycle *of length k or k-dicycle.*

A directed walk may also be denoted by

$$a_0 \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_{k-1} \longrightarrow a_k.$$

Note that preference reversals correspond to directed cycles of length 2 or *2-dicycles* and intransitive triples are equivalent to *3-dicycles.*

Two vertices a and b are called *strongly connected* provided there are directed walks from a to b and from b to a. A single vertex is regarded as strongly connected to itself. Strong connectivity between vertices is reflexive, symmetric, and transitive. Hence, strong connectivity defines an equivalence relation on the vertices of D and yields a partition

$$V_1 \cup V_2 \cup \ldots \cup V_t$$

of the vertices V. The subdigraphs $D_1, D_2, \ldots, D_t$ formed by taking the vertices in an equivalence class and the arcs incident to them are called the *strong components* of D. The digraph D is *strong* if it has exactly one strong component.

[1] 'Directed cycles' are also called 'circuits' in the context of networks. Directed cycles should not be confused with undirected cycles that are called 'cycles' when dealing with undirected graphs.

A *tournament* is a digraph where every pair of vertices is connected by a single arc. Typically, such a digraph is the result of a pair comparison or round-robin tournament, in which each contestant battles every other contestant and with wins or losses as the only outcomes. There are $2^{\binom{n}{2}}$ possible tournaments of order n.

Two tournaments of order 5 are depicted in Figure 2.1. Tournament A differs from Tournament B by a single arc ($1 \to 2$ and $2 \to 1$, respectively). Tournament A is strong and has three 3-dicycles ($3 \to 4 \to 5 \to 3$; $1 \to 2 \to 5 \to 1$; $1 \to 4 \to 5 \to 1$), two 4-dicycles ($1 \to 2 \to 4 \to 5 \to 1$; $1 \to 3 \to 4 \to 5 \to 1$), and one 5-dicycle ($1 \to 2 \to 3 \to 4 \to 5 \to 1$) whereas Tournament B has two 3-dicycles ($1 \to 4 \to 5 \to 1$; $3 \to 4 \to 5 \to 3$), one 4-dicycle ($1 \to 3 \to 4 \to 5 \to 1$) and no 5-dicycle. This simple example illustrates that a single arc quickly changes the number of dicycles.

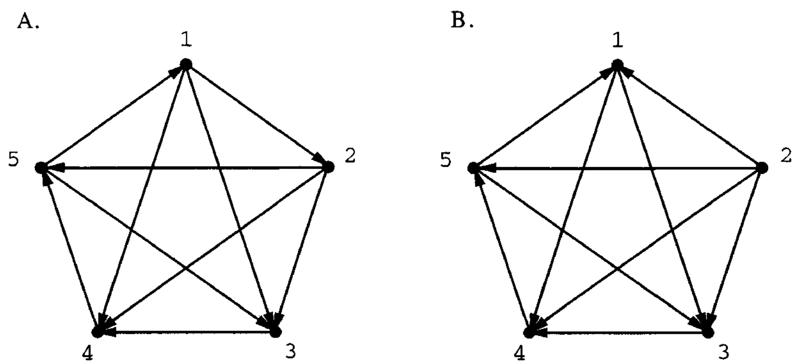

Figure 2.1: Tournament A differs from B by a single arc.

According to Reid and Beineke (1978) very little is known about dicycles of length greater than 4. However, Bermond and Thomassen (1981) provide a survey on dicycles in digraphs. It is concluded that dicycles of length greater than 3 do describe inconsistencies in a pair comparison and should be taken into account if a measure of inconsistency is established.

The following lemma was obtained by Moon (1968) and characterizes tournaments which are strongly connected abbreviated to *strong tournaments.* It shows that for the special case of a strong tournament each vertex in a strong component is part of some directed cycles of length greater than 3.

Lemma 2.1.2 *Each vertex of a strong tournament of order n is contained in a k-dicycle, for* $(3 \leq k \leq n)$.

Proof. This is proven by induction over k. Let D be a tournament of order n, and let a_0 be a vertex in V. First, a_0 must be on a 3-dicycle, since there must be a vertex b so that $(a_0, b) \in D$, and a vertex c so that (c, a_0) and $(b, c) \in D$ because D is strong. Now assume that a_0 lies on a k-dicycle $C = (a_0, a_1), (a_1, a_2), \ldots, (a_{k-1}, a_0)$ with $k < n$. We define the sets of vertices X, Y and Z as $X = \{x \in V \mid (x, a_i) \in D\}$, $Y = \{y \in V \mid (a_i, y) \in D\}$, and $Z = \{z \in V \mid z \notin C, X, Y\}$. If $Z \neq \emptyset$, then since D is strong, X and Y cannot be empty, and for some vertex $b \in X$ and $c \in Y$ it must hold $(b, c) \in D$. In this case a_0 lies on a $k+1$-dicycle $(a_0, b), (b, c), (c, a_2), \ldots, (a_{k-1}, a_0)$. On the other hand, if there exists a vertex $z \in Z$, then there must be vertices a_i and a_{i+1} in C such that (a_i, z) and $(z, a_{i+1}) \in D$; then again a_0 lies on a $k+1$-dicycle. The result then follows by induction. □

However, not every arc in a strong tournament belongs to a dicycle of length k with $k \in \{1, 2, \ldots, n\}$.

Let A be the adjacency matrix of a tournament, J the all 1's matrix, and I the identity matrix. Then A is a (0,1)-matrix satisfying the equation

$$A + A^T = J - I \tag{2.1}$$

A more detailed description of tournaments and their graph-theoretical properties can be found in Moon (1968) and Reid and Beineke (1978).

2.2 Decomposition into Strong Components

Let $\mathcal{M}_n$ be the set of all square $(0,1)$-matrices of size $n \times n$ and $A \in \mathcal{M}_n$. Let $\mathcal{B}$ denote a certain class of matrices. A decomposition is an expression of the form

$$A = B_1 + B_2 + \ldots + B_t + X \tag{2.2}$$

where $B_1, B_2, \ldots, B_t$ are in the class $\mathcal{B}$. Usually, X is required to be the zero matrix O. The purpose of such a decomposition is to minimize or maximize t or another quantity associated with the decomposition.[2]

Two simple decompositions of tournaments are presented as examples. A trivial decomposition results if the adjacency matrix A of a tournament is decomposed into acyclic matrices $B_1, \ldots, B_t$ $(t \leq i \times j)$ where each matrix contains a single entry $a_{ij} \neq 0$ of A and 0's otherwise. Another simple decomposition of an adjacency matrix A into matrices without directed cycles

[2] Three goals are usually achieved in a decomposition theorem: (1) The decomposition is well defined, (2) the existence of the decomposition, and (3) the uniqueness of the decomposition is shown.

is given by $A = B_1 + B_2$ where B_1 contains the non-zero entries of the upper triangular matrix of A and 0's elsewhere, and B_2 contains the non-zero entries of the lower triangular matrix of A and 0's elsewhere. The decomposition holds for any simultaneous permutation of A, written as PAP^T, where P is a permutation matrix. This minor result is summarized in the following lemma.

Lemma 2.2.1 *Let $A \in \mathcal{M}_n$. Then there exists a decomposition*

$$A = B_1 + B_2 \tag{2.3}$$

where B_1 and B_2 are acyclic matrices (without directed cycles). B_1 and B_2 are unique up to simultaneous permutations of A.

Proof. A matrix with 0's in the lower (upper) triangular matrix and the main diagonal implies that the vertices of A can be ordered $a_1, a_2, \ldots a_n$ so that each arc of B_1 (B_2) is of the form (a_i, a_j) for some i and j with $1 \leq i < j \leq n$ $(1 \geq i > j \geq n)$. It follows that neither B_1 nor B_2 contain a directed cycle for this would require an arc (a_j, a_i). □

The next lemma shows the equivalence between irreducible adjacency matrices and strong components of a digraph. This result is well known (e.g., Reid & Beineke, 1978) but is presented here with proof because it will be needed later.

Lemma 2.2.2 *Let $A \in \mathcal{M}_n$ be an adjacency matrix of the digraph D. Then A is irreducible if and only if D is strongly connected.*

Proof. Assume that A is reducible. Then the vertex set V of D can be partitioned into two nonempty sets V_1 and V_2 in such a way that there is no arc from a vertex in V_1 to a vertex in V_2. If $a \in V_1$ and $b \in V_2$ there is no directed walk from a to b. Hence D is not strongly connected. Now assume that D is not strongly connected. Then there are distinct vertices a and b in V for which there is no directed walk from a to b. Let W_1 consist of b and all vertices of V from which there is a directed walk to b, and let W_2 consist of a and all vertices to which there is a directed walk from a. The sets W_1 and W_2 are nonempty and disjoint. Let W_3 be the set consisting of those vertices which belong to neither W_1 and W_2. By simultaneously permuting the lines of A so that the lines corresponding to the vertices in W_2 come first followed by those corresponding to the vertices in W_3 we obtain:

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Since there is no directed walk from a to b there is no arc from a vertex in W_2 to a vertex in W_1. Also there is no arc from a vertex c in W_3 to a vertex in W_1, because such an arc implies that c belongs to W_1. Hence $A_{13} = O$ and $A_{23} = O$, and A is reducible. □

The decomposition into irreducible components is introduced next. It simplifies the study of any adjacency matrix A by breaking it down into submatrices which contain all directed cycles. The decomposition theorem can be derived from the specific arrangement of a matrix also known as the *Frobenius normal form* (Frobenius, 1912). In the case of tournaments it is shown that this form is unique up to simultaneous permutations within the irreducible submatrices. The following theorem and proof is due to Brualdi and Ryser (1991).

Theorem 2.2.3 *Let A be a matrix of size $n \times n$. Then there exists a permutation matrix P of order n and an integer $t \geq 1$ such that*

$$PAP^T = \begin{bmatrix} A_1 & A_{12} & \dots & A_{1t} \\ O & A_2 & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_t \end{bmatrix}$$

where $A_1, A_2, \dots, A_t$ are square irreducible submatrices. The irreducible submatrices are uniquely determined up to simultaneous permutation of their rows and columns, but their ordering as diagonal components is not necessarily unique.

Proof. see Brualdi and Ryser (1991), Chapter 3, pp. 57–58 □

Note that Theorem 2.2.3 guarantees that any (0,1)-matrix can take on the Frobenius normal form, and that a permutation matrix P exists. The problem, however, of how to obtain such a permutation matrix is not answered by the theorem.[3] The following decomposition of tournaments is a special case of Theorem 2.2.3 and the proof provides a simple way to determine a permutation matrix P for the adjacency matrix of a tournament.

Corollary 2.2.4 *Let A be an adjacency matrix of a tournament of order n. Then there exists a permutation matrix P of order n and an integer $t \geq 1$*

[3]Tarjan (1972) developed a depth-first search algorithm to determine strong components and their ordering in a digraph.

such that

$$PAP^T = \begin{bmatrix} A_1 & J_{12} & \dots & J_{1t} \\ O & A_2 & \dots & J_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_t \end{bmatrix}$$

where $A_1, A_2, \dots, A_t$ are square irreducible submatrices. The irreducible submatrices are uniquely determined up to simultaneous permutation of their rows and columns, and their ordering as diagonal components is unique.

Proof. The Frobenius normal form PAP^T of a tournament A has a descending order of row sums $s_1 \leq s_2 \leq \dots \leq s_n$. Therefore, permuting the rows so that the row sums of A decrease monotonically determines a permutation matrix P.

Only the special form of the Frobenius normal form has to be shown; the uniqueness of the order of the diagonal components follows immediately. If A is a tournament and PAP^T the Frobenius normal form with $A_1, \dots, A_t$ strong subtournaments then the matrices denoted by O contain 0's only. It follows from the asymmetry and connectedness of the tournament A that the matrices $A_{ij} = J_{ij}$ for $i < j \in \{1, \dots, t\}$. □

It is easy to see that the decomposition has a minimal number of irreducible subdigraphs. This decomposition of adjacency matrices can reduce the computational effort for determining polynomial expressions. A polynomial expression which has directly interpretable coefficients is defined in the following section.

2.2.1 Polynomials

Associated with the adjacency matrix A of an arbitrary digraph D is the characteristic polynomial $\phi(x)$ as defined in Equation A.5. The characteristic polynomial of a tournament or a digraph without loops is a monic polynomial or monom. Research on the characteristic polynomial of digraphs has been concerned with eigenvalues of A or their nearest bounds in the complex plane (Wilkinson, 1988). But the coefficients of the characteristic polynomial also have an interesting interpretation in terms of directed cycles: There is a correspondence between collections of disjoint dicycles whose lengths add up to k and the k-th coefficient of the characteristic polynomial. The explicit relationship was shown by Mowshowitz (1972) in the context of cospectral graphs. For an arbitrary digraph D of order k, let $f_D(\{i_1, i_2, \dots, i_r\})$ denote the number of collections of disjoint directed cycles in D of lengths

$i_1, i_2, \ldots, i_r$, where $i_j \geq 1\,(1 \leq j \leq r)$ and $i_1 + i_2 + \ldots + i_r = k$. Using the formula for the determinant of the adjacency matrix of a digraph (see Theorem A.4.6) Mowshowitz obtained the following result.

Theorem 2.2.5 *Let D be a digraph of order n and A its adjacency matrix. Then for $1 \leq k \leq n$, the k-th coefficient z_k of the characteristic polynomial $\phi(x)$ of A is given by*

$$z_k = \sum \left[\prod_{j=1}^{r} (-1)^{i_j+1} \right] f_D\left(\{i_1, i_2, \ldots, i_r\}\right)$$

where the summation is taken over all rank r partitions $\{i_1, i_2, \ldots, i_r\}$ of k with $(1 \leq r \leq k)$, and $z_0 = 1$.

Proof. Only a sketch of the proof is presented. It is known that the k-th coefficient z_k $(1 \leq k \leq n)$ is equal to the sum of all principal minors of order k. Since each k order principal submatrix of A is the adjacency matrix of a subdigraph of D containing k vertices, it is clear that any principal minor of A is the determinant of the adjacency matrix of a subdigraph of D. Thus, the coefficients of the characteristic polynomial of A can be expressed in terms of determinants of matrices belonging to subdigraphs of D. $\square$

In the following a modification of the characteristic polynomial is suggested. It is shown that the k-th coefficient of this modified polynomial equals the number of k-dicycles in any digraph.

To establish this polynomial the characteristic polynomial is first written as the signed sum of products

$$\phi(x) = \sum_{\pi} (\operatorname{sign} \pi) \prod_{i=1}^{n} (xI - A)_{i\pi(i)} \tag{2.4}$$

Each fixed-point i of π will contribute either x or $-a_{ii}$ to the product, while each non-fixed point i will contribute $-a_{i\pi(i)}$. Reordering the sum and organizing it by $|S|$ gives

$$\phi(x) = \sum_{k=0}^{n} x^{n-k} \sum_{|S|=k} \sum_{\pi \in P(S)} \operatorname{sign}(\pi)(-1)^k \prod_{i \in S} a_{i\pi(i)} \tag{2.5}$$

where $P(S)$ denotes the set of permutations on S. If the sign function is dropped and the permutations $\pi \in P(S)$ are replaced by permutations $\tau \in P(S)$ with a *single* permutation cycle, the expression becomes

$$\psi(x) = \sum_{k=0}^{n} x^{n-k} \sum_{|S|=k} \sum_{\tau \in P(S)} \prod_{i \in S} a_{i\tau(i)} \tag{2.6}$$

The so defined polynomial counts only k-dicycles as shown in the following proposition.

Proposition 2.2.6 *Let D be a digraph of order n and A its adjacency matrix. Then for $(1 \le k \le n)$, the k-th coefficient z_k of the polynomial $\psi(x)$ of A is given by*

$$z_k = f_D(\{i_k\})$$

where f_D is the collection of all directed cycles of length k and $z_0 = 1$.

Proof. Follows immediately from the definition of $\psi(x)$, Theorem 2.2.5, and the one-to-one correspondence between directed cycles and permutations with a single permutation cycle as shown in Lemma A.4.2. □

Although the polynomial $\psi(x)$ describes any digraph in terms of dicycles, it does not factor into irreducible polynomials.[4] In the following, polynomials are discussed which factor into irreducible polynomials, each corresponding to a strong component.

Definition 2.2.7 *Let A be the adjacency matrix of a directed graph, X the diagonal matrix*

$$X = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{bmatrix}$$

and f a combinatorial matrix function. Then

$$\phi = \phi(x_1, \dots, x_n) = f(A + X)$$

is a polynomial *in the polynomial ring $R[x_1, \dots, x_n]$.*

The factorization can be established by showing that a strong component corresponds to an irreducible polynomial. The following result holds for polynomials where the combinatorial matrix function f is the determinant (Frobenius, 1912; Schneider, 1977), and for polynomials where f takes on a form related to the permanent (König, 1936; Ryser, 1973).

Theorem 2.2.8 *Let D be a directed graph. Then A is the adjacency matrix of a strong directed graph if and only if ϕ is an irreducible polynomial in the polynomial ring $R[x_1, \dots, x_n]$.*

[4]An unsolved question in this context is: Which type of digraph has strong components that correspond to irreducible polynomials of $\phi(x)$ in $\mathbf{Z}[x]$, the characteristic polynomial in the polynomial ring over the integers. It follows from Proposition 2.2.6 that a digraph whose strong components have no vertex-disjoint partition into dicycles is a sufficient condition.

Proof. see Schneider (1977). □

If the polynomial ring is defined over the integers then the uniqueness of the result follows from the fact that $\mathbf{Z}[x_1, \ldots, x_n]$ is a unique factorization domain (Herstein, 1975, Chapter 3, and Corollary A.4.11).

Although the polynomial ϕ has a factorization which corresponds to the strong components its coefficients neither equal nor determine the number of k-dicycles. To overcome this limitation a different approach is suggested next. In the following, a polynomial in the polynomial ring over the integers is defined. This polynomial can be partitioned in its indeterminates describing the strong components of a digraph in terms of k-dicycles.

Definition 2.2.9 *Let A be the adjacency matrix of a digraph. The* combinatorial matrix function $g : \mathcal{M} \to \mathbf{Z}$ *is given by the formula*

$$g(A) = \sum_{k=0}^{n} \sum_{|S|=k} \sum_{\tau \in P(S)} \prod_{i \in S} a_{i\tau(i)} a_{ii} \tag{2.7}$$

where the third summation extends over all permutations τ in S with a single permutation cycle.

Notice that in contrast to the determinant as defined in Equation 2.4 or the permanent only permutations with a *single* permutation cycle appear in Definition 2.2.9 and that the factor a_{ii} in the product determines the support of each dicycle.

Definition 2.2.10 *Let D be a digraph of order n and A the associated adjacency matrix. The* polynomial expression $\psi(x_1, \ldots, x_n)$ *of A is given by*

$$\psi = \psi(x_1, \ldots, x_n) = g\,(A + X) \tag{2.8}$$

where the combinatorial matrix function g and matrix X are defined as before.

Proposition 2.2.11 (Partition) *Let A be the adjacency matrix of a digraph of order n, and ψ the non-zero polynomial expression associated with the digraph. Then A is irreducible if and only if ψ cannot be partitioned into non-constant polynomial expressions in disjoint sets of indeterminates.*

Proof. Assume that A is reducible. Then there exist positive integers r and $n - r$ and a permutation matrix P such that

$$P(A+X)P^T = \begin{bmatrix} A_r + X_r & O \\ * & A_{n-r} + X_{n-r} \end{bmatrix}$$

where $A_r + X_r$ and $A_{n-r} + X_{n-r}$ are square matrices of order r and $n - r$, respectively. The union of their diagonal elements is $\{x_1, x_2, \ldots, x_n\}$. It follows that ψ can be written as a partition of two nonconstant polynomials in $\mathbf{Z}[x_1, \ldots, x_n]$.

Now assume that the matrix $A = [a_{ij}]$ is irreducible. Suppose that ψ has a partition

$$\psi = p + q \tag{2.9}$$

into two nonconstant polynomials in $\mathbf{Z}[x_1, \ldots, x_n]$. Without loss of generality let p be a polynomial in the indeterminates $\{x_1, \ldots, x_r\}$ and q a polynomial in the indeterminates $\{x_{r+1}, \ldots, x_n\}$.

Consider the digraph D associated with the adjacency matrix A. By applying Lemma 2.2.2 we know that A is irreducible if and only if the associated digraph D is strongly connected. It follows that there is at least one dicycle whose support has at least one vertex in $\{x_1, \ldots, x_r\}$ and at least one in $\{x_{r+1}, \ldots, x_n\}$. Consequently, ψ cannot be expressed as the sum of two polynomials in distinct sets of indeterminates, which renders a contradiction. Hence, ψ has no partition in its indeterminates. □

The partition of ψ in its indeterminates is uniquely determined. The development of an algorithm for the computation of the polynomial and its partition is limited by the complexity of the digraph. Explicit considerations on the computational complexity are beyond the scope of this thesis and have been omitted. However, counting all dicycles in a polynomial is not an efficient technique although complexity can be reduced by the partition of the polynomial expression. A slightly better way to assess all directed cycles in a digraph was implemented in the Prolog program listed in Appendix C.1 which employs a depth-first strategy. Directed cycles in digraphs have been subject to extensive research in connection with the 'linear ordering problem' and the corresponding 'acyclic subgraph problem' (see for example Fishburn, 1992; Jünger, 1985; Reinelt, 1985). These problems among many others are known to be $\mathcal{NP}$-complete which means that if there exists an algorithm which solves one of them in polynomial time, there would exist a polynomial time algorithm for solving them all. Garey and Johnson (1979) provide a comprehensive treatment of the theory of $\mathcal{NP}$-completeness and a compilation of all contemporary $\mathcal{NP}$-complete problems.[5]

In a straightforward application of results the adjacency matrix A of a tournament is first brought into Frobenius normal form. Then the polyno-

[5]There is an occasional survey on recent developments in the field by Johnson in *Journal of Algorithms.*

mial ψ is computed for each irreducible submatrix $A_1, \ldots, A_t$ using single indeterminates. According to Proposition 2.2.11 the resulting polynomials can be written in a sum which contains equivalent information to the polynomial $\psi(x_1, \ldots, x_n)$. Compared to an implementation of the polynomial ψ in n indeterminates this technique reduces the computational complexity and size of the suggested polynomial expression and determines the number of k-dicycles.

2.2.2 Critical Arcs in Tournaments

So far we have established a general algebraic decomposition which breaks down a digraph (tournament) into subdigraphs (subtournaments) on the basis of its dicycles. To take this approach a step further, it appears promising to explore which arcs are particularly responsible or critical for dicycles. This question leads to a fundamental optimization problem. The following discussion is mostly restricted to tournaments.

The following proposition suggests how to perform simple matrix operations on an adjacency matrix to find the number of 3-dicycles intersecting in each arc.

Proposition 2.2.12 *Let $A \in \mathcal{M}$ be the adjacency matrix of a directed graph without loops. Let $R_3^T = [r_{ij}]$ be the matrix*

$$R_3^T = A^2 * A^T$$

where $*$ denotes the Hadamard product. Then each entry in R_3^T records the number of times arc a_{ij} belongs to a directed cycle of length 3.

Proof. The Hadamard product is defined as $A * B = [a_{ij}b_{ij}]$ for $(1 \leq i, j \leq n)$. A^2 records all directed walks of length 2 from a_i to a_j. The transposed matrix of the Hadamard product between A^2 and A^T, has entry $r_{ij} > 0$ if there is an arc $a_i \to a_j$ completing r_{ij} directed 3-cycles. Consequently, arc a_{ij} belongs to r_{ij} different 3-dicycles. □

Note that this result does not hold for multigraphs where 2-dicycles and loops are allowed. However, the matrix

$$R_k^T = A^{k-1} * A^T \tag{2.10}$$

for $3 \leq k \leq 5$ identifies k-dicycles in a tournament and $R_2^T = A * A^T$ can be used to determine preference reversals between pair comparisons.

Lemma 2.2.13 *Let A be the adjacency matrix of a tournament. Then the following two statements are equivalent.*

(i) *A is acyclic*

(ii) $\left(A^2\right)^T * A = O$

Note that in this context an acyclic tournament is equivalent to a transitive tournament without directed cycles.

Proof. That (i) implies (ii) is clear. Now assume that A has no 3-dicycles but contains an n-dicycle. If $a_{i-1} \to a_i \to a_{i+1}$ is a directed walk within the n-dicycle then there is an arc $a_{i-1} \to a_{i+1}$ because A represents a tournament, hence there is a directed cycle of length $n-1$. By induction a 3-dicycle remains and a contradiction is reached. Consequently, (ii) implies (i). □

This result does not ensure that a tournament is acyclic or without inconsistent choices if arcs from 3-dicycles in a tournament are successively removed until no 3-dicycle remains. This is illustrated in Figure 2.2. If the 3-dicycles in Tournament A are eliminated by removing arcs $1 \to 3$ and $2 \to 4$ from Tournament A then the subdigraph is acyclic. If the same arcs are removed from Tournament B then the 4-dicycle $1 \to 4 \to 3 \to 2 \to 1$ remains. In this case the removal of arc $3 \to 2$ which belongs to both 3-dicycles and

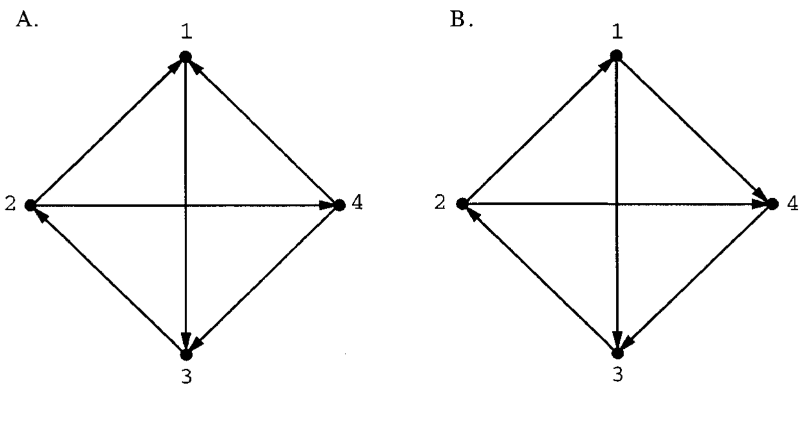

Figure 2.2: Tournament A differs from B by a single arc.

the 4-dicycle in Tournament B results in acyclic digraphs.

It is conjectured that an iterative algorithm which takes into account the number of intersecting k-dicycles for each arc of a digraph eventually leads to an acyclic digraph with a minimal set of removed arcs. For example, consider the following iterative lexicographic heuristic: First select the highest entry $r_{ij} \in R_3^T$, delete the corresponding entry $a_{ij} \in A$ then recompute R_3^T from the modified matrix A and continue this process until all dicycles in A are resolved. If the entries of R_4^T are employed whenever the entries in R_3^T are tied and the entries of R_5^T whenever the entries of R_3^T and R_4^T are tied and so on should lead to a minimal solution.

Example. Consider the adjacency matrix A representing a tournament of size 8 and R_3^T from the first step in this process.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}; \quad (A^2)^T * A = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & \boxed{4} \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 1 & 0 \end{bmatrix}$$

Four 3-dicycles are removed if entry $a_{38} = \boxed{1}$ in A is deleted. This entry corresponds to the highest entry $r_{38} = \boxed{4}$ in R_3^T. In the next step and the following steps there is no unique highest entry in R_3^T. If we delete in subsequent steps $a_{12}, a_{65}, a_{32}, a_{42}$, and a_{52} the highest entries of R_3^T in this order then all directed cycles are resolved but if $a_{26}, a_{27}, a_{28}, a_{65}$ the highest entries in R_3^T and R_4^T are selected no sixth arc has to be removed. □

Thomassen (1989) showed that strong tournaments with identical 4-dicycles are isomorphic or anti-isomorphic (Definition A.1.5) solving a problem raised by Goldberg and Moon (1971). It is not possible to conclude from this result that an iterative heuristic which takes into account all 4-dicycles in a strong tournament is indeed an algorithm which finds a minimal number of arcs because the first removal renders a digraph and not a strong tournament. In general this problem is linked to the 'acyclic subgraph problem' which is known to be $\mathcal{NP}$-complete. This problem arises in any digraph where the removal of a number of arcs may be minimal for 3-dicycles or 4-dicycles but does not necessarily guarantee an optimal solution for all dicycles. However, heuristic methods and approximate algorithms have been developed which can find solutions in polynomial time (e.g. Jünger, 1985).

2.2.3 Examples of Tournaments

The theoretical concepts are explained and summarized in the following examples. The correspondence between strong components of a tournament and the partition of the associated polynomial expression is illustrated. This section also gives an idea how quantitative and qualitative information can be extracted from adjacency matrices if they are decomposed into irreducible submatrices.

Note that the following $(0,1)$-matrix A does not contain artificial data but describes the preferences of Subject 1 in Session 1 of Experiment 2B (see Chapter 3).

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The index of the rows and columns refers to the number of the twelve lotteries as listed in Table B.10. The subject made choices between all possible pairs of lotteries in a forced choice pair comparison resulting in a tournament. The entry a_{ij} in matrix A is 1 if the subject preferred the Lottery i in row i over the Lottery j in column j and 0 otherwise. By simultaneous permutations of the rows and columns according to their row sums, the following Frobenius normal form of the tournament can be found.

$$PAP^T = \begin{bmatrix} [0] & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The irreducible submatrices along the diagonal, which correspond to the strong components of the tournament, are indicated by square brackets.

The characteristic polynomial is computed by

$$\phi(x) = \det(xI - A) = z_0 x^n + z_1 x^{n-1} + \ldots + z_{n-1}x + z_n \tag{2.11}$$

where x is indeterminate in $\mathbf{Z}$ and I is the identity matrix. The coefficients z_k of the characteristic polynomial correspond to certain collections of dicycles within the matrix. The characteristic polynomial $\phi(x)$ of matrix A has the form

$$\phi(x) = x^{12} - 7x^9 - 11x^8 - 10x^7 + 5x^6 + 24x^5 + 30x^4 + 20x^3 + 7x^2 + x \tag{2.12}$$

The factorization of the polynomial $\phi(x)$ results in three irreducible polynomials.[6] The disadvantages of the characteristic polynomial are twofold: First, the coefficients of the characteristic polynomial detect dicycles of any given length together with collections of disjoint dicycles which add up to the same given length. Consequently, if there are disjoint dicycles within strong components then the coefficients (before and after factorization) do not equal the number of dicycles. Second, the factorization of $\phi(x)$ does not necessarily correspond to the strong components of a tournament.

To overcome both problems the polynomial $\psi(x_1, \ldots, x_n)$ is used instead. It counts only directed cycles and has a partition in its indeterminates. The partitioned polynomial corresponds to the strong components of any digraph including tournaments. For computational reasons the polynomials ψ in single indeterminates are computed for the irreducible submatrices of the Frobenius normal form. According to Theorem 2.2.11 the resulting polynomials can be written as a sum.

The Hasse-like diagram in Figure 2.3 illustrates the preferences of matrix A in a special way. The alternatives are arranged in descending order according to their preference scores. As in a Hasse diagram preference decreases from top to bottom along the solid lines without arrow-heads. In the example of Figure 2.3 Lottery 11 is preferred over any other lottery, followed by Lotteries 3, 10, 6, 9, 7, 4 and 1. The arc pointing from Lottery 1 to 3 indicates a preference which violates the top-down order and creates many intransitivities among the seven lotteries in this strong component. However, all these lotteries are preferred to Lottery 8. Finally, Lottery 8 is preferred over Lottery 5, followed by Lottery 2, and 12. Here the arc pointing from Lottery 12

[6] In this special case the three factors correspond to the decomposition of the tournament A into three strong components A_1, A_2, and A_3. Moreover, the coefficients of the three irreducible polynomials equal the number of directed cycles within the strong components. But we know from Theorem 2.2.5 and Proposition 2.2.6 that both results do not hold for arbitrary tournaments.

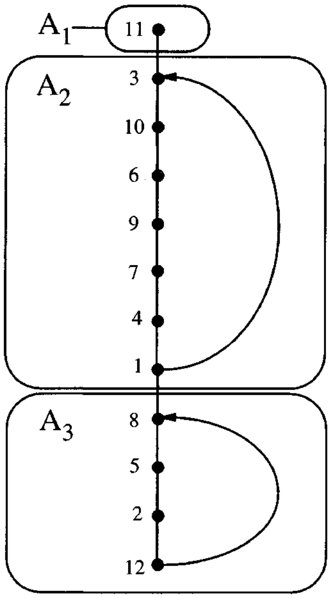

Figure 2.3: Hasse-like diagram of Subject 1 in Session 1 (Exp 2B)

to 8 introduces intransitivities among the four lotteries in this strong component. The three strong components are indicated by surrounding boxes. They correspond to the partition of the polynomial ψ.

By applying ψ to the irreducible matrices in the Frobenius normal form and by writing the resulting polynomials in a sum the following polynomial expression results:

$$\psi = (x^7 + 5x^6 + 10x^5 + 10x^4 + 5x^3) + (y^4 + 2y^3) \tag{2.13}$$

This expression is interpreted as follows: The tournament of order 12 has three strong components, one of order 1 (A_1), another of order 7 (A_2), and the third of order 4 (A_3). Note that the strong components are the smallest subdigraphs, which are linearly ordered within the tournament. The strong component of order 7 has a single directed 7-cycle (x^7), five directed 6-cycles ($5x^6$), ten directed 5-cycles ($10x^5$), ten directed 4-cycles ($10x^4$), and five directed 3-cycles ($5x^3$), whereas the strong component of order 4 has one directed 4-cycle (y^4), and two directed 3-cycles ($2y^3$). No directed 2-cycles or loops (directed 1-cycles) were detected because the subject chose in a pair comparison.

The strong components of the pair comparisons of the same subject in

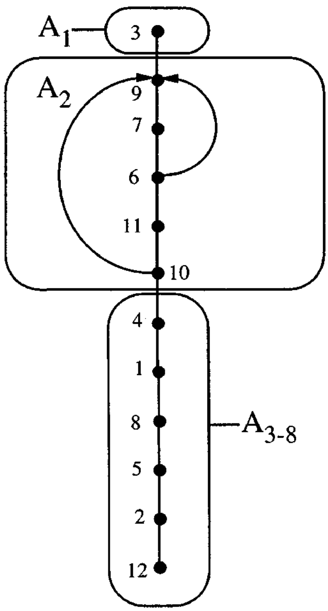

Figure 2.4: Diagram of Subject 1 in Session 2 (Exp 2B)

Session 2 and 3 are illustrated in Figures 2.4 and 2.5, respectively. The preference structures are more consistent in the second and third session than in the first. This improvement is reflected in the decreasing number of strong components with size greater than 1 as well as the total number of k-dicycles. The partition of ψ results in the following expression for the tournament of Session 2:

$$\psi = x^5 + 2x^4 + 3x^3 \tag{2.14}$$

The tournament has seven strong components of order 1 ($A_1, A_3, \ldots, A_8$) and one strong component of order 5 (A_2). This strong component has a single directed 5-cycle (x^5), two directed 4-cycles ($2x^4$), and three directed 3-cycles ($3x^3$). This example shows how consistency of choices improved over Session 1 and 2, and were even perfect in the last session.

