A DENSITY THEOREM FOR ASYMPTOTICALLY HYPERBOLIC INITIAL DATA SATISFYING THE DOMINANT ENERGY CONDITION

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ABSTRACT. When working with asymptotically hyperbolic initial data sets for general relativity it is convenient to assume certain simplifying properties. We prove that the subset of initial data with such properties is dense in the set of physically reasonable asymptotically hyperbolic initial data sets. More specifically, we show that an asymptotically hyperbolic initial data set with non-negative local energy density can be approximated by an initial data set with strictly positive local energy density and a simple structure at infinity, while changing the mass arbitrarily little. The argument follows an argument used by Eichmair, Huang, Lee, and Schoen in the asymptotically Euclidean case.

1. INTRODUCTION

In general relativity Einstein's equations read

$$\mathrm{Ric}^\gamma - \tfrac{1}{2}\mathrm{Scal}^\gamma \gamma = \mathrm{T}. \tag{1}$$

Here Ric^γ and Scal^γ denote respectively the Ricci tensor and the scalar curvature of a spacetime $(\mathcal{M}, \gamma)$, and the symmetric divergence-free 2-tensor T is the stress-energy tensor of the spacetime. A spacetime $(\mathcal{M}, \gamma)$ satisfying (1) is said to obey the dominant energy condition if for any future directed timelike vector ν the vector $-\mathrm{T}(\nu, \cdot)^\sharp$ is either future directed timelike or null. This condition means that the energy density of $(\mathcal{M}, \gamma)$ is non-negative and that the energy cannot travel faster than the speed of light.

Let (M, g) be a Riemannian submanifold of the spacetime $(\mathcal{M}, \gamma)$ satisfying the Einstein equations with unit normal denoted by η and second fundamental form denoted by K. In this case (M, g) can be viewed as a "constant time slice" of $(\mathcal{M}, \gamma)$. The dominant energy condition for $(\mathcal{M}, \gamma)$ at points of M is equivalent to the inequality $\mu \geq |J|_g$ everywhere on M. Here the energy density $\mu := \mathrm{T}(\eta, \eta)$ and the momentum density $J := \mathrm{T}(\eta, \cdot)$ can be computed from g and K using the *constraint equations*

$$-2\mu = -\mathrm{Scal}^g - (\mathrm{tr}^g K)^2 + |K|_g^2, \tag{2}$$

$$J = \mathrm{div}^g K - d(\mathrm{tr}^g K). \tag{3}$$

By an *initial data set* for the Einstein equations we will mean a triple (M, g, K) consisting of a Riemannian n-manifold (M, g) and a symmetric 2-tensor K defined on M. We will say that (M, g, K) satisfies the *dominant energy condition* if $\mu \geq |J|_g$ holds everywhere on M, where μ and J are defined through (2)–(3). By the above discussion, this means that (M, g, K) arises as a constant time slice of a spacetime satisfying the dominant energy condition.

An initial data set (M, g, K) is said to be *asymptotically Euclidean* if outside some compact set M is diffeomorphic to the complement of a ball in the Euclidean space $\mathbb{R}^n$, and if under this diffeomorphism (g, K) approaches $(\delta, 0)$ sufficiently fast at infinity. For asymptotically Euclidean initial data sets *asymptotic integrals* at infinity can be defined. They are integrals which arise as boundary terms when integrating the constraint operator

$$\Phi : (g, K) \mapsto \left(-\mathrm{Scal}^g - (\mathrm{tr}^g K)^2 + |K|_g^2, \ \mathrm{div}^g K - d(\mathrm{tr}^g K) \right)$$

against elements in the kernel of $D\Phi^*_{(\delta,0)}$, which correspond to Killing vectors of the Minkowski spacetime. In particular, this gives rise to the Arnowitt-Deser-Misner energy E and linear momentum P. The *positive mass conjecture* asserts that $E \geq |P|_g$ provided that the dominant energy condition holds. In particular, $E \geq 0$ is expected to hold under the same assumption, which is the statement of the *positive energy conjecture*. For recent progress on these conjectures, see [13].

For many applications in mathematical general relativity it is an advantage to work with initial data which has simple asymptotics at infinity. For example, an asymptotically Euclidean initial data set (M, g, K) is said to have *harmonic asymptotics* if in asymptotically Euclidean coordinates at infinity we have

$$g = u^{\frac{4}{n-2}} \delta, \qquad \pi := K - (\mathrm{tr}^g K)g = u^{\frac{2}{n-2}} (\mathcal{L}_Y \delta - \mathrm{div}^\delta Y),$$

where $\mathcal{L}$ denotes the Lie derivative, and the positive function u and the vector field Y are such that

$$u(x) = 1 + A|x|^{2-n} + O(|x|^{1-n}), \qquad Y_j(x) = B_j |x|^{2-n} + O(|x|^{1-n}),$$

for $A \in \mathbb{R}$ and $B \in \mathbb{R}^n$. In this case the Arnowitt-Deser-Misner energy and linear momentum can be easily recovered from the asymptotic expansions of u and Y at infinity, namely,

$$E = 2A, \qquad P_j = -\tfrac{n-2}{n-1} B_j.$$

Further, many arguments can be simplified by working with initial data with strictly positive local energy density, that is such that the *strict dominant energy condition* $\mu > |J|_g$ is satisfied. This condition is preserved under "small" perturbations of the initial data, whereas the standard dominant energy condition $\mu \geq |J|_g$ might get violated by a perturbation. This observation is crucial for the proof of the positive energy conjecture in dimension $n = 3$ by Schoen and Yau [24] and for its extension to dimensions $3 \leq n \leq 7$ by Eichmair [12].

In [13, Theorem 18] Eichmair, Huang, Lee, and Schoen prove that an asymptotically Euclidean initial data set satisfying dominant energy condition can be slightly perturbed to an initial data with harmonic asymptotics which obeys strict dominant energy condition while changing the energy E and the linear momentum P arbitrarily little. That is, the set of asymptotically Euclidean initial data with these preferred properties is dense in the set of asymptotically Euclidean initial data satisfying the dominant energy condition. This result is used in the proof of the positive mass theorem by the above authors [13, Theorem 1].

The goal of the current paper is to prove the analogue of this density result for asymptotically hyperbolic initial data sets. Roughly speaking, an initial data set (M, g, K) is asymptotically hyperbolic if the Riemannian metric g approaches the hyperbolic metric b on hyperbolic space $\mathbb{H}^n$ in a chart covering everything outside a compact set. For K, there are two natural choices: either $K \to 0$ at infinity (as

for spacelike totally geodesic hypersurfaces in asymptotically anti-de Sitter space-times) or $K \to g$ at infinity (as for "hyperboloidal" hypersurfaces in asymptotically Minkowski spacetimes). In this paper we adopt the second approach and consider "hyperboloidal" initial data, see Definition 2.2. Then similar considerations as in the asymptotically Euclidean case give rise to the notion of *mass* for asymptotically hyperbolic initial data, which is a linear functional on a certain finite dimensional vector space.

The main result of this paper is that a given asymptotically hyperbolic initial data set (M, g, K) satisfying the dominant energy condition can be approximated by an initial data set with *conformally hyperbolic asymptotics* in the sense of Definition 2.3 which obeys the strict dominant energy condition, while changing the value of the mass functional by an arbitrary small amount. While the method of the proof requires $g - b = O(e^{-\tau r})$ and $K - b = O(e^{-\tau r})$ for $\frac{n}{2} < \tau < n$, a slight modification of the argument yields a useful density result in the critical case $\tau = n$. The applications of our results are similar to those of [13, Theorem 18]. In particular, the results are used in the proof of the positive mass theorem for asymptotically hyperbolic initial data by the second author (in preparation).

The paper is organized as follows. The definition of mass and its continuity with respect to the initial data is discussed in Section 2. In Section 3 we show that a given asymptotically hyperbolic initial data set satisfying the dominant energy condition can be perturbed slightly to satisfy the strict dominant energy condition, while changing the mass arbitrarily little. Then in Section 4 we make a further perturbation to conformally hyperbolic asymptotics, while preserving the strict dominant energy condition. In Section 5 we prove a density result in the critical case, concerning asymptotically hyperbolic initial data sets that have *Wang's asymptotics* (see Definition 5.1). Finally, in Section 6 we give comments on extensions of the results of this paper. Some supplementary results concerning differential operators on asymptotically hyperbolic manifolds are contained in Appendices A, B, and C.

2. Preliminaries

2.1. **Asymptotically hyperbolic initial data.** We denote hyperbolic space of dimension n by $\mathbb{H}^n$ and the hyperbolic metric by b. We choose a point in $\mathbb{H}^n$ as the origin. In polar coordinates around this point we have $b = dr^2 + \sinh^2 r \, \sigma$ on $(0, \infty) \times S^{n-1}$, where σ is the round metric on the unit sphere S^{n-1} and r is the distance to the origin. The open ball of radius R centered at the origin is denoted by B_R, and its closure is denoted by $\overline{B}_R$.

We first give the definition of an asymptotically hyperbolic Riemannian manifold.

Definition 2.1. We say that (M, g) is a $C_\tau^{l, \beta}$-*asymptotically hyperbolic manifold* for a non-negative integer l, $0 \leq \beta < 1$ and $\tau > 0$, if there exists a compact set $K_0 \subset M$, $R_0 > 0$ and a diffeomorphism

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0},$$

such that $\Psi_* g - b \in C_{loc}^{l, \beta}(\mathbb{H}^n; S^2 \mathbb{H}^n)$ and it satisfies

$$\|\Psi_* g - b\|_{C_\tau^{l,\beta}(\mathbb{H}^n \setminus B_{R_0}; S^2\mathbb{H}^n)} := \sup_{x \in \mathbb{H}^n \setminus B_{R_0+1}} e^{\tau r(x)} \|\Psi_* g - b\|_{C^{l,\beta}(B_1(x); S^2\mathbb{H}^n)} < \infty.$$

The diffeomorphism Ψ introduced in this definition is called a *chart at infinity* for the asymptotically hyperbolic manifold.