With the above decomposition technique at hand tournaments can be characterized in great detail on ordinal level. The number of directed cycles of different length as well as the number and size of strong components can be assessed at once. With these measures inconsistency of individual choice behavior can be studied quantitatively in an exhaustive way. On the other hand, the enumeration of all directed cycles in a digraph can be very

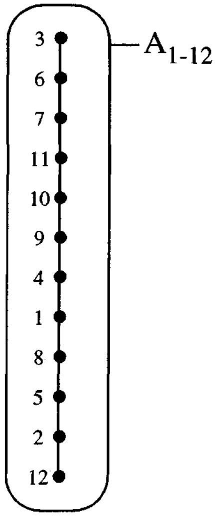

Figure 2.5: Hasse-like diagram of Subject 1 in Session 3 (Exp 2B)

time consuming and almost impossible for strong digraphs of order $n \geq 30$. Furthermore, these measures do not reflect the adaptive nature of individual choice behavior.

It was suggested that matrix operations may help to identify critical arcs which can explain all inconsistencies in a tournament. If the matrix operations are applied to A of the previous example then we observe

$$R_3 = (A^2)^T * A = \begin{bmatrix} 0 & 0 & \boxed{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

The two entries $r_{1,3} = \boxed{5}$ and $r_{12,8} = \boxed{2}$ of matrix R indicate that the removal of arcs $1 \rightarrow 3$ and $12 \rightarrow 8$ in A will break up all directed 3-cycles. However, resolving all 3-dicycles in a tournament does not necessarily lead to a digraph without dicycles of greater length. For a graphical representation of the tournament A we refer to Figure 2.3: The arcs belonging to directed cycles which do not fit into the top-down arrangement of the Hasse diagram are drawn with an arrow. In this example two arcs are easily identified which are responsible for all directed cycles in this tournament. They correspond to the arrows pointing from $1 \rightarrow 3$, and from $12 \rightarrow 8$. The same applies to the example in Figure 2.4. The arcs $6 \rightarrow 9$ and $10 \rightarrow 9$ resolve all inconsistencies within this tournament. As described in Section 2.2.2 matrix operations may help to determine these 'critical arcs'. In general, however, solutions are hard to determine and not necessarily unique. There can be several minimal sets of arcs explaining all directed cycles or inconsistencies within a tournament. A different approach which can overcome some of the problems is suggested in the next two sections.

2.3 Ear Decomposition

In the following a technique is described which decomposes the strong components of a digraph. This technique is known as ear decomposition. Folklore in mathematics tells that the name was derived from a graphical illustration of a simple ear decomposition of an undirected graph as pictured in Figure 2.6. The idea of an ear structure for undirected graphs was first suggested by Whitney (1932) and has been useful in proving many graph theoretical results (Lovász & Plummer, 1986).

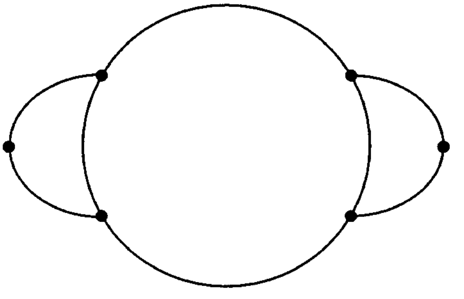

Figure 2.6: Illustration of 'ears' in an undirected graph.

The sequence of choices in a forced choice pair comparison is employed to ensure an ear decomposition with uniquely determined ear bases. Each ear basis describes a minimal collection of dicycles in a strong component generating all dicycles together with all cycles of the associated undirected graph which are not dicycles. If all possible combinations (modulo 2 sums) of these dicycles are generated then a lattice results which covers the strong component and has a minimal number of dicycles as a basis.

The psychological interpretation of ear bases in a pair comparison is straightforward. Each ear basis describes a minimal set of intransitive preferences of a strong subtournament that can account for all intransitive preferences and all cyclic comparisons which are not intransitive. This means if all possible combinations of the minimal set of intransitive preferences are generated, that is by combining preferences that belong to either of two preference cycles but not both, then a structure results covering all preferences in a strong subtournament. The ear decomposition is unique if the minimal set of dicycles is determined by the sequence of intransitive choices.

2.3.1 Cycle Space and Dicycle Basis

The *cycle space* of an undirected graph G is the subspace $GF[2]^{|E|}$ generated by the incidence vectors of the cycles of G. The cycle space of a directed graph D is the cycle space of the underlying undirected graph. A *cycle basis* is a basis of the cycle space of G, equivalently a minimal set of elements of the cycle space such that any cycle of G is a modulo 2 sum of these elements. A set $\mathcal{D}$ of directed cycles of D which is a cycle basis is called a directed cycle basis, i.e. any incidence vector of a cycle of the underlying graph of D is a modulo 2 sum of incidence vectors of some directed cycles of $\mathcal{D}$. The modulo 2 sum of incidence vectors may be interpreted as the combination of edges in the undirected graph.

The lattice generated by a set A of vectors is the set of all integer linear combinations of vectors of A. It is a well-known fact (e.g. Schrijver, 1986) that each lattice generated by a finite set of rational vectors has a basis, i.e. a set of linearly independent vectors (over rationals) such that any other element of the lattice is an integer linear combination. Since the elements of a lattice are integer linear combinations of elements of each lattice basis, each element of the lattice is also an integer linear combination *mod* q of vectors of each lattice basis, for any $q > 1$. For short, an integer linear combination *mod* q is sometimes called a *q-combination.*

The directed cycle bases which are naturally defined from an ear decomposition of a digraph are bases of the lattice generated by the directed cycles as well. Galluccio and Loebl (1996) showed that the lattices generated by the cycles of undirected graphs have bases consisting of directed cycles. This result can be applied to a lattice whose elements are cycles with arcs instead of edges.

A digraph is strongly connected if and only if it may be built up from a vertex by sequentially adding arcs and by subdividing arcs. This property leads to the concept of an *ear decomposition.*

Definition 2.3.1 *An* ear decomposition *of a digraph D is a successive extension $D_0, \ldots, D_t = D$ of subdigraphs of D such that D_0 consists of a single vertex and no arc, and D_i arises from D_{i-1} by adding a directed path P_i whose endvertices (which are not necessarily distinct) belong to D_{i-1} while the arcs and intermediate vertices of P_i do not.*

The paths P_i are called *ears* and the endvertices of P_i are called *initial vertices* of the ear. A digraph is strongly connected if and only if it has an ear decomposition (Whitney, 1932). It is well known that from each ear decomposition of a strongly connected digraph it is possible to obtain a directed cycle basis by simply completing each new ear to a directed cycle using a directed path in the already built subdigraph. Such directed cycle bases are called *directed ear-bases.*

The following result concerning the lattice generated by the directed cycles of a digraph D is due to Galluccio and Loebl (1996).

Theorem 2.3.2 *Let D be a strongly connected digraph. Any directed ear basis of D is a basis of the lattice generated by the directed cycles of D. Hence for any $q > 0$, the incidence vector of each directed cycle of D is a q-combination of incidence vectors of a directed ear basis of D.*

Proof. Let $\mathcal{B} = \chi_{C_1}, \ldots, \chi_{C_k}$ denote a directed cycle basis of digraph D, i.e. the set of incidence vectors of the directed cycles C_i obtained from an ear decomposition $D_0, D_1, \ldots, D_k = D$ by completing the directed path P_i into a directed cycle of D_i. Let $\mathcal{B}_i = \chi_{C_1}, \ldots, \chi_{C_i}$ be the set of incidence vectors from the ear decomposition of D_i. In order to prove the theorem it has to be shown that the vectors of $\mathcal{B}$ are linearly independent over the rationals and that the characteristic vectors of directed cycles of D are integer linear combinations of them.

The linear independence follows from the construction of the directed ear-basis: for each $j < i \leq k$ the directed cycle C_j contains no arc of the path P_i while C_i contains P_i.

Let $\mathcal{E}$ be the set of eulerian vectors of D whose components are integers. Define $\mathcal{E}_i$ to be the set of vectors of $\mathcal{E}$ having non-zero components only on arcs of D_i. Hence, $\mathcal{E} = \mathcal{E}_k$. It is proved by induction on i that each element of $\mathcal{E}_i$ is an integer linear combination of elements of $\mathcal{B}_i$. This finishes the proof of the theorem since the incidence vector of each directed cycle of D is eulerian. Let ζ be any vector in $\mathcal{E}_i - \mathcal{E}_{i-1}$. Since ζ is eulerian, the components of ζ corresponding to the arcs of P_i are equal, say p. Hence, the vector $\zeta - p\chi_{C_i}$, belongs to $\mathcal{E}_{i-1}$, and the result follows from the induction hypothesis. □

In the following this result is applied to the special case of strong tournaments and integer linear combinations with $q = 2$ only.

Each element in such a lattice is a collection of arcs whose underlying undirected graph G forms a cycle. Therefore, such an element is either a dicycle or an acyclic subset of arcs. Consequently, determining a basis in this lattice is the minimal set of directed cycles generating all the elements which form directed cycles or not. Such a basis offers a characterization of a strong tournament in terms of directed cycles and acyclic subsets.

2.3.2 Ear Decomposition by Sequence

In a straightforward application an ear decomposition is established which determines a directed ear basis for a given sequence of arcs in a strong digraph. The directed ear basis of such a labeled digraph is unique.

Thereto, we define a sequence on the arcs or preference relations. When considering the trials of a pair comparison their sequence may be expressed as an n-tuple. Hence, a sequence of trials in a pair comparison describes a single outcome in the sample space of all possible sequences. Together with the observed choices a simple pair comparison leads to a $\binom{n}{2}$-tuple of preferences ordered in time.

$$((a,b),(c,d),\ldots,(v,w))$$

If a neighborhood relation $\rhd \subseteq S^2 \times S^2$ is introduced to describe the sequence in time, then the tuple can be expressed as a chain K_0 of pairs:

$$K_0 := (a,b) \rhd (c,d) \rhd \ldots \rhd (v,w) \quad \text{for all } (x,y) \in S \times S \tag{2.15}$$

The chain K_0 can be understood as a set of ordered pairs ordered in time.

Definition 2.3.3 (Ear Decomposition by Sequence) *Let K_0 be a sequence defined on the arcs of a strong digraph. An* ear decomposition by sequence *of digraph D is a successive extension $D_1, \ldots, D_t = D$ of subdigraphs of D such that D_1 consists of the first arcs in K_0 which form a directed cycle, and D_i arises from D_{i-1} by adding the first directed path P_i in K_0 whose endvertices (which are not necessarily distinct) belong to D_{i-1} while the arcs and intermediate vertices of P_i do not.*

The definition above results in a unique collection of directed ears each of which can be completed to an ear dicycle by adding arcs, according to their sequence, from the already built subdigraph. From this collection of ear dicycles the dicycle space can be generated. Because these directed ears subdivide arcs which already belong to the subdigraph but appeared earlier in the sequence than any other directed ears they are believed to reflect adaptiveness in individual decision behavior. The ear decomposition by sequence is adaptive because it can model the process in which the strong components and their dicycles evolved across trials in a pair comparison.

Corollary 2.3.4 *The ear decomposition by sequence of a strongly connected digraph D is uniquely determined.*

Proof. It follows from Theorem 2.3.2 that the ear decomposition by sequence exists and is well defined. Since every ear has at least one distinctive labeled arc their sequence is uniquely determined. They can be uniquely completed to dicycles using the first directed path in the already built subdigraph. Hence, the resulting ear dicycles form a directed ear basis and are uniquely determined. □

In the next section the ear decomposition by sequence is applied to real data from a forced choice pair comparison. The data corresponds to a tournament with more than one strong component. The underlying graph of a tournament is a complete graph of order n where each arc belongs to a cycle of length k with ($3 \leq k \leq n$). Because the strong components of a tournament are strong subtournaments the directed cycle basis of a strong component generates all cycles and thus all dicycles within this subtournament. Collecting the directed ear bases of all strong components in a tournament leads to disjoint dicycle spaces for each strong subtournament. The number of subtournaments and their sizes can be derived from the directed ear bases.

2.3.3 Example of Ear Decomposition by Sequence

If the preferences of Subject 1 in Session 1 (Exp 2B) are ordered by their sequence the following ear decomposition results. In the first fifteen steps an ear decomposition by sequence of the strong component A_2 is obtained. Because A_1 has a single vertex it cannot be decomposed (see Figure 2.3). By finding the first dicycle in the sequence of preferences K_0 the first and second step of Definition 2.3.3 are executed in one step. The resulting fifteen directed ears and the corresponding completed ear dicycles from each step are listed below.

In the first step the dicycle of intransitive preferences with the lowest trial numbers is found. In the following step the arcs of a directed path are collected that start and end in a vertex of the subdigraph D_1 and that have the lowest trial numbers. This ear is then attached to the subdigraph D_1 extending it to D_2. In the next step a directed path is searched that starts and ends in a vertex of the extended subdigaph D_2 and has the lowest trial number. The search is terminated when no further ear is found that can be attached to the subdigraph. Figure 2.7 illustrates the first four steps of this

Step	Ear | Ear Dicycle	Step	Ear | Ear Dicycle
1.	$10 \to 1 \to 3 \to 10$	9.	$7 \to 4 \to 1 \mid \to 3 \to 10 \to 7$
2.	$10 \to 7 \to 1 \mid \to 3 \to 10$	10.	$9 \to 4 \mid \to 1 \to 3 \to 10 \to 9$
3.	$3 \to 7 \mid \to 1 \to 3$	11.	$6 \to 4 \mid \to 1 \to 3 \to 10 \to 6$
4.	$10 \to 9 \to 1 \mid \to 3 \to 10$	12.	$10 \to 4 \mid \to 1 \to 3 \to 10$
5.	$3 \to 9 \mid \to 1 \to 3$	13.	$6 \to 1 \mid \to 3 \to 10 \to 6$
6.	$9 \to 7 \mid \to 1 \to 3 \to 10 \to 9$	14.	$3 \to 4 \mid \to 1 \to 3$
7.	$10 \to 6 \to 7 \mid \to 1 \to 3 \to 10$	15.	$3 \to 6 \mid \to 7 \to 1 \to 3$
8.	$6 \to 9 \mid \to 1 \to 3 \to 10 \to 6$		

process. D_1 displays the first ear which is a dicycle in K_0, D_2 corresponds to D_1 extended by the second ear, D_3 corresponds to D_2 extended by the third ear, and so forth. The decomposition proceeds until D_{15} (not shown) is found which corresponds to the strong component A_2 in Figure 2.3.

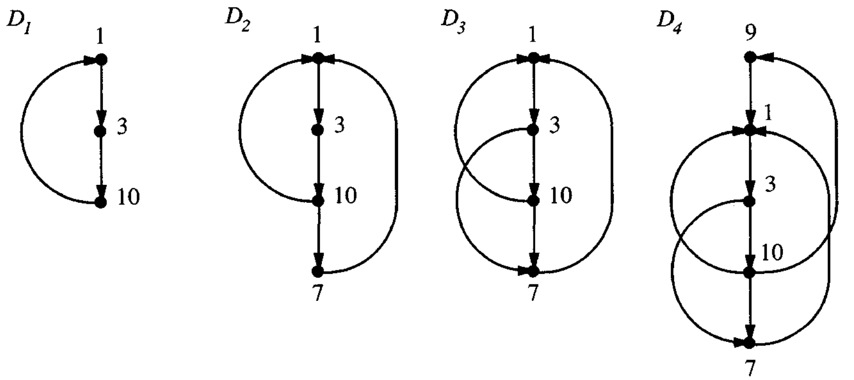

Figure 2.7: Ear decomposition of tournament for Subject 1 in Session 1 (Exp 2B). Only the first four steps are illustrated. See text for explanation.

The ear decomposition is continued if another disjoint dicycle is found as in this example. The disjoint dicycle initiates the ear decomposition of the strong component A_3. The three ear dicycles which are found in successive steps are listed below.

The associated subdigraphs are illustrated in Figure 2.8. E_1 corresponds to the first ear or dicycle in K_0 which is disjoint from D_{15}, E_2 corresponds to E_1 extended by the second ear, and E_3 finally corresponds to E_2 extended by the third and last ear. The decomposition terminates because no more ears can be found and no more dicycles or strong components are left. E_3 in Figure 2.8

Step	Ear | Ear Dicycle
1.	$8 \rightarrow 5 \rightarrow 12 \rightarrow 8$
2.	$8 \rightarrow 2 \rightarrow 12 \mid \rightarrow 8$
3.	$5 \rightarrow 2 \mid \rightarrow 12 \rightarrow 8 \rightarrow 5$

is equivalent to the strong component A_3 in Figure 2.3. Note that the ear dicycles are a subset of all dicycles leading to a smaller number e_k of k-dicycles in comparison with the coefficients z_k of ψ. A Prolog program is listed in Appendix C.1 that performs a depth- and breadth-search on a given sequence of preferences. It determines the ear dicycles of the ear decomposition by sequence. This unique collection of ear dicycles corresponds to the directed ear basis of each strong component.

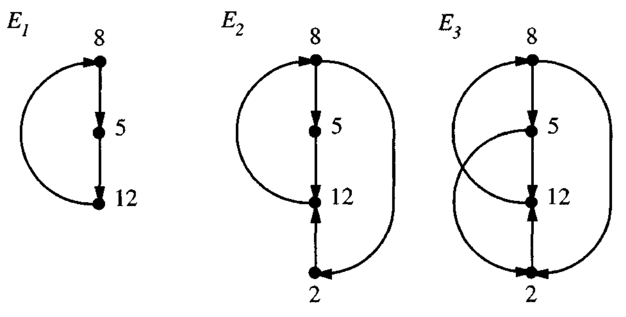

Figure 2.8: Continued ear decomposition of tournament for Subject 1 in Session 1 (Exp 2B). See text for explanation.

The ear decomposition by sequence solves the problems encountered earlier. Unlike the polynomial ψ and other techniques it works efficiently, that is in polynomial time. It provides a minimal solution in the sense that the ear dicycles form a basis of the cycle space. If the sequence of choices in a pair comparison are taken into account then the solution is also unique and can be studied experimentally.

2.4 Completion by Cuts

In this digression a slightly stronger assumption about the sequence of choice-trials is made. The following technique does not follow from the algebraic decompositions studied in the previous sections but it offers a more detailed explanation of inconsistent choice behavior. The completion by cuts is applied extending the idea of a sequence of intransitive choices. A technique is suggested which is based on latent preference orders and the embedding of intransitive subchains into a lattice structure. However, a computerized implementation needs to be developed before it can be applied to empirical data. Therefore, this technique was not applied to the experimental data presented in the next chapter.

The technique is motivated by the following simple observation. If the choices in a pair comparison are brought into a sequence then intransitive choices may appear anywhere in this sequence. This means, they are often separated by a number of choices which might or might not belong to other intransitivities. Hence, a change in information processing or shift of viewpoint yielding a different preference order may have occurred not just in the intransitive choices but in any choice-trial in between. It is assumed that a consistent preference order persists until the next shift of viewpoint in a given sequence of choices occurs.

As in the previous section, suppose the chain K_0 of pairs represents the outcome of a pair comparison of a single subject. Every subchain of K_0 can be understood as a set of choices with the usual set operations, union and intersection, working upon them. The intersection of subchains is again a subchain, but the union of subchains is not necessarily a subchain. Structures which are closed under intersection are well known in computer science (domain theory) and sometimes denoted as algebraic $\bigcap$-structures. The properties of such an algebraic structure are employed in a simple model which suggests that a shift of perspective occurs in the intersection of subchains enclosed by intransitive choices.

Intransitive and transitive choices may belong to different numbers of latent order structures (see Section 1.2.1). Within a pair comparison intransitivities may occur if pairs belong to different orders associated with the alternatives. It is emphasized that the use of a linear order is not essential for the development of ideas and may be replaced by weak orders, semiorders or partial orders.

As mentioned before, consistent choice in a pair comparison should lead to a transitive order of its alternatives. The order of the alternatives can be understood as a permutation of its elements. The number of permutations for k of n objects ($k \leq n$) is $n\,(n-1)\cdots(n-k+1)$. The number of permutations of length n on n objects is given by $(n!)$.

Example. The three elements $\{a, b, c, \ldots\} \subset S$ can be brought into $3! = 6$ different linear orders. The orders $R_1, \ldots, R_6$ are presented here as sets of pairs.

$$\begin{array}{ll} R_1 = \{(a,b),(b,c),(a,c),\ldots\} & R_4 = \{(b,a),(b,c),(a,c),\ldots\} \\ R_2 = \{(a,b),(c,b),(a,c),\ldots\} & R_5 = \{(b,a),(b,c),(c,a),\ldots\} \\ R_3 = \{(a,b),(c,b),(c,a),\ldots\} & R_6 = \{(b,a),(c,b),(c,a),\ldots\} \end{array}$$

□

2.4.1 Families of Intransitive Subchains

In the following, it is postulated that in a pair comparison a change in information processing may occur from one choice-trial to the next. Consequently, intransitivities can be the result of different latent preference orders where a single intransitivity in a pair comparison indicates a change of the preference order.

As in the previous section the sequence of trials in a pair comparison can be expressed as an n-tuple describing a single outcome in the sample space of all possible sequences. The neighborhood relation $\rhd \subseteq S^2 \times S^2$ describes the sequence of choice-trials in a chain K_0 of pairs:

$$K_0 := (a,b) \rhd (c,d) \rhd \ldots \rhd (v,w) \quad \text{for all } (x,y) \in S \times S \tag{2.16}$$

The chain can be understood as a set of ordered pairs ordered in time. Instead of the decomposition of a preference matrix A into submatrices the partition of such a chain into subchains with acyclic preferences is considered. For $K_0 \subseteq S^2 \times S^2$ we write $K_0 = \bigcup_{i \in I} K_i$. Thereby K_i denotes a subchain which is a sequence of choices of any length in K_0. For the chain K_0 on n alternatives in a pair comparison there are

$$|\mathcal{K}| = \sum_{l=1}^{\binom{n}{2}} l = \frac{\binom{n}{2}\left[\binom{n}{2}+1\right]}{2} \tag{2.17}$$

possible subchains which are collected in the set of all subchains $\mathcal{K}$ with $i \in I$ as the index set for the elements of $\mathcal{K}$. All subchains $K_i \in \mathcal{K}$ are sets for which intersections and subsets can be defined in the usual way. This is legitimate because the sequence of pairs is the same for all subchains of a given chain K_0.

In order to establish weak assumptions about the sequence of choices in a pair comparison a shift of perspective or *shift* for short, and an unaltered perspective or *link* for short, are defined. A shift of perspective simply designates any change of viewpoint or information processing which causes an alteration of the underlying preference order R_i.

Definition 2.4.1 *Let $(a,b) \rhd (c,d) \subsetneq K_0$ be a subchain where $(a,b) \in R_i$ and $(c,d) \in R_j$ are elements of orders on S with $i,j \in \{1,\ldots,n!\}$. A subchain $(a,b) \rhd (c,d)$ is called* shift *if and only if $(a,b) \in R_i$, $(c,d) \in R_j$ and $R_i \neq R_j$ and* link *otherwise.*

It is stressed here that the linear orders R_i are not directly observable. Consequently, shifts and links are not directly observable and their locations have to be inferred from intransitive choices.
Every ordered pair $(a,b) \in S \times S$ in a pair comparison can be thought of belonging to an order R_i on S. The definition states that *subsequent* pairs are links and therefore belong to the same order unless a shift has occurred. Therefore, chains of subsequent pairs belong to the most recent order as long as no shift has occurred.

In the previous section we have encountered the basis of an ear decomposition as a family of dicycles. These dicycles naturally form a family of intransitive subchains $\mathcal{I} = \{K_j\}_{j\in J}$. The set of all dicycles can also be translated into the family of all intransitive subchains. Both families are proper subsets of $\mathcal{K}$. They are eligible to the following operations because each subchain is just enclosed by intransitive choices and therefore has at least a single shift. The *intransitive subchain* K_j is defined as

$$K_j := \bigcap\{K_i \in \mathcal{K} \mid C_j \in K_i\} \tag{2.18}$$

where C_j denotes the choices of a k-dicycle. Some straightforward consequences of the definitions are collected in the following corollary

Corollary 2.4.2 *Let $\{K_j\}_{j\in J} \subsetneq \mathcal{I}$ be intransitive subchains in $\mathcal{K}$. Then the following statements are true:*

(i) *Every intransitive subchain contains at least one shift.*

(ii) *Let* $|\bigcap_{j\in J} K_j| > 1$ *with* $\{K_j\}_{j\in J}$ *intransitive subchains in* $\mathcal{I}$*. Then exactly* one *shift in* $\bigcap_{j\in J} K_j$ *is sufficient for the intransitive subchains* K_j, $j \in J$.

Proof. (i) is obvious, It follows from (i) that a shift occurs in every K_j. Because a single shift in $\bigcap_{j\in J} K_j$ splits every K_j into two subchains without directed cycles a single shift is sufficient. □

Statement (ii) motivates the investigation of the algebraic intersection structure of subchains which is discussed in the next section. Thereto, the following lemma is already provided.

Lemma 2.4.3 *Every family of subchains* $\mathcal{A} \subseteq \mathcal{K}$ *ordered by* $\subseteq$ *is a partial order.*

Proof. It is easily established that $\subseteq$ is reflexive, transitive, and antisymmetric. □

This result is needed for the family of intransitive subchains $\mathcal{I}$.

If exactly $t-1$ shifts occurred in a chain K_0 then there are exactly t disjoint successive subchains $K_1 \cup K_2 \cup \ldots \cup K_t = K_0$ whose elements belong to different orders; it follows that there are at least 2 alternating and at most t different orders involved. If the elements in a subchain belong to t orders then at least $t-1$ shifts must have occurred.

Example. The intransitive subchain $(a,b) \rhd \ldots \rhd (b,c) \rhd \ldots \rhd (c,a)$ may be explained by a single shift with $(a,b) \rhd \ldots \rhd (b,c) \rhd \ldots \in R_1$ and $\ldots \rhd (c,a) \in R_4$. Correspondingly, $(a,b) \rhd \ldots \in R_5, \ldots \rhd (b,c) \rhd \ldots \in R_3$ and $\ldots \rhd (c,a) \in R_6$ is an example for an intransitive subchain which contains two shifts of perspective. □

2.4.2 Closure and Lattice

In this section it is shown that the Dedekind-MacNeille completion or completion by cuts on a family of intransitive subchains defines a closure operator. This closure on a set of intransitive subchains corresponds to a lattice structure which can be used to investigate subchains in more detail.

Before we can derive some results the following notation is introduced: The set of all upper bounds and the set of all lower bounds of a set A (see also Definition A.2.4) is written as A^u and A^l, respectively. In the case of intransitive subchains they are defined by

$$\begin{aligned}\mathcal{A}^u &:= \{K_i \in \mathcal{I} \,|\, (\forall K \in \mathcal{A}) K \subseteq K_i\} \\ \mathcal{A}^l &:= \{K_i \in \mathcal{I} \,|\, (\forall K \in \mathcal{A}) K_i \subseteq K\}\end{aligned}$$

First we establish the closure operator on the partially ordered set of subchains.

Lemma 2.4.4 *Let $\mathcal{A} = \{K_i\}_{i \in I}$ be a family of subchains in $\mathcal{I}$ the partially ordered set of intransitive subchains including K_0. Then the mapping $\mathcal{C} : 2^{\mathcal{I}} \longrightarrow 2^{\mathcal{I}}$ with*

$$\mathcal{C}(\mathcal{A}) := \mathcal{A}^{ul}$$

defines a closure operator on $\mathcal{I}$.

Proof. We have to show (i) to (iii) of Definition A.2.6, the properties of a closure operator.

(i) We have $K_i \subseteq K_j$ for all $K_i \in \mathcal{A}$ and all $K_j \in \mathcal{A}^u$, therefore $\mathcal{A} \subseteq (\mathcal{A}^u)^l = \mathcal{A}^{ul}$. Dually, $\mathcal{A} \subseteq \mathcal{A}^{lu}$.

(ii) If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{B}^u \subseteq \mathcal{A}^u$, hence every lower bound of $\mathcal{A}^u$ is also a lower bound of $\mathcal{B}^u$, thus belongs to $\mathcal{B}^{ul}$.

(iii) It follows from (i) that $\mathcal{A}^{ul} \subseteq (\mathcal{A}^{ul})^{ul}$, clearly, $(\mathcal{A}^{ul})^{ul} \subseteq \mathcal{A}^{ul}$ because (i) applied to $\mathcal{A}^u$ gives $\mathcal{A}^u \subseteq (\mathcal{A}^u)^{lu}$ and every lower bound of $\mathcal{A}^{ulu}$ is also a lower bound of $\mathcal{A}^u$ hence belongs to $\mathcal{A}^{ul}$.

□

The following theorem is well known (e.g., Davey & Priestley, 1990) and stresses the relationship between various definitions of lattices.

Theorem 2.4.5 *Let $\mathcal{L}$ be a non-empty, partially ordered set. Then the following statements are equivalent.*

(i) *$\mathcal{L}$ is a lattice.*

(ii) *inf$\{A\}$ in $\mathcal{L}$ for every subset $\mathcal{A}$ of $\mathcal{L}$.*

(iii) *$\mathcal{L}$ has a maximum and inf$\{A\}$ exists in $\mathcal{L}$ for every non-empty subset $\mathcal{A}$ of $\mathcal{L}$.*

Proof. (i) implies (ii) and (ii) implies (iii) because the infimum of the empty subset of $\mathcal{L}$ exists only if $\mathcal{L}$ has a maximum. Finally, (iii) implies (i) by the use of Lemma A.2.5. $\square$

The following result holds for any partial order and is equivalent to the completion by cuts. It is shown here for a family of intransitive subchains on the chain of choice-trials.

Proposition 2.4.6 *Let $\mathcal{I} = \{K_j\}_{j\in J}$ be the family of intransitive subchains of K_0 including K_0 itself. Then*

$$\mathcal{L} := \{\mathcal{A} \subseteq \mathcal{I} \mid \mathcal{A}^{ul} = \mathcal{A}\}$$

is a lattice in which, when ordered by $\subseteq$, infimum and supremum is defined as

$$\begin{aligned} inf\{K_i \mid i \in I\} &= \bigcap_{i\in I} K_i \\ sup\{K_i \mid i \in I\} &= \left[\bigcup_{i\in I} K_i\right]^{ul} . \end{aligned}$$

Proof. By Theorem 2.4.5 it suffices to show that $\mathcal{L}$ has a maximum and that the infimum of every non-empty subset of $\mathcal{L}$ exists in $\mathcal{L}$.

Clearly, K_0 is the maximal element in $\mathcal{L}$. Let $\{K_i\}_{i\in I}$ be a non-empty subset of $\mathcal{L}$ and assume that $\bigcap\{K_i\}_{i\in I}$ is *not* in $\mathcal{L}$. For $\bigcap\{K_i\}_{i\in I} \subseteq \mathcal{I}$ and

$$\begin{aligned} \left[\bigcap\{K_i\}_{i\in I}\right]^{ul} &= \bigcap[\{K_i\}]^{ul} \\ &= \bigcap\{K_i\}_{i\in I} \end{aligned}$$

a contradiction is reached. Hence, $\bigcap\{K_i\}_{i\in I} \in \mathcal{L}$. Since $\bigcap_{i\in I} K_i \subseteq K_j$ for all $j \in I$ it follows that $\bigcap_{i\in I} K_i$ is a lower bound of $\{K_i\}_{i\in I}$. If $K \in \mathcal{L}$ is a lower bound of $\{K_i\}_{i\in I}$ then $K \subseteq K_i$ for all $i \in I$ and therefore $K \subseteq \bigcap_{i\in I} K_i$. Hence, $\bigcap_{i\in I} K_i$ is the greatest lower bound of $\{K_i\}_{i\in I}$ in $\mathcal{L}$, which can be expressed as

$$\inf\{K_i \mid i \in I\} = \bigcap_{i\in I} K_i .$$

Thus $\langle \mathcal{L}, \subseteq\rangle$ is a lattice. Since $K_0 \in \mathcal{L}$ is an upper bound of $\{K_i\}_{i\in I}$ in $\mathcal{L}$, it follows from Lemma A.2.5 that

$$\begin{aligned} \sup\{K_i \mid i \in I\} &= \inf\{K_i \mid i \in I\}^u \\ &= \bigcap\{K \in \mathcal{L} \mid (\forall i \in I) K_i \subseteq K\} \\ &= \bigcap\{K \in \mathcal{L} \mid \bigcup_{i\in I} K_i \subseteq K\} \end{aligned}$$

$\square$

The closure system on a set of intransitive subchains with K_0 as the maximum, $\mathcal{L} := \mathcal{C}(\mathcal{I})$, forms a lattice in regard of $\subseteq$.

The minimal elements of the lattice can be investigated further. The minimal elements are subchains containing sets of arcs ordered in time. It is assumed that the smallest sets of subchains are responsible for the inconsistent choice behavior.[7]

2.4.3 Example of Completion by Cuts

In the following example the hypothetical outcome of a pair comparison is studied. The example illustrates the theoretical concepts discussed in the previous two sections.

Example. Let $S = \{a, b, c, d, e, f, g\}$ be a set of alternatives and

$$\begin{aligned} R = \ & \{(c,f),(b,d),(a,g),(d,e),(b,f),(a,b),(c,d),\\ & (d,g),(b,c),(e,c),(b,g),(d,a),(b,e),(a,c),\\ & (a,f),(e,f),(c,g),(a,e),(f,g),(b,f),(g,e)\} \end{aligned}$$

the result of a pair comparison written as a tuple of $\binom{7}{2}$=21 preference relations. First we define the chain $K_0 := (c,f) \rhd \ldots \rhd (g,e)$ and identify a family of intransitive subchains $\mathcal{I} = \{K_j\}_{j \in J}$ as:

$$\begin{aligned} K_1 &= \underline{(b,d)} \rhd (a,g) \rhd (d,e) \rhd (b,f) \rhd \underline{(a,b)} \rhd (c,d) \rhd (d,g) \rhd (b,c) \rhd \\ & \quad (e,c) \rhd (b,g) \rhd \underline{(d,a)},\\ K_2 &= \underline{(d,e)} \rhd (b,f) \rhd (a,b) \rhd \underline{(c,d)} \rhd (d,g) \rhd (b,c) \rhd \underline{(e,c)},\\ K_3 &= \underline{(c,d)} \rhd (d,g) \rhd (b,c) \rhd (e,c) \rhd (b,g) \rhd \underline{(d,a)} \rhd (b,e) \rhd \underline{(a,c)},\\ K_4 &= \underline{(e,f)} \rhd (c,g) \rhd (a,e) \rhd \underline{(f,g)} \rhd (b,f) \rhd \underline{(g,e)}\} \end{aligned}$$

In each subchain $K_1, \ldots, K_4$ the pairs which belong to the defining intransitive triples are underlined. On the left-hand side of Figure 2.9 the Hasse diagram of intransitive subchains is shown. The subchains are partially ordered as subsets of K_0. In the next step the closure $\mathcal{C}$ is applied to $\mathcal{I}$ and the lattice structure $\mathcal{L}$ is obtained. The lattice then contains all possible subchains closed under intersections including two new subchains

$$\begin{aligned} K_5 &= (c,d) \rhd (d,g) \rhd (b,c) \rhd (e,c) \rhd (b,g) \rhd (d,a),\\ K_6 &= (c,d) \rhd (d,g) \rhd (b,c) \rhd (e,c) \end{aligned}$$

[7]Note that unlike the ear decomposition by sequence an overlap between intransitive subchains representing dicycles from different strong components suggests a common source of intransitivity.