Let (M, g) be a $C_\tau^{l,\beta}$-asymptotically hyperbolic manifold for l, β, τ as in Definition 2.1. Suppose that u is a locally integrable section of a geometric tensor bundle E (see [18, Chapter 3] for the definition of geometric tensor bundles) over $M \setminus K_0$. In this case we say that $u \in W_\delta^{k,p}(M \setminus K_0)$ for $0 \le k \le l$, and $1 < p < \infty$ if

$$\|u\|_{W_\delta^{k,p}(M\setminus K_0)} := \left\| e^{\delta r} \Psi_* u \right\|_{W^{k,p}(\mathbb{H}^n \setminus B_{R_0})} < \infty.$$

Note also the following equivalent definition of the $W_\delta^{k,p}(M \setminus K_0)$ norm,

$$\|u\|_{W_\delta^{k,p}(M\setminus K_0)} = \sum_{0 \le j \le k} \|e^{\delta r} \nabla^j (\Psi_* u)\|_{L^p(\mathbb{H}^n \setminus B_{R_0})}.$$

Building on these definitions, it is straightforward to define *weighted Sobolev spaces* $W_\delta^{k,p}(M)$ for any δ, $0 \le k \le l$, and $1 < p < \infty$. The *weighted Hölder spaces* $C_\delta^{k,\alpha}(M)$ are defined in a similar fashion. We say that $u \in C_\delta^{k,\alpha}(M \setminus K_0)$ for $0 \le k + \alpha \le l + \beta$, and $0 \le \alpha < 1$ if

$$\|u\|_{C_\delta^{k,\alpha}(M\setminus K_0)} := \sup_{x \in \mathbb{H}^n \setminus B_{R_0+1}} e^{\tau r(x)} \|\Psi_* u\|_{C^{k,\alpha}(B_1(x))} < \infty.$$

The following equivalent definition of the $C_\delta^{k,\alpha}(M \setminus K_0)$ norm is often useful,

$$\|u\|_{C_\delta^{k,\alpha}(M\setminus K_0)} = \sum_{0 \le j \le k} \sup_{\mathbb{H}^n \setminus B_{R_0}} |e^{\delta r} \nabla^j (\Psi_* u)| + \|e^{\delta r} \nabla^k (\Psi_* u)\|_{C^{0,\alpha}(\mathbb{H}^n \setminus B_{R_0})}.$$

Again, these definitions can be extended to define weighted Hölder spaces $C_\delta^{k,\alpha}(M)$.

The weighted Sobolev and Hölder spaces that we have just defined are analogues of the respective spaces defined by J. Lee in [18] on conformally compact manifolds. It is easy to check that standard facts such as embedding theorems, the Rellich lemma, and density theorems hold for these spaces and that the statements of these results repeat verbatim the respective statements in [18]. In particular Lemma 3.6 and Lemma 3.9 of [18] hold for $W_\delta^{k,p}(M)$ and $C_\delta^{k,\alpha}(M)$ as defined above and we will refer to [18] for these results throughout the text.

It is straightforward to check that classical interior elliptic regularity as formulated in [18, Lemma 4.8] holds for asymptotically hyperbolic manifolds and weighted function spaces as defined in Section 2.1. In Appendix A we show that improved elliptic regularity [18, Proposition 6.5] holds in the current setting. As a consequence, Fredholm theory for geometric elliptic operators on asymptotically hyperbolic manifolds in the sense of Definition 2.1 can be established, since the proof of [18, Theorem C] can be adapted. The reader is referred to Appendix A for details.

We can now give the definition of an asymptotically hyperbolic initial data set. Recall that the energy density μ and the momentum density J are defined via the constraint equations (2)–(3).

Definition 2.2. A triple (M, g, K) is an *asymptotically hyperbolic initial data set of class* (α, τ) for $0 \le \alpha < 1$ and $\tau > 0$ if

- (M, g) is a $C_\tau^{2,\alpha}$-asymptotically hyperbolic manifold in the sense of Definition 2.1,
- a symmetric 2-tensor K is such that $K - g \in C_\tau^{1,\alpha}(M; S^2 M)$.

If, in addition, $(\mu, J) \in C_{n+\tau_0}^{0,\alpha}$ for some $\tau_0 > 0$ then (M, g, K) is an *asymptotically hyperbolic initial data set of class* (α, τ, τ_0).