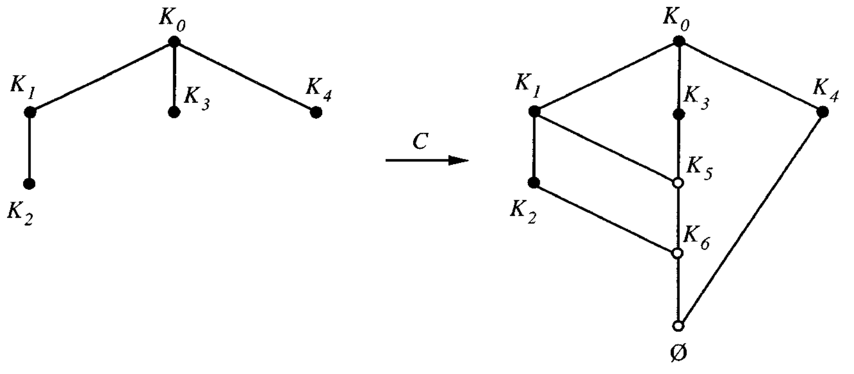

Figure 2.9: Hasse diagram of $\mathcal{I}$ and $\mathcal{L}$. See text for explanation.

together with the empty set $\emptyset$. In Figure 2.9 the lattice $\mathcal{L}$ is represented in a Hasse diagram on the right-hand side. The newly obtained subchains are denoted by open circles. The minimal elements in $\mathcal{L}$ are the subchains K_6 and K_4. It follows from Corollary 2.4.2 that a single shift in both subchains would explain all intransitive triples within the pair comparison. □

In this example the minimal elements are uniquely determined. In general the minimal elements in a lattice are not necessarily unique and subchains may have to be selected, according to their sequence in K_0, until all intransitive triples are covered. The lattice structure may be investigated further by exploiting properties of lattices (Birkhoff, 1967). The completion by cuts can be applied to any family of intransitive subchains such as the family of subchains defined by ear dicycles or dicycles.

It is emphasized that the completion by cuts is not unique in terms of critical choice-trials and therefore predictions of the model may not be very precise. However, the completion by cuts provides a general model and more specific assumptions can be incorporated. The model is based on the testable assumption of a sequential dependency between trials as stated in Definition 2.4.1.

2.5 Summary

In this chapter we defined intransitive choice as k-dicycles and discussed the general algebraic decomposition of digraphs into strong components which is related to the identification of all k-dicycles. The decomposition into strong components can be achieved by matrix operations and corresponds to the factorization or partition of associated polynomials. The partition has the advantage that it completely characterizes the intransitivities within a digraph. The coefficients of the partitioned polynomial ψ equal the number of dicycles in each strong component. Associated with the problem of identifying a minimal set of critical arcs that are responsible for all dicycles in a digraph is the so called acyclic subgraph problem. For tournaments a simple matrix technique was suggested and examples of decompositions into strong components were presented.

In a more specific model the ear decomposition was introduced. This technique is known to be efficient, and has a minimal solution. The ear decomposition by sequence finds a unique set of dicycles that constitutes a directed ear basis in a suitable space of incidence vectors. By using the sequence of intransitive choices in a pair comparison a unique basis can be identified.

In a digression related to the ear decomposition by sequence the completion by cuts on the sequence of choice-trials was suggested. This is a simple technique which can be performed on subchains associated with any family of intransitive dicycles. It leads to subsets of choice-trials which may be responsible for most intransitivities. Some aspects of the completion by cuts were discussed which may have implications on the detection of critical choices.

One might presume that we are now well-equipped with empirically testable assumptions which can be derived from the theoretical models. At this point, however, a few cautious words may be appropriate. The algebraic decomposition as outlined in Section 2.2 is not a model of choice but an exhaustive descriptive characterization of individual choice behavior. It therefore offers limited opportunity for testing except for comparative purposes. As mentioned before, the completion by cuts introduced in Section 2.4 has no computer-based implementation and was not applied. The ear decomposition is based on the sequence of intransitive choice-trials. Consequently, it should be sensitive to systematic changes of the sequence of choice-trials in a pair comparison. This specific assumption was tested in the next chapter. The completion by cuts makes a slightly stronger assumption about the sequence of choice-trials in Definition 2.4.1. It states that subsequent choice-trials belong to the same point of view unless a shift of perspective has occurred. It is stressed here that in contrast to assumptions about the sequence of choice-trials the independence of choice-trials is an implicit or explicit assumption of all classical algebraic and probabilistic models as discussed in Chapter 1.

In the next chapter an experimental design is proposed which tests the independence of choices in regard of intransitive choice behavior. If we can show that inconsistent choice behavior systematically adapts to different arrangements of choice-trials then this would favor the ear decomposition by sequence and reject classical algebraic or probabilistic models. Algebraic decomposition models are designed to study individual decision behavior. However, experimental testing is usually based on group data. Because individual choice varies additional qualitative information provided by the decompositions had to be excluded. As a consequence, the experimental testing in the next chapter focuses on a comparison of quantitative measures of inconsistency, that is Kendall's ζ and τ, with new measures derived from the algebraic decompositions: the number of k-dicycles, ear dicycles, and the size of strong components. A qualitative validation of the decomposition techniques is left to future studies.

Chapter 3

Experiments

The main objective of this chapter is to produce empirical evidence for the algebraic decompositions introduced in the previous chapter. More specifically, it is tested if inconsistent choice behavior is sensitive to changes in the design of pair comparisons.

If intransitive choices are not random errors but are related to changes in the decision process, then it should be possible to induce systematic differences experimentally. It was claimed that the Ellsberg Paradox is an example of a persistent change in the information process induced by carefully constructed alternatives. A more subtle factor which has theoretical and empirical implications is tested by changing the sequence of choice-trials in a pair comparison. If no differences in intransitive choice between specially designed pair comparisons are observed, then the assumption of independent information processing in successive choice-trials cannot be replaced by weak assumptions about adapting choice behavior. If, however, inconsistency of choice behavior is systematically affected by the sequence of choice-trials and if there is an effect over sessions then an alternative approach such as the ear decomposition by sequence is recommended.

The ear decomposition by sequence offers an explanation of adaptive individual choice behavior because it operates on the sequence of intransitive choices. The finding that intransitive choices are not random but are related to the sequence of choice-trials would be difficult, if not impossible, to explain within the framework of traditional algebraic or probabilistic models.

Six experiments on individual decision behavior in three different domains are reported in this chapter to explore whether or not this new approach can be applied to different domains. The first and second experiment investigate riskless choice between named alternatives in the domain of chocolate bars. The third and fourth experiment study risky choice between described

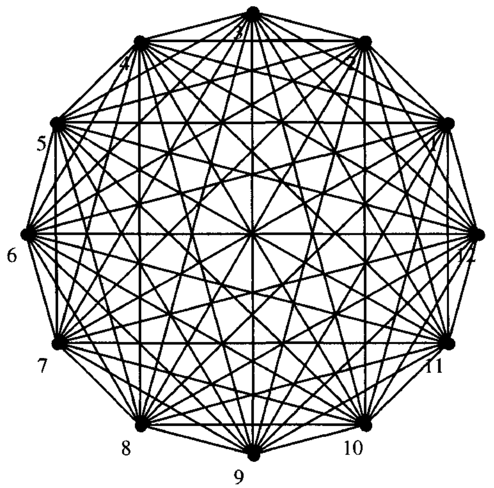

Figure 3.1: Complete graph of order 12

alternatives in the domain of lotteries, and the fifth and sixth experiment investigate psychophysical discrimination between disks varying in brightness contrast. The aim of the experiments is to decide in which of the three domains algebraic decomposition models offer an alternative to classical algebraic and probabilistic decision models. In particular, the ear decomposition by sequence is expected to trace adapting intransitive decision behavior providing a better measure of inconsistency. The completion by cuts as outlined in Section 2.4 was not applied to the data sets because a computerized implementation has yet to be developed.

3.1 General Method

In the following experiments subjects chose in two or three sessions between the same set of alternatives. In a single session subjects chose in an incomplete pair comparison between 12 alternatives comprising 66 choice-trials. The sequence of trials was arranged in different block designs which are explained below. In Figure 3.1 a complete graph of order 12 is depicted that also describes a pair comparison. Each of the 66 lines connecting two vertices refer to a single pair comparison or binary choice. If an orientation is assigned to each connecting line a tournament of order 12 results, which completely describes the preference relations in a pair comparison.

The block designs were constructed so that in each block of six choice-trials

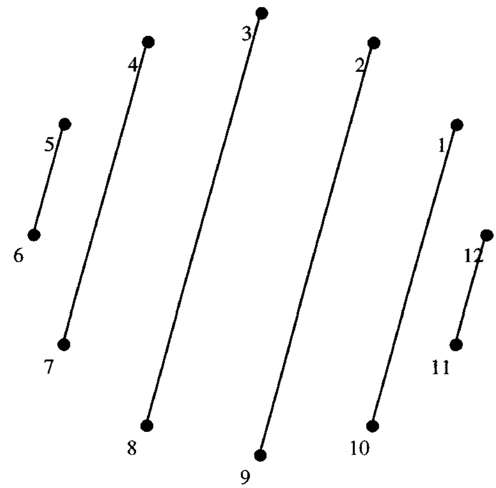

Figure 3.2: Graph of resolution block B_0

an alternative of the preceding trial was either repeated in the successive trial or appeared only once in each block. In the following the resulting two pair comparisons are called *repetition block design* and *resolution block design*, respectively. In combinatorial theory the latter is also known as the resolution of a balanced incomplete block design (Street & Street, 1987, Chapter 2).

Figure 3.2 and Figure 3.3 illustrate the difference between the two block designs and Table 3.1 contains the arrangement of choice-trials used in the experiments. In the resolution block design all alternatives appear in each block whereas the repetition block design covers only half of the alternatives in a single block. The construction of the resolution block design can be expressed in mathematical terms as follows: Let $V = \{\omega, 0, 1, \ldots, 2n-2\}$ be the set of alternatives in a pair comparison (resulting in a complete graph of order $2n$). Define addition on the set V to be carried out modulo $2n-1$, except that $\omega + i = \omega$ for all $i \in V$.

Definition 3.1.1 *Let $V = \{\omega, 0, 1, \ldots, 2n-2\}$ be a set where addition is defined as before. Then for $i = 0, 1, 2, \ldots, 2n-2$*

$$B_i := \{\omega, i\}, \{1+i, 2n-2+i\}, \{2+i, 2n-3+i\}, \ldots, \{n-i+1, n+i\}$$

is called resolution block.

Collecting the blocks $\{B_0, B_2, \ldots, B_{2n-2}\}$ constitutes a resolution block design.

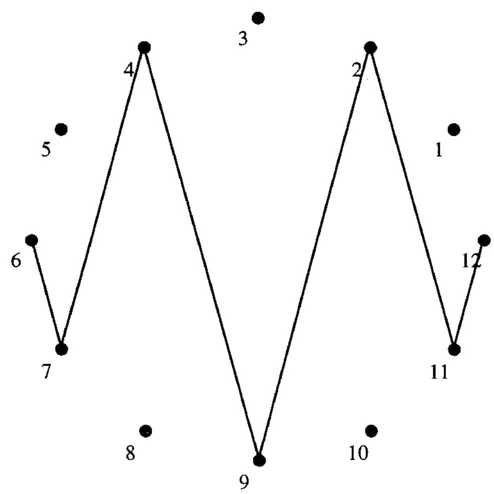

Figure 3.3: Graph of repetition block B'_0

It is emphasized here that both block designs are incomplete pair comparisons. They differ only by their arrangement of choice-trials. The repetition blocks B'_i are constructed by replacing the trials of every second column in the resolution block B_i by trials of every second column in B_{i+1}. Consequently, half of the trials appear at the same position or trial number in both designs (see Table 3.1). The important feature of the two block designs is that they are optimal in the sense that the number of repetitions between successive trials is maximal under the repetition block design and minimal under the resolution block design.

In the third block design subjects chose in an individually randomized sequence of trials leading to repetition and resolution blocks of random length between one and six choice-trials. Randomization is a standard and widely used technique in the social sciences because it is convenient to assume that a randomized sequence of trials rules out unwanted sequential effects. Putting aside the problems associated with generating (pseudo-) random sequences, this assumption is challenged here for small samples. It is argued that randomized choice-trials are subject to similar adaptive choice behavior as the repetition block design. A design with randomized trials appeared in Experiment 2A and 2B, and Experiment 3A and 3B and is called *random block design*. Note that the number of repetitions of alternatives in successive trials under the random block design reaches on average an intermediate level compared to the repetition and resolution block design.

Table 3.1: *Resolution and Repetition Block Designs*

Block	Resolution Block Design					
B_0:	{12,11}	{1,10}	{2,9}	{3,8}	{4,7}	{5,6}
B_1:	{1,12}	{11,2}	{10,3}	{9,4}	{8,5}	{7,6}
B_2:	{12,2}	{3,1}	{4,11}	{5,10}	{6,9}	{7,8}
B_3:	{3,12}	{2,4}	{1,5}	{11,6}	{10,7}	{9,8}
B_4:	{12,4}	{5,3}	{6,2}	{7,1}	{8,11}	{9,10}
B_5:	{5,12}	{4,6}	{3,7}	{2,8}	{1,9}	{11,10}
B_6:	{12,6}	{7,5}	{8,4}	{9,3}	{10,2}	{11,1}
B_7:	{7,12}	{6,8}	{5,9}	{4,10}	{3,11}	{2,1}
B_8:	{12,8}	{9,7}	{10,6}	{11,5}	{1,4}	{2,3}
B_9:	{9,12}	{8,10}	{7,11}	{6,1}	{5,2}	{4,3}
B_{10}:	{12,10}	{11,9}	{1,8}	{2,7}	{3,6}	{4,5}
Block	Repetition Block Design					
B'_0:	{12,11}	{11,2}	{2,9}	{9,4}	{4,7}	{7,6}
B'_1:	{1,12}	{3,1}	{10,3}	{5,10}	{8,5}	{7,8}
B'_2:	{12,2}	{2,4}	{4,11}	{11,6}	{6,9}	{9,8}
B'_3:	{3,12}	{5,3}	{1,5}	{7,1}	{10,7}	{9,10}
B'_4:	{12,4}	{4,6}	{6,2}	{2,8}	{8,11}	{11,10}
B'_5:	{5,12}	{7,5}	{3,7}	{9,3}	{1,9}	{11,1}
B'_6:	{12,6}	{6,8}	{8,4}	{4,10}	{10,2}	{2,1}
B'_7:	{7,12}	{9,7}	{5,9}	{11,5}	{3,11}	{2,3}
B'_8:	{12,8}	{8,10}	{10,6}	{6,1}	{1,4}	{4,3}
B'_9:	{9,12}	{11,9}	{7,11}	{2,7}	{5,2}	{4,5}
B'_{10}:	{12,10}	{1,10}	{1,8}	{3,8}	{3,6}	{5,6}

Note: The numbers 1 to 12 refer to alternatives. Their position in each pair indicates the actual presentation (left and right, up and down) on screen.

Two experiments were conducted in each domain of alternatives. In the first experiment block designs were tested between subjects. In the second experiment block designs were tested within subjects. It was hoped that the test of block designs between and within subjects would help to validate results and increase the understanding of adapting choice behavior. In Experiment 1A the block designs varied *between subjects* of Group S and P, and in Experiment 2A, and 3A between subjects of Group S, P, and N. Each group label corresponds to the third letter of the block design, that is Group P stands for rePetition block design, Group S for reSolution block design and Group N for raNdomized block design. In these experiments pair comparisons under the same block designs were repeated over two or three sessions. Note that the repetition of a whole pair comparison with the same block design can be understood as an extension of the same block design. The block designs in Experiment 1B, 2B, and 3B varied *within subjects* over successive sessions. Experiment 1B investigates the effect of exchanging sessions for the resolution and repetition block design in Group SP and PS. In Experiment 2B and 3B it was tested if an exchange of the resolution and repetition block design in the first and last session of Group SNP and PNS has an effect.[1]

3.2 Hypotheses and Statistical Testing

In the following the hypotheses for the experiments are summarized. In general it was predicted that individuals interact with block designs by adapting to the sequence of choice-trials. How subjects interact with the block design is believed to be highly domain-specific and hard to predict. Therefore, the alternative hypothesis remained unspecified in the first experiment of each domain and was tested against the null hypothesis of no effect of block design. This leads to two-tailed testing on a reduced significance level of 2.5 percent (Winer, Brown & Michels, 1991). The following hypotheses were tested in the first experiment of each domain.

(1) The resolution and repetition block design should lead to differences in inconsistent choice behavior. Random blocks are a mixture of repetition and resolution blocks. Therefore, inconsistency should reach an intermediate level under the random block design.

(2) It is expected that over sessions increasing familiarity with task and alternatives reduces inconsistency and shortens mean response times.

Conducting two experiments in each domain also had the purpose of confirming results from the first experiment in the second. In particular, the effect of block designs in the first experiment was used to specify the alternative hypothesis in the first session of the second experiment. Therefore planned (one-tailed) t-tests were conducted on a 5 percent significance level to detect differences between block designs in the first session of the second experiment.

From the construction of the block designs the following more specific predictions about inconsistent choice behavior were made. It can be predicted that under the resolution block design familiarity with initially unfamiliar

[1] In a factorial design all possible sequences of block designs in sessions would have required nine groups. In order to cut down the number of subjects, the random block design was only presented in the second session of Experiment 2B and 3B.

alternatives is achieved earlier because all alternatives appear in each block. Therefore, resolution blocks should facilitate exhaustive or independent information processing over successive trials leading to more consistent choice behavior. Accordingly, one would predict that under the repetition block design familiarity with initially unfamiliar alternatives is achieved later because all the alternatives appear only in every two blocks. Moreover, repetition of multiattribute alternatives is likely to encourage selective information processing because subjects can reduce the cognitive effort when the same alternative appears in successive choice-trials. Thereby an important attribute of the alternatives may have been disregarded which comes into play in a later choice-trial. Consequently, a repetition block design should cause more inconsistent choice behavior.

Furthermore, if selective information processing occurs in successive trials then this should produce more and longer dicycles whereas complete information processing between trials should lead to less and shorter dicycles. More specifically, the ear decomposition by sequence should produce directed ear bases with more and longer ear dicycles under the repetition block design than under the resolution block design.

The three block designs were compared to investigate the effect of repeated alternatives in successive trials on inconsistent choice behavior. Accordingly, measures of intransitive choice behavior should discriminate between the three block designs. The experiments reported in this chapter employ pair comparisons and statistical analyses focus on quantitative measures of intransitive choice such as the number of 3-dicycles, number of k-dicycles and number of ear dicycles of length k, $(3 \leq k \leq n)$. Analyses of variance were conducted on univariate measures of inconsistency (Kendall's ζ, and size of strong components) whereas stepwise discriminant analyses were applied to find the best discriminatory variables among multivariate measures of inconsistency (k-dicycles and ear dicycles of length k). Further evidence was expected from analyses of variance on preference reversals between pair comparisons and mean response times.

At this point some remarks about the statistical methods appear necessary. Statistical testing of the number of k-dicycles and ear dicycles creates its own problems because the distributions for these variables are unknown. The use of group data for statistical analysis has the advantage that the mean number of k-dicycles and ear dicycles is approximately normal distributed regardless of their underlying distribution (Central Limit Theorem, cf. Rice, 1988). The same argument applies to the analyses of mean response times. Considering the small sample sizes (N=10) in the experiments, it is likely that this assumption was violated in some of the analyses. However, the analysis of variance is known to be robust even if the assumption of a normally distributed variable is not fulfilled (Winer, Brown & Michels, 1991). Therefore, this assumption remained untested.

As the analysis of variance the discriminant analysis is related to linear multiple regression and hypothesis testing requires normally distributed variables and homogeneous covariances. The stepwise discriminant analysis selects a subset of quantitative variables to produce a good discrimination model using stepwise selection (Klecka, 1980). The set of variables that make up each class is assumed to be multivariate normal with a common covariance matrix. Variables are chosen to enter or leave the model according to the following criterion: The significance level of an F test from an analysis of covariance, where the variables already chosen act as covariates and the variable under consideration is the dependent variable. Because the partial F statistics in successive steps are not independent it is not advisable to use critical values of the F distribution. In most applications, all variables considered have some discriminatory power, however small. To choose the model that provides the best discrimination using the sample estimates one has to guard against estimating more parameters than can be reliably estimated with the given sample size; hence, a moderate significance level of 15 percent is appropriate.[2]

Stepwise selection started like forward selection with no variables in the model. At each step during the analysis the model was examined. If the variable in the model that contributed least to the discriminatory power of the model as measured by Wilk's Λ failed to meet the criterion to stay, then that variable was removed. Otherwise, the variable not in the model that contributed most to the discriminatory power of the model was entered. When all variables in the model met the criterion to stay and none of the other variables met the criterion to enter, the stepwise selection process stopped.

It is important to realize that in the selection of variables for entry, only one variable can be entered into the model at each step. The selection process does not take into account relationships between variables that have not yet been selected. Thus, some important variables could have been excluded in

[2]Costanza and Afifi (1979) used Monte Carlo studies to compare alternative stopping rules for the forward selection method in a two-group multivariate normal classification problem. They conclude that the use of a moderate significance level, in the range of 10 percent to 25 percent, often performs better than the use of a much larger or a much smaller significance level. Hence, a significance level of 15 percent was the criterion for entering or removing variables in stepwise discriminant analyses.

the process.

The stepwise discriminant analyses on the number of all dicycles and the number of ear dicycles can be compared in terms of their discriminatory power. It is hypothesized that the ear decomposition by sequence discriminates better between block designs because the mean number of ear dicycles have smaller variances and are based on the sequence of intransitive choices.

All statistical analyses were conducted using procedures of the statistical package SAS/STAT (Version 6.0) on a VAX computer.

3.3 Experiment 1A: Riskless Choice

In the first experiment it was investigated if individual choice behavior is affected by the resolution and repetition block design. More specifically, it was asked if forced choice pair comparisons with successive binary choices display different inconsistencies when a labeled or named alternative is repeated from trial to trial (repetition block design) or when no alternative is repeated over blocks of six choices (resolution block design).

3.3.1 Method

By using forced choice pair comparisons, subjects chose in two separate sessions between the same set of named alternatives under different fixed block designs. The block designs were constructed so that in each block either an alternative of the preceding trial was repeated in the successive trial (repetition block design) or each alternative appeared only once in each block (resolution block design).

Design

In a 2×2 design (block design by session) with repeated measurement on the second factor two groups of subjects chose in forced choice pair comparisons with different block designs in two consecutive sessions. The block designs were varied between subjects.

- Group S: In the first and second session subjects chose under the reSolution block design.

- Group P: In the first and second session subjects chose under the rePetition block design.

Dependent variables were inconsistencies and mean response times. Measures of inconsistency were derived from the choices in terms of preference reversals (Kendall's τ), intransitive triples (Kendall's ζ), directed k-cycles (coefficients z_k of polynomial ψ), and ear dicycles of length k (of a directed ear basis). Kendall's τ is a standardized measure for the number of reversed preferences in a set of $|S| = n$ alternatives. It is defined as

$$\tau = 1 - (4R/R_n) \tag{3.1}$$

where $R_n = n(n-1)$, and R is the number of reversed relations (Kendall, 1970). Values of Kendall's τ may vary between -1.0 and +1.0, for the maximal and minimal number of reversals, respectively.

Depending on $|S| = n$, the number of intransitive triples in a pair comparison can range from $T = 0$ up to a maximal number T_n of

$$\begin{aligned} T_n &= \frac{n(n^2-1)}{24} \qquad \text{for } n \text{ odd and} \\ T_n &= \frac{n(n^2-4)}{24} \qquad \text{for } n \text{ even.} \end{aligned} \tag{3.2}$$

Kendall's ζ is then defined as

$$\zeta = 1 - T/T_n \tag{3.3}$$

where T is the observed number of intransitive triples or 3-dicycles (Kendall, 1970; Kendall & Babington-Smith, 1940). The standardized measure of intransitive triples ζ can range from 0 to 1.0, for maximal and minimal number of 3-dicycles, respectively. Listings of programs which recorded the number of reversals between pair comparisons, dicycles (expressed as coefficients z_k for each pair comparison), and ear dicycles are presented in Appendix C.

Response times of 66 choice-trials were averaged for each subject and session and are referred to as mean response times.

Subjects

A total of 20 subjects participated in this experiment. Half of them were students at the University of Heidelberg and the other half had various occupations. They were assigned to two groups: Group S (average 26.4 years of age, range 18-33), Group P (average 25.5 years of age, range 17-32). Subjects in each group were balanced in gender. They received 12.- Deutsche mark per hour for their participation in an unrelated experiment which took place between the two sessions.

Materials and Apparatus

Twelve chocolate bars and their names (see Table B.1) were presented to the subjects. If the subject was unfamiliar with one of the chocolate bars then the experimenter offered a slice of the chocolate bar to the subject who tasted it. During choice-trials only the names for a pair of chocolate bars were displayed on a computer screen in 9×16 dot characters with a refresh rate of 70 Hz. The screen intensity was adjusted to an easy reading level and was maintained at this level throughout the experiment. The experiment was programmed in MEL and run on an Tandon 386 (IBM-AT compatible) computer allowing response time measures from the keyboard with an accuracy of up to 6–8 milliseconds (Schneider, 1988; 1990).

Procedure

The subject had two sessions each with $\binom{12}{2} = 66$ choice-trials. The two sessions were separated by an unrelated experiment in the field of social psychology which lasted between half an hour and an hour. Before the first session all twelve chocolate bars were presented to the subject. The subject was asked to taste any of the chocolate bars they were unfamiliar with. The experimenter recorded which bars were tasted (see Table B.1). Subsequently, the subject was seated in front of a computer screen and keyboard. First, the subject read the general instruction explaining the experiment (Appendix B.1.1). On each choice-trial the following question was displayed on screen: Which chocolate bar tastes better? Two seconds later the names of two chocolate bars were displayed on the left- and righthand side of the screen and response timing was initiated. Then the subject chose either the left or right alternative by pressing the 'F' or 'J' key with their left and right index finger, respectively. Each session started with three training trials so that the subject knew how to express their preference by pressing the appropriate key. Each session lasted between six and twelve minutes.

3.3.2 Results

This section is divided according to the three dependent variables response times, preference cycles and preference reversals.

The hypotheses are summarized as follows: The resolution and repetition block design should show different inconsistent choice behavior as measured by Kendall's ζ and τ, number of dicycles, and ear dicycles. In comparison the number of ear dicycles should discriminate better between block designs than the total number of dicycles. Moreover, a decrease in response times and inconsistency was expected over sessions.

Response Times

Mean response times were analyzed to compare groups and to test effects of learning and familiarity over sessions. In Table 3.2 the mean and standard deviations of mean response times are tabulated for each group and session.

Table 3.2: *Mean Response Times of Choices (Exp 1A)*

	Session 1		Session 2	
	Mean[a]	*SD*	*Mean*	*SD*
Group S (N=10)	2.40	*.76*	1.45	*.43*
Group P (N=10)	2.50	*.82*	1.83	*.55*

Note: Statistical significant effect between Session 1 and 2, $F(1,18) = 25.86, p = 0.0001$).

[a]in seconds

For the two block designs mean response times are significantly longer in the first session than in the second. On average the mean response times are reduced by 0.81 secs from 2.45 secs in the first session to 1.64 secs in the second.

Mean response times were entered into a 2 by 2 analysis of variance (ANOVA) with repeated measurement on the second factor. Block design (resolution, repetition) served as fixed factor between subjects, session (first, second) as fixed factor within subjects and subjects acted as random factor (see Table 3.3 for detailed results).[3] The analysis revealed no statistically significant effect between block designs ($F[1,18] < 1$, ns). However, there was a highly significant effect of session and the hypothesis of no effect across sessions is rejected ($F[1,18] = 25.86$, $p = 0.0001$). No other effects approached statistical reliability.

Preference Cycles

Table 3.4 lists the mean and standard deviation of Kendall's ζ for each group and session. Kendall's ζ is a standardized measure of the number of intransitive triples (3-dicycles) within each preference matrix. As mentioned before, if there are no intransitive triples in a pair

[3]For the reason that only two levels of the within-subject variable were present no test of homogeneity is required. The normality assumption remained untested because the F-statistic is considered as sufficiently robust against violations and response times were averaged across 66 trials for each subject (Winer, Brown, & Michels, 1991).

Table 3.3: *ANOVA on Mean Response Times (Exp 1A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	581.2	*1*	581.2	0.94
Error	11120.3	*18*	617.8	
Within subjects:				
Session	6495.8	*1*	6495.8	25.86***
BD by Session	195.7	*1*	195.7	0.78
Session by Error	4520.8	*18*	251.2	

Note: ***$p < 0.0001$.

comparison then ζ should equal 1, and 0 if there is a maximal number of intransitivities (Kendall, 1970).

Table 3.4: *Intransitive Triples of Preference (Exp 1A)*

	Kendall's ζ			
	Session 1		Session 2	
	Mean	*SD*	*Mean*	*SD*
Group S	.90	*.09*	.89	*.15*
Group P	.97	*.04*	.96	*.06*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2-4)/24 = 70$ for $n = 12$, and T number of 3-dicycles.

In their first and second session subjects chose under the resolution block design in Group S and under the repetition block design in Group P. In summary, the ζ-values for the first sessions suggest that the repetition block design leads to more consistent choices. On the other hand, subjects displayed more intransitive triples if they chose under the resolution block design. The same type of analysis as for mean response times was applied to Kendall's ζ (Table 3.5).

Although the effect of block design nearly reached the significance level of 2.5 percent the null hypothesis of no effect between block designs could not be rejected ($F[1,18] = 5.86$, $p = 0.026$). No effect across sessions ($F[1,18] < 1$, ns) was detected and no other effect approached statistical significance.

Stepwise discriminant analyses were performed on the number of k-dicycles, i.e. the coefficients z_k of the polynomial ψ, as well as the number of ear dicycles from the ear decomposition by sequence.[4] The number of k-dicycles were entered stepwise as dependent variables starting with dicycles who discriminated best between the two groups. Table 3.6 and 3.7 gives the results of the analyses. Variables were entered and excluded at a significance level of $\alpha = 0.15$.

[4]Two Prolog programs are listed in Appendix C.1. The first detects all dicycles in a coded digraph and the second the ear dicycles of an ear bases.

Table 3.5: *ANOVA on Kendall's ζ (Exp 1A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	0.046	*1*	0.046	5.86*
Error	0.141	*18*	0.008	
Within subjects:				
Session	0.0003	*1*	0.0003	0.03
BD by Session	0.00001	*1*	0.00001	0.00
Session by Error	0.177	*18*	0.0098	

Note: $^*p < 0.05$.

Table 3.6: *Stepwise Discriminant Analysis on Dicycles (Exp 1A)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
Session 1						
1.	3-cycles	-	1	0.199	4.477	0.049
2.	4-cycles	-	2	0.135	2.650	0.122
Session 2						
No variables entered						

In Session 1 the two groups can be discriminated by the number of 3-dicycles, and in addition by the number of 4-dicycles. No variable reached the significance level in Session 2. The number of k-dicycles for each subject and session are listed in Table B.2 and B.3 together with their mean and standard deviation. A comparison of Subject 39 in Session 1 and Subject 45 in Session 2 illustrates that 3-dicycles alone are not sufficient to describe inconsistency in a pair comparison.

The two groups can be discriminated by the number of ear dicycles of length 3 and 4. The considerably higher F-values in this analysis show that the use of ear dicycles instead of dicycles adds discriminatory power to the model. Accordingly, the ear dicycles of length 3 also reached the significance level in Session 2. The number of ear dicycles for each subject and session are listed in Table B.4 and B.5 together with mean and standard deviation.

Preference Reversals

A (two-tailed) t-test was performed on the number of reversed preferences between the two sessions using Kendall's τ as dependent variable.[5] Kendall's τ is a standardized measure for the number of reversed preferences and may vary between values of -1 and $+1$.

In Table 3.8 the mean and standard deviation of Kendall's τ are shown for both groups. There was no statistical significant effect between the two groups ($t = 1.62$, *df*=18, $p = 0.12$).

In general, mean response times are significantly shorter for both groups in the second session but there was no significant difference between block designs. The analyses on mean response times and Kendall's ζ suggest that no improvement of consistency took place over

[5] The number of preference reversals were detected by a program written in C listed in Appendix C.3.

Table 3.7: *Stepwise Discriminant Analysis on Ear Dicycles (Exp 1A)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1.	3-dicycles	-	1	0.355	9.90	0.006
2.	4-dicycles	-	2	0.374	10.17	0.005
			Session 2			
1.	3-dicycles	-	1	0.140	2.94	0.104

Table 3.8: *Reversals of Preference (Exp 1A)*

	Kendall's τ	
	Mean	*SD*
Group S	.77	*.11*
Group P	.84	*.09*

Note: $\tau = 1 - (4R/R_n)$ with $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed preferences.

the two sessions. In their second session subjects performed quicker but not more consistent than in their first session.

A nearly significant difference between the two block designs was detected. Subjects who chose under the resolution block design (Group S) displayed a more inconsistent choice behavior in terms of Kendall's ζ than subjects under the repetition block design (Group P). This was also reflected in the discriminant analyses on the number of dicycles and the number of ear dicycles. It was shown that 3-dicycles and 4-dicycles together with ear dicycles of length 3 and 4 distinguished between groups. The number of ear dicycles discriminated considerably better between groups than the total number of dicycles. It is believed that high familiarity with the alternatives may have contributed to the reversed intransitivities under the two block designs:

As mentioned earlier, subjects who chose under the resolution block design encountered all alternatives in a single block whereas under the repetition block design they made decisions involving all alternatives every two blocks. As a consequence, they should have been able to establish a preference structure at an earlier stage under the resolution block design. It was hypothesized that this results in more consistent choice behavior under the resolution block design than under the repetition block design. However, the presentation and tasting of all chocolate bars before the actual pair comparisons and the fact that alternatives were non-artificial, highly familiar objects probably reversed the hypothesized effect.

It was hypothesized that the repetition of alternatives from one trial to the next would facilitate selective information processing thereby increasing inconsistent choice behavior. Although the alternatives in this domain may be considered as multiattribute as the taste of chocolate bars is closely related to the ingredients of each chocolate bar (e.g. plain or milk chocolate, peanuts, hazelnuts, caramel, raisins, waffle), it is argued that subjects employed named alternatives in a simpler decision process. Encouraged by the presentation and tasting of chocolate bars before the choices, subjects might have used a preference value of the alternatives rather than comparing the two alternatives in an information process. Hence, inconsistency is not decreased under the resolution block design but consistency is increased under the repetition block design because it is likely that the same preference values are employed in a decision process if highly familiar alternatives are repeated in successive trial.

It is assumed that these arguments together with the fact that alternatives covered a wide range of attractiveness can account for the highly consistent choice behavior under both block designs and the significant difference between the number of ear dicycles of length 3 and 4. This effect is a consequence of the advantage of the repetition block design over the resolution block design in this domain of named and highly familiar alternatives.

3.4 Experiment 1B: Riskless Choice

In Experiment 1A it was tested if inconsistency of individual choice is affected by the resolution and repetition block design. The results indicated an advantage for Group P where subjects chose under the repetition block design. This experiment investigates if this effect can be replicated when the block designs are varied within rather between subjects. It was expected that in the first session similar ζ-values as in the first session of Experiment 1A would occur and that reversed values would appear in the second session due to the interchanged block designs.

3.4.1 Method

By using pair comparisons, subjects chose in two separate sessions between the same set of named alternatives under different fixed block designs. The block designs were constructed in the same way as in Experiment 1A.

Design

In a 2×2 design (sequence by session) with repeated measurement on the second factor, the same subject chose under different pair comparisons in two consecutive sessions. The block designs were varied within subjects and the sequence of the block designs between subjects.

- Group SP: In their first session subjects chose under the reSolution block design and in their second session under the rePetition block design.
- Group PS: In their first session subjects chose under the rePetition block design and in their second session under the reSolution block design.

Dependent variables were the binary choices and their response times. The same measures of inconsistency as in Experiment 1A were employed.

Subjects A total of 22 subjects participated in this experiment.[6] Half of the subjects were students of the University of Heidelberg and the other half had various occupations. 10 subjects were assigned to each group: Group SP (average 29.0 years of age, range 22-43 years), and Group PS (average 26.5 years of age, range 21-38 years). The subjects for each group were balanced in gender. They received 12.- Deutsche mark per hour for their participation in an unrelated experiment which took place between the two sessions.

Materials and Apparatus

The same twelve chocolate bars as in Experiment 1A (see Table B.1) were presented to the subjects. Before the first pair comparison subjects were allowed to taste any of the chocolate bars if they were unfamiliar with them. The same set-up and equipment as in Experiment 1A was employed. During choice-trials only the names of the chocolate bars were displayed on screen.

[6]Two subjects were excluded and replaced because they were unfamiliar with more than five alternatives.

Procedure

The same procedure as in Experiment 1A was applied. Each session lasted between six and twelve minutes.

3.4.2 Results

This section is divided according to the dependent variables response times, preference cycles and preference reversals. The hypotheses were the same as for Experiment 1A with the exception that the alternative hypothesis for the effect of block design was specified and tested against the null hypothesis. According to the findings in Experiment 1A choice behavior should be more consistent under the repetition block design than under the resolution block design and should appear in the first session.

Response Times

In Table 3.9 mean and standard deviations of mean response times are tabulated for both groups and sessions.

Table 3.9: *Mean Response Times of Choice Trials (Exp 1B)*

	Session 1		Session 2	
	Mean[a]	*SD*	*Mean*	*SD*
Group SP (N=10)	2.94	*.89*	2.16	*.51*
Group PS (N=10)	2.94	*1.19*	2.05	*.64*

Note: Statistical significant effect between Session 1 and 2, $F[1,18] = 28.3$, $p < 0.0001$).

[a]in seconds

For every group mean response times were significantly longer in the first session than in the second. On average the mean response times were reduced by 0.84 secs from 2.94 secs in the first session to an average of 2.1 secs in the second session.

Mean response times were entered into a 2 by 2 analysis of variance with repeated measures on the second factor. Sequence of block designs (resolution-repetition, repetition-resolution) served as fixed factor between subjects, session (first, second) as fixed factor within subjects and subjects acted as random factor (see Table 3.10 for detailed results).[7] The analysis revealed no statistically significant effect between groups ($F[1,18] < 1$, ns). However, the hypothesis of no effect across sessions was rejected ($F[1,18] = 25.31$, $p = 0.0001$). No other effects approached statistical reliability.

Preference Cycles

Table 3.11 shows the mean and standard deviation of Kendall's ζ for each group and session as a standardized measure of the number of intransitive triples in each pair comparison.

In the first session Group SP chose under the resolution block design whereas Group PS made choices under the repetition block design. As in Experiment 1A the ζ-values in the first

[7]For the reason that only two levels of the within-subject variable were present no test of homogeneity is required.

Table 3.10: *ANOVA for Mean Response Times (Exp 1B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	29.9	*1*	29.9	0.02
Error	21550.7	*18*	1197.3	
Within subjects:				
Session	7055.6	*1*	7055.6	28.31***
S by Session	35.4	*1*	35.4	0.14
Session by Error	4485.5	*18*	249.2	

Note: $^{***}p = 0.0001$.

Table 3.11: *Intransitive Triples of Preference (Exp 1B)*

	Kendall's ζ			
	Session 1		Session 2	
	Mean	*SD*	*Mean*	*SD*
Group SP	.94	*.08*	.98	*.02*
Group PS	.97	*.02*	.97	*.03*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2-4)/24 = 70$ for $n = 12$, and T number of 3-dicycles.

session suggest that the repetition block design enhanced consistent choice. On the contrary, subjects displayed more inconsistent choice behavior under the resolution block design. In the second session Group SP chose under the repetition block design and Group PS under the resolution block design. There was no difference between the two block designs and subjects displayed approximately the same ζ-values.

The same analysis was applied to the number of intransitive triples for each preference matrix using Kendall's ζ as a standardized measure. Table 3.12 gives the results of the analysis in full detail.

As can be expected from the results of Experiment 1A and the design of this experiment there was no significant main effect between groups ($F[1,18] < 1$, ns) and no significant effect between sessions ($F[1,18] = 2.11, p = 0.16$). A planned t-test between the first session of Group SP and PS did not reveal a significant effect of block design (minimal significant difference = 0.056).

As in Experiment 1A stepwise discriminant analyses were performed on the number of k-dicycles, i.e. the coefficients z_k of the polynomial ψ, and the number of ear dicycles from the ear decomposition by sequence. None of the variables reached a significance level of $p = 0.15$ in Session 1 or Session 2. The number of k-dicycles for each subject and session are listed in Table B.6 and B.7. When comparing the number of k-dicycles in Table B.2 with B.3 only a few subjects had dicycles with length greater than 3. The number ear dicycles together with means and standard deviation are presented in Table B.8 and B.9

Table 3.12: *ANOVA for Kendall's ζ (Exp 1B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	0.011	*1*	0.011	0.47
Error	0.044	*18*	0.002	
Within subjects:				
Session	0.0037	*1*	0.0037	2.11
S by Session	0.0023	*1*	0.0023	1.28
Session by Error	0.0317	*18*	0.0018	

Preference Reversals

A (one-tailed) t-test was performed for the number of reversed preferences between the two sessions using Kendall's τ as a standardized measure.

Table 3.13: *Reversals of Preference (Exp 1B)*

	Kendall's τ	
	Mean	*SD*
Group SP	.82	*.08*
Group PS	.81	*.08*

Note: $\tau = 1 - (4R/R_n)$ where $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed preferences.

In Table 3.13 the mean and standard deviation of Kendall's τ is shown for each group. There was no statistical significant effect between groups ($t = 0.25$, *df 18*, $p = 0.80$).

As in Experiment 1A the analysis of variance on mean response times suggests that subjects performed quicker in the first session regardless of the block designs. From the already high ζ-values with small standard deviations in the first session it was expected that only Group SP may improve consistency of their choice behavior in the second session. Accordingly, the average ζ-values increased from .94 to .98 in Group SP but remained at a value of .97 in Group PS. If choice behavior were not adaptive then the block designs should have an independent effect in each session and the ζ-value for Group PS in the second session should have dropped to a value around .94. The unchanged values point to the conclusion that a more consistent preference structure persists and is hardly affected by the change of block designs.

In the first session subjects in Group PS established a more consistent preference structure while choosing under a repetition block design. The same explanation as in Experiment 1A is offered for the differences between block designs. It is very plausible that the preference structure from the first session was maintained in the second session leading to similar ζ-values and suppressing effects of the block design. This explanation is supported by the comparable τ-values in Group SP and PS.

3.5 Discussion

The results for the first two experiments may indicate that contrary to the original hypotheses no significant effect between block designs occurred for Kendall's ζ and that no improvement in consistency took place over the two sessions. Subjects performed quicker but not more consistent in the second session regardless of the block designs.

However, from the already high ζ-values with small standard deviations in the first session of Experiment 1B it can be expected that only Group SP would improve in consistency due to a ceiling effect. Accordingly, the average ζ-values increased from .94 to .98 but remained at .97 for Group PS.

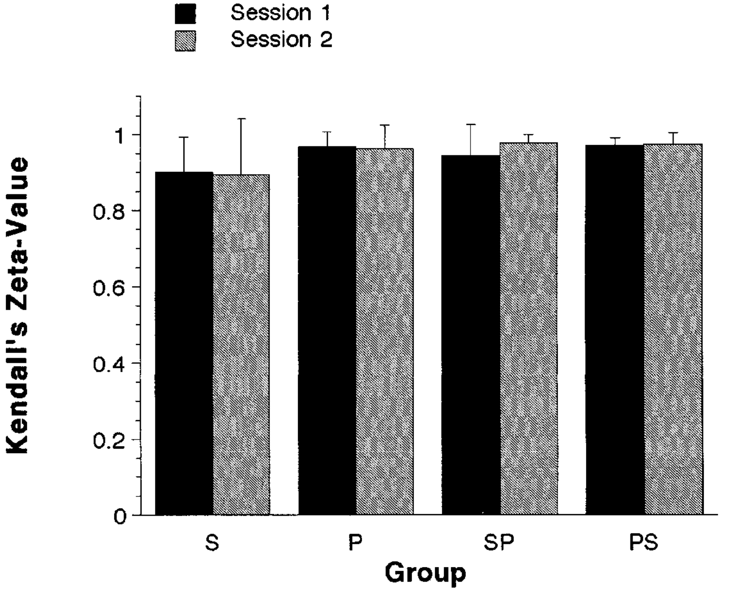

Figure 3.4: Kendall's ζ for all groups of Experiment 1A and 1B

In line with our hypothesis about the effect of the block designs, there were systematic lower average ζ-values for the resolution blocks of Group S and SP compared to the repetition blocks of Group P and PS in the first session. This effect also appeared for 4-dicycles in Experiment 1A as shown by the stepwise discriminant analysis on dicycles and ear dicycles. The same difference was present in the second session of Experiment 1A but did not reverse when the block designs were interchanged between groups in the second session of Experiment 1B. Therefore, improvement in consistency is regarded as irreversible unless more drastic alterations are introduced.

From these results the following question arises: Why did subjects exhibit less intransitivities under the repetition block design when a named alternative is repeated from trial to trial? As mentioned earlier, the answer probably lies in the way these highly familiar alternatives were perceived and compared. It was claimed that under the resolution block design early familiarity with multiattribute alternatives leads to more exhaustive and therefore more independent decision processes facilitating consistent choice whereas under the repetition block design information processing is more likely to be selective when the same alternative is repeated in successive choice-trials. Selective information processing would lead to more inconsistent choices.

In the domain of familiar and named alternatives, however, the repetition of one alternative from trial to trial might facilitate consistent choice behavior because the subject simplifies the decision process by recalling preference values only. It is emphasized that identification of alternatives is a preliminary and necessary part of the decision process in the domain of named alternatives. The identification is likely to be accompanied by a recall of preference values if the alternatives are sufficiently familiar.

More specifically, it is believed that in the domain of chocolate bars identification led to reduced inconsistencies in a repetition block design because names of chocolate bars were familiar and easily recognizable. In fact, in each choice-trial the alternatives had to be identified by their names before a choice could be made. Their preference value[8] could have been recalled from preceding trials simplifying the decision process. It is also likely that subjects already evaluated the chocolate bars when they were introduced to all alternatives and tasted unfamiliar bars at the beginning of the first session.

The main objective of our investigation was to identify inconsistency as a discriminatory variable of individual choice behavior. The systematic differences between block designs for 3 and 4-dicycles in Experiment 1A illustrate that at least some subjects interacted with the arrangement of the choice-trials. This contradicts the assumption of independent choice-trials which is essential to traditional probabilistic or non-adaptive algebraic models. Its violation supports the algebraic decomposition approach in this domain, even though results were not convincingly confirmed in Experiment 1B.

[8]Note that the term 'preference value' may be replaced by 'holistic evaluation', 'utility', 'information integration', or 'accumulative value' depending on the terminology of the theoretical framework it refers to.

3.6 Experiment 2A: Risky Choice

In the following experiment it was tested if the inconsistency of choice behavior is affected by different block designs in the domain of risky alternatives. The study of decision making under risk or uncertainty, mostly by using gambles or lotteries, has a long tradition and still is a popular reserach topic.

The design of the previous experiments was extended. In addition to the fixed resolution and repetition block design a random block design was introduced which should have an intermediate effect on inconsistency. Each subject had three sessions and the alternatives were generated so that they were almost indifferent in terms of expected utility.

3.6.1 Method

By using pair comparisons subjects chose in three sessions between the same set of lotteries[9] varying in probability and amount of winning. The fixed block designs, repetition and resolution block design, were constructed as in Experiment 1A and 1B. The random block design had a sequence of choice-trials which was randomized for each subject and each pair comparison.

Design

The experiment had a 3×3 design with repeated measurement on the second factor. In three consecutive sessions subjects chose between risky alternatives in different block designs. The three different block designs were varied between subjects:

- Group S: In all three sessions subjects chose in a reSolution block design.
- Group P: In all three sessions subjects chose in a rePetition block design.
- Group N: In all three sessions subjects chose in a raNdom block design.

Dependent variables were the choices and their response times as well as different measures of inconsistency in terms of preference cycles and preference reversals.

Subjects

A total of 30 undergraduate and graduate students were recruited from the subject panel of the Department of Experimental Psychology, Oxford University. There were 10 subjects in each group (Group S: average age 24.5 years, SD=5.21; Group P: average age 27.3 years, SD=6.43; Group N: average age 21.9 years, SD 3.21). The subjects for each group were balanced in gender. Each subject received £3 per hour plus traveling expenses for their participation in this experiment and in Experiment 3A or 3B. In each experiment subjects were assigned to different groups.

[9]Gambles are usually called *lotteries* if the decision maker is provided with probabilities rather than uncertain events and if the consequences are stated in terms of money.

Materials and Apparatus

Twelve lotteries with equivalent expectancy values were constructed by varying probability of winning (in percentages) and payoff (in pounds). Any possible loss x_2 was fixed to £1 with probability $1 - p$ whereas the probability of winning p and the win itself x_1 was varied to gain an approximate constant expectancy value of £9.1 (see Table B.10). The expectancy values of the lotteries varied around £9.1 with standard deviation SD=0.28.[10]

$$E(X) = px_1 + (1 - p)(-1) = 9.1 \tag{3.4}$$

The probabilities for each lottery were displayed in a pie chart with percentages written out. A legend above the pie chart gave information about possible win and loss in pounds. During choice-trials two lotteries differing in probability and payoff were displayed on a 21 inch Two-Page Monochrome Macintosh computer screen as illustrated in Figure 3.5. In Figure 3.6 the

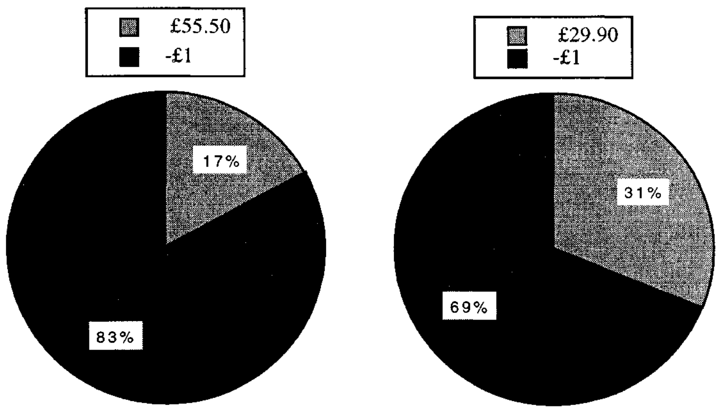

Figure 3.5: Display of two lotteries in a single choice-trial

payoff of the twelve lotteries is plotted on a logarithmic scale (in log £) against the chance of winning (in percent). If the probability scale were logarithmic, too, then the fitted function would be linear. The experiment was programmed in SuperLab and run on a Macintosh IIcx (enhanced by a floating point unit) allowing time measures on the keyboard with accuracy of up to 4 millisecs.

Procedure

Each subject had three sessions with $\binom{12}{2} = 66$ choice-trials plus 3 training trials with different alternatives. Sessions were separated by a session of Experiment 3A or 3B which lasted between six and twelve minutes.

The subject was seated in front of the computer screen and the keyboard. First, they read a general instruction about the experiment, followed by more detailed instructions (see Appendix B.2.1). In three training trials they learned how to give a response by pressing key

[10]The 12 lotteries were selected from a pool of 24 lotteries. In a pre-test four additional subjects computed roughly the expectancy value for each of the 24 lotteries following an instruction. Only lotteries which had on average approximately the same estimated expectancy value were selected.

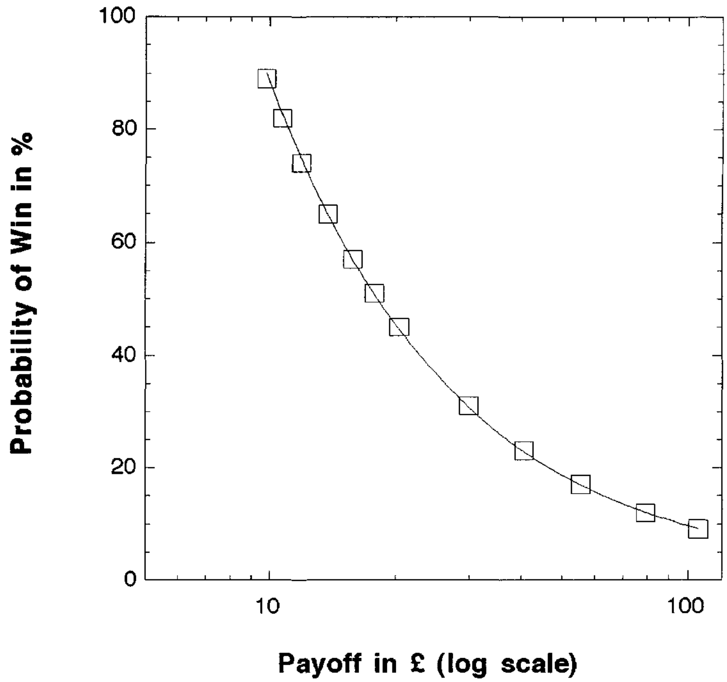

Figure 3.6: Lotteries plotted by payoff and chance of win on a log-linear scale. The fitted function is linear when plotted on log-log scales.

'D' or 'K' on the keyboard. In each choice-trial the following question was displayed first: 'Which gamble do you prefer to play?'.[11] After 3.0 seconds two lotteries were displayed on the left- and righthand side of the screen and response timing was initiated. The two lotteries stayed visible until the subject chose the left or right lottery by pressing the appropriate key. Each session lasted between ten and fifteen minutes.

3.6.2 Results

This section is divided according to the dependent variables response times, intransitive preferences, size of strong components, and preference reversals. The size of strong components is an example of an additional consistency measure which can be derived from the algebraic decompositions.

The hypotheses can be summarized as follows: The resolution and repetition block design should show different consistent choice behavior as measured by Kendall's ζ and τ, number of dicycles and ear dicycles. The random block design should lead to intermediate consistent choices. In comparison the number of ear dicycles should discriminate better between block designs than the total number of dicycles. Moreover, a decrease in response times and inconsistency can be expected over sessions.

[11]The question remained on the screen for 1.7 seconds.

Response Times

As in the previous experiments mean response times and standard deviations are listed in Table 3.14 for each group and session.

Table 3.14: *Mean Response Times of Choices (Exp 2A)*

	Session 1		Session 2		Session 3	
	Mean[a]	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group S (N=10)	3.33	*2.16*	2.69	*2.17*	2.20	*1.48*
Group P (N=10)	4.15	*2.27*	3.28	*2.32*	2.87	*2.12*
Group N (N=10)	4.13	*1.37*	2.79	*1.01*	2.07	*.87*

Note: Statistical significant effect between sessions, $F[2,54] = 36.79$, $p < 0.0001$) adjusted for heterogeneity.

[a]in seconds

The mean response times were significantly shorter in the second and third session than in the first session. On average mean response times were reduced by 0.95 secs from Session 1 to 2 and again by 0.54 secs from Session 2 to 3.

Mean response times were entered into a 3 by 3 analysis of variance (ANOVA) with repeated measurement on the second factor. Block design (resolution, repetition, random) served as fixed factor between subjects, session (first, second) as fixed factor within subjects and subjects acted as random factor (see Table 3.15 for complete results).

Table 3.15: *ANOVA for Mean Response Times (Exp 2A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	737.7	*2*	368.9	0.40
Error	24720.2	*27*	915.6	
Within subjects:				
Session	3422.9	*2*	1711.5	36.79***
BD by Session	271.0	*4*	67.8	1.46
Session by Error	2511.8	*54*	46.5	

Note: $***p = 0.0001$.

A sphericity test revealed a significant violation of homogeneity (approximate $\chi^2[2] = 16.76, p < 0.0002$) so that a conservative test statistic is recommended. Accordingly, degrees of freedom were adjusted by $\epsilon = 0.754$ as suggested by Huynh and Feldt (1976) and Huynh (1978).

The analysis showed no statistical significant effect between block designs ($F[2,27] < 1$, ns). However, the hypothesis of no effect across sessions was rejected ($F[2,54] = 36.79$, with $p < 0.0001$ adjusted for heterogeneity). No other effects approached statistical reliability.

Preference Cycles

The mean ζ-values are listed in Table 3.16 for each block design and session. The three groups show distinctive average ζ-values in the first session. During the first session subjects in Group S made choices under a resolution block design reaching average ζ-values of .64, whereas subjects in Group P had a value of .48. As expected, the average ζ-value of .56 for subjects choosing in the random block design lies in between.

Table 3.16: *Intransitive Triples of Preference (Exp 2A)*

	Kendall's ζ					
	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group S	.64	*.25*	.70	*.20*	.77	*.16*
Group P	.48	*.24*	.64	*.23*	.65	*.22*
Group N	.56	*.26*	.71	*.29*	.73	*.27*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2-4)/24 = 70$ for $n = 12$, and T number of 3-dicycles.

The same analysis as for mean response times was applied to the number of intransitive triples for each preference matrix using Kendall's ζ as a standardized measure. A sphericity test revealed a significant violation of homogeneity for the repeated measurements over sessions (approximate $\chi^2[2] = 15.66, p < 0.001$) and degrees of freedom were adjusted by $\epsilon = 0.767$.

Table 3.17: *ANOVA for Kendall's ζ (Exp 2A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	0.191	*2*	0.095	0.85
Error	3.015	*27*	0.112	
Within subjects:				
Session	0.393	*2*	0.196	6.88**
BD by Session	0.028	*4*	0.007	0.25
Session by Error	1.542	*54*	0.029	

Note: $^{**}p < 0.01$.

There was no significant effect between block designs ($F[2,27] < 1$, ns) but the analysis revealed a highly significant effect across sessions ($F[2,54] = 6.88$, $p = 0.005$, adjusted for heterogeneity). Table 3.17 shows the detailed results of the analysis.

For each session stepwise discriminant analyses were performed on the total number of dicycles, i.e. the coefficients z_k of the polynomial ψ, and number of ear dicycles from the ear decomposition by sequence. The number of k-dicycles were selected stepwise as dependent variables starting with dicycles whose number discriminated best between groups. For the analysis of all dicycles no variables could be entered in any of the sessions. The number of k-dicycles for each subject and session are listed in Table B.11 for Group S, in Table B.12 for Group P and in Table B.13 for Group N characterizing all intransitivities in the pair comparisons. A comparison of polynomials with the same z_3 coefficients, i.e. number of 3-dicycles, illustrates that in some cases the coefficients z_k with $k > 3$ can be considerably different.

The stepwise discriminant analysis on the number of ear dicycles revealed differences between groups. The results are summarized in Table 3.18. The ear dicycles of length 10

Table 3.18: *Stepwise Discriminant Analysis on Ear Dicycles (Exp 2A)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1.	10-dicycles	-	1	0.286	5.40	0.011
			Session 2			
1.	10-dicycles	-	1	0.141	2.21	0.129
			Session 3			
1.	3-dicycles	-	1	0.159	2.54	0.097

had considerable discriminatory power in the first session but less so in the second session. Ear dicycles of length 3 did also discriminate in the last session. The number of ear dicycles from the ear decomposition by sequence for each subject and session are listed in Table B.14 for Group S, in Table B.15 for Group P, and in Table B.16 for Group N.

Strong Components

The average size of strong components was determined for each pair comparison (see Table 3.19) as an additional consistency measure. The size of strong components can be derived from the ear decomposition and a larger size indicates inconsistent choices between more alternatives.

Table 3.19: *Average Size of Strong Components (Exp 2A)*

	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group S	9.77	*3.77*	7.43	*4.92*	6.77	*4.78*
Group P	10.97	*3.25*	9.22	*4.67*	8.87	*4.23*
Group N	8.04	*4.26*	5.66	*4.59*	5.61	*4.62*

An analysis of variance was conducted on the average size of strong components for each pair comparison per session. Subjects from the three different groups were compared over three sessions leading to a 3 by 3 design with repeated measurement on the second factor as in the previous analyses.[12]

[12]A sphericity test revealed no significant violation of homogeneity with approximate $\chi^2[2] = 1.79, p = 0.41$.

Table 3.20: *ANOVA for Size of Strong Components (Exp 2A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	158.6	*2*	79.3	2.28
Error	940.9	*27*	34.8	
Within subjects:				
Session	110.7	*2*	55.4	4.91**
BD by Session	2.8	*4*	0.7	.06
Session by Error	608.9	*54*	11.3	

Note: $^{**}p = 0.01$.

As for the analysis of the ζ-values, there was no significant effect between block designs ($F[2,27] = 2.28$, $p = 0.12$) but the analysis revealed a significant effect across sessions ($F[2,54] = 4.91$, $p = 0.01$). Table 3.20 provides the results of the analysis in detail.

Preference Reversals

The number of preference reversals which occurred between each of the three sessions are listed in Table 3.21 using Kendall's τ as a standardized measure. Table 3.21 shows the pattern of τ-values for the three comparisons indicating that more reversals occurred between Session 2 and 3 than Session 1 and 2 or Session 1 and 3.

Table 3.21: *Reversals of Preference (Exp 2A)*

	Kendall's τ					
	Session 1 vs 2		Session 2 vs 3		Session 1 vs 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	SD
Group S	.51	*.21*	.58	*.25*	.47	*.23*
Group P	.34	*.34*	.52	*.27*	.34	*.32*
Group N	.48	*.20*	.68	*.22*	.43	*.23*

Note: $\tau = 1 - (4R/R_n)$ where $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed relations.

An analysis of variance with repeated measurement was performed on the number of preference reversals which occurred between each of the three sessions. Kendall's τ served as dependent variable. A sphericity test revealed a significant violation of homogeneity for the repeated measurements (approximate $\chi^2[2] = 13.40$, $p < 0.001$) and the degrees of freedom were adjusted by $\epsilon = 0.797$.

Again, there was no statistical significant effect between block designs ($F[2,27]$=1.04, p=0.37) but there was a highly significant effect for the three pairwise comparisons of sessions ($F[2,54]$=12.23, p=0.0002, adjusted for heterogeneity).

Although not significant the different τ-values between block designs correspond to the ζ-values and number of ear dicycles. Especially in Session 1 the pattern of results indicates that

Table 3.22: *ANOVA for Preference Reversals (Exp 2A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	0.310	2	0.155	1.04
Error	4.042	27	0.150	
Within subjects:				
Combination (C)	0.551	2	0.276	12.23***
BD by C	0.072	4	0.018	.79
C by Error	1.217	54	0.023	

Note: $***p = 0.0002$.

subjects chose more consistent under the resolution block design (Group S), less consistent under the random block design (Group N), and worst under the repetition block design (Group P).

In general, mean response times are significantly shorter for all three groups in the second and third session. As for Experiment 1A there was no significant difference between block designs. The analyses of mean response times and Kendall's ζ suggest that some form of improvement in consistency took place over the three sessions. Subjects performed quicker and more consistent in the second and third session. This dynamic was also reflected in the significantly smaller size of strong components in later sessions. Surprisingly, the steady improvement over sessions was accompanied by a different number of reversals between sessions. This means that despite the increasing ζ-values over Session 1, 2, and 3 the adjacency matrices in Session 3 were more similar to the matrices in Session 1 than to the matrices in Session 2 as shown by the values of τ.

No significant difference between the block designs was detected except for the number of ear dicycles. There is a tendency in the first two sessions that subjects who chose under the resolution block design in Group S displayed a more consistent choice behavior especially in terms of longer ear dicycles than subjects under the random block design in Group N and the repetition block design in Group P.

As mentioned earlier, subjects who chose under the resolution block design encountered all alternatives in every single block of six choice-trials whereas under the repetition block design they encountered all alternatives in every two blocks. As a consequence subjects may have established a preference structure earlier when choosing under the resolution blocks. Hence, it can be expected that subjects are choosing more consistent under a resolution block design.

The alternatives in this domain are explicitly defined by their chance of winning and their payoff. After a sufficient number of trials it is possible that lotteries were identified by their pie charts thereby simplifying the choice process and facilitating consistency in the repetition blocks as in Experiment 1A and 1B.

3.7 Experiment 2B: Risky Choice

The experimental design was changed similar to the design in Experiment 1B. The three block designs were varied within subjects and two different sequences of block designs were investigated. From the results of Experiment 2A a similar effect of the block designs on the measures of inconsistency and a comparable learning effect over sessions was expected.

3.7.1 Method

The same method and equipment as in Experiment 2A was employed.

Design

The experiment had a 2×3 design (sequence by session) with repeated measurement on the second factor. In three consecutive sessions each subject chose between risky alternatives under different block designs. Two different sequences of block designs in three sessions were tested in Group SNP and PNS:

- Group SNP: In the first session subjects chose under a reSolution block design, in the second under a raNdom block design, and in the third under a rePetition block design.

- Group PNS: In the first session subjects chose under a rePetition block design, in the second under a raNdom block design, and in the third under a reSolution block design.

Dependent variables were the choices and their response times as well as measures of inconsistency derived from preference cycles and preference reversals.

Subjects

A total of 20 undergraduate and graduate students were recruited from the subject panel of the Department of Experimental Psychology, Oxford University. There were 10 subjects in each group (Group SNP: average age 24.9 years, SD 4.09; Group PNS: average age 21.5 years, SD 3.5). Subjects for both groups were balanced in gender. Each subject received £3 per hour plus traveling expenses for their participation in this experiment and in Experiment 3A or 3B in a different group.

Materials and Apparatus

The same stimuli (lotteries) and the same equipment as in Experiment 2A was used.

Procedure

The same procedure as in Experiment 2A was employed. Each session lasted between ten and fifteen minutes.

3.7.2 Results

This section is divided according to the dependent variables response times, preference cycles, size of strong components, and preference reversals. The hypotheses were the same as for Experiment 2A with the exception that the alternative hypothesis for the effect of block design was specified and tested against the null hypothesis. According to the findings in Experiment 2A choice behavior in the first session should be more consistent under the resolution block design than under the random block design and more consistent under the random block design than under the resolution block design.

Response Times

Table 3.23 shows mean response times and standard deviations for each group and session. As in Experiment 2A mean response times were significantly shorter in the second and third session than in the first session. On average mean response times are reduced by 1.7 secs from Session 1 to 2 and again by 1.23 secs from Session 2 to 3.

Table 3.23: *Mean Response Times of Choices (Exp 2B)*

	Session 1		Session 2		Session 3	
	Mean[a]	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group SNP (N=10)	5.69	*4.93*	4.54	*4.41*	2.43	*1.86*
Group PNS (N=10)	5.48	*4.52*	3.23	*1.74*	2.88	*1.82*

Note: Statistical significant effect between sessions, $F[2, 36] = 8.40$, $p < 0.005$ adjusted for heterogeneity.

[a]in seconds

Mean response times were entered into a 2 by 3 analysis of variance (ANOVA) with repeated measurement on the second factor. Sequence of block designs (resolution-random-repetition, repetition-random-resolution) served as fixed factor between subjects, session (first, second, third) as fixed factor within subjects and subjects acted as random factor (see Table 3.24).

Table 3.24: *ANOVA for Mean Response Times (Exp 2B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	303.7	*1*	303.7	0.28
Error	19491.5	*18*	1082.9	
Within subjects:				
Session	2365.5	*2*	1182.7	24.97***
S by Session	98.1	*2*	49.1	1.04
Session by Error	1705.4	*36*	47.4	

Note: $^{***}p = 0.0001$.

A sphericity test revealed a significant violation of homogeneity (approximate $\chi^2[2] = 15.29, p < 0.001$) and the degrees of freedom were adjusted by $\epsilon = 0.69$. The analysis showed no statistical significant effect between groups ($F[1,18] < 1$, ns) but the hypothesis of no effect across sessions was rejected ($F[2,36] = 8.40$, with $p = 0.004$ adjusted for heterogeneity).

Intransitive Preferences

Mean ζ-values for each group and session are displayed in Table 3.25. During the first session subjects in Group SNP made choices in a resolution block design resulting in an average ζ-value of .65 whereas subjects in Group PNS reached a value of .41. A planned t-test revealed a significant difference between the two values (minimal significant difference = 0.199).

Table 3.25: *Intransitive Triples of Preference (Exp 2B)*

	Kendall's ζ					
	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group SNP	.65	*.28*	.71	*.29*	.82	*.23*
Group PNS	.41	*.11*	.55	*.24*	.64	*.26*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2 - 4)/24 = 70$ for $n = 12$, and T number of intransitivities.

The same type of analysis as for mean response times was applied to number of intransitive triples in each preference matrix using Kendall's ζ as a standardized measure (see Table 3.26). A sphericity test revealed a significant violation of homogeneity for the repeated measurements of factor session (approximate $\chi^2[2]$=6.50, p=0.039) and degrees of freedom were adjusted by $\epsilon = 0.767$.

Table 3.26: *ANOVA for Kendall's ζ (Exp 2B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	0.536	*1*	0.536	4.12
Error	2.341	*18*	0.130	
Within subjects:				
Session	0.397	*2*	0.199	9.06***
S by Session	0.016	*2*	0.007	0.36
Session by Error	0.789	*36*	0.022	

Note: ***$p < 0.001$.

As before, there was a significant effect across sessions ($F[2,36]$=9.06, p=0.001, adjusted for heterogeneity) but no significant effect between groups ($F[1,18]$=4.12, p=0.057).

For each session stepwise discriminant analyses were performed on the number of dicycles i.e. the coefficients z_k of the polynomial ψ, and the number of ear dicycles derived from the ear decompositions by sequence. The number of k-dicycles were entered stepwise as dependent variables starting with dicycles whose number discriminated best between groups. Table 3.27 and Table 3.28 show the results of the analyses. Variables were entered and excluded at a significance level of $p < 0.15$.

Table 3.27: *Stepwise Discriminant Analysis on all Dicycles (Exp 2B)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1	3-cycles	-	1	0.254	6.121	0.024
			Session 2			
			No variables entered			
			Session 3			
1	3-cycles	-	1	0.124	2.538	0.129

In Session 1 and 3 the two groups can be discriminated by the number of 3-dicycles whereas in Session 2 no variable could be entered. The number of k-dicycles for each subject and each session are listed in Table B.17 for Group SNP and in Table B.18 for Group PNS summarizing all inconsistencies within pair comparisons. A comparison of polynomials with equal coefficients z_3, i.e. number of 3-dicycles, shows that in some cases the coefficients z_k for $k > 3$ are considerably different.

A stepwise discriminant analysis on the number of ear dicycles also showed differences between groups. The results are summarized in Table 3.28.

Table 3.28: *Stepwise Discriminant Analysis on Ear Dicycles (Exp 2B)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1.	8-dicycles	-	1	0.412	12.62	0.002
			Session 2			
1.	7-dicycles	-	1	0.268	6.60	0.019
			Session 3			
1.	6-dicycles	-	1	0.243	5.76	0.027

The ear dicycles of length 8 discriminated best between groups in the first session. Ear dicycles of length 7 discriminated in the second session and finally ear dicycles of length 6 discriminated in the last session. The number of ear dicycles from the ear decomposition by sequence for each subject and session are listed in Table B.19 for Group SNP, in and in Table B.20 for Group PNS.

Strong Components

The average size of strong components was determined for each pair comparison as an additional consistency measure which can be derived from the algebraic decompositions (Table 3.29).