Abusing notation slightly we may summarize the content of this definition as follows: (M, g, K) is an asymptotically hyperbolic initial data set of class (α, τ) for $0 \leq \alpha < 1$ and $\tau > 0$ if $(g - b, K - g) \in C_\tau^{2,\alpha} \times C_\tau^{1,\alpha}$. In this case $(\mu, J) \in C_\tau^{0,\alpha}$. The necessity for the faster decay $(\mu, J) \in C_{n+\tau_0}^{0,\alpha}$ will become clear in Section 2.2.

Given an asymptotically hyperbolic initial data set (M, g, K) of class (α, τ) it is convenient to set

$$\pi := (K - g) - \mathrm{tr}^g(K - g)g.$$

Note that $\pi \in C_\tau^{1,\alpha}$ and that π contains the same information as K, since $K = \pi + g - \frac{1}{n-1}(\mathrm{tr}^g \pi)g$. It is therefore equivalent to work with (M, g, π) as an initial data set, and in this paper we will only work with initial data given in this form.

In terms of (g, π) the constraint equations (2)–(3) are written as

$$-2\mu = -(\mathrm{Scal}^g + n(n - 1)) + 2\,\mathrm{tr}^g \pi - \frac{(\mathrm{tr}^g \pi)^2}{n - 1} + |\pi|_g^2,$$

$$J = \mathrm{div}^g \pi.$$

By the *constraint map* we mean the map

$$\Phi : (g, \pi) \mapsto \left(-(\mathrm{Scal}^g + n(n - 1)) + 2\,\mathrm{tr}^g \pi - \frac{(\mathrm{tr}^g \pi)^2}{n - 1} + |\pi|_g^2, \mathrm{div}^g \pi \right). \quad (4)$$

Finally, we define initial data sets with conformally hyperbolic asymptotics. Recall that the *conformal Killing operator* $\mathring{\mathcal{L}}$ is defined by

$$(\mathring{\mathcal{L}}_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i - \tfrac{2}{n}(\mathrm{div}^g Y)g_{ij},$$

that is, $(\mathring{\mathcal{L}}_Y g)_{ij}$ is the trace-free part of the Lie derivative $(\mathcal{L}_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i$.

Definition 2.3. We say that an initial data set (M, g, π) has *conformally hyperbolic asymptotics* if there exists a compact set K_0, a radius $R_0 > 0$, and a diffeomorphism

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0},$$

such that

$$\Psi_* g = (1 + v)^{\frac{4}{n-2}} b, \quad \Psi_* \pi = (1 + v)^{\frac{2}{n-2}} \mathring{\mathcal{L}}_Y b,$$

where the function v and the radial and tangential components of the 1-form Y can be written in the form

$$\begin{aligned}
v &= v_0 e^{-nr} + v_1, \\
Y_r &= (Y_0)_r e^{-nr} + (Y_1)_r, \\
Y_\varphi &= (Y_0)_\varphi e^{-(n-1)r} + (Y_1)_\varphi,
\end{aligned} \qquad (5)$$

where $(v_0, Y_0) \in C_{loc}^{2,\alpha}$ is independent of r and $(v_1, Y_1) \in C_{n+1}^{2,\alpha}$ for $0 \leq \alpha < 1$.

2.2. **The mass functional for asymptotically hyperbolic initial data.** In this section we review the concept of mass in the asymptotically hyperbolic setting and discuss the continuity of mass with respect to the initial data. We first recall how the asymptotic charge integrals are defined, following Michel [20].

Let (M, g, π) be an asymptotically hyperbolic initial data set of type (α, τ) for $0 \leq \alpha < 1$ and $\tau > 0$, and let Ψ be the chart at infinity as in Definition 2.1. Clearly, in this case we have $e := \Psi_* g - b \to 0$ and $\eta := \Psi_* \pi \to 0$ at infinity. Let the constraint map Φ be defined by (4). Since $\Phi(b, 0) = 0$, linearization gives us

$$\Phi(\Psi_*(g, \pi)) = D\Phi|_{(b,0)}(e, \eta) + \mathcal{Q}(e, \eta), \qquad (6)$$

where $\mathcal{Q}(e, \eta)$ is a remainder term of second order. For any function V and 1-form ϖ there is a unique 1-form $\mathbb{U}_{(V,\varpi)}(e, \eta)$ such that

$$\langle D\Phi|_{(b,0)}(e, \eta), (V, \varpi)\rangle = \text{div}^b\, \mathbb{U}_{(V,\varpi)}(e, \eta) + \langle (e, \eta), D\Phi^*_{(b,0)}(V, \varpi)\rangle,$$

where $D\Phi^*_{(b,0)}$ is the formal adjoint of $D\Phi|_{(b,0)}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product induced by b on geometric tensor bundles over $\mathbb{H}^n$. Contracting (6) with $(V, \varpi) \in \ker D\Phi^*_{(b,0)}$ we obtain

$$\langle \Phi(\Psi_*(g, \pi)), (V, \varpi)\rangle = \text{div}^b\, \mathbb{U}_{(V,\varpi)}(e, \eta) + \langle \mathcal{Q}(e, \eta), (V, \varpi)\rangle. \qquad (7)$$