An analysis of variance was conducted on the average size of strong components for each pair comparison per session. Subjects from the two different groups were compared over

Table 3.29: *Average Size of Strong Components (Exp 2B)*

	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group SNP	6.64	*4.69*	6.17	*5.20*	3.76	*4.39*
Group PNS	11.40	*1.90*	8.70	*3.59*	6.90	*4.58*

three sessions leading to a 2 by 3 design with repeated measurement on the second factor as in the previous analyses.[13]

Table 3.30: *ANOVA for Size of Strong Components (Exp 2B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	181.2	*1*	181.2	4.93*
Error	660.9	*18*	36.7	
Within subjects:				
Session	137.1	*2*	68.6	8.48***
S by Session	13.2	*2*	6.6	0.8
Session by Error	291.1	*36*	8.1	

Note: $^*p = 0.04$, $^{***}p = 0.001$.

In contrast to the analysis on Kendall's ζ a significant effect of sequence ($F[1,18] = 4.93$, $p = 0.04$) and a highly significant effect of session ($F[2,36] = 8.48$, $p = 0.001$) was detected. Table 3.30 gives detailed results of this analysis.

Preference Reversals

The pattern of τ-values, as shown in Table 3.31, indicates that more preference reversals occurred between Session 2 and 3 than Session 1 and 2 or Session 1 and 3. The significant difference of τ-values for reversals between Session 1 to 2 corresponds with the ζ-values and supports the finding that subjects chose more consistent under the resolution block design in Group SNP and less consistent under the repetition block design in Group PNS.

An analysis of variance with repeated measurement was performed on the number of preference reversals which occurred between each pair of the three sessions. Kendall's τ is a standardized measure for the number of reversals and served as dependent variable. A sphericity test showed no significant violation of homogeneity for the repeated measurements (approximate $\chi^2[2] = 4.24, p = 0.12$).

Despite the non-significant overall test between-subjects ($F[1,18] = 4.18$, $p = 0.056$) a Scheffe test showed a significant difference (minimal significant difference = 0.207) between the τ-values .54 and .30 for Group SNP and PNS, respectively. As for Experiment 2A there was a highly significant effect for the three combinations of sessions ($F[2,54] = 10.56, p = 0.0003$).

[13]A sphericity test showed no significant violation of homogeneity with approximate $\chi^2[2] = 3.79, p = 0.15$.

Table 3.31: *Reversals of Preference (Exp 2B)*

	Kendall's τ					
	Session 1 vs 2		Session 2 vs 3		Session 1 vs 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	SD
Group SNP	.54	*.28*	.65	*.27*	.52	*.32*
Group PNS	.30	*.13*	.50	*.22*	.32	*.17*

Note: $\tau = 1 - (4R/R_n)$ where $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed relations.

Table 3.32: *ANOVA for Preference Reversals (Exp 2B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	0.624	*1*	0.624	4.18
Error	2.687	*18*	0.149	
Within subjects:				
Combination (C)	0.270	*2*	0.135	10.56***
S by C	0.008	*2*	0.004	.32
C by Error	0.461	*36*	0.013	

Note: ***$p = 0.0003$.

As in Experiment 2A mean response times were significantly shorter in the second and third session but displayed no significant difference between groups. The analyses on mean response times and Kendall's ζ suggest that some form of learning or improvement in consistency took place over the three sessions. Regardless of the block designs, subjects performed quicker and also more consistently in the second and especially in the third session.

An analysis of the size of strong components showed a significant effect for the sequence of block designs. The effect is most likely due to the large difference between the block designs in the first session and corresponds to the decreasing length of discriminating ear dicycles over sessions.

Confirming the results of Experiment 2A similar differences between block designs appeared in the first session of both experiments. Subjects who chose under the resolution block design in Group SNP displayed a more consistent choice behavior in terms of intransitive triples and ear dicycles than subjects under the repetition block design in Group PNS. Accordingly, groups differ by the number of intransitive triples in the first session and more significantly by the number of ear dicycles. In Session 3 ζ-values for Group SNP are still higher than for Group PNS, suggesting that as in Experiment 1B improvement in consistency is irreversible.

If consistent choice persists and cannot be reversed by block designs then the significant difference of preference reversals between Session 2 and 3 also supports the finding that subjects chose more consistently under resolution blocks in the first session.

3.8 Discussion

In both experiments mean response times decreased considerably over sessions. Subjects in each group learned to solve the decision task quicker due to increasing familiarity with the task and the alternatives. At the same time the number of intransitivities decreased across sessions. This result suggests that some form of learning or improvement took place so that from session to session subjects were able to choose more consistently in later sessions. It is assumed that the strong effect across sessions covers up differences between block designs, especially in the second and third session of Experiment 2A. Nevertheless, in both experiments the measures of inconsistency in terms of ζ-values, size of strong components, number of dicycles, and number of ear dicycles show marked differences between block designs in the first session.

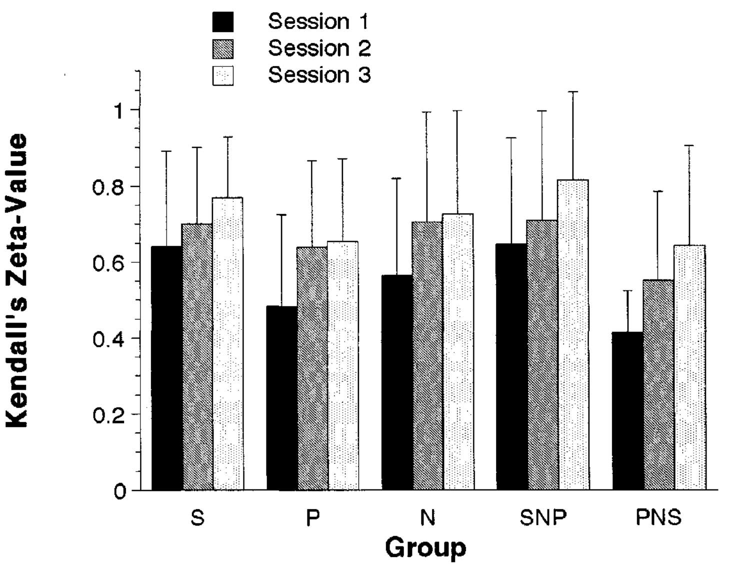

Figure 3.7: Kendall's ζ for all groups of Experiment 2A and 2B.

The differences between block designs in the first sessions were strongest for ear dicycles confirming the predictions for the effect of block designs on individual choice behavior. These effects are interpreted as follows: Under the resolution block design the subject encountered all twelve unfamiliar alternatives in each block of six choices. This enabled the subject to establish a more consistent set of preferences compared with the repetition block design where only half of the alternatives are introduced in each block.

Furthermore, the repetition of multiattribute alternatives possibly favored selective information processing which leads to more inconsistent choices. The resolution blocks on the other hand should have enhanced more complete and therefore independent information processing resulting in more consistent choices.

Although pointing in the opposite direction, the differences between block designs in the first session strengthen the conclusion from Experiment 1A and 1B that subjects interact with the arrangement of choices. This was reflected in all measures of inconsistency. As in Experiment 1A and 1B this finding is incompatible with classical algebraic or probabilistic models. The highly significant effect of session is also incompatible with algebraic or probabilistic decision models and supports adaptive behavior as modeled by algebraic decomposition. Probabilistic and non-adaptive algebraic decision models do not permit any systematic changes of choice behavior over time.

No serious attempt is made to offer an explanation for the finding that in both experiments significantly more preference reversals occurred between the second and third session although intransitivities decreased over all three sessions. It is possible, however, that subjects decided to adopt a simpler and therefore more consistent strategy in the last session. A strategy they had used occasionally in the first session. The steadily decreasing response times across sessions seem to support this explanation.

3.9 Experiment 3A: Discrimination

By using discrimination tasks in the domain of visual contrast perception this experiment investigates if different block designs can have an effect on inconsistency of discrimination.

If two center fields have the same luminance[14] but surrounding fields of different luminances then the brightness of the center field with a bright surround is perceived darker than the center field with a dark surround. This effect is known as brightness contrast and was quantified in classical experiments by Heinemann (1955). In the following experiment the brightness contrast was used to create indifferent stimuli which were indifferent in perceived brightness near the discrimination threshold although they were quite different in absolute luminance. If two center fields with different luminance are surrounded by fields of different luminance then the brightness contrast can create the impression of equivalent brightness between the center fields. Because each stimulus consisted of an indifferent center field and a different surround it was assumed that repetition of stimuli with the same surround over subsequent trials would influence individual discrimination performance similar to the choice tasks in Experiment 2A. The same unspecified hypotheses were investigated in this first experiment.

3.9.1 Method

As in Experiment 2A all stimuli remained constant. Subjects chose in pair comparisons (two-alternative forced choice) in three separate sessions between the same set of visual stimuli which differed in luminance but were near-threshold in perceived brightness. The fixed block designs (repetition and resolution block design) and the random block design were used as before. The sequence of trials in the random block design was randomized for each subject and each pair comparison.

Design

The same design as in Experiment 2A was employed. In a 3×3 design (block design by session) with repeated measurement on the second factor each subject discriminated between square disks in three sessions under different block designs. The block designs in three sessions were varied between subjects leading to three groups labeled S, P and N.

- Group S: In all three sessions subjects chose under the reSolution block design.

- Group P: In all three sessions subjects chose under the rePetition block design.

[14]In the following luminance always refers to physical brightness measured in candela per square meter (cd/m^2).

- Group N: In all three sessions subjects chose under the raNdom block design.

Dependent variables were the choices and response times as well as measures of inconsistency (reversals, intransitive triples, dicycles, and ear dicycles). As in the previous experiments it was hypothesized that the different block designs would affect the consistency of the discrimination performance.

Subjects

A total of 30 undergraduate and graduate students who also took part in Experiment 2A or 2B were recruited from the subject panel of the Department of Experimental Psychology, Oxford University. Each group had 10 subjects and was balanced in gender. (Group S: average age 24.9 years SD 4.09; Group P: average age 21.5 years SD 3.5; Group N: average age 21.9 years, SD 3.21). All subjects had normal or corrected-to-normal vision. Each subject received £3 per hour plus traveling expenses for their participation in Experiment 2A or 2B and this experiment.

Materials and Apparatus

Twelve visual stimuli with different luminance of center and surround were generated. The center field was a square disk within a square surround. The luminances of center and surround were adjusted in luminance so that all centers appeared to have approximately the same perceived brightness (see Appendix B.3.2). All stimuli appeared monochromatic. The

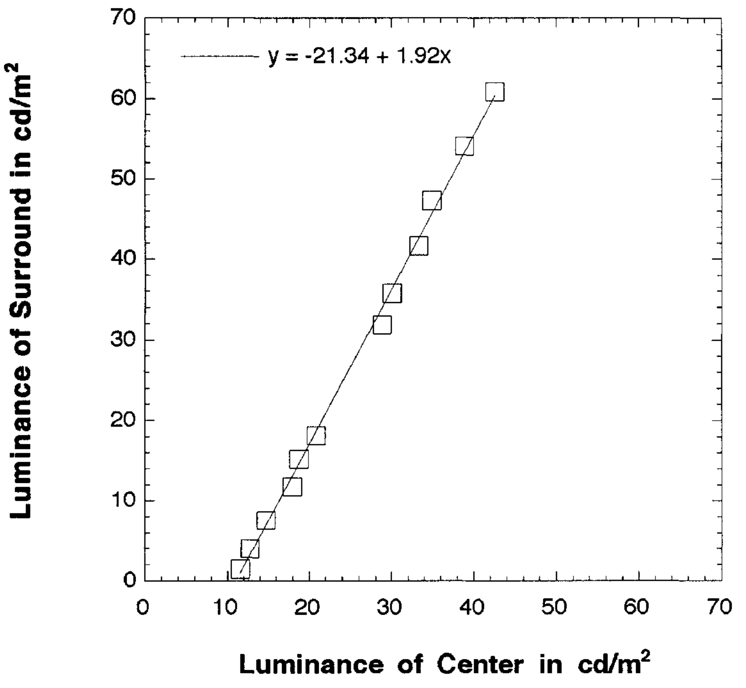

Figure 3.8: Luminances of center and surround. A linear function is fitted to the two luminances of the stimuli.

luminances of center and surround of the stimuli were varied along a linear axis with stimuli close to indifference (see Figure 3.8). It is well documented that small linear differences in contrast do not produce a linear simultaneous contrast effect (Takasaki, 1966; Semmelroth, 1970). Therefore, only stimuli were selected which had a sufficient difference in luminance between center and surround. The fitted regression line has a slope of 1.92 and an intercept of -21.34 ($R^2 = 0.997$).

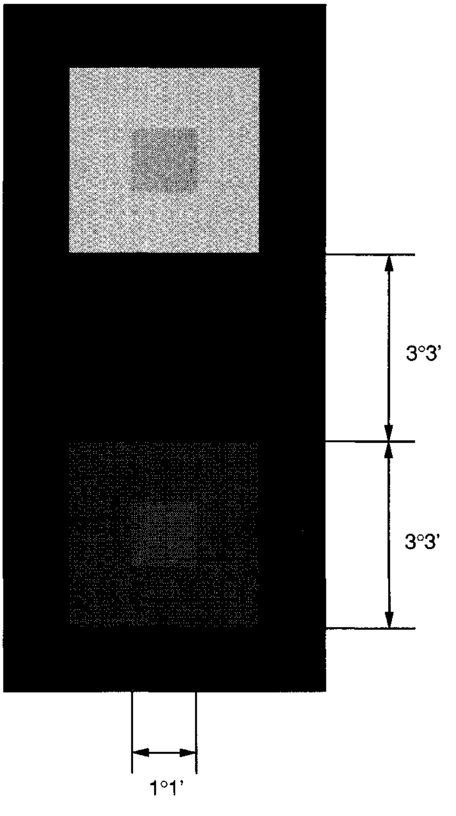

Figure 3.9: Display of stimuli on screen.

During choice-trials two stimuli with centers of different luminance were displayed on a 21 inch Two-page Macintosh monochrome monitor with a frame rate of 66.7 Hz. The center field subtended $1°1'$ degree visual angle and the surround had a visual angle of $3°3'$. The two stimuli were displayed vertically $3°3'$ apart on a black background as illustrated in Figure 3.9.[15] No fixation point was displayed and the screen was observed binocularly from a distance of 114 cm while the head was supported by a chin-rest. The images were drawn on screen in an interval of 3.0 secs between last response and next display. During this interval

[15]The square disks were displayed vertically rather than horizontally to reduce the effect of voltage droop (Pelli & Zhang, 1991). This bias on luminance was revealed during calibration of the monitor. The effect was stronger for a horizontal than a vertical display.

the screen remained black. The program waited until the image was completed which took between 300 and 500 msec.[16] Each stimulus pair was displayed on screen by changing the color look-up table (CLUT) in frame rate so that the whole image became instantly visible.

The experiment was programmed in SuperLab and run on a Macintosh IIcx (enhanced by a floating-point unit) allowing time measures on the keyboard with an accuracy of up to 4 millisecs.

Procedure

Each subject had three sessions with $\binom{12}{2} = 66$ choice-trials plus 3 training trials. Each session was preceded by a session of Experiment 2A or 2B which lasted between ten and fifteen minutes. The background for the instructions and stimuli of Experiment 2A and 2B had approximately the same average luminance of 12 cd/m^2 as the stimuli so that the subjects were sufficiently dark adapted before each session.

Subject were seated in front of the computer screen and keyboard in a completely darkened cubicle. They were asked to place their head on a chin-rest to keep viewing angle and distance constant.

First, they read a general instruction about the experiment, which was followed by a more detailed instruction (see Appendix B.3.1). In three training trials they learned how to give a response by pressing the 'U' and 'N' key on the keyboard. In each trial they were asked to decide which disk in the center of the two squares appeared brighter. A black background was shown for three seconds before response timing was initiated and the two stimuli were displayed in the upper and lower half of the screen. The two stimuli stayed visible until the subject chose the upper or lower alternative by pressing the appropriate key. Each session lasted between six and twelve minutes.

Table 3.33: *Mean Response Times for Discrimination (Exp 3A)*

	Session 1		Session 2		Session 3	
	Mean[a]	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group S (N=10)	2.03	*1.10*	1.65	*.87*	1.29	*.55*
Group P (N=10)	2.02	*.72*	1.74	*.86*	1.39	*.53*
Group N (N=10)	1.94	*.60*	1.46	*.55*	1.32	*.42*

Note: Statistical significant effect between sessions, $F[2,54] = 34.41, p < 0.0001$).

[a]in seconds

3.9.2 Results

In this section the results of analyses on the dependent variables response times, discrimination cycles, and discrimination reversals are reported.

The hypotheses were the same as before and are summarized as follows: The resolution block design should result in different discrimination performance (measured by Kendall's

[16]In each trial the varying delay was subtracted from the time interval between trials.

Table 3.34: *ANOVA for Mean Response Times (Exp 3A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	293.2	*2*	146.6	0.11
Error	34682.7	*27*	1284.5	
Within subjects:				
Session	6578.8	*2*	3289.4	24.16***
BD by Session	200.7	*4*	50.2	0.37
Session by Error	7352.2	*54*	136.1	

Note: $^{***}p = 0.0001$.

ζ, τ, number of dicycles and ear dicycles) than the repetition block design. The random block design should lead to an intermediate level of consistent discrimination performance. In comparison the number of ear dicycles should discriminate better between block designs than the total number of dicycles. Moreover, a decrease in response times and inconsistency can be expected over sessions.

Response Times

Mean response times and standard deviations are listed in Table 3.33. In every group mean response times decreased over sessions. On average response times were reduced by 0.49 secs from Session 1 to 2 and again by 0.17 secs from Session 2 to 3.

Mean response times were entered into a 3 by 3 analysis of variance (ANOVA) with repeated measurement on the second factor. Block design (resolution, repetition, random) served as fixed factor between subjects, session (first, second, third) as fixed factor within subjects and subjects acted as random factor (see Table 3.34 for results). A sphericity test showed no significant violation of homogeneity (approximate $\chi^2[2] = 2.03, p = 0.36$).

The analysis revealed no statistical significant effect between block designs ($F[2, 27] < 1$, ns). However, the null hypothesis of no effect over sessions was rejected ($F[2, 54] = 24.16$, $p < 0.0001$). No other effects approached statistical reliability.

Table 3.35: *Intransitive Triples of Discrimination*

	Kendall's ζ					
	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group S	.73	*.23*	.59	*.22*	.66	*.24*
Group P	.66	*.20*	.69	*.28*	.68	*.22*
Group N	.77	*.18*	.82	*.21*	.79	*.17*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2 - 4)/24 = 70$ for $n = 12$, and T number of 3-dicycles.

Table 3.36: *ANOVA for Kendall's ζ (Exp 3A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	0.255	*2*	0.128	0.99
Error	3.479	*27*	0.129	
Within subjects:				
Session	0.088	*2*	0.044	2.11
BD by Session	0.028	*4*	0.007	0.25
Session by Error	1.542	*54*	0.029	

Discrimination Cycles

In Table 3.35 the average ζ-values with standard deviations for each group and session are listed. All groups had comparable average ζ-values in the first and more or less in the following sessions. In the first session subjects in Group S made choices under a resolution block design reaching an average ζ-value of .73, whereas subjects in Group P had an average value of .66 under the repetition block design. The subjects of Group N made choices under the random block design and reached a higher average ζ-value of .77.

Table 3.37: *Stepwise Discriminant Analysis on Dicycles (Exp 3A)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
			No variables entered			
			Session 2			
1	3-dicycles	-	1	0.143	2.250	0.125
			Session 3			
			No variables entered			

The same type of analysis was applied to the number of intransitive triples using Kendall's ζ as a standardized measure (see Table 3.36).[17] There was no significant effect of block design ($F[2,27] < 1$, ns) and no effect of session ($F[2,54] = 2.11$, $p = 0.131$).

For each session stepwise discriminant analyses were performed on the number of dicycles, i.e. the coefficients ω_k of the polynomial ψ, and the number of ear dicycles from the ear decompositions by sequence. The numbers of k-dicycles were entered stepwise as dependent variables starting with dicycles which discriminated best between groups. The results of the analysis are given in Table 3.37. Variables were entered and excluded at a significance level of $p < 0.15$.

Only in Session 2 the three block designs could be weakly discriminated by the number of 3-dicycles whereas in Session 1 and 3 no variables were entered. The stepwise discriminant analysis on the number of ear dicycles revealed some differences between groups in all three sessions. The results are summarized in Table 3.38.

[17] A sphericity test revealed no significant violation of homogeneity for the repeated measurements over sessions (approximate $\chi^2[2] = 5.25, p = 0.07$).

Table 3.38: *Stepwise Discriminant Analysis on Ear Dicycles (Exp 3A)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1.	5-dicycles	-	1	0.150	2.39	0.111
			Session 2			
1.	7-dicycles	-	1	0.273	5.06	0.014
2.	12-dicycles	-	1	0.143	2.18	0.134
			Session 3			
1.	9-dicycles	-	1	0.159	2.54	0.097
2.	10-dicycles	-	1	0.141	2.21	0.129

The ear dicycles of length 7 had considerable discriminatory power in the second session. Ear dicycles of length 5 discriminated in the first session and dicycles of length 9 and 10 in the third session. The number of ear dicycles together with mean and standard deviation are listed for each subject and session in Table B.25 for Group S, in Table B.26 for Group P and in Table B.27 for Group N.

Table 3.39: *Reversals of Discrimination (Exp 3A)*

	Kendall's τ					
	Session 1 vs 2		Session 2 vs 3		Session 1 vs 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	SD
Group S	.38	*.30*	.43	*.27*	.33	*.33*
Group P	.48	*.18*	.53	*.20*	.43	*.22*
Group N	.57	*.22*	.65	*.13*	.52	*.17*

Note: $\tau = 1 - (4R/R_n)$ where $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed relations.

Discrimination Reversals

The average τ-values and standard deviations are listed for each group and session in Table 3.39. The pattern of τ-values for the three comparisons indicates that slightly more reversals occurred between Session 2 and 3 than Session 1 and 2 or Session 1 and 3. Although not significant the τ-values for Group N, the random block design, were higher than for Group S or P which corresponds to the higher ζ-values in Group N.

An analysis of variance with repeated measurement was performed on the number of preference reversals between each combination of the three sessions (see Table 3.40). Kendall's τ was used as standardized dependent variable. A sphericity test showed no significant violation of homogeneity for the repeated measurements (approximate $\chi^2[2] = 4.24, p = 0.12$). There was no statistical significant effect of block design ($F[2,27] = 2.19, p = 0.13$) but a significant effect for the three combinations of sessions ($F[2,54] = 5.03, p = 0.01$).

No systematic effects between groups were detected suggesting that the block designs had no significant effect on consistency of discrimination. Beside the session effect for mean

Table 3.40: *ANOVA on Discrimination Reversals (Exp 3A)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Block Design (BD)	0.564	*2*	0.282	2.19
Error	3.475	*27*	0.129	
Within subjects:				
Combination (C)	0.172	*2*	0.086	5.03*
BD by C	0.004	*4*	0.001	.06
C by Error	0.925	*54*	0.017	

Note: $^*p < 0.01$.

response times there was a similar effect for reversals between sessions as in Experiment 2A. Subjects reversed more discriminations between Session 2 to 3 than between Session 1 and 2 and Session 1 and 3. No explanation for this effect is offered. Although not significant, ζ- and τ-values in this experiment appeared to be higher for the random block design. The ear dicycles of length 5 in the first session and some combination of ear dicycles in Session 2 and 3 discriminated between block designs but these results are inconclusive because numbers are pointing in different directions (see Table B.25 B.26 and B.27).

3.10 Experiment 3B: Discrimination

The results from Experiment 3A suggest that there is no effect of block design on consistency of discrimination. Hence, the alternative hypothesis could not be specified and no systematic effects were expected from the variation of block designs across sessions.

3.10.1 Method

The same method and stimuli as in Experiment 3A was applied.

Design

The same design as in Experiment 2B was applied. In a 2×3 design (sequence by session) with repeated measurement on the second factor each subject chose between stimuli in three sessions under different block designs. Two different sequences of block designs over three sessions were tested in Group SNP and PNS.

- Group SNP: In the first session subjects chose under a reSolution block design, in the second under a raNdom block design, and in the third under a rePetition block design.

- Group PNS: In the first session subjects chose under a rePetition block design, in the second under a raNdom block design, and in the third under a reSolution block design.

Dependent variables were the choices and response times as well as measures of inconsistency (reversals, intransitive triples, dicycles, and ear dicycles). As in the previous experiments it was hypothesized that the different block designs would affect the consistency of the discrimination performance.

Subjects

A total of 20 undergraduate and graduate students who also took part in Experiment 2A or 2B were recruited from the subject panel of the Department of Experimental Psychology, Oxford University. There were 10 subjects in each group (Group SNP: average age 24.5 years, SD 5.21; Group PNS: average age 27.3 years, SD 6.43). Both groups were balanced in gender. All subjects had normal or corrected-to-normal vision. Each subject received £3 per hour plus traveling expenses.

Materials and Apparatus

The same equipment, stimuli and set-up as in Experiment 3A was used.

Procedure

The procedure was the same as in Experiment 3A. Each session was preceded by a session of Experiment 2A or 2B and lasted between six and twelve minutes.

3.10.2 Results

In the following the results from analyses on the dependent variables response times, discrimination cycles and reversals are presented. The hypothesis are the same as for Experiment 3A with the exception that differences between block designs should appear over sessions.

Response Times

Mean response times and standard deviations are listed in Table 3.41. In both groups mean response times decreased over sessions. On average response times were reduced by 0.29 secs from Session 1 to 2 and again by 0.20 secs from Session 2 to 3.

Table 3.41: *Mean Response Times for Discrimination (Exp 3B)*

	Session 1		Session 2		Session 3	
	Mean[a]	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group SNP (N=10)	1.75	*.61*	1.47	*.55*	1.18	*.48*
Group PNS (N=10)	1.84	*.76*	1.54	*.72*	1.44	*.59*

Note: Statistical significant effect between sessions, $F(2,90) = 35.6, p < 0.0001$).

[a]in seconds

Mean response times were entered into a 2 by 3 analysis of variance (ANOVA) with repeated measurement on the second factor. Sequence of block designs (resolution-random-repetition, repetition-random-resolution) served as fixed factor between subjects, session (first, second, third) as fixed factor within subjects and subjects acted as random factor (see Table 3.42). A sphericity test showed no significant violation of homogeneity (approximate $\chi^2[2] = 2.32, p = 0.31$).

Table 3.42: *ANOVA for Mean Response Times (Exp 3B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	303.7	*1*	303.7	0.28
Error	19491.5	*18*	1082.9	
Within subjects:				
Session	2365.5	*2*	1182.7	24.97***
S by Session	98.1	*2*	49.1	1.04
Session by Error	1705.4	*36*	47.4	

Note: $^{***}p = 0.0001$.

There was no statistical significant effect between groups ($F[1,18]<1$, ns) but the null hypothesis of no effect of session was rejected ($F[2,36] = 25.0$, $p < 0.0001$).

Discrimination Cycles

In Table 3.43 the average ζ-values are given for each group and session. As shown in Table 3.43 subjects in Group SNP first made discriminations under a resolution block design reaching an average ζ-value of .64, whereas subjects in Group PNS discriminated under the repetition block design and displayed a value of .68. The values increase slightly in the last session.

Table 3.43: *Intransitive Triples of Discrimination (Exp 3B)*

	Kendall's ζ					
	Session 1		Session 2		Session 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*
Group SNP	.64	*.25*	.63	*.26*	.76	*.16*
Group PNS	.68	*.26*	.64	*.33*	.73	*.27*

Note: $\zeta = 1 - T/T_n$ with $T_n = n(n^2 - 4)/24 = 70$ for $n = 12$, and T number of intransitivities.

The same type of analysis was applied to the number of intransitive triples using Kendall's ζ as a standardized measure (see Table 3.44). A sphericity test detected no significant violation of homogeneity for the repeated measurements (approximate $\chi^2[2] = 0.95, p = 0.15$). Again, there was no significant effect of sequence of block designs ($F[1,18] < 1$, ns) and no effect of session ($F[2,36] < 1$, ns).

Table 3.44: *ANOVA for Kendall's ζ (Exp 3B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	0.005	*1*	0.005	0.04
Error	2.123	*18*	0.118	
Within subjects:				
Session	0.031	*2*	0.015	0.65
S by Session	0.070	*2*	0.035	1.49
Session by Error	0.851	*36*	0.024	

For each session stepwise discriminant analyses were performed on the number of dicycles, i.e. the coefficients z_k of the polynomial ψ, as well as the number of ear dicycles from the ear decomposition by sequence. The numbers of k-dicycles were entered stepwise as dependent variables starting with dicycles which discriminated best between the two groups. For the number of dicycles none of the variables reached a significance level of $p < 0.15$. The number of k-dicycles together with mean and standard deviation are listed for each subject and each session in Table B.28 for Group SNP and in Table B.29 for Group PNS. The results of the stepwise discriminant analysis on the number of ear dicycles are shown in Table 3.45.

Only in the first session ear dicycles of length 3 discriminated weakly between the block designs. The number of ear dicycles from the ear decomposition by sequence are listed for

Table 3.45: *Stepwise Discriminant Analysis on Ear-Dicycles (Exp 3B)*

Step	Entered	Removed	Number	Partial R^2	Partial F	$p > F$
			Session 1			
1.	3-dicycles	-	1	0.126	2.60	0.124
			Session 2			
			No variables entered			
			Session 3			
			No variables entered			

each subject and session in Table B.30 for Group SNP, and in Table B.30 for Group PNS. In the next section the number of dicrimination reversals between sessions are discussed.

Discrimination Reversals

In Table 3.46 mean Kendall's τ are listed for each group and session as a standardized measure of reversals between pair comparisons. The τ-values do not exhibit any systematic variations. Especially, the average τ-values in Group PNS remained almost constant.

Table 3.46: *Reversals of Discrimination (Exp 3B)*

	Kendall's τ					
	Session 1 vs 2		Session 2 vs 3		Session 1 vs 3	
	Mean	*SD*	*Mean*	*SD*	*Mean*	SD
Group SNP	.32	*.29*	.46	*.22*	.36	*.33*
Group PNS	.50	*.32*	.50	*.33*	.50	*.33*

Note: $\tau = 1 - (4R/R_n)$ where $R_n = n(n-1) = 132$ for $n = 12$, and R number of reversed relations.

An analysis of variance with repeated measurement was performed on the number of reversals between each combination of the three sessions (see Table 3.47). Kendall's τ was used as standardized dependent variable.[18]

There is no statistical significant effect of sequence of block designs ($F[1,18] < 1$, ns) and no significant effect of combination of sessions ($F[2,36] = 1.51, p = 0.23$).

The results are quickly summarized. In Experiment 3B no effects other than the session effect for mean response times were present. In general the block designs had no systematic effect on intransitive triples or dicycles. The number of ear dicycles of length 3 discriminated weakly between block designs in the first session but their means do not correspond to the findings in Experiment 3A.

The ζ-values for discrimination under the random block design in Session 2 of Group SNP and PNS were lower than or equal to the values of Session 1 and 3. Hence, the tendency for more consistent discrimination under the random block design in the results of Experiment 3A was not confirmed. In Experiment 3A the τ-values for reversals between Session 2

[18]A sphericity test showed no significant violation of homogeneity for the repeated measurements (approximate $\chi^2[2] = 1.10, p = 0.58$).

Table 3.47: *ANOVA on Preference Reversals (Exp 3B)*

Source	*SS*	*df*	*MS*	*F*
Between subjects:				
Sequence (S)	0.220	*1*	0.220	0.92
Error	4.325	*18*	0.240	
Within subjects:				
Combination (C)	0.060	*2*	0.030	1.51
S by C	0.055	*2*	0.028	1.38
C by Error	0.721	*36*	0.020	

and 3 were slightly higher but no corresponding effect appeared in Experiment 3B. We conclude that no systematic changes of consistency occurred, neither over sessions nor between block designs.

3.11 Discussion

In general mean response times for Experiment 3A and 3B are shorter in comparison to Experiment 1A and 1B as well as Experiment 2A and 2B but they also decrease significantly across sessions. Although subjects in each group solve the discrimination tasks quicker probably due to increasing familiarity with the task but not the stimuli, increasing consistency over sessions and varying consistency between groups was not observed: Intransitivities seem to vary unsystematically across groups and sessions (see for example Figure 3.10). In contrast to Experiment 1A and 1B as well as Experiment 2A and 2B the results suggest that subjects did not interact with the arrangement of choice-trials and did not improve in consistency over sessions. Hence, subjects were not able to choose more consistently under a particular block design or in a later session.

This result suggests that in this domain subjects perform discrimination tasks independently of block designs. Therefore, the results do not support the application of algebraic decompositions and are compatible with probabilistic decision models as favored in psychophysics. However, much more data needs to be accumulated before probabilities can be reliably estimated and before any conclusions about probabilistic choice behavior can be drawn.

Although Experiment 3A and 3B was conducted under the same design as Experiment 2A and 2B there are two important differences between these experiments. First, the alternatives in Experiment 3A and 3B were physical stimuli, more precisely, visual stimuli which were matched in perceived brightness leading to indifferent alternatives or stimuli near a discrimination threshold (or point of subjective equality). Although the stimuli can be described as two-dimensional just as the lotteries in Experiment 2A and 2B, they were identifiable only by the perceived brightness of their surround which is a different and more difficult task than remembering a pie chart with numbers. Second, the choice task was a discrimination task which does not involve preferences, i.e. an evaluation of the stimuli.

Mean ζ-values varied unsystematically between sessions and groups as shown in Figure 3.10 and the same holds for the total number of dicycles and ear dicycles. It is assumed that the indifferent or near-threshold stimuli did not allow subjects to improve in consistency. Subjects could not benefit from previous trials nor simplify the discrimination task over sessions as in Experiment 2A and 2B. This is confirmed by the absence of any systematic differences between block designs and sessions. The subjects were unable

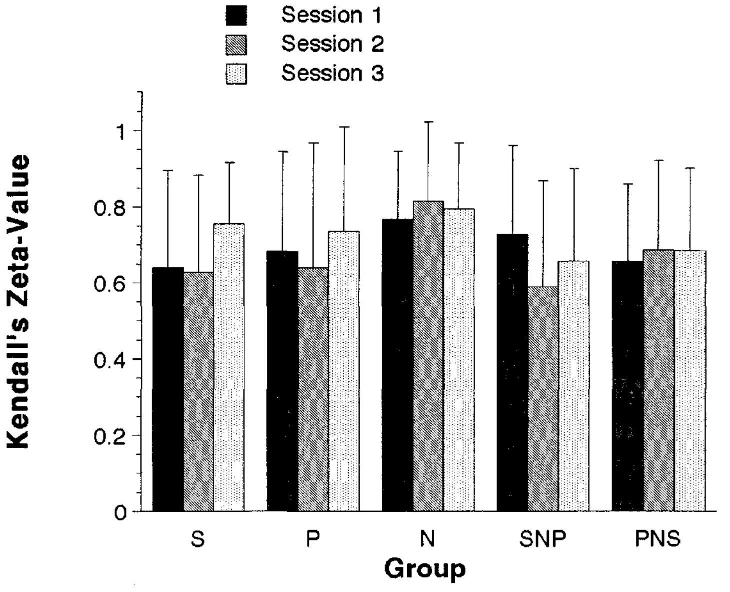

Figure 3.10: Kendall's ζ for all groups of Experiment 3A and 3B

to take advantage of the repetition of a stimulus in successive trials simply because the stimulus was not recognized and repetitions went by unnoticed. The subject certainly noticed that the stimuli had surrounds which differed in brightness but from the present results it seems unlikely that they were able to identify stimuli by their surround. However, effects similar to the results in Experiment 2A and 2B may occur in a different domain where visual stimuli are easily distinguishable by attributes irrelevant to the discrimination task.

3.12 General Discussion

Six experiments in three domains were conducted leading to different results. Named and familiar alternatives under riskless choice showed more consistent choice behavior under the repetition block design and an effect of block design on 3-dicycles (intransitive triples) and 4-dicycles was observed. An explanation was offered which takes into account that the named alternatives were highly familiar so that existing preference values were simply recalled if alternatives were repeated from one trial to another thereby increasing consistency. Although these results were not in line with the original predictions about information processing in multiattribute alternatives they reject the null hypothesis which is linked to the assumption that choice behavior should be independent of arrangements of the choice-trials.

The effect of block design was not very prominent for indifferent alternatives under risky choice but appeared consistently in the first session of both experiments. It was argued that the effect of block designs on consistency was overshadowed by a stronger effect across sessions. The consistency effect over sessions proved to be irreversible and increased regardless of the initial block design. This improvement is not seen as a pure learning effect because subjects did not receive feedback on their choice behavior. It is suggested that subjects eventually adopted a simpler decision strategy which shortened response times and improved consistency over sessions. The strong session effect also casts doubt on the assumption of non-adaptive choice behavior, particularly in domains with multiattribute alternatives.

Finally, no significant effects of block design or session on consistency were found for visual near-threshold stimuli in a psychophysical discrimination task. Discrimination of stimuli according to their induced brightness contrast showed unsystematic variations of inconsistency between groups and sessions.

The polynomial ψ which assessed all inconsistencies in a pair comparison occasionally had the same number of 3-cycles in one pair comparison or session but distinctive numbers of k-dicycles for $k > 3$ in another. Because the number of k-dicycles with $k > 4$ increased exponentially in cases of very inconsistent choice behavior causing huge standard deviations (see for example Subject 39 and Subject 4 in Experiment 3A and 3B, respectively) statistical tests on the number of dicycles are not reliable. On the other hand, it was not possible to establish an adequate test statistic because the distributional properties of the k-dicycles are unknown. Nevertheless, the polynomials provide an exhaustive way to characterize individual choice behavior and may prove fruitful in domains with few indifferent alternatives such as named alternatives. In general, the number of ear dicycles proved to be a better measure of inconsistency. Their numbers increased linearly with the size of the strong components and discriminated better between block designs than dicycles. It is concluded that the sequence of intransitive choices as employed by the ear decomposition offers a valid way to capture adaptive choice behavior in a quantitative way.

In summary, there is evidence that adaptive choice behavior is reflected in inconsistent choice behavior. Inconsistency may be induced by systematic arrangements of choice-trials but different results were obtained for familiar named alternatives and unfamiliar multiattribute alternatives. It is concluded that inconsistency is efficiently characterized in terms of ear dicycles.

Chapter 4

Conclusions

In this chapter the theoretical and experimental results are briefly discussed and their implications on decision making theory are evaluated. Recommendations for experimental studies and the analyses of empirical data are given and further theoretical developments are encouraged. The chapter concludes with remarks on the concept of error in human decision behavior.

It is emphasized that algebraic decomposition models can serve as a universal tool to analyze adaptiveness in binary data. In the case of choice behavior it is not necessary to specify a particular decision rule that is supposed to apply to all decisions in successive choice-trials. Often this is not even desirable because the specification of a decision rule is only possible if all attributes of the alternatives are known and remain unchanged over time. This describes a rather domain-specific and artificial choice situation. In this context it is important to realize that the specification of an underlying knowledge structure of attributes usually favors a certain type of decision rule. In essence this means that declarative knowledge implies procedural knowledge and vice versa; a simple fact that has not been fully appreciated and hampers progress in psychological decision research.

4.1 Theoretical and Empirical Implications

Further developments on the algebraic decomposition of individual choice behavior are necessary to find appropriate representations. The identification of a directed ear basis as suggested in Chapter 2 is one of several possible ramifications. For example, the decomposition of strong components into acyclic substructures such as spanning trees are investigated in problems of linear optimization and opens up promising paths for future research. In addition, the rich theory of algebra and developments on oriented matroids may offer different and exciting ways to decompose digraphs and strong components (e.g., Björner, Las Vergnas, Sturmfels, White, & Ziegler, 1993).

More generally, algebraic decompositions can be linked to research issues in theories of knowledge representation. The theory of knowledge structures as proposed by Doignon and Falmagne (1985) shares some interesting features with algebraic decomposition models. For an introduction to the theory of knowledge structures consult Falmagne, Koppen, Villano, Doignon, and Johannesen (1990). The assessment of knowledge states in a knowledge space can be translated into the assessment of preference orders in a family of possible preference orders. A theory which may bring the two approaches together is currently under investigation (e.g., Doignon & Falmagne, 1997). At present the theory of knowledge structures is based on group data because a knowledge space has to be assessed and verified by experts before it can be applied. Moreover, such a knowledge space is believed to be universal and therefore it should be applicable to any subject and knowledge state. Algebraic decompositions, on the other hand, have the advantage that they are based on individual preference data with no universal structure superimposed.

In this work a basic assumption of an algebraic decomposition model has been investigated which contradicts independence of choice trials. The ear decomposition by sequence makes weak assumptions about the sequence of intransitive choices. A stronger dependency between choice-trials was suggested in Definition 2.4.1 for the completion by cuts. This dependency may be regarded as a deterministic counterpart of the Markov property in stochastic processes. These or similar assumptions might offer a way to extend algebraic decompositions to an adaptive probabilistic model.

In the domain of named alternatives the experimental study of block designs revealed differences between choice behavior in terms of consistency. The data and results of experiments in the domain of risky alternatives showed a steady improvement of consistent choice behavior over three consecutive sessions as well as differences between block designs in the first session. It is believed that these results hint at the intricate domain-specificity of individual decision behavior which is imminent for any study of choices between single- or multiattribute alternatives. The results demonstrate that subjects do not express their preferences independently trial after trial. It is assumed that they interact with the alternatives, experimental task and paradigm instead. If slight experimental manipulations such as a different sequence of choice-trials and the repetition of a pair comparison show significant effects on inconsistent choice behavior then it seems likely that other alterations bring about more drastic changes. Consequently, an adaptive analysis is needed to understand and explain individual choice behavior.

Three categories of possible experimental manipulations with several dichotomous subcategories are listed below already leading to 2^{12} possible combinations. Each of these combinations may have a different impact on choice behavior. More categories are easily obtained extending the number of possible combinations. Obviously, only some of the combinations have been studied whereas others have been ignored. Experimental studies which use the suggested decompositions might not elucidate governing principles of adapting choice behavior. On the contrary, it is believed that decision research carried out with different alternatives, choice tasks or choice paradigms would illustrate the limitations of normative or non-adaptive decision models. If only some of the categories listed below interact with each other then this would also limit efforts to derive general principles from the observed choice behavior. Any experimental indication of higher order effects between these

(a) choice alternatives:
   i. real or artificial
   ii. certain or uncertain outcomes
   iii. single- or multiattribute
   iv. similar or dissimilar
(b) choice task:
   i. forced or non-forced choice
   ii. binary or n-ary choice
   iii. with or without time-limit
   iv. with or without cover-story
(c) choice paradigm:
   i. small or big set size
   ii. complete or incomplete pair comparison
   iii. fixed or random sequence of choice-trials
   iv. same or different sessions

categories would emphasize the difficulty to explain choice behavior in an non-adaptive choice model.

On the basis of weak assumptions algebraic decomposition techniques are able to detect where individual choices have adapted. No costly tracing methods (e.g., Ericsson & Simon, 1984) are necessary to single out choices or subsets of choices which are at the center of inconsistent choice behavior. As mentioned before, the question of how the choice process has adapted cannot be answered without detailed knowledge about the particular domain and the underlying information process. This is not only difficult to assess but also difficult to generalize because the changes in information processing are known to be context and domain-specific. It has been shown that at least in some domains the hierarchical organisation of semantic memory which is recalled in binary decision tasks (Lages, 1991; Albert & Lages, in preparation) makes the study of human decision making even more demanding.

4.2 Individual vs Group Data

The algebraic decomposition of choice behavior into critical sets of choices is an excellent tool to study individual choice behavior in greater detail without imposing strong assumptions upon the choice process. In Chapter 2 some techniques were developed that might help to extract not only quantitative but also qualitative information from a set of choices.

In Chapter 3 we have employed standard and new measures of inconsistency for the statistical analyses of experimental data. The average size of strong components together with the number of ear dicycles was analyzed in Experiment 2A and 2B as an example of another measure of inconsistency. The size of strong components is related to the longest dicycles and can be derived from the decomposition into strong components as well as the ear decomposition by sequence (see Table 3.19 and 3.29). The analyses on size of strong components revealed effects between groups which were not discovered by standard measures of inconsistency as provided by Kendall's ζ.

As mentioned before, the stepwise discriminant analysis is a crude statistical method for investigating the number of dicycles and ear dicycles. Unlike the analysis of variance it only allowed testing between subjects but not within subjects. If testing across sessions were possible it is assumed that the strong learning effect for 3-dicycles across sessions as found in Experiment 2A and 2B would have been confirmed by a corresponding analysis of dicycles and ear dicycles.

Determining all k-dicycles in a polynomial expression provides limited additional information about individual choice behavior. For group data it might be sufficient to assess only 3-dicycles in a pair comparison. For the comparison of individual choice behavior, however, assessment of all k-dicycles is recommended. The tabulated coefficients of every subject and session in Appendix B show that intransitive triples alone are often not sufficient to capture the extent of inconsistent choice behavior. The assessment of ear dicycles, however, is efficient and showed more discriminatory power than dicycles. It is concluded that ear dicycles give a clearer picture of inconsistent choice because they do not increase exponentially and because they take into account how inconsistencies evolved in a pair comparison.

As discussed in the introduction an untested assumption of any pair comparison is asymmetry. It is believed that an incomplete pair comparison does not sufficiently describe individual choice behavior unless all possible ordered pairs are presented, that is all pairs in the cartesian product of a set except for the diagonal elements in the adjacency matrix. Under such a paradigm the adjacency matrix can either remain a $(0, 1)$-matrix if preferences are counted only once, or the matrix may be extended to a multigraph with weighted arcs. Allowing 2-cycles together with k-dicycles provides a more detailed picture of inconsistent choices and choices can be analyzed in an ear decomposition by sequence in the same way as the choices in an incomplete pair comparison.

In a straightforward application the preference matrices of different subjects or sessions may be added together creating a multigraph. A difficulty lies in the fact that such a multigraph may have only one strong component due to individual differences in preference. This would considerably increase the complexity of the multigraph and a characterization in terms of all dicycles would be no longer suitable. Again, an ear decomposition is not restricted to adjacency matrices of moderate complexity and size and the same type of analysis can be performed. In general, the ear decomposition seems to be a robust technique which also works for preferences in a non-forced choice task or pair comparisons with missing data. It is highly recommended for small samples such as a pair comparison.

4.3 Toward a Qualitative Theory of Error

One of the major challenges in the field of social sciences is the understanding of error in human choice behavior (e.g., Luce, 1996). The probabilistic viewpoint is unsatisfying because it identifies error with randomness.

A probabilistic choice model, for example, describes preferences between lotteries or gambles by a probabilistic preference relation between alternatives. Thereby, the alternatives themselves are represented in terms of probabilities, thus subjecting a single preference to randomness in choice and randomness in utility. In this case and for most decisions of everyday's life a probabilistic model of individual decision making does not appeal.

On the other hand, if sensitivity to a physical intensity varies randomly and affects the decision process as in psychophysical tasks then probabilistic models appear justified. In Experiment 3A and 3B, for example, the energy modulation of luminances may have had probabilistic properties so that perceived contrast varies randomly due to the light source, the optical characteristics of the eye and some low-level visual processing.

The information processing approach including conjoint measurement, although successful under some circumstances, requires vector-like representations of alternatives which always results in artificial and domain-specific applications.

In general, deterministic and probabilistic theories address the problem of structure and variability in behavioral data. The dilemma is that classical algebraic approaches are inflexible and cannot account for variability whereas the probabilistic approach often does not provide a desirable algebraic or qualitative representation. An algebraic decomposition model, however, incorporates variability without loosing its qualitative appeal.

Appendix A

Mathematical Background

A.1 Notations and Basics

The following mathematical notations and abbreviations are used throughout the text without further comment.

$X, Y, \{\ldots\}$	sets
$\emptyset$	empty set
$X_1 \times \ldots \times X_n$	cartesian product of the sets $X_1, \ldots, X_n$
$(x_1, \ldots, x_n)$	ordered n-tuple
$\in$, $\subset$, $\subseteq$	element of, proper subset of, subset of
$\wedge$, $\vee$, $\neg$	logical and, or, negation
$\Rightarrow$	implies
$\forall$, $\exists$	for all, exists
$\Leftrightarrow$	equivalent
$>$, $\geq$	greater than, greater or equal than
A	matrix
A^T	transposed matrix
I (I_n)	identity matrix (of order n)
J (J_n)	all 1's matrix (of order n)
O (O_n)	zero matrix (of order n)
$\mathcal{M}_n$	set of all $(0,1)$-matrices (of order n)
$\mathbf{Z}, \mathbf{R}, \mathbf{C}$	set of natural, real, and complex numbers
$\mathbf{Z}[x]$	polynomial ring on $\mathbf{Z}$ in one indeterminate x

The symbol $\square$ indicates the end of an example or proof.

Some elementary mathematical definitions are introduced next.

Definition A.1.1 *Let $X_1, \ldots, X_n \neq \emptyset$ be sets. A subset R of the cartesian product $X_1 \times \ldots \times X_n$ is called n-ary* relation *on $X_1, \ldots, X_n$. A 2-ary relation on X, Y is called binary relation on X, Y, and if $X = Y$ binary relation on X.*

Definition A.1.2 *Let $R \subseteq X \times Y$ be a binary relation on X, Y. The* domain *and* image *of R is defined as:*

$$D(R) = \{x \in X \,|\, y \in Y (x, y) \in R\} \tag{A.1}$$
$$I(R) = \{y \in Y \,|\, x \in X (x, y) \in R\} \tag{A.2}$$

Instead of $(x, y) \in R$ we can also write xRy.

If $(x, y) \in R$ then $(y, x) \in \overline{R}$ defines the *complement* $\overline{R}$ of a binary relation R.

Definition A.1.3 *A binary relation F on X, Y is called a* mapping *from X to Y ($F : X \to Y$), if $\forall x \in X$ there is exactly one $y \in Y$ so that xFy. In this case we write $F(x) = y$ instead of xFy.*

Definition A.1.4 *A mapping $F : X \to Y$ is called*

(i) injective $:\Leftrightarrow \forall x_1, x_2 \in X$ (*if* $x_1 \neq x_2$ *then* $F(x_1) \neq F(x_2)$)

(ii) surjective $:\Leftrightarrow \forall y \in Y \, \exists x \in X \, (F(x) = y)$

(iii) bijective $:\Leftrightarrow$ *F is injective and surjective*

Definition A.1.5 *Let $X, Y \neq \emptyset$ be sets, $R_1, \ldots, R_n$ relations on X and $S_1, \ldots, S_n$ relations on Y. A mapping $F : X \to Y$ is called* homomorphism *of $\langle X, R_1, \ldots, R_n \rangle$ into $\langle Y, S_1, \ldots, S_n \rangle$ if and only if $\forall x_1, x_2 \in X$ and $\forall i = (1, \ldots, n)$ $[(x_1, x_2) \in R_i \Rightarrow (F(x_1), F(x_2)) \in S_i]$.*

A bijective homomorphism is called *isomorphism* and an *anti-isomorphic* mapping is a bijective homomorphism of the complement $\overline{R}$.

A.2 Order Theory

In this section some basic definitions of order theory are presented. For a complete introduction to order theory and its application consult Davey and Priestley (1990).

In the following $X \neq \emptyset$ is a set.

Definition A.2.1 *A binary relation $\succeq$ on X is called*

(i) reflexive $:\Leftrightarrow \forall x \in X \, (x \succeq x)$

(ii) irreflexive $:\Leftrightarrow \forall x \in X \, \neg(x \succeq x)$

(iii) transitive $:\Leftrightarrow \forall x, y, z \in X \, ((x \succeq y \land y \succeq z) \Rightarrow x \succeq z)$

(iv) negatively transitive $:\Leftrightarrow \forall x, y, z \in X \, (x \succeq y \Rightarrow (x \succeq z \lor z \succeq y))$

(v) antisymmetric $:\Leftrightarrow \forall x, y \in X\,((x \succeq y \wedge y \succeq x) \Rightarrow x = y)$

(vi) symmetric $:\Leftrightarrow \forall x, y \in X\,(x \succeq y \Rightarrow y \succeq x)$

(vii) asymmetric $:\Leftrightarrow \forall x, y \in X\,(if x \succeq y \Rightarrow y \not\succeq x)$

(viii) connected $:\Leftrightarrow \forall x, y \in X\,(x \succeq y \vee y \succeq x)$

Definition A.2.2 *A reflexive and transitive binary relation on X is called* partial quasiorder *on X. An antisymmetric partial quasiorder on X is called* partial order *on X. A transitive and connected binary relation on X is called a* weak order *on X. An antisymmetric weak order is called a* linear order *on X.*

Definition A.2.3 (Equivalence) *A reflexive, symmetric and transitive binary relation on X is called* equivalence relation *on X. If $\approx$ denotes an equivalence relation on X we define for any $x \in X$ the* equivalence class $[x]$ *of x by:*

$$[x] := \{y \in X \mid x \approx y\}$$

For $x, y \in X$ holds either $[x] = [y]$ or $[x] \cap [y] = \emptyset$, which means that two equivalence classes are either identical or disjoint.

We write $\sim$ for the *symmetric complement* of relation $\succ$, so that $x \sim y$ iff neither $x \succ y$ nor $y \succ x$, and $\succeq$ is the union of $\succ$ and $\sim$.

Definition A.2.4 *Let X be an ordered set and let $A \subseteq X$. Then*

(i) *$a \in A$ is called* minimal element *of A if $a \geq x \in A$ implies $a = x$, and* minimum *of A if $a \leq x$ for every $x \in A$.*

(ii) *$x \in X$ is called* lower bound *of A, if $x \leq a$ for all $a \in A$.*

(iii) *Let x be a lower bound of A, so that $y \leq x$ for all lower bounds y of A, then x is the* greatest lower bound *or* infimum *of A.*

A maximal element, maximum, upper bound, *and the* least upper bound *or* supremum *is defined dually.*

The following notation for the infimum and supremum of a set A is used in the text: $\inf\{A\}$ and $\sup\{A\}$.

Lemma A.2.5 *Let X be a partially ordered set such that $inf\{A\} \in X$ for every non-empty subset A of X. Then $sup\{A\}$ exists in X for every subset A of X which has an upper bound in X; in fact $sup\{A\} = inf\{x \in X \mid a \leq x \text{ for all } a \in A\}$ the infimum of the set of all upper bounds of A.*

Proof. see David and Priestley (1990), p.33. □

Definition A.2.6 (Closure) *Let X be a set. A mapping $\mathcal{C} : 2^X \to 2^X$ is a* closure operator *on X if, for all $A, B \subseteq X$,*

(i) $A \subseteq \mathcal{C}(A)$,

(ii) *if* $A \subseteq B$, *then* $\mathcal{C}(A) \subseteq \mathcal{C}(B)$,

(iii) $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$.

A subset A *of* X *is called* closed *(with respect to* $\mathcal{C}$*) if* $\mathcal{C}(A) = A$.

Definition A.2.7 (Lattice) *Let* X *be a finite and non-empty partially ordered set. If* $\sup\{A\} \in X$ *and* $\inf\{A\} \in X$ *for all* $A \subseteq X$, *then* X *is called a* lattice.

A.3 Graph Theory

Ore (1962) and Harary (1969) are classic textbooks on graph theory. A more extensive account on graph theory is contained in Bergé (1976). More specifically, Moon (1968) wrote a small book on tournaments and Reid and Beineke (1978) included a chapter on tournaments in their book.

A *graph* G consists of a set $V = \{a_1, a_2, \ldots, a_n\}$ of elements called *vertices* together with a prescribed set E of *unordered* pairs of distinct vertices of V. The number n of vertices is called the *order* of the graph G. Every unordered pair $\{a, b\}$ of vertices in E is called an *edge* of the graph G. Two vertices on the same edge or two distinct edges with a common vertex are *adjacent*. Also, an edge and a vertex are *incident* with one another if the vertex is contained in the edge. A *complete graph* is one in which all possible pairs of vertices are edges. A *digraph (directed graph)* D consists of a set V of vertices together with a prescribed set E of *ordered* pairs of *not necessarily distinct* vertices of V. Every ordered pair (a, b) of vertices in E is called an *arc* (or directed edge) of the digraph D.

A *directed walk* is of the form

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{k-1}, a_k).$$

If $a_k = a_0$ this is a closed directed walk. Moreover, if a_k and a_0 are the only identical vertices it is said to be a *directed cycle of length* k or k*-dicycle* for short. Thereby, the length of the walk $k > 0$ is the length of the directed cycle.[1] Note that intransitive pairs are directed cycles of length 3 or *3-dicycles*.

Two vertices a and b are called *strongly connected* provided there are directed walks from a to b and from b to a. A single vertex is regarded as strongly connected to itself. Strong connectivity between vertices is reflexive, symmetric, and transitive. Hence, strong connectivity defines an equivalence relation on the vertices of D and yields a partition

$$V_1 \cup V_2 \cup \ldots \cup V_t$$

of the vertices V. The subdigraphs $D(V_1), D(V_2), \ldots, D(V_t)$ formed by taking the vertices in an equivalence class and the arcs incident to them are called the *strong components* of D. The digraph D is *strong* if it has exactly one strong component.

[1] Directed cycles are sometimes called *circuits*.

A.4 Algebra

Only two topics from algebra are touched in the following subsections: (combinatorial) matrix theory and polynomial rings. For a proper introduction to algebra consult for example Allerby (1991) or Herstein (1975). The main focus here is on (ir)reducibility of matrices in $\mathcal{M}_n$, the set of square $(0,1)$-matrices, and the factorization of polynomials in $\mathbf{Z}[x]$, the polynomial ring of the integers.

A.4.1 Matrix Theory

The presentation here is due to Brualdi and Ryser (1991), Biggs (1993) and Cameron (1994). First some algebraic aspects of permutations are discussed.

There are two ways of regarding a *permutation*.

Definition A.4.1 *Let $X = \{1, 2, \ldots, n\}$ be a finite set. A* permutation $\pi : X \to X$ *is a one-to-one mapping from X onto itself.*

For the second representation, we assume that there is a natural ordering of the elements of X, say $\{1, 2, \ldots, n\}$. Then the permutation π can be represented as as ordered n-tuple $(\pi(1), \pi(2), \ldots, \pi(n))$. If $i \in X$ does not change position in the n-tuple, $\pi(i) = i$, then it is called a *fixpoint*. The set of all permutations of $\{1, 2, \ldots, n\}$, equipped with the operation of composition, is a group. It is known as the *symmetric group* of degree n, denoted by S_n.

A permutation π can be represented in so-called two-line notation as

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ 1\pi & 2\pi & \ldots & n\pi \end{pmatrix}$$

The top row can be in any order, as long as $x\pi$ is directly under x for all $x \in X$.

There is another representation of a permutation, called the cycle form. A *permutation cycle*, or *cyclic permutation*, is a permutation of a set X which maps

$$x_1 \to x_2 \to \ldots \to x_n \to x_1 \tag{A.3}$$

where $x_1, \ldots, x_n$ are all the elements of X in some order. If the cyclic permutation has $n - k$ fixpoints then the permutation cycle has length k. A permutation cycle (of any length) is not unique and can start at any point.

Lemma A.4.2 *Any permutation can be written as the composition of permutation cycles on pairwise disjoint subsets. The representation is unique, apart from the order of the factors, and the starting-points of the cycles.*

Proof. for example Cameron (1994), p.30. □

The one-to-one correspondence of permutation cycles and directed cycles follows immediately.

A simple indicator function is defined which is needed to establish a homomorphism between digraphs and adjacency matrices. Let D be a digraph of order n with $a, b \in V$ the set of vertices and whose set of arcs is $E \subseteq V \times V$. The indicator function $\iota : V \times V \to \{0, 1\}$ is given by

$$\iota(a_{ij}) = \begin{cases} 1 : (a, b) \in E \\ 0 : (a, b) \notin E \end{cases}$$

Next, the adjacency matrix of a digraph is defined by the indicator function ι.

Definition A.4.3 *Let D be a digraph of order n whose set of vertices is $V = \{a_1, a_2, \ldots, a_n\}$. The* adjacency matrix *is a $(0,1)$-matrix of size $n \times n$ $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ whose entries a_{ij} are given by the indicator function $\iota : V \times V \to \{0,1\}$,*

Let D be a digraph and let $\mathcal{C}$ be a collection of directed cycles $C_i \subset E$. Define an indicator function κ by

$$\kappa(c_{ij}) = \begin{cases} 1 : (a,b) \in C_i \\ 0 : (a,b) \notin C_i \end{cases}$$

An incidence matrix for the dicycles of a digraph can now be defined by the indicator function κ.

Definition A.4.4 *Let D be a digraph whose set of arcs is $E \subset V \times V$ and $\mathcal{C}$ the family of directed cycles $C_i \subseteq E$. The incidence matrix of dicycles is a $(0,1)$-matrix $C = [c_{ij}], (i, j = 1, 2, \ldots, n)$ whose entries c_{ij} are given by the indicator function $\kappa : \mathcal{C} \times E \to \{0,1\}$,*

Definition A.4.5 *Let x be an integer vector indexed by the arcs of a digraph D. The indegree of a vertex of D in x is the sum of the entries of x corresponding to the arcs entering that vertex. The outdegree of a vertex of D in x is the sum of the entries of x corresponding to the arcs leaving that vertex. A vector x is eulerian if each vertex of D has equal indegree and outdegree.*

The $n!$ permutation matrices of order n are obtained from the identity matrix I_n by arbitrary permutations of rows and columns of I_n. A permutation matrix P of order n satisfies the matrix equations

$$PP^T = P^T P = I_n \tag{A.4}$$

A matrix A of order n is called *reducible* if by simultaneous permutations of its lines a matrix can be obtained of the form

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices of order at least one. If A is not reducible, then A is called *irreducible*. Notice that a matrix of order 1 is irreducible.

The *determinant* of a matrix A is a matrix function defined by the formula

$$\det(A) = \sum_{\pi} (\operatorname{sign} \pi) a_{1\pi(1)} a_{2\pi(2)} \ldots a_{n\pi(n)} \tag{A.5}$$

where the summation extends over all permutations π of $\{1, 2, \ldots, n\}$. Suppose that the permutation π consists of k permutation cycles of sizes $l_1, l_2, \ldots, l_k$, respectively, where $l_1 + l_2 + \ldots + l_k = n$. Then $\operatorname{sign} \pi$ can be computed by

$$\operatorname{sign} \pi = (-1)^{l_1 - 1 + l_2 - 1 + \ldots + l_k - 1} = (-1)^{n-k} = (-1)^n (-1)^k. \tag{A.6}$$

The following formula can be applied to compute the determinant of the adjacency matrix of a digraph.

Theorem A.4.6 *If D is a digraph whose linear subgraphs are D_i, $i = 1, \ldots, n$ and D_i has e_i even cycles, then*

$$\det(A) = \sum_{i=1}^{n} (-1)^{e_i} \tag{A.7}$$

Proof. see Harary (1969), p.151. □

A.4.2 Polynomial Rings

It is impossible to give a self-contained and short account on polynomials and polynomial rings. Therefore, only a definition, and two theorems are presented here. The omitted proofs can be found in most textbooks on (abstract) algebra (e.g., Herstein, 1975, chap. 3). Let R be a commutative ring with unit element. By *the polynomial ring in x over R*, denoted as $R[x]$ we shall mean an expression of the form

$$\phi(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n$$

where $c_0, \ldots, c_n$ are coefficients in R, $1, \ldots, n$ are positive integers, and x is indeterminate. The degree of a polynomial $\phi(x) \in R[x]$ is the highest exponent n with a non-zero coefficient. A polynomial is called *monic*, if c_0 equals unity and *primitive* if the greatest common divisor of the coefficients equals one. The polynomial $\phi(x) = \sum_{i=0}^{n} c_i x^i$ of degree n is called *reducible* if and only if it can be expressed by the product of nonzero polynomials of lower degree, and *irreducible* otherwise. Whether or not a polynomial is irreducible depends on which polynomial ring it is considered as belonging to.[2]

Definition A.4.7 *Let R be an integral domain. R is a* unique factorization domain *if and only if*

(i) *every non-zero non-unit element p of R can be written as $p = q_1q_2\ldots q_m$, the q_i being irreducibles, and*

(ii) *if $p = q_1q_2\ldots q_m = q_1'q_2'\ldots q_n'$ then $m = n$ and $q_i = uq_j'$ pair off with u some unit of R.*

Theorem A.4.8 *If R is a unique factorization domain and if $p(x)$ is a primitive polynomial in $R[x]$, then it can be factored in a unique way as the product of irreducible elements in $R[x]$.*

Proof. see Herstein (1975), chap. 3, p.164. □

The next theorem offers a very helpful tool to decide if a polynomial ring is a unique factorization domain with the desired factorization properties.

Theorem A.4.9 *If R is a unique factorization domain, then so is $R[x]$.*

Proof. see Herstein (1975), chap. 3, p.165. □

We conclude this section with two corollaries. The first states that $\mathbf{Z}[x]$ the polynomial ring in x over the integers $\mathbf{Z}$ is a unique factorization domain.

Corollary A.4.10 *The polynomial ring $\mathbf{Z}[x]$ over the integers $\mathbf{Z}$ is a unique factorization domain and every monic polynomial can be factored in a unique way as the product of irreducible elements in $\mathbf{Z}[x]$.*

[2]For example every polynomial in $\mathbf{C}[x]$ the polynomial ring over the complex numbers $\mathbf{C}$ has a unique representation as linear factors. This is known as *the fundamental theorem of algebra.*

Proof. Only a sketch of a proof is given. From the fundamental theorem of arithmetic it is known that the integers $\mathbf{Z}$ are a unique factorization domain. It follows from Theorem A.4.9 that the polynomial ring $\mathbf{Z}[x]$ is a unique factorization domain. Clearly every monic polynomial $\phi(x)$ in $\mathbf{Z}[x]$ is primitive. Hence, according to Theorem A.4.8 $\phi(x)$ can be uniquely factored into irreducible polynomials. □

It follows that every $\phi(x)$ in $\mathbf{Z}[x]$ has a unique representation in terms of irreducible polynomials in $\mathbf{Z}[x]$.

Corollary A.4.11 *The polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$ over the integers $\mathbf{Z}$ is a unique factorization domain and every monic polynomial can be factored in a unique way as the product of irreducible elements in $\mathbf{Z}[x_1, \ldots, x_n]$.*

The *characteristic polynomial* of an adjacency matrix is given by

$$\phi(x) = \det(xI_n - A) \tag{A.8}$$

where x is indeterminate and I_n is the identity matrix.

If the characteristic polynomial is set equal to zero the solutions for the indeterminate are called *eigenvalues* of the matrix (e.g., Wilkinson, 1988).

Appendix B

Supplements

B.1 Experiment 1A and 1B

B.1.1 Instructions

Instruction[1]

This is a simple study on preferences between chocolate bars. In the following you are asked:'Which chocolate bar tastes better?' Shortly afterwards the name of a chocolate bar is displayed on the left and right side of the screen. By pressing key 'F' you choose the left chocolate bar, and by pressing key 'J' you choose the right bar. There are no correct or incorrect choices. After reading this instruction, please press key 'F' or 'J' to exercise the choice tasks in three examples.
After three training trials a similar instruction was displayed.

[1] Instructions of Experiment 1 were translated from German.

B.1.2 Stimuli

Stimuli were names of twelve chocolate bars which are listed in Table B.1. The code for each chocolate bar is given in the left column, the name of the chocolate bar in the middle column, and in the rightmost column it is listed how many of the 40 subjects in Experiment 1A and 1B tasted a chocolate bar prior to the forced choice comparison. Each subject was encouraged to taste any of the chocolate bars they were unfamiliar with before they chose between names of chocolate bars in two pair comparisons.

Table B.1: *Chocolate Bars*

Code	Name	Tasted
1	Balisto	9
2	Banjo	7
3	Mars	1
4	Milky Way	2
5	Lion	11
6	Snickers	1
7	Bounty	1
8	Duplo	0
9	Nussini	19
10	Nuts	4
11	Twix	3
12	Kitkat	5

B.1.3 Results of Experiment 1A

The following tables list the number of all dicycles and ear dicycles for each subject and session as determined by the prolog programs in Appendix C.1. The number of all dicycles can also be described by the coefficients z_k of polynomial ψ. The number of ear dicycles e_k which constitute directed ear bases are listed in separate tables.

Table B.2: *Coefficients of ψ for Each Subject and Session (Exp 1A)*

Group S (Resolution Block Designs)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
35	4	2	0	0	0	0	0	0	0	0
	4	3	1	0	0	0	0	0	0	0
39	24	79	206	482	1032	1936	2986	3504	2725	1031
	3	3	2	1	0	0	0	0	0	0
41	8	10	13	12	8	2	0	0	0	0
	3	3	2	0	0	0	0	0	0	0
43	7	9	9	6	2	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
45	4	1	0	0	0	0	0	0	0	0
	36	105	267	621	1239	1958	2291	1747	628	0
47	8	10	8	4	1	0	0	0	0	0
	4	3	2	0	0	0	0	0	0	0
49	2	1	0	0	0	0	0	0	0	0
	4	4	1	0	0	0	0	0	0	0
51	1	0	0	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
53	7	11	16	15	9	3	0	0	0	0
	10	17	26	31	24	9	0	0	0	0
54	5	4	1	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
Mean	7.0	12.7	25.3	51.9	105.2	194.1	298.6	350.4	272.5	103.1
	7.2	14.3	30.3	65.3	126.3	196.7	229.1	174.7	62.8	0
SD	6.4	23.7	63.8	151.2	325.7	612.0	944.3	1108	861.7	326.0
	10.4	32.2	83.5	195.5	391.0	618.9	724.5	552.4	198.6	0

Note: Subject numbers in the leftmost column were assigned to each subject in an unrelated experiment between sessions. The first and second line for each subject refers to Session 1 and 2, respectively.

Table B.3: *Coefficients of ψ for Each Subject and Session (Exp 1A)*

Group P (Repetition Block Designs)										
Subject	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
34	4	3	2	1	0	0	0	0	0	0
	4	2	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
40	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
42	1	0	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
44	8	10	14	10	4	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
46	2	1	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
50	1	0	0	0	0	0	0	0	0	0
	14	26	44	67	79	66	38	12	0	0
52	6	8	8	7	4	1	0	0	0	0
	4	6	4	1	0	0	0	0	0	0
Mean	2.3	2.2	2.4	1.8	0.8	0.1	0	0	0	0
	2.6	3.5	4.8	6.8	7.9	6.6	3.8	1.2	0	0
SD	2.8	3.7	4.8	3.6	1.7	0.3	0	0	0	0
	4.3	8.1	13.8	21.2	25.0	20.9	12.0	3.8	0	0

Note: Subject numbers in the leftmost column were assigned to each subject in an unrelated experiment between sessions. The first and second line for each subject refers to Session 1 and 2, respectively.