In this way we assign to every $(V, \varpi) \in \ker D\Phi^*_{(b,0)}$ the *charge integral*

$$\mathbb{Q}_{(V,\varpi)}(g, \pi) := \lim_{R\to\infty} \int_{S_R} \mathbb{U}_{(V,\varpi)}(e, \eta)(\nu)\, d\mu^b,$$

where ν is the outer unit normal of the $(n-1)$-dimensional sphere S_R in $\mathbb{H}^n$.

The structure of the kernel of $D\Phi^*_{(b,0)}$ is well understood, see Moncrief [21]. Namely, $(V, \varpi^\sharp)$ corresponds to the normal-tangential (or lapse-shift) decomposition of the restriction along the unit hyperboloid of a Killing vector field of Minkowski spacetime. In other words, $(V, \varpi^\sharp)$ is a *Killing initial data* (or *KID*) for Minkowski spacetime given on the unit hyperboloid.

In particular, we have $(V, -dV) \in \ker D\Phi^*_{(b,0)}$ for $V \in \mathcal{N}$, where the vector space $\mathcal{N}$ is spanned by the functions

$$V_{(0)} = \cosh r, \quad V_{(1)} = x^1 \sinh r, \quad \ldots, \quad V_{(n)} = x^n \sinh r$$

expressed in polar coordinates on $\mathbb{H}^n = (0, \infty) \times S^{n-1}$. Here $x^1, \ldots, x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}.

For these KIDs we have the following result.

Proposition 2.4. *Let (M, g, π) be asymptotically hyperbolic initial data of type (α, τ, τ_0) for $0 \le \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. Then for every $V \in \mathcal{N}$ the charge integral $\mathbb{Q}_{(V,-dV)}(g, \pi)$ is well-defined and can be computed by the formula*

$$\mathbb{Q}_{(V,-dV)}(g, \pi)$$
$$= \lim_{R\to\infty} \int_{S_R} \left(V(\text{div}^b\, e - d\,\text{tr}^b\, e) + (\text{tr}^b\, e)dV - (e + 2\eta)(\nabla^b V, \cdot) \right)(\nu)\, d\mu^b. \qquad (8)$$

Proof. Integrating (7) over $\mathbb{H}^n \setminus B_{R_0}$ and using the divergence theorem we obtain

$$\mathbb{Q}_{(V,-dV)}(g, \pi)$$
$$= \int_{\mathbb{H}^n \setminus B_{R_0}} \langle \Phi(\Psi_*(g, \pi)) - \mathcal{Q}(e, \eta), (V, -dV)\rangle\, d\mu^b + \int_{S_{R_0}} \mathbb{U}_{(V,-dV)}(e, \eta)(\nu)\, d\mu^b.$$

Estimating $\mathcal{Q}(e, \eta)$ as in [20, Equation (12)] and using our assumptions on the decay of the initial data, we see that the integral over $\mathbb{H}^n \setminus B_{R_0}$ converges, hence $\mathbb{Q}_{(V,-dV)}(g, \pi)$ is well-defined.

We refer to [20, Section IV.2.B] and references therein for the derivation of the formula (8). $\qquad \square$

Definition 2.5. Let (M, g, π) be an asymptotically hyperbolic initial data set. Then the *mass* of (M, g, π) is the linear functional $\mathcal{M}_{(g,\pi)} : \mathcal{N} \to \mathbb{R}$ given by

$$\mathcal{M}_{(g,\pi)}(V) = \tfrac{1}{2(n-1)\omega_{n-1}} \mathbb{Q}_{(V,-dV)}(g, \pi),$$

where ω_{n-1} denotes the volume of the unit sphere (S^{n-1}, σ).

This is the same as the expression for the Bondi mass obtained by Chruściel, Jesierski, and Lęski in [7], under asymptotic decay conditions that however do not allow for gravitational radiation. See [20] for a discussion of coordinate covariance.

As Proposition 2.4 shows, the mass functional is well defined for asymptotically hyperbolic initial data of type (α, τ, τ_0) for $0 \le \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. It is also straightforward to check that the mass functional is trivial for asymptotically hyperbolic initial data sets of type (α, τ) with $\tau > n$. The following are two examples of the "critical" case $\tau = n$.

Example 2.6. The Anti-de Sitter Schwarzschild Riemannian metric is given by

$$g_{\text{AdSS}} = \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 \sigma$$

on $[a, \infty) \times S^{n-1}$, where the inner radius a depends on m, see for example [10, Appendix A]. It can be realized as an *umbilic* (that is, $g = K$) asymptotically hyperbolic initial data set for Schwarzschild spacetime, see Brendle and Wang [6]. In this case

$$\mathcal{M}(V_{(0)}) = m, \qquad \text{and} \qquad \mathcal{M}(V_{(i)}) = 0,$$

for $i = 1, \ldots, n$, where m coincides with the mass parameter of the Schwarzschild metric.