Table B.4: *Number of Ear Dicycles for Each Subject and Session (Exp 1A)*

Group S (Resolution Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
35	4	2	0	0	0	0	0	0	0	0
	3	3	1	0	0	0	0	0	0	0
39	6	4	7	5	3	14	7	6	3	0
	4	3	2	1	0	0	0	0	0	0
41	6	5	5	3	2	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
43	7	6	2	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
45	4	1	0	0	0	0	0	0	0	0
	6	4	7	6	13	6	2	1	0	0
47	8	6	5	2	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
49	2	1	0	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
51	1	0	0	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
53	6	2	3	5	4	1	0	0	0	0
	6	6	7	2	0	0	0	0	0	0
54	5	3	1	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
Mean	4.9	3.0	2.3	1.5	0.9	1.5	0.7	0.6	0.3	0
	3.6	2.7	2.2	0.9	1.3	0.6	0.2	0.1	0	0
SD	2.2	2.2	2.6	2.1	1.5	4.4	2.2	1.9	0.9	0
	1.3	1.4	2.6	1.9	4.1	1.9	0.6	0.3	0	0

Table B.5: *Number of Ear Dicycles for Each Subject and Session (Exp 1A)*

Group P (Repetition Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
34	4	3	2	1	0	0	0	0	0	0
	4	2	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
40	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
42	1	0	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
44	4	5	3	2	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
46	2	1	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
50	1	0	0	0	0	0	0	0	0	0
	8	8	6	6	5	3	0	0	0	0
52	6	7	5	3	0	0	0	0	0	0
	4	4	2	0	0	0	0	0	0	0
Mean	1.9	1.6	1.0	0.6	1.0	0	0	0	0	0
	2.0	1.5	0.8	0.6	0.5	0.3	0	0	0	0
SD	2.1	2.5	1.8	1.1	0.3	0	0	0	0	0
	2.6	2.6	1.9	1.9	1.6	0.9	0	0	0	0

B.1.4 Results of Experiment 1B

Table B.6: *Coefficients of ψ for Each Subject and Session (Exp 1B)*

Group SP (Resolution-Repetition Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
1	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
15	19	43	90	178	295	405	430	317	125	0
	3	2	1	0	0	0	0	0	0	0
17[a]	6	7	6	2	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
21	6	10	10	9	4	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
23	4	4	1	0	0	0	0	0	0	0
	3	1	0	0	0	0	0	0	0	0
25	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0
	4	3	1	0	0	0	0	0	0	0
29	3	2	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
33	1	0	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
Mean	3.8	5.9	10.2	18.7	29.9	40.5	43.0	31.7	12.5	0
	1.5	0.8	0.2	0	0	0	0	0	0	0
SD	5.7	13.4	28.2	56.0	93.2	128.1	136.0	100.2	39.5	0
	1.5	1.0	0.4	0	0	0	0	0	0	0

Note: Subject numbers in the leftmost column were assigned to each subject in an unrelated experiment between sessions (see Experiment 1A). Therefore some subject numbers are missing. The first and second line for each subject refers to Session 1 and 2, respectively.

[a]Subject 17 was excluded because he was unfamiliar with more than 5 chocolate bars.

Table B.7: *Coefficients of ψ for Each Subject and Session (Exp 1B)*

Group PS (Repetition-Resolution Block Designs)										
Subject	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
4	3	3	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
6[a]	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
12	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
14	2	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
18	3	3	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
24	4	3	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
28	4	3	3	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
30	1	0	0	0	0	0	0	0	0	0
	5	3	3	0	0	0	0	0	0	0
32	2	0	0	0	0	0	0	0	0	0
	6	6	7	3	0	0	0	0	0	0
Mean	2.1	1.4	0.6	0	0	0	0	0	0	0
	1.8	1.1	1.0	0.3	0	0	0	0	0	0
SD	1.4	1.4	1.0	0	0	0	0	0	0	0
	2.1	2.0	2.3	0.9	0	0	0	0	0	0

Note: Subject numbers in the leftmost column were assigned to each subject in an unrelated experiment between sessions (see Experiment 1A). Therefore some numbers are missing. The first and second line for each subject refers to Session 1 and 2, respectively.

[a]Subject 6 was excluded because he was unfamiliar with more than 5 chocolate bars.

Table B.8: *Number of Ear Dicycles for Each Subject and Session (Exp 1B)*

Group SP (Resolution-Repetition Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
1	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
15	6	7	7	7	5	6	6	1	0	0
	3	2	1	0	0	0	0	0	0	0
17[a]	4	2	3	1	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
21	4	5	3	3	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
23	4	2	0	0	0	0	0	0	0	0
	3	1	0	0	0	0	0	0	0	0
25	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0
	4	2	1	0	0	0	0	0	0	0
29	3	2	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
33	1	0	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
Mean	2.3	1.6	1.1	1.0	0.5	0.6	0.6	0.1	0	0
	1.5	0.7	0.2	0	0	0	0	0	0	0
SD	1.9	2.5	2.3	2.3	1.6	1.9	1.9	0.3	0	0
	1.5	0.8	0.4	0	0	0	0	0	0	0

[a]Subject 17 was excluded because he was unfamiliar with more than 5 chocolate bars.

Table B.9: *Number of Ear Dicycles for Each Subject and Session (Exp 1B)*

Group PS (Repetition-Resolution Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
4	3	2	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
6[a]	2	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
12	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
14	2	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
18	3	2	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
24	3	3	1	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
28	2	2	2	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
30	1	0	0	0	0	0	0	0	0	0
	3	1	3	0	0	0	0	0	0	0
32	2	0	0	0	0	0	0	0	0	0
	4	2	2	2	0	0	0	0	0	0
Mean	1.8	1.1	0.5	0	0	0	0	0	0	0
	1.4	0.5	0.5	0.2	0	0	0	0	0	0
SD	1.1	1.1	0.7	0	0	0	0	0	0	0
	1.3	0.7	1.1	0.6	0	0	0	0	0	0

[a]Subject 6 was excluded because he was unfamiliar with more than 5 chocolate bars.

B.2 Experiment 2A and 2B

B.2.1 Instructions

Instruction

This is a simple study on preferences between playing gambles. You are asked to choose between two gambles which are described by their possible win and loss (in pounds) and by their chance of winning and losing (in percentages). For your convenience the chance of winning and losing is also displayed as a pie chart for each gamble.

After reading the instructions there will be 3 training trials followed by 66 trials. In each trial you are asked:"Which gamble do you prefer to play?" Shortly afterwards two gambles are displayed on the left and right hand side and you are asked to make your decision as quickly and as accurately as possible.

On the next page you will see an example and you will learn how to give a response.

-Press 'SPACEBAR' to continue-

As soon as you know which gamble you prefer to play press key 'D' if you prefer to play the gamble on the left, and press key 'K' if you prefer to play the gamble on the right.

- Please always place your left index finger on key 'D' and your right index finger on key 'K' to ensure an undelayed response.

- Please respond as quickly and as accurately as possible.

B.2.2 Stimuli

In a pre-test four additional subjects computed roughly the expectancy value for each of 24 lotteries with approximately the same expectancy. From this pool 12 lotteries were selected which had equivalent expectancy values in spite of possible rounding errors by the subjects and which covered the widest range of probabilities and payoffs. The lotteries are listed in Table B.10 together with their expectancy values.

Table B.10: *Description of Lotteries*

Lottery Code	Chance of Win (in %)	Payoff (in £)	Expectancy Value
1	31	29.90	9.27
2	12	79.10	9.49
3	89	9.80	8.72
4	45	20.40	9.18
5	9	105.70	9.51
6	82	10.70	8.77
7	74	11.90	8.81
8	23	40.70	9.36
9	51	17.80	9.08
10	57	15.80	9.01
11	65	13.80	8.97
12	17	55.50	9.44

B.2.3 Results of Experiment 2A

Table B.11: *Coefficients of ψ for Each Subject and Session (Exp 2A)*

Group S (Resolution Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
3	44	136	398	1060	2452	4783	7462	8729	6785	2633
	24	57	122	228	329	363	275	106	0	0
	19	46	75	114	149	125	46	0	0	0
7	28	104	255	495	791	987	881	520	181	28
	27	79	191	378	617	800	798	559	235	43
	12	14	12	4	0	0	0	0	0	0
14	17	47	103	203	348	486	510	366	157	30
	23	88	219	435	766	1129	1227	877	362	65
	19	53	119	236	414	597	643	447	146	0
19	3	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
	4	6	4	1	0	0	0	0	0	0
22	14	24	38	56	69	68	50	22	4	0
	9	10	10	6	2	0	0	0	0	0
	20	45	100	205	362	549	673	620	375	107
28	13	16	22	21	13	0	0	0	0	0
	15	25	43	58	65	61	39	15	0	0
	16	39	96	195	320	409	394	258	96	15
33	26	66	171	374	710	1123	1429	1374	882	280
	34	92	239	533	1030	1615	1987	1820	1143	388
	32	87	232	548	1157	2049	2853	2881	1843	559
40	64	238	753	2299	6264	14419	27019	38369	36631	17589
	0	0	0	0	0	0	0	0	0	0
	3	3	1	0	0	0	0	0	0	0
46	20	43	95	184	304	408	438	368	210	52
	40	138	363	904	1997	3673	5447	6037	4356	1511
	3	2	1	0	0	0	0	0	0	0
47	22	58	144	282	433	512	456	269	78	0
	36	96	264	625	1296	2247	3096	3159	2117	710
	34	100	255	589	1143	1820	2210	1809	743	0
Mean	25.1	73.3	197.9	497.4	1138	2279	3825	5002	4493	2061
	21.0	58.6	145.1	316.7	610.2	988.8	1287	1257	821.3	271.7
	16.2	39.5	89.5	189.2	354.5	554.9	681.9	601.5	320.3	68.1
SD	17.4	70.6	228.6	705.4	1939	4494	8452	12022	11484	5517
	14.1	47.5	128.8	312.1	673.7	1221	1791	1973	1422	495.0
	11.1	34.5	93.0	220.6	449.1	765.4	1022	977.8	586.3	175.7

Table B.12: *Coefficients of ψ for Each Subject and Session (Exp 2A)*

Group P (Repetition Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
4	44	138	400	1072	2491	4923	7784	9248	7293	2834
	30	84	203	468	917	1486	1945	1929	1276	425
	32	91	232	555	1140	1962	2724	2893	2075	719
9	26	60	129	262	452	652	765	669	397	98
	19	35	51	79	120	167	189	154	80	21
	14	21	32	42	56	62	52	28	8	0
15	6	9	5	1	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	4	6	4	1	0	0	0	0	0	0
20	21	40	67	119	194	280	329	285	166	48
	21	41	79	124	163	177	163	129	78	26
	26	69	150	304	563	869	1048	919	497	118
21	30	92	235	545	1101	1857	2424	2261	1331	369
	0	0	0	0	0	0	0	0	0	0
	6	5	3	1	0	0	0	0	0	0
34	61	207	710	2150	5761	13142	24351	34272	32426	15474
	45	147	417	1114	2604	5113	8028	9323	6929	2212
	45	140	422	1137	2679	5375	8711	10767	9009	3816
41	41	127	385	1004	2322	4495	7039	8282	6482	2579
	26	69	170	391	786	1299	1665	1528	862	205
	15	33	65	111	144	141	98	44	9	0
48	55	184	597	1735	4430	9604	16872	22405	19943	8916
	27	70	169	365	647	914	928	583	144	0
	19	49	109	197	289	322	260	145	50	8
50	50	164	507	1420	3430	7053	11684	14572	12158	5058
	38	105	296	721	1541	2727	3870	4112	2903	1014
	47	159	435	1167	2743	5323	8261	9441	6948	2444
51	28	71	172	367	680	1024	1224	1088	635	172
	45	145	419	1128	2674	5346	8682	10622	8734	3585
	34	98	256	604	1225	2081	2818	2790	1782	547
Mean	36.2	109.2	320.7	867.5	2086	4303	7247	9308	8083	3555
	25.2	69.6	180.4	439.0	945.2	1723	2547	2838	2101	748.8
	24.2	67.1	170.8	411.9	883.9	1614	2397	2703	2038	765.2
SD	17.0	65.2	236.5	730.0	1960	4459	8213	11467	10756	5088
	15.8	52.5	156.0	425.9	1016	2035	3290	3980	3180	1220
	15.2	54.3	161.5	444.2	1060	2115	3386	4069	3260	1313

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.13: *Coefficients of ψ for Each Subject and Session (Exp 2A)*

Group N (Random Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
5	31	85	211	453	832	1209	1284	869	271	0
	31	91	243	552	1054	1648	1981	1661	836	183
	23	51	112	198	273	260	127	0	0	0
10	56	183	626	1828	4736	10352	18327	24606	22140	9993
	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
13	15	34	54	82	112	110	69	22	0	0
	30	86	193	422	777	1108	1171	775	228	0
	18	37	64	109	165	219	239	191	89	18
18	38	128	356	914	2067	3977	6191	7237	5618	2160
	13	30	60	98	127	116	63	15	0	0
	28	73	178	373	659	937	984	672	216	0
24	8	28	56	70	56	28	8	1	0	0
	7	21	35	35	21	7	1	0	0	0
	5	5	5	3	1	0	0	0	0	0
27	37	110	300	722	1519	2670	3742	3909	2706	943
	15	27	47	58	52	20	0	0	0	0
	11	18	23	25	19	8	0	0	0	0
32	13	22	36	50	55	48	25	5	0	0
	7	6	2	0	0	0	0	0	0	0
	9	11	12	11	10	8	4	1	0	0
37	48	152	470	1311	3225	6742	11463	14873	13053	5767
	64	247	713	2145	6048	14011	26202	37732	36822	18183
	64	267	709	2039	5996	13946	25420	35985	34758	16886
43	50	173	531	1464	3517	7095	11501	13941	11157	4386
	38	119	343	863	1922	3542	5197	5625	3900	1263
	30	85	203	475	963	1602	2099	1988	1180	319
45	10	17	22	26	21	9	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	3	1	0	0	0	0	0	0	0	0
Mean	30.6	93.2	266.2	692.0	1614	3224	5261	6546	5495	2325
	20.6	62.7	163.6	417.3	1000	2045	3462	4581	4179	1963
	19.1	54.8	130.6	323.2	808.6	1698	2887	3884	3624	1722
SD	18.0	65.0	224.9	666.0	1714	3703	6470	8569	7617	3397
	20.1	77.2	226.5	675.1	1886	4359	8158	11780	11534	5713
	18.9	80.5	216.6	626.2	1852	4336	7946	11297	10945	5329

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.14: *Numbers of Ear Dicycles for Each Subject and Session (Exp 2A)*

Group S (Resolution Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
3	11	7	10	8	6	3	0	1	0	0
	8	4	6	4	5	5	4	0	0	0
	7	7	5	3	2	3	1	0	0	0
7	11	10	11	7	9	5	1	1	0	0
	6	8	9	11	6	8	5	2	0	0
	7	7	4	2	0	0	0	0	0	0
14	7	9	9	10	8	9	3	0	0	0
	8	12	13	11	7	4	0	0	0	0
	7	6	5	7	5	7	3	3	2	0
19	3	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
	3	5	2	0	0	0	0	0	0	0
22	7	7	9	9	5	4	3	1	0	0
	8	4	4	2	0	0	0	0	0	0
	7	10	12	9	10	6	1	0	0	0
28	9	5	4	3	0	0	0	0	0	0
	7	6	7	5	5	3	2	1	0	0
	4	6	5	9	9	9	6	5	2	0
33	10	12	11	11	5	3	2	1	0	0
	6	7	10	9	9	7	4	2	1	0
	3	10	6	10	7	7	7	3	2	0
40	5	12	10	8	13	4	2	1	0	0
	0	0	0	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
46	7	8	8	7	10	8	4	2	1	0
	7	7	8	9	11	7	5	1	0	0
	3	1	0	0	0	0	0	0	0	0
47	6	7	9	10	7	5	1	0	0	0
	7	7	7	12	7	8	3	3	1	0
	4	7	7	8	5	6	4	3	1	0
Mean	7.6	7.8	8.1	7.3	6.3	4.1	1.6	0.7	0.1	0
	5.9	5.6	6.4	6.3	5.0	4.2	2.3	0.9	0.2	0
	4.8	6.1	4.7	4.8	3.8	3.8	2.2	1.4	0.7	0
SD	2.6	3.3	3.5	3.4	4.1	2.9	1.4	0.7	0.3	0
	2.7	3.5	4.1	4.7	3.9	3.3	2.2	1.1	0.4	0
	1.9	2.9	3.4	4.2	3.9	3.6	2.7	1.9	0.9	0

Table B.15: *Number of Ear Dicycles for Each Subject and Session (Exp 2A)*

Group P (Repetition Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
4	9	7	10	6	8	6	5	1	3	0
	9	10	10	10	7	6	3	0	0	0
	10	10	8	6	9	8	3	1	0	0
9	6	10	10	8	4	5	5	5	1	1
	9	11	8	7	6	6	4	4	0	0
	8	7	7	7	5	4	5	2	0	0
15	2	3	4	1	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	4	3	3	0	0	0	0	0	0	0
20	6	4	8	9	7	9	7	2	3	0
	10	9	10	7	5	5	4	3	2	0
	8	9	8	11	9	7	3	0	0	0
21	8	12	11	10	6	5	3	0	0	0
	0	0	0	0	0	0	0	0	0	0
	6	4	3	0	0	0	0	0	0	0
34	6	6	8	6	6	7	5	5	2	4
	11	9	7	7	4	7	4	5	1	0
	4	8	7	6	10	5	7	5	3	0
41	7	7	11	9	10	7	0	3	1	0
	7	5	8	7	7	6	9	5	1	0
	10	12	9	5	1	0	0	0	0	0
48	7	4	9	7	8	8	5	6	1	0
	11	4	7	6	7	10	0	0	0	0
	5	9	11	12	10	5	3	0	0	0
50	5	8	7	7	8	7	7	3	2	1
	10	8	12	11	7	3	2	2	0	0
	9	9	13	8	5	3	4	3	1	0
51	4	9	11	10	9	6	4	2	0	0
	6	7	6	7	6	10	3	4	6	0
	9	10	10	9	7	4	3	2	1	0
Mean	6.0	7.0	8.9	7.3	6.6	6.0	4.1	2.7	1.3	0.6
	7.4	6.3	6.8	6.2	4.9	5.3	2.9	2.3	1.0	0
	7.3	8.1	7.9	6.4	5.6	3.6	2.8	1.3	0.5	0
SD	2.0	2.9	2.2	2.7	2.9	2.4	2.5	2.1	1.2	1.3
	4.0	3.9	4.0	3.6	2.8	3.5	2.7	2.2	1.9	0
	2.4	2.8	3.2	4.0	4.1	2.9	2.3	1.7	1.0	0

Table B.16: *Number of Ear Dicycles for Each Subject and Session (Exp 2A)*

Group N (Random Block Designs)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
5	9	12	8	7	3	3	3	0	0	0
	6	12	15	5	5	5	5	1	1	0
	8	7	6	4	1	2	0	0	0	0
10	8	7	6	8	9	5	6	3	3	0
	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
13	10	8	7	2	3	4	2	0	0	0
	8	6	5	6	7	6	7	0	0	0
	8	6	6	4	5	6	7	10	3	0
18	5	6	7	14	6	10	3	3	1	0
	6	6	6	8	7	3	0	0	0	0
	7	7	10	8	7	5	1	0	0	0
24	4	7	7	6	4	4	3	1	0	0
	7	13	7	1	0	0	0	0	0	0
	4	3	5	2	1	0	0	0	0	0
27	6	8	13	9	10	5	4	0	0	0
	4	5	4	3	4	1	0	0	0	0
	4	7	4	3	3	0	0	0	0	0
32	8	10	7	3	3	5	0	0	0	0
	6	4	2	0	0	0	0	0	0	0
	8	8	7	5	4	3	1	0	0	0
37	8	7	12	8	9	5	4	2	0	0
	11	8	4	5	7	7	6	5	2	0
	10	9	4	9	7	6	2	5	2	1
43	8	15	11	10	4	4	2	1	0	0
	6	10	10	6	8	8	3	3	1	0
	7	15	11	12	6	2	0	1	1	0
45	4	6	4	3	3	1	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	3	1	0	0	0	0	0	0	0	0
Mean	7.0	8.6	8.2	7.0	5.4	4.6	2.7	1.0	0.4	0
	5.5	6.4	5.3	3.4	3.8	3.0	2.1	0.9	0.4	0
	5.9	6.3	5.3	4.7	3.4	2.4	1.1	1.6	0.6	0.1
SD	2.1	2.9	2.9	3.7	2.9	2.3	1.8	1.2	1.0	0
	3.2	4.5	4.6	3.0	3.5	3.2	2.9	1.7	0.7	0
	3.0	4.3	3.6	3.9	2.8	2.5	2.2	3.3	1.1	0.3

B.2.4 Results of Experiment 2B

Table B.17: *Coefficients of ψ for Each Subject and Session (Exp 2B)*

Group SNP (Resolution-Random-Repetition Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
1	7	11	10	5	1	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
6	30	73	188	387	667	877	785	355	0	0
	30	89	221	470	793	1056	993	562	127	0
	21	51	99	157	193	163	70	0	0	0
12	49	153	473	1333	3347	7154	12559	16834	15376	7179
	32	89	225	534	1120	1995	2896	3203	2407	940
	2	1	0	0	0	0	0	0	0	0
16	43	133	371	948	2114	4001	5997	6661	4857	1732
	23	69	165	345	626	923	1022	778	363	79
	11	22	34	43	31	11	0	0	0	0
23	14	32	49	58	46	22	5	0	0	0
	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
30	34	90	225	528	1058	1834	2549	2691	1919	691
	61	212	705	2131	5677	12846	23634	33013	31045	14772
	39	109	323	801	1731	3114	4474	4839	3512	1288
35	9	11	12	14	13	10	5	1	0	0
	8	11	11	10	5	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
36	4	6	4	1	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
39	53	178	570	1613	3991	8346	13995	17615	14745	6151
	37	111	318	795	1703	3063	4393	4715	3363	1192
	42	141	410	1083	2507	4900	7712	8945	6510	2102
49	3	1	0	0	0	0	0	0	0	0
	3	3	1	0	0	0	0	0	0	0
	8	12	12	8	2	0	0	0	0	0
Mean	24.6	68.8	190.2	488.7	1124	2224	3590	4416	3690	1575
	20.0	58.8	164.8	428.5	992.4	1988	3294	4227	3731	1698
	12.6	33.8	87.9	209.2	446.4	818.8	1226	1378	1002	339.0
SD	19.5	66.8	214.3	607.5	1511	3188	5455	7074	6191	2749
	20.1	69.5	224.4	663.5	1750	3956	7301	10245	9672	4615
	16.1	51.2	151.4	394.9	902.3	1733	2677	3063	2228	740.0

Table B.18: *Coefficients of ψ for Each Subject and Session (Exp 2B)*

Group PNS (Repetition-Random-Resolution Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
2	47	150	437	1166	2702	5313	8341	9745	7502	2826
	22	66	183	432	890	1547	2156	2189	1344	359
	9	14	16	11	4	0	0	0	0	0
8	37	106	268	621	1208	1930	2317	1880	761	0
	41	130	379	972	2126	3909	5788	6500	4942	1907
	31	88	208	469	878	1349	1649	1486	892	260
11	46	145	436	1206	2935	6113	10295	13091	11066	4597
	51	169	520	1445	3493	7182	11855	14663	12048	4897
	60	207	696	2086	5456	12166	21985	30205	27911	13011
17	35	97	230	547	1175	2082	2982	3248	2361	857
	52	172	525	1487	3633	7613	12933	16669	14398	6226
	38	107	301	747	1601	2894	4144	4397	3061	1056
25	50	159	502	1411	3466	7244	12251	15715	13509	5831
	13	25	48	73	87	69	30	0	0	0
	13	20	29	35	26	11	0	0	0	0
26	40	125	353	899	1959	3595	5301	5874	4315	1561
	21	60	151	333	639	1019	1253	1037	419	0
	0	0	0	0	0	0	0	0	0	0
31	47	153	439	1215	2989	6179	10469	13596	12005	5430
	55	187	568	1659	4211	9248	16595	22535	20630	9544
	45	133	394	1047	2484	4983	8128	10056	8404	3521
38	33	91	251	591	1220	2089	2823	2749	1633	406
	21	55	119	218	309	329	241	105	20	0
	18	46	94	143	156	115	51	10	0	0
42	48	152	476	1313	3162	6421	10532	13058	10772	4403
	23	51	113	214	331	405	368	228	75	0
	20	43	95	171	252	292	250	146	38	0
44	27	69	164	350	612	872	962	775	418	122
	15	30	55	86	120	136	106	47	9	0
	15	28	47	70	83	72	42	14	0	0
Mean	41.0	124.7	355.6	931.9	2143	4184	6627	7973	6434	2603
	31.4	94.5	266.1	691.9	1584	3146	5133	6397	5389	2293
	24.9	68.6	188.0	477.9	1094	2188	3625	4631	4031	1785
SD	7.7	31.9	118.8	377.9	1027	2323	4193	5674	5110	2299
	16.5	63.1	208.9	633.6	1633	3576	6322	8430	7558	3416
	18.3	65.0	221.4	666.5	1745	3877	6976	9548	8803	4096

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.19: *Number of Ear Dicycles for Each Subject and Session (Exp 2B)*

Group SNP (Resolution-Random-Repetition Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
1	6	6	6	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
6	5	4	6	8	4	4	5	0	0	0
	7	10	9	7	4	4	2	2	0	0
	7	4	3	4	3	5	2	0	0	0
12	8	8	8	9	7	2	4	6	2	1
	9	11	11	11	6	5	2	0	0	0
	3	1	0	0	0	0	0	0	0	0
16	10	9	9	7	11	4	4	1	0	0
	10	11	7	10	9	3	4	1	0	0
	6	4	5	3	2	1	0	0	0	0
23	7	7	4	4	3	3	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
30	6	7	9	8	7	7	7	3	1	0
	7	6	7	4	7	4	10	4	3	3
	6	9	4	10	9	9	6	2	0	0
35	6	7	6	6	5	3	2	1	0	0
	5	5	3	1	1	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
36	4	3	3	0	0	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
39	9	9	7	6	7	4	7	4	2	0
	8	13	9	10	6	6	1	2	0	0
	6	8	10	6	4	3	12	5	1	0
49	3	1	0	0	0	0	0	0	0	0
	3	3	0	0	0	0	0	0	0	0
	6	5	3	1	0	0	0	0	0	0
Mean	6.4	6.1	5.8	4.8	4.4	2.7	2.9	1.5	0.5	0.1
	5.6	6.3	4.8	4.3	3.3	2.2	1.9	0.9	0.3	0.3
	3.7	3.3	2.6	2.4	1.8	1.8	2.0	0.7	0.1	0
SD	2.2	2.6	2.8	3.6	3.7	2.3	2.9	2.1	0.8	0.3
	3.0	4.6	4.2	4.7	3.5	2.4	3.1	1.4	0.9	0.9
	2.9	3.3	3.2	3.4	2.9	3.0	4.0	1.6	0.3	0

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.20: *Number of Ear Dicycles for Each Subject and Session (Exp 2B)*

Group PNS (Repetition-Random-Resolution Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
2	11	7	10	5	6	2	5	5	3	1
	6	10	9	10	9	3	5	1	2	0
	4	5	2	3	1	0	0	0	0	0
8	10	6	5	5	5	7	4	3	0	0
	10	11	10	8	5	6	3	1	1	0
	8	9	10	10	6	5	7	0	0	0
11	6	8	6	8	6	12	6	2	1	0
	5	7	6	5	6	7	10	5	3	1
	5	4	10	6	5	6	8	8	1	2
17	6	7	6	5	7	9	3	3	6	3
	5	6	11	9	4	7	6	4	3	0
	11	8	6	10	7	7	4	2	0	0
25	5	10	5	8	7	6	8	2	3	1
	5	5	4	4	4	0	0	0	0	0
	7	4	4	4	2	1	0	0	0	0
26	6	6	6	8	8	5	6	6	3	1
	6	5	6	4	8	5	7	4	0	0
	0	0	0	0	0	0	0	0	0	0
31	10	7	4	11	4	7	6	3	3	0
	11	8	6	9	9	4	3	2	2	1
	5	7	5	9	12	8	5	2	1	1
38	9	6	7	7	9	6	6	4	0	1
	6	7	11	13	5	2	1	0	0	0
	6	9	8	8	3	2	0	0	0	0
42	10	7	11	12	7	7	1	0	0	0
	7	9	9	12	7	0	1	0	0	0
	5	5	8	6	10	6	5	0	0	0
44	9	11	11	10	5	5	3	1	0	0
	6	9	8	6	9	5	1	1	0	0
	5	8	6	4	5	4	2	2	0	0
Mean	8.2	7.5	7.1	7.9	6.4	6.6	4.8	2.9	1.9	0.7
	6.7	7.7	8.0	8.0	6.6	3.9	3.7	1.8	1.1	0.2
	5.6	5.9	5.9	6.0	5.1	3.9	3.1	1.4	0.2	0.3
SD	2.2	1.7	2.6	2.5	1.5	2.6	2.0	1.8	2.0	0.9
	2.1	2.1	2.4	3.2	2.1	2.6	3.2	1.9	1.3	0.4
	2.8	2.8	3.3	3.3	3.8	3.0	3.1	2.5	0.4	0.7

B.3 Experiment 3A and 3B

B.3.1 Instructions

Instruction

This is a simple study on the perception of brightness. Therefore, the light in your room should be switched off. In this study you are asked to compare two squares which are surrounded by grey frames.

After reading the instructions there will be 3 training trials followed by 66 trials. In each trial you are asked:"Which square does look brighter to you?" Shortly afterwards two framed squares are displayed on the upper and lower part of the screen and you are asked to make your decision as quickly and as accurately as possible. On the next page you will see an example and you will learn how to give a response.

-Press 'SPACEBAR' to continue-

Every square is surrounded by a frame of different grey, but you are asked to compare only the brightness of the square in the center. As soon as you know which square in the center looks brighter to you, press key 'U' if the square within the frame on the upper part of the screen looks brighter, and press key 'N' if the square within the frame on the lower part of the screen looks brighter.

- Please place your head on the chin-rest during all trials. Don't try to adjust the seat or the chin-rest during the trials.
- Please always place your left index finger on key 'U' and your right index finger on key 'D' to ensure an undelayed response.
- Please respond as quickly and as accurately as possible.

-Press 'U' or 'N' to start first training trial-

- The following 66 trials are run without a break. (The question "Which square does look brighter to you?" does not appear on the screen any more.)
- The light has to be switched off.
- Please place your head on the chin-rest during all trials. Don't try to adjust the seat or the chin-rest during the session.
- Please place your left index finger on key 'U' and your right index finger on key 'N' to ensure an undelayed response.
- Please respond as quickly an as accurately as possible.

-Press 'U' or 'N' to start-

B.3.2 Stimuli

The stimuli used in Experiment 3 were square disks in the center of a square surround. The luminance of the center and surround of each stimulus was chosen in such a way that the center was almost indiscriminate in brightness among stimuli. Figure B.1 gives only a impression of the stimuli and the levels of gray because the resolution of dots per inch (dpi) is varied rather than the luminance of pixels on screen (see Table B.21). The unattenuated monitor which appeared monochromatic was calibrated using a Minolta LS-110 photometer. Each pixel on the monitor was controlled by three 8 bit digital to analog converters (DAC) and a color look-up table (CLUT) allowing for 256 luminances (or levels of gray) on screen. Consequently, there were 256 by 256 possible combinations of gray levels for center and surround. The stimuli were generated by using the method of adjustment.