Example 2.7. For initial data with conformally hyperbolic asymptotics as in Definition 2.3 it is not complicated to compute that

$$\mathcal{M}(V_{(0)}) = \frac{2(n+1)}{(n-2)\omega_{n-1}} \int_{S^{n-1}} v_0 \, d\mu^\sigma + \frac{2(n+1)}{n\omega_{n-1}} \int_{S^{n-1}} (Y_0)_r \, d\mu^\sigma,$$

and

$$\mathcal{M}(V_{(i)}) = \frac{2(n+1)}{(n-2)\omega_{n-1}} \int_{S^{n-1}} x^i v_0 \, d\mu^\sigma + \frac{2(n+1)}{n\omega_{n-1}} \int_{S^{n-1}} x^i (Y_0)_r \, d\mu^\sigma$$

for $i = 1, \ldots, n$.

Concluding this section, we confirm that the mass is continuous as a function of asymptotically hyperbolic initial data sets of type (α, τ, τ_0), where $0 \le \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. For simplicity, the charts at infinity are suppressed in the statement of the result and in the proof.

Proposition 2.8. *Let (g, π) and $(\bar{g}, \bar{\pi})$ be asymptotically hyperbolic initial data sets of type (α, τ, τ_0) for $0 \le \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. Given $\varepsilon > 0$ there exists $\delta > 0$ depending only on (g, π) and ε, such that if*

$$\|g - \bar{g}\|_{C_\tau^2} \le \delta, \quad \|\pi - \bar{\pi}\|_{C_\tau^1} \le \delta, \tag{9}$$

and

$$\|(\mu, J) - (\bar{\mu}, \bar{J})\|_{C_{n+\tau_0}^0} \le \delta, \tag{10}$$

then for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ we have

$$\left| \mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V) \right| \le \varepsilon.$$

Proof. Fix $R \geq R_0$. Arguing as in the proof of Proposition 2.4 we find that

$$2(n-1)\omega_{n-1}\left(\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)\right)$$

$$= \int_{\mathbb{H}^n \setminus B_R} \langle \Phi(g,\pi) - \Phi(\bar{g},\bar{\pi}), (V, -dV) \rangle \, d\mu^b$$

$$- \int_{\mathbb{H}^n \setminus B_R} \langle \mathcal{Q}(e,\eta) - \mathcal{Q}(\bar{e},\bar{\eta}), (V, -dV) \rangle \, d\mu^b$$

$$+ \int_{S_R} \left(\mathbb{U}_{(V,-dV)}(e,\eta) - \mathbb{U}_{(V,-dV)}(\bar{e},\bar{\eta})\right)(\nu) \, d\mu^b.$$

Now suppose that (g,π) is fixed and that δ and $(\bar{g},\bar{\pi})$ are such that (9) and (10) hold. Then by assumption (10) the absolute value of the first integral over $\mathbb{H}^n \setminus B_R$ is bounded by $C\delta$ for some $C > 0$ depending only on (g,π). The same is true for the second integral over $\mathbb{H}^n \setminus B_R$ by assumption (9) combined with the fact that the remainder term $\mathcal{Q}(e,\eta)$ in (6) is at least quadratic in e and η and their derivatives of order up to 2 and 1 respectively. As for the inner boundary integral, we see that its absolute value is bounded by $C\delta e^{(n-\tau)R}$ for $C > 0$ depending only on (g,π). From this it is clear that δ can be chosen so that the sum of the absolute values of these three integrals is less than ε.   □

3. Perturbation to strict inequality in the dominant energy condition

This section is devoted to the following result.

Theorem 3.1. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type (α, τ, τ_0) for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$, and $\tau_0 > 0$. Assume that (M, g, π) satisfies the dominant energy condition, $\mu \geq |J|_g$. Then for every $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$ of type (α, τ, τ_0') for some $\tau_0' > 0$ such that*

$$\|g - \bar{g}\|_{C_\tau^{2,\alpha}} < \varepsilon, \qquad \|\pi - \bar{\pi}\|_{C_\tau^{1,\alpha}} < \varepsilon,$$

and the strict dominant energy condition

$$\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$$

holds for $\gamma > 0$, and

$$\left|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)\right| < \varepsilon$$

for $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

The argument follows [13, Proof of Theorem 22]. In simple terms it can be described as follows. We would like to choose symmetric 2-tensors h and w so that the perturbed initial data $\bar{g} = g + th$ and $\bar{\pi} = \pi + tw$ satisfies $\bar{\mu} > |\bar{J}|_{\bar{g}}$ for sufficiently small $t > 0$. From the Taylor expansion $\Phi(\bar{g}, \bar{\pi}) = \Phi(g,\pi) + tD\Phi|_{(g,\pi)}(h,w) + O(t^2)$, we see that $\bar{\mu} = \mu + \frac{t}{2}f + O(t^2)$ and $\bar{J} = J + tX + O(t^2)$, where $(-f, X) = D\Phi|_{(g,\pi)}(h,w)$. Further,

$$|\bar{J}|_{\bar{g}}^2 = \bar{g}^{ij}\bar{J}_i\bar{J}_j$$

$$= (g^{ij} - th^{ij} + O(t^2))(J_i + tX_i + O(t^2))(J_j + tX_j + O(t^2)) \qquad (11)$$

$$= |J|_g^2 + t(2X^j - h^{ij}J_i)J_j + O(t^2),$$

where indices are raised using the metric g. Hence if we set $X^j = \frac{1}{2}h^{ij}J_i$ then $|\bar{J}|_{\bar{g}} = |J|_g + O(t^2)$, as long as the decay of $|J|_g^2$ at infinity is not faster than that of

the $O(t^2)$ term in the last line of (11). This leads to the expectation that $\bar{\mu} > |\bar{J}|_{\bar{g}}$ will be achieved if we can find a pair (h, w) such that $D\Phi|_{(g,\pi)}(h, w) = (-f, X)$, where $X^j = \frac{1}{2}h^{ij}J_i$ and $f > 0$. Indeed, in this case we have

$$\bar{\mu} - |\bar{J}|_{\bar{g}} = \mu - |J|_g + tf + O(t^2) \geq tf + O(t^2) > 0$$

provided that the $O(t^2)$ term above decays at least as fast as f at infinity.

However, $D\Phi|_{(g,\pi)}$ is not a determined elliptic operator (see Delay [11]), and this makes it difficult to ensure that the solutions of the equation $D\Phi|_{(g,\pi)}(h, w) = (-f, X)$ will have good asymptotic behaviour. This problem can be overcome by combining the above considerations with a certain construction introduced by Corvino and Schoen in their proof of the density result in [9, Theorem 1]. The idea is similar in spirit to the conformal method of solving the constraint equations (see for example [5, Section 4.1]) and is based on the observation that by suitably choosing a first order differential operator $\mathcal{D}$ one can ensure that the linearization $DT|_{(1,0)}$ of the operator

$$T : (u, Z) \mapsto \Phi\left(u^\kappa g, u^{\kappa/2}(\pi + \mathcal{D}Z)\right), \tag{12}$$

where $\kappa := \frac{4}{n-2}$, is a second order elliptic operator with nice properties.

We begin the proof of Theorem 3.1 with some preliminaries. For $(u-1, Y) \in C_\tau^{2,\alpha}$ we let

$$\tilde{g} = u^\kappa g, \qquad \text{and} \qquad \tilde{\pi} = u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g), \tag{13}$$

where $\mathring{\mathcal{L}}$ is the conformal Killing operator described in Section 2.1. Our choice of the operator $\mathcal{D} = \mathring{\mathcal{L}}$ in (12) is motivated by the fact that the *vector Laplacian* $\Delta_L = \text{div } \mathring{\mathcal{L}}$ is a well-known elliptic operator on asymptotically hyperbolic manifolds whose Fredholm properties (see Appendix A) fit nicely into the context of the current argument. Let $\tilde{\mu}$ and $\tilde{J}$ be the energy and momentum densities of $(\tilde{g}, \tilde{\pi})$ computed via the constraint equations (2)–(3) and consider the operator

$$T(u, Y) = (-2u^\kappa \tilde{\mu}, u^{\kappa/2}\tilde{J}).$$

This conformal rescaling of the constraint equations is needed to ensure that the dominant energy condition scales correctly when we pass to the deformed initial data (13), see (23) and (24) below.

It is straightforward to check that

$$-2u^\kappa \tilde{\mu} = \tfrac{4(n-1)}{n-2}u^{-1}\Delta^g u - \text{Scal}^g - n(n-1)u^\kappa + 2u^{\kappa/2}\,\text{tr}^g\,\pi - \tfrac{1}{n-1}(\text{tr}^g\,\pi)^2$$
$$+ \left(|\pi|_g^2 + 2\langle \pi, \mathring{\mathcal{L}}_Y g\rangle + |\mathring{\mathcal{L}}_Y g|_g^2\right), \tag{14}$$
$$u^{\kappa/2}\tilde{J}_j = (\Delta_L Y + \text{div}^g\,\pi)_j + \tfrac{2(n-1)}{n-2}u^{-1}(\pi + \mathring{\mathcal{L}}_Y g)_j^k \nabla_k u - \tfrac{2}{n-2}u^{-1}\nabla_j u\,\text{tr}^g\,\pi,$$