Table B.21: *Description of Disks*

Disk Code	Luminance of Center (in cd/m^2)	Luminance of Surround (in cd/m^2)
1	20.80	18.08
2	33.29	41.67
3	14.70	7.54
4	42.52	60.86
5	29.99	35.71
6	17.92	11.77
7	28.79	31.88
8	18.74	15.23
9	38.88	54.07
10	12.76	4.00
11	11.61	1.46
12	34.87	47.31

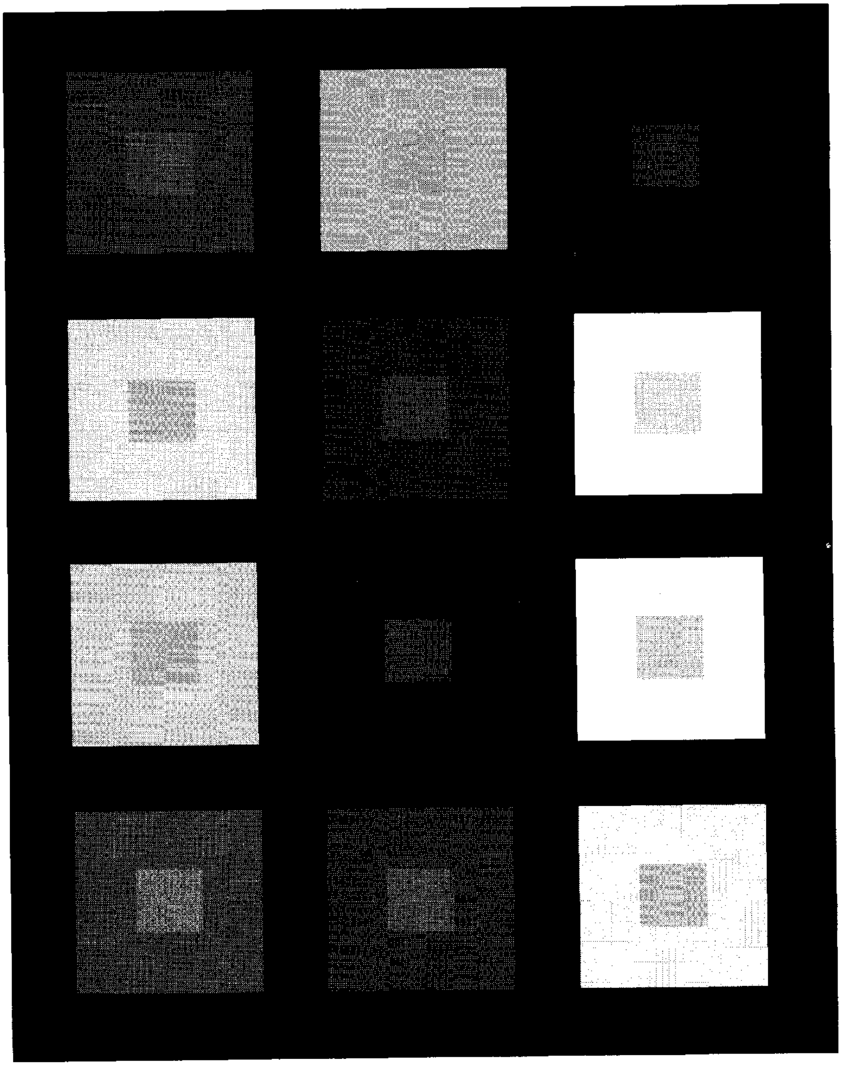

Figure B.1: Illustration of stimuli: The square disks in the center appear indifferent in brightness due to brightness contrast of center and surround

B.3.3 Results of Experiment 3A

Table B.22: *Coefficients of ψ for Each Subject and Session (Exp 3A)*

Group S (Resolution Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
1	24	57	126	254	443	665	792	674	361	90
	30	79	184	407	778	1210	1431	1131	446	0
	37	100	284	719	1627	3130	4910	5845	4685	1891
6	31	92	240	588	1273	2358	3483	3805	2695	884
	50	166	523	1471	3593	7414	12279	15230	12487	5000
	51	162	512	1423	3500	7226	12051	15129	12642	5237
12	2	1	0	0	0	0	0	0	0	0
	15	26	43	71	101	120	113	77	33	6
	4	3	2	0	0	0	0	0	0	0
16	17	29	52	85	133	185	202	161	83	20
	29	86	227	549	1187	2177	3164	3338	2203	656
	10	21	41	67	83	76	48	18	3	0
23	5	6	5	4	1	0	0	0	0	0
	12	21	37	49	50	36	18	5	0	0
	34	107	285	719	1592	3025	4725	5582	4371	1680
30	24	60	146	312	557	810	918	761	427	132
	30	78	189	422	826	1371	1808	1768	1113	335
	15	31	61	110	158	180	157	87	21	0
35	7	7	4	1	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
36	4	3	2	1	0	0	0	0	0	0
	28	75	190	438	891	1532	2100	2120	1376	416
	13	24	35	41	29	9	0	0	0	0
39	56	197	594	1757	4520	10007	18104	24827	22874	10607
	48	167	500	1372	3372	6995	11632	14565	12174	5075
	36	97	263	607	1191	1869	2216	1740	677	0
49	20	46	102	197	321	421	398	234	65	0
	43	124	364	970	2305	4719	7949	10282	9051	4060
	37	116	319	806	1795	3372	5059	5623	4071	1416
Mean	19.0	49.8	127.1	319.9	724.8	1445	2390	3046	2651	1173
	28.7	82.3	225.7	574.9	1310	2557	4049	4852	3888	1555
	23.9	66.2	180.2	449.2	997.5	1889	2917	3402	2647	1022
SD	16.4	60.0	182.3	539.3	1391	3094	5622	7740	7154	3326
	15.7	57.3	184.7	529.8	1330	2813	4773	6096	5197	2206
	17.0	56.5	175.4	480.4	1162	2366	3903	4866	4053	1679

Table B.23: *Coefficients of ψ for Each Subject and Session (Exp 3A)*

Group P (Repetition Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
2	49	159	499	1412	3498	7401	12691	16526	14488	6381
	53	172	562	1594	4008	8509	14651	19128	16761	7391
	46	145	426	1157	2688	5346	8559	10268	8148	3149
8	49	161	504	1382	3274	6524	10441	12582	10137	4104
	4	4	1	0	0	0	0	0	0	0
	18	38	83	152	237	301	288	186	70	11
11	26	64	140	307	592	970	1297	1303	864	284
	14	26	40	49	41	15	0	0	0	0
	46	144	425	1147	2791	5664	9370	11786	10028	4346
17	14	32	55	70	63	40	16	3	0	0
	8	11	13	12	7	2	0	0	0	0
	4	3	2	0	0	0	0	0	0	0
25	21	47	97	184	282	335	277	122	0	0
	50	161	486	1360	3323	6955	11808	15156	13004	5601
	30	79	210	490	999	1716	2406	2579	1870	681
26	12	31	68	117	150	143	100	46	10	0
	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
31	19	41	82	145	212	242	207	128	50	10
	21	49	91	162	257	354	395	326	167	37
	21	48	97	194	333	483	570	506	308	99
38	24	65	148	287	444	538	461	205	0	0
	11	17	20	27	26	21	12	3	0	0
	19	50	112	219	356	463	445	263	68	0
42	9	13	15	17	20	21	15	6	1	0
	12	28	58	96	124	123	93	51	18	3
	11	15	21	24	24	17	6	0	0	0
44	16	25	38	57	78	102	114	102	64	24
	44	141	423	1173	2844	5884	9859	12504	10593	4508
	24	61	142	287	486	647	627	385	108	0
Mean	23.9	63.8	164.6	397.8	861.3	1632	2562	3102	2561	1080
	21.9	61.0	169.4	447.3	1063	2186	3682	4717	4054	1754
	22.1	58.4	151.8	367.0	791.4	1464	2227	2597	2060	828.6
SD	14.2	53.2	182.3	534.9	1343	2831	4790	6119	5247	2260
	19.5	68.7	225.5	650.2	1632	3460	5924	7671	6649	2897
	15.2	51.7	158.2	439.6	1069	2189	3625	4522	3773	1578

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.24: *Coefficients of ψ for Each Subject and Session (Exp 3A)*

Group N (Random Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
5	44	141	383	978	2122	3811	5243	4905	2351	0
	7	8	10	7	3	0	0	0	0	0
	7	12	15	15	11	5	1	0	0	0
10	9	28	58	80	76	48	18	3	0	0
	13	27	43	63	68	52	24	5	0	0
	5	6	4	1	0	0	0	0	0	0
13	22	68	186	438	904	1576	2196	2225	1398	393
	6	8	8	4	1	0	0	0	0	0
	13	25	47	72	83	65	23	0	0	0
18	7	9	7	3	0	0	0	0	0	0
	9	10	9	4	0	0	0	0	0	0
	27	70	152	320	575	850	966	749	300	0
24	24	57	139	309	605	992	1297	1299	909	335
	34	90	242	562	1135	1944	2665	2707	1799	582
	21	43	92	157	222	245	202	114	34	0
27	8	11	14	16	15	11	5	1	0	0
	0	0	0	0	0	0	0	0	0	0
	3	3	1	0	0	0	0	0	0	0
32	16	32	62	102	140	139	93	34	0	0
	5	2	1	0	0	0	0	0	0	0
	6	8	8	4	1	0	0	0	0	0
37	2	0	0	0	0	0	0	0	0	0
	10	12	15	13	5	0	0	0	0	0
	18	34	61	95	124	131	110	67	22	0
43	8	10	14	12	6	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	4	2	1	0	0	0	0	0	0	0
45	23	71	180	401	793	1309	1625	1377	705	165
	44	134	402	1050	2409	4660	7269	8521	6636	2556
	40	120	330	882	2094	4301	7228	9259	7980	3398
Mean	16.3	42.7	104.3	233.9	466.1	788.6	1048	984.4	536.3	89.3
	12.9	29.1	73.0	170.3	362.1	665.6	995.8	1123	843.5	313.8
	14.4	32.3	71.1	154.6	311.0	559.7	853.0	1019	833.6	339.8
SD	12.4	43.0	121.0	311.8	680.5	1225	1689	1595	811.6	154.3
	14.5	45.6	137.1	354.7	801.7	1530	2358	2735	2112	808.8
	12.1	37.6	103.4	275.0	651.5	1340	2260	2905	2513	1075

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.25: *Number of Ear Dicycles for Each Subject and Session (Exp 3A)*

Group S (Resolution Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
1	6	8	8	15	11	7	0	0	0	0
	8	5	10	4	5	6	6	1	0	0
	7	7	9	10	7	8	5	2	0	0
6	9	7	4	9	7	6	3	6	4	0
	7	8	9	15	10	6	0	0	0	0
	8	7	6	9	7	8	7	2	1	0
12	2	1	0	0	0	0	0	0	0	0
	8	10	8	10	5	5	4	4	1	0
	3	2	1	0	0	0	0	0	0	0
16	7	9	4	1	1	0	0	0	0	0
	7	12	10	7	7	6	3	2	1	0
	8	9	9	7	5	4	3	0	0	0
23	4	4	3	3	1	0	0	0	0	0
	8	8	12	6	2	0	0	0	0	0
	9	7	5	6	5	11	8	2	2	0
30	7	8	11	11	6	7	4	1	0	0
	5	10	8	7	12	6	6	1	0	0
	5	8	7	8	6	6	3	2	0	0
35	6	5	4	1	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
36	4	3	2	1	0	0	0	0	0	0
	9	8	6	6	8	7	5	3	3	0
	5	4	6	5	1	0	0	0	0	0
39	9	8	6	7	9	4	8	2	1	1
	7	7	6	7	5	9	12	2	0	0
	10	8	7	6	7	3	3	1	0	0
49	8	12	9	5	8	2	0	1	0	0
	8	10	9	7	6	6	3	5	1	0
	9	9	4	10	12	4	4	1	2	0
Mean	6.2	6.5	5.1	5.3	4.3	2.6	1.5	1.0	0.5	0.1
	6.9	7.9	7.8	6.9	6.0	5.1	3.9	1.8	0.6	0
	6.6	6.2	5.4	6.1	5.0	4.4	3.3	1.0	0.5	0
SD	2.3	3.2	3.4	5.1	4.3	3.1	2.7	1.9	1.3	0.3
	2.0	3.1	3.3	3.8	3.5	2.9	3.7	1.8	1.0	0
	2.7	2.9	3.0	3.6	3.8	3.8	2.8	0.9	0.8	0

Table B.26: *Number of Ear Dicycles for Each Subject and Session (Exp 3A)*

Group P (Repetition Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
2	7	8	5	7	7	10	3	4	3	1
	5	3	4	9	9	10	7	7	1	0
	11	10	10	11	6	2	4	1	0	0
8	6	8	13	11	8	5	2	2	0	0
	3	3	0	0	0	0	0	0	0	0
	9	8	11	9	7	5	3	2	1	0
11	6	5	7	9	10	7	2	0	0	0
	5	5	8	2	1	0	0	0	0	0
	5	3	8	11	11	11	2	3	1	0
17	7	9	6	6	4	3	1	0	0	0
	6	5	5	4	1	0	0	0	0	0
	3	2	1	0	0	0	0	0	0	0
25	6	8	7	6	5	1	2	1	0	0
	8	9	10	7	2	8	3	7	0	1
	8	7	11	9	7	5	4	1	3	0
26	7	12	11	9	6	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0	0
31	6	8	9	8	7	5	6	5	1	0
	8	11	9	10	11	4	2	0	0	0
	8	7	8	12	8	6	5	1	0	0
38	5	6	6	4	4	6	3	2	0	0
	6	8	9	7	4	2	0	0	0	0
	5	7	5	6	9	7	5	1	0	0
42	8	11	7	8	5	5	1	0	0	0
	7	9	11	11	9	7	1	0	0	0
	7	7	5	4	4	1	0	0	0	0
44	8	8	9	7	6	5	6	2	3	1
	5	5	10	3	6	9	4	7	5	1
	8	6	11	11	5	3	1	0	0	0
Mean	6.6	8.3	8.0	7.5	6.2	4.7	2.6	1.6	0.7	0.2
	5.5	5.9	6.6	5.3	4.3	4.0	1.7	2.1	0.6	0.2
	6.6	5.8	7.0	7.3	5.7	4.0	2.4	0.9	0.5	0
SD	1.0	2.1	2.5	2.0	1.9	2.9	2.0	1.8	1.3	0.4
	2.0	3.2	4.1	4.1	4.2	4.1	2.4	3.4	1.6	0.4
	2.8	2.9	4.1	4.6	3.6	3.5	2.1	1.0	1.0	0

Table B.27: *Number of Ear Dicycles for Each Subject and Session (Exp 3A)*

Group N (Random Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
5	1	9	9	4	10	5	4	1	2	0
	4	3	4	3	1	0	0	0	0	0
	4	6	6	5	3	1	0	0	0	0
10	7	12	8	8	1	0	0	0	0	0
	6	10	7	9	3	1	0	0	0	0
	5	4	1	0	0	0	0	0	0	0
13	8	8	11	6	3	7	8	3	1	0
	5	6	4	0	0	0	0	0	0	0
	6	7	6	8	1	0	0	0	0	0
18	4	4	2	0	0	0	0	0	0	0
	8	5	5	2	0	0	0	0	0	0
	7	6	4	7	7	4	5	5	0	0
24	10	9	7	10	6	6	5	1	1	0
	10	9	8	7	4	6	5	4	2	0
	7	10	10	10	5	2	1	0	0	0
27	6	7	6	5	4	5	2	1	0	0
	0	0	0	0	0	0	0	0	0	0
	3	3	0	0	0	0	0	0	0	0
32	6	6	6	7	5	4	2	0	0	0
	5	2	1	0	0	0	0	0	0	0
	5	5	4	1	0	0	0	0	0	0
37	2	0	0	0	0	0	0	0	0	0
	8	6	3	1	0	0	0	0	0	0
	6	7	8	11	7	4	1	1	0	0
43	4	5	4	1	1	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	4	2	1	0	0	0	0	0	0	0
45	7	11	6	6	5	7	5	3	4	1
	7	7	6	7	5	12	6	3	2	0
	7	9	5	8	6	10	2	6	1	1
Mean	5.5	7.1	5.9	4.7	3.5	3.4	2.6	0.9	0.8	0.1
	5.4	4.8	3.8	2.9	1.3	1.9	1.1	0.7	0.4	0
	5.4	5.9	4.5	5.0	2.9	2.1	0.9	1.2	0.1	0.1
SD	2.8	3.5	3.2	3.4	3.2	3.1	2.8	1.2	1.3	0.3
	3.1	3.5	2.8	3.5	1.9	4.0	2.3	1.5	0.8	0
	1.4	2.5	3.2	4.4	3.1	3.2	1.6	2.3	0.3	0.3

B.3.4 Results of Experiment 3B

Table B.28: *Coefficients of ψ for Each Subject and Session (Exp 3B)*

Group SNP (Resolution-Random-Repetition Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
3	49	166	513	1459	3680	7836	13513	17645	15397	6718
	33	90	214	466	842	1160	1085	512	0	0
	10	12	7	1	0	0	0	0	0	0
7	20	43	80	133	178	186	133	49	0	0
	17	37	67	118	169	178	130	46	0	0
	25	60	134	272	470	660	701	489	163	0
14	40	121	359	976	2312	4649	7514	9169	7454	3046
	2	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
19	31	81	186	403	750	1122	1276	992	403	0
	57	200	605	1773	4653	10221	18186	24691	22578	10409
	33	91	233	521	967	1443	1585	1122	376	0
22	9	11	13	13	9	4	0	0	0	0
	31	83	206	464	895	1423	1737	1443	598	0
	12	31	70	121	159	153	103	44	9	0
28	52	168	528	1510	3811	8192	14322	19046	17066	7770
	51	176	516	1407	3370	6825	11049	13190	10184	3736
	32	85	211	449	787	1062	959	450	0	0
33	8	12	16	20	21	15	6	1	0	0
	8	12	17	19	15	9	3	0	0	0
	16	35	68	114	152	155	116	58	14	0
40	29	73	186	414	791	1268	1611	1496	884	224
	15	30	55	76	74	47	13	0	0	0
	21	49	112	231	418	625	736	642	364	97
46	10	11	10	6	2	0	0	0	0	0
	16	37	84	159	234	258	212	119	38	5
	3	1	0	0	0	0	0	0	0	0
47	3	2	1	0	0	0	0	0	0	0
	31	93	238	501	890	1283	1398	1010	360	0
	19	45	90	166	246	281	234	127	35	0
Mean	25.1	68.8	189.2	493.4	1155	2327	3838	4840	4120	1776
	26.1	75.9	200.2	498.3	1114	2140	3381	4101	3376	1415
	17.1	40.9	92.5	187.5	319.9	437.9	443.4	293.2	96.1	9.7
SD	17.8	64.3	207.9	603.6	1538	3314	5785	7653	6791	3042
	17.9	67.3	208.5	611.8	1598	3502	6189	8292	7454	3371
	11.2	31.9	82.6	183.8	339.2	499.5	535.2	376.3	152.6	30.7

Table B.29: *Coefficients of ψ for Each Subject and Session (Exp 3B)*

Group PNS (Repetition-Random-Resolution Block Design)										
Subj	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
4	55	183	587	1698	4326	9363	16389	21701	19313	8630
	65	240	761	2326	6414	14911	28132	40455	39192	19114
	60	207	678	2029	5358	12014	21885	30372	28421	13472
9	25	60	139	268	415	480	353	119	0	0
	27	67	161	339	613	911	1077	957	581	184
	27	66	163	345	624	920	1079	941	551	161
15	15	34	71	124	181	201	157	78	19	0
	7	15	22	21	12	3	0	0	0	0
	7	10	10	11	10	5	1	0	0	0
20	54	190	564	1645	4238	9332	16788	22743	20592	9317
	59	195	678	2012	5314	11818	21346	29213	26914	12476
	43	147	397	1025	2321	4352	6378	6649	4241	1191
21	13	22	38	53	58	47	27	8	0	0
	10	13	14	12	5	0	0	0	0	0
	6	7	6	4	0	0	0	0	0	0
34	22	47	104	203	344	486	542	463	272	80
	46	147	416	1146	2787	5759	9764	12636	11148	5030
	13	24	42	68	94	105	87	50	18	3
41	4	3	1	0	0	0	0	0	0	0
	13	23	34	50	57	54	42	20	4	0
	6	6	6	2	0	0	0	0	0	0
48	18	45	79	127	199	222	140	37	0	0
	15	30	47	66	78	66	35	9	0	0
	15	25	43	66	80	80	66	44	17	0
50	3	0	0	0	0	0	0	0	0	0
	7	9	10	7	3	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
51	14	24	41	56	56	42	16	0	0	0
	4	3	2	1	0	0	0	0	0	0
	8	14	17	19	17	9	2	0	0	0
Mean	22.3	60.8	162.4	417.4	981.7	2017	3441	4515	4020	1803
	25.3	74.2	214.5	598.0	1528	3352	6040	8329	7784	3680
	18.6	50.6	136.2	356.9	850.4	1749	2950	3806	3325	1483
SD	18.3	68.9	222.0	666.5	1745	3867	6932	9337	8403	3783
	23.0	87.3	294.7	901.7	2453	5614	10429	14746	14060	6757
	19.1	70.5	226.7	668.5	1741	3852	6943	9561	8916	4229

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Table B.30: *Number of Ear Dicycles for Each Subject and Session (Exp 3B)*

Group SNP (Resolution-Random-Repetition Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
3	5	6	5	9	8	5	8	5	2	2
	3	9	8	6	4	2	4	0	0	0
	8	7	1	0	0	0	0	0	0	0
7	8	5	6	8	6	3	0	0	0	0
	8	10	6	6	5	1	0	0	0	0
	6	6	8	8	5	7	1	3	1	0
14	8	7	8	9	8	8	6	0	1	0
	2	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
19	10	8	8	6	5	5	3	0	0	0
	6	9	11	6	9	6	3	1	3	1
	10	12	10	7	2	2	2	0	0	0
22	7	7	5	2	1	0	0	0	0	0
	8	7	8	9	5	7	0	1	0	0
	7	6	6	5	3	5	8	4	1	0
28	7	3	12	6	5	7	4	6	3	2
	6	9	8	9	5	5	10	3	0	0
	6	6	8	7	3	4	1	1	0	0
33	6	6	9	8	5	2	0	0	0	0
	6	6	5	5	4	1	1	0	0	0
	5	10	5	8	7	6	4	0	0	0
40	9	9	6	8	6	9	8	0	0	0
	4	5	4	6	6	2	1	0	0	0
	8	9	10	10	7	5	3	2	1	0
46	8	6	4	2	1	0	0	0	0	0
	7	11	14	13	7	3	0	0	0	0
	3	1	0	0	0	0	0	0	0	0
47	3	2	1	0	0	0	0	0	0	0
	4	11	10	6	6	4	2	2	0	0
	6	11	9	7	8	4	0	0	0	0
Mean	7.1	5.9	6.4	5.8	4.5	3.9	2.9	1.1	0.6	0.4
	5.4	7.8	7.4	6.6	5.1	3.1	2.1	0.7	0.3	0.1
	5.9	6.8	5.7	5.2	3.5	3.3	1.9	1.0	0.3	0
SD	2.0	2.1	3.0	3.3	2.9	3.4	3.4	2.3	1.1	0.8
	2.1	3.1	3.9	3.3	2.3	2.3	3.1	1.1	0.9	0.3
	2.8	4.0	4.0	3.8	3.1	2.6	2.6	1.5	0.5	0

Table B.31: *Number of Ear Dicycles for Each Subject and Session (Exp 3B)*

Group PNS (Repetition-Random-Resolution Block Design)										
Subj	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
4	7	7	7	6	4	7	7	5	5	0
	5	6	12	6	4	9	8	4	1	0
	4	11	8	8	7	6	6	1	3	1
9	7	4	9	5	9	2	0	0	0	0
	10	8	9	6	7	4	5	3	2	1
	9	10	12	9	6	4	4	1	0	0
15	7	7	9	8	5	3	3	2	1	0
	5	7	3	1	0	0	0	0	0	0
	6	7	5	5	3	2	0	0	0	0
20	6	7	8	6	11	6	4	3	4	0
	12	7	8	7	5	10	4	2	0	0
	10	11	7	9	4	4	4	4	1	1
21	7	10	7	6	4	1	1	0	0	0
	4	4	4	3	1	0	0	0	0	0
	3	2	3	2	0	0	0	0	0	0
34	7	8	6	7	7	8	4	5	2	1
	10	12	9	13	5	5	1	0	0	0
	8	15	12	11	5	2	1	1	0	0
41	4	2	1	0	0	0	0	0	0	0
	9	8	7	6	8	3	3	1	0	0
	4	3	2	1	0	0	0	0	0	0
48	4	9	8	3	8	3	1	0	0	0
	6	8	10	4	3	4	1	0	0	0
	6	7	6	7	6	5	5	2	1	0
50	3	0	0	0	0	0	0	0	0	0
	5	4	3	3	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
51	6	8	6	5	3	0	0	0	0	0
	4	3	2	1	0	0	0	0	0	0
	8	8	8	4	0	0	0	0	0	0
Mean	5.8	6.2	6.1	4.6	5.1	3.0	2.0	1.5	1.2	0.1
	7.0	6.7	6.7	5.0	3.3	3.5	2.2	1.0	0.3	0.1
	5.9	7.4	6.3	5.6	3.1	2.3	2.0	0.9	0.5	0.2
SD	1.5	3.2	3.1	2.8	3.7	3.0	2.4	2.1	1.9	0.3
	2.9	2.6	3.5	3.5	3.0	3.7	2.7	1.5	0.7	0.3
	2.9	4.6	4.0	3.8	2.9	2.3	2.4	1.3	1.0	0.4

Note: The first, second and third line for each subject refers to Session 1, 2 and 3, respectively.

Appendix C

Program–Listings

C.1 Prolog

The following two listings are written in Open Prolog (Version 1.0.2). The first program performs a depth-first search (Bratko, 1990) for directed cycles of any length in a directed graph. The program reads in the arcs (binary relations) of a coded digraph and lists all dicycles within the digraph. The algorithm is slow and does not operate in polynomial time. Consequently, depending on the complexity of the digraph, the application can be space and time consuming. The second program executes an ear decomposition by sequence. It finds directed ears of the coded digraph in a depth-breadth search by using a given sequence of the arcs. The ears are completed to ear dicycles using the same search procedure. The ear dicycles of each strong component form a directed ear basis as described in Section 2.3.

```
%********* OPEN PROLOG (Version 1.0.2d0) ********
% last change 29/8/95
% by M. Lages
% program finds dicycles in coded digraph

% running the program:
% start program by typing "start.<ENTER>"
% input-file    digraph1/2/3 (coded digraphs for Exp1/2/3)
% help-file     helpfile     (temporary clauses)
% output-file   circuit1/2/3 (list of dicycles)

start :-
  see(digraph3),
  read(graph(Inf,Arcs)),
  process(graph(Inf,Arcs)).

process(end of file) :-
  !.

process(graph([],[])) :-
  !,
  search.
```

```
process(graph(Inf,Arcs)) :-
  tell(helpfile),
  writelist(Inf),
  writelist(Arcs),nl,
  told,
  start.

%****** HELP-ROUTINES ******

writelist([]).              % write list
writelist([X|L]) :-
  write(X),write('.'),nl,
  writelist(L).

writelist2([]).
writelist2([X|L]) :-
  write(X),nl,
  writelist2(L).

conc([],L,L).               % concatenate two lists
conc([X|L1],L2,[X|L3]) :-
  conc(L1,L2,L3).

firstm(X,[X|L]).            % first member in list

maxlist([X|Y],Z) :- maxOfList(Y,X,Z). % maximum in list
maxOfList([],X,X).
maxOfList([H|T],M,X) :- H>M,!,maxOfList(T,H,X).
maxOfList([_|T],X,Y) :- maxOfList(T,X,Y).

member(X,[X|L]).            % member of list
member(X,[Y|L]) :-
  member(X,L).

reverse(X,Y) :- reverse(X,[],Y).
reverse([],X,X).
reverse([X|Y],Z,A) :- reverse(Y,[X|Z],A).

tellall(Query,Template) :-
        call(Query),
        write(Template),
        write('.'),
        nl,fail.
tellall(_,_).

%****** MAIN PROGRAM *******

search :-
  reconsult(helpfile),
  plain(Inf),
```

```
  tell(circuit3),
  write('graph(['),
  write(Inf),write('],'),
  tellall(solve(Node1,Resolution),Resolution),
  tell(circuit3),
  writelist2(List),write(').'),nl,   % List is a list of lists
  write('graph([],[]).'),nl,nl,
  start.

solve(Node1,Resolution) :-
  depth([],Node1,Node2,Solution),
  reverse(Solution,Resolution).

plain(i(Vp,Sess,Exp)) :-
  i(Vp,Sess,Exp).

depth(Path,Node1,Node2,[Node1|Path]) :-
  reverse(Path,Repath),
  firstm(Node2,Repath),          % condition for cycle
  maxlist(Path,M),
  Node2 >= M,                    % unique cycle
  Node2 >= Node1.

depth(Path,Node1,Node2,Sol) :-
  p(Node1,Node2,Trial1),
  p(Node2,Node3,Trial2),
  not member(Node2,Path),
  depth([Node1|Path],Node2,Node3,Sol).


%*********** OPEN PROLOG (Version 1.0.2d0) *************
% Last Change 7/2/97
% by M. Lages
% program performs ear decomposition by sequence

% Finds directed ears of coded digraph in depth-breadth
% search by using sequence of arcs. Ears are completed
% to directed ear cycles using the same search procedure

% start program by typing "start."<ENTER>
% input-file    digraph1/2/3 (coded digraphs)
% help-file     helpfile     (temporary clauses)
% output-file   dears1/2/3   (list of ear dicycles)
%*****************************************************

start :-
  see(digraph3),
  read(graph(Inf,Arcs)),
  process(graph(Inf,Arcs)).
```

```
process(end_of_file) :-
  !.

process(graph([],[])) :-
  !,
  search.

process(graph(Inf,Arcs)) :-
  tell(helpfile),
  writelist(Inf),
  writelist(Arcs),nl,
  told,
  start.

%******** HELP-ROUTINES ********

writelist([]).

writelist([X|L]) :-
  write(X),write('.'),nl,
  writelist(L).

conc([],L,L).             % concatenate two lists

conc([X|L1],L2,[X|L3]) :- conc(L1,L2,L3).

maxlist([X|Y],Z) :- maxOfList(Y,X,Z). % maximum in list

maxOfList([],X,X).
maxOfList([H|T],M,X) :- H>M,!,maxOfList(T,H,X).
maxOfList([_|T],X,Y) :- maxOfList(T,X,Y).

member(X,[X|L]).          % member of list

member(X,[Y|L]) :-
  member(X,L).

add(X,[],[X]).
add(X,L,L) :- member(X,L),!.
add(X,L,[X|L]).

del([X|L],X,L).

firstm(X,[]) :- !,fail.
firstm(X,[X|L]).             % first member in list

last(X,[]) :- !,fail.
last(X,L) :-                 % last member in list
  conc(_,[X],L).
```

```
delete(L,R) :-                 % remove directed ear
  s(S),
  delete(L,[],N1,S,R),
  retract(s(S)),!.             % ! prevents backtracking

delete([],L,N1,S,S).

delete(L,L2,N1,S,Z) :-
  firstm(N1,L2),
  firstm(N2,L),
  retract(p(N1,N2,Trial)),       % remove p-clause
  assert(q(N1,N2,Trial)),        % assume q-clause
  write(N1),write(N2),write('+'),
  del(L,First,L1),
  add(First,S,S1),
  delete(L1,[First|L2],First,S1,Z).

delete(L,L2,N1,S,Z) :-         % test if arcs in Sub
  firstm(N1,L2),
  firstm(N2,L),
  q(N1,N2,Trial),
  write(N1),write(N2),write('+'),
  del(L,First,L1),
% add(First,S,S1),
  delete(L1,[First|L2],First,S,Z).

delete(L,L2,N1,S,Z) :-         % add first vertex
  del(L,First,L1),
  add(First,S,S1),
  delete(L1,[First|L2],First,S1,Z).

retractq :-
   retract(q(_,_,_)),
   retractq.

%reverse([],Y).
reverse(X,Y) :- reverse(X,[],Y).

reverse([],X,X).
reverse([X|Y],Z,A) :- reverse(Y,[X|Z],A).

shell(Q,T) :-
      iterate3(N1,S,Sol),
      reverse(Sol,Resol),
      delete(Resol,NS),nl       % remove ear (p-clauses)
      assertz(s(NS)),           % assert new Sub
      tellall(Q,T),
      retract(s(X)),            % start new component
      assertz(s([])),
      retractq,                 % remove all q-clauses
```

```
      fail.
shell(_,_).

tellall(Query,Template) :-
        call(Query),
%        write(Template),
%        write('.'),
         nl,fail.
tellall(_,_).

%****** MAIN PROGRAM *******

search :-
  reconsult(helpfile),
  plain(Inf),
  tell(dears3),
  write('graph(['),
  write(Inf),write('],'),nl,
  assertz(s([])),
  shell(solve(Sub,Resolution),Resolution),
  write('graph([],[]).'),nl,nl,
  retract(s(Sub)),
  start.

solve(NSub,Resolution) :-                % do Query
  iterate(Sub,Solution),
  reverse(Solution,Resolution),
  delete(Resolution,NSub),               % remove arcs
  assertz(s(NSub)).                      % assert new Sub
% write(NSub),nl.

plain(i(Vp,Sess,Exp)) :-
  i(Vp,Sess,Exp).

iterate(Sub,Solution) :-
  trydepth(Sub,Sol,1),
% write(Sol),nl,
  firstm(N1,Sol),
  depth2([],Sol,N1,Solution).

trydepth(Sub,Sol,Trial) :-
  depth1([],Sub,Node1,Sol,Trial)
  ;
  Trial<66,
  NTrial is Trial+1,
  trydepth(Sub,Sol,NTrial).

depth1(Path,Sub,Node1,[Node1|Path],Trial) :-
  firstm(N,Path),          % N is last vertex in Path
  p(N,Node1,T),            % test if arc exists
```

```
  last(Nod,Path),
  s(Sub),
  member(Nod,Sub),        % Nod first vertex is in Sub
  member(Node1,Sub).      % Node1 last vertex is in Sub
% write('e').

depth1(Path,Sub,Node1,[Node1|Path],Trial) :-
  last(Node1,Path),
  s(Sub),
  member(Node1,Sub).       % first and last vertex is in Sub
% write('x').

depth1(Path,Sub,Node1,Sol,Trial) :-
% search for Path in p-clauses
  p(Node1,Node2,Trial1),
  Trial>=Trial1,
  not member(Node2,Path),
  depth1([Node1|Path],Sub,Node2,Sol,Trial).

depth2(Path,Sol,N1,[N1|Sol]) :-
  reverse(Sol,Resol),
  firstm(N1,Resol),                 % directed cycle
  firstm(N,Path),                   % last vertex
  q(N,N1,T),
  del(Resol,N1,Res),
  firstm(N2,Res),
  p(N1,N2,_).
% write('b').

depth2(Path,Sol,N1,Solution) :-
% search for Path in q-clauses
  q(N1,N2,Trial1),
  not member(N2,Path),
  depth2([N1|Path],[N1|Sol],N2,Solution).

iterate3(Node1,Sub,Sol) :-
  trydepth3(Node1,Sub,Sol,1).

trydepth3(Node1,Sub,Sol,Trial) :-
  depth3([],Sub,Node1,Node2,Sol,Trial)
  ;
  Trial<66,
  NTrial is Trial+1,
  trydepth3(Node1,Sub,Sol,NTrial).

depth3(Path,Sub,Node1,Node2,[Node2,Node1|Path],Trial) :-
   last(Node2,Path).               % directed cycle
%  write('c').

depth3(Path,Sub,Node1,Node2,Sol,Trial) :-
```

```
% search for Path in p-clauses
  p(Node1,Node2,Trial1),
  Trial>=Trial1,
  p(Node2,Node3,Trial2),
  Trial>=Trial2,
  not member(Node2,Path),
  depth3([Node1|Path],Sub,Node2,Node3,Sol,Trial).
```

C.2 Mathematica

The following program was written for Mathematica 2.0. The listing of the function ReadGraph1[] is given that reads in the adjacency matrix of an arbitrary digraph of order 12 and prints out the factorization of its characteristic polynomial. Useful functions for doing discrete mathematics are provided by Skiena (1990) and Wolfram (1991).

```
(* writen by ML based on ReadGraph[], last change 9.5.95 *)

ReadGraph1[fileName0_String, fileName1_String] :=
Module[{file0, expr, m1, poly, coef, facs},
file0 = OpenRead[fileName0];
file1 = OpenWrite[fileName1];
If[ file0 === $Failed, Return[] ];
While[ True,
expr = Read[ file0, {Number, Number, Number} ];
If[ expr === EndOfFile, Break[] ];
m1 = Read[ file0, Table[{Number, Number, Number, Number,
                  Number, Number, Number, Number,
                  Number, Number, Number, Number},
                 {12}] ];
            poly = Det[ m1 - x IdentityMatrix[12] ];
            coef = CoefficientList[poly, x];
            facs = Factor[poly];
Write[file1, expr]
Write[file1, coef]
Print["Subject ", expr]
Print[facs]
(*Print[poly]*)
(* Print[M1] *)
(*Print[facs]*)
];
Close[file0]
Close[file1]
]
```

C.3 C-Program

The following C-program was written for Think C 6.0 and counts preference reversals between two adjacency matrices using standard ANSI C language (Kernighan & Ritchie, 1988).

Another program was developed in Mathematica utilizing the built-in function for the Hadamard product and the function ReadGraph1[].

```
/*********************************
 * Last revision 5.2.94, M. Lages
 *
 * reads  adjacency.o
 * writes reverse.o
 *********************************/
#include <stdio.h>

/* #include <math.h> */

FILE  *filein;
FILE  *fileout;

main ()
     {
     int  i,j,m,n;                 /* row and column indices */
     int  a1[12][12], a2[12][12];  /* preference matrix */
     int  subj, sess, exp, deci, coun;

     /* Read data */

     filein = fopen("adjacency.o", "r");
     if (filein == NULL)
        printf("Couldnt open file adjacency.o\n");
     fileout = fopen ("reverse.o", "w");
     if (fileout == NULL)
        printf("Couldnt open file reverse.o\n");  /* fp); */
     else {
           while (!feof(filein)) {
           /* feof() returns non-zero at end of file */
                 fscanf(filein,"%d %d %d", &subj, &sess, &exp);
                 printf("%d %d %d\n", subj, sess, exp);
                 if (sess == 1) {
                    for (m=1;m<13;++m) {
                        for (n=1;n<13;++n) {
                            fscanf(filein,"%2d ", &a1[m][n]);
                        /*    a1[m][n] = getc (filein);  */
                            printf("%2d ", a1[m][n]);
                        };
                    printf("\n");
                    };
                 };
                 fscanf(filein,"%d %d %d", &subj, &sess, &exp);
```

```
                printf("%2d %d %d\n", subj, sess, exp);
                if (sess == 2) {
                   for (m=1;m<13;++m) {
                       for (n=1;n<13;++n) {
                           fscanf(filein,"%2d ", &a2[m][n]);
                       /*    a2[m][n] = getc(filein);            */
                           printf("%2d ", a2[m][n]);
                       };
                   printf("\n");
                   };
                };

                /* comparing matrices */
                coun = 0;
                for (m=1;m<13;++m) {
                     for (n=1;n<13;++n) {
                        if ((m!=n) && (a1[n][m]==0) && (a2[m][n]==0) ) {
                           coun++;
                           fprintf(fileout,"%2d %d %2d %2d %2d %2d %2d\n",
                           subj, exp, coun, m, n, a1[m][n], a2[n][m]);
                        };
                     };
                };
                /* fprintf(fileout, "%2d %d %2d\n", subj, exp, coun); */
          };
     };
     fclose(filein);
     fclose(fileout);

return;
}
```

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