for $j = 1, 2, \ldots, n$. Consequently, the linearization of T at $(1, 0)$ is

$$DT|_{(1,0)}(v, Z) = \left(\tfrac{4(n-1)}{n-2}(\Delta^g v - nv) + \tfrac{4}{n-2}(\text{tr}^g\,\pi)v + 2\langle \pi, \mathring{\mathcal{L}}_Z g\rangle,\right.$$
$$\left.(\Delta_L Z)_j + \tfrac{2(n-1)}{n-2}\pi_j^k \nabla_k v - \tfrac{2}{n-2}(\text{tr}^g\,\pi)\nabla_j v\right), \tag{15}$$

for $j = 1, 2, \ldots, n$. The following lemma concerns Fredholm properties of the operator $DT|_{(1,0)}$.

Lemma 3.2. *If (M, g, π) is an asymptotically hyperbolic initial data set of type (α, τ) for $0 < \alpha < 1$ and $\tau > 0$ then $DT|_{(1,0)}$ is a Fredholm operator with index zero in the following cases:*

- *as a map $C_\delta^{2,\beta} \to C_\delta^{0,\beta}$ for $0 < \beta \le \alpha$, $-1 < \delta < n$,*
- *as a map $W_\delta^{2,p} \to W_\delta^{0,p}$ for $1 < p < \infty$, $-1 < \delta + \frac{n-1}{p} < n$.*

Proof. We give the proof in the case of weighted Hölder spaces, the case of weighted Sobolev spaces is treated similarly. Write $DT|_{(1,0)} = P_0 + P_1$, where

$$P_0 : (v, Z) \mapsto \left(\tfrac{4(n-1)}{n-2}(\Delta^g v - nv), \Delta_L Z \right),$$

and

$$P_1 : (v, Z) \mapsto \left(\tfrac{4}{n-2}(\mathrm{tr}^g \pi)v + 2\langle \pi, \mathring{\mathcal{L}}_Z g \rangle, \tfrac{2(n-1)}{n-2}\pi_j^k \nabla_k v - \tfrac{2}{n-2}(\mathrm{tr}^g \pi)\nabla_j v \right).$$

Here $P_0 : C_\delta^{2,\alpha} \to C_\delta^{0,\alpha}$ is a Fredholm operator of index zero for $\delta \in (-1, n)$, see Proposition A.2. By [18, Lemma 3.6 (a)] the map $P_1 : C_\delta^{2,\alpha} \to C_{\delta+\tau}^{1,\alpha}$ is continuous, whereas by the Rellich Lemma, [18, Lemma 3.6 (d)], the inclusion $C_{\delta+\tau}^{1,\alpha} \hookrightarrow C_\delta^{0,\alpha}$ is compact. We conclude that $P_1 : C_\delta^{2,\alpha} \to C_\delta^{0,\alpha}$ is compact for $-1 < \delta < n$, and the claim follows. $\qquad\square$

Recall that the constraint map Φ is defined by the formula (4). A direct computation shows that the linearization of Φ is

$$D\Phi|_{(g,\pi)}(h, w) = \left(\Delta^g(\mathrm{tr}^g h) - \mathrm{div}^g \mathrm{div}^g h + \langle h, \mathrm{Ric}^g \rangle \right.$$

$$+ 2\left(1 - \frac{\mathrm{tr}^g \pi}{n-1} \right)(\mathrm{tr}^g w - \langle h, \pi \rangle) - 2\langle h, \pi \circ \pi \rangle + 2\langle \pi, w \rangle,$$

$$\left. (\mathrm{div}^g w)_k - h^{ij}\nabla_i \pi_{jk} - (\mathrm{div}\, h)_j \pi_k^j + \tfrac{1}{2}\nabla_j(\mathrm{tr}^g h)\pi_k^j - \tfrac{1}{2}\pi^{ij}\nabla_k h_{ij} \right),$$

where $(\pi \circ \pi)_{ij} = g^{kl}\pi_{ik}\pi_{jl}$. The formal adjoint of $D\Phi$ is given by

$$D\Phi_{(g,\pi)}^*(V, X) = \left((\Delta V)g_{ij} - \nabla_i\nabla_j V + V\mathrm{Ric}_{ij} - 2V\left(1 - \frac{\mathrm{tr}^g \pi}{n-1} \right)\pi_{ij} \right.$$

$$- 2V\pi_{ik}\pi_j^k + \tfrac{1}{2}(\pi_{jk}\nabla_i X^k + \pi_{ik}\nabla_j X^k) - \tfrac{1}{2}(\mathrm{div}\,\pi)_k X^k g_{ij}$$

$$- \tfrac{1}{4}\langle \pi, \mathcal{L}_X g \rangle g_{ij} + \tfrac{1}{2}X^k\nabla_k\pi_{ij} + \tfrac{1}{2}(\mathrm{div}\, X)\pi_{ij}, \tag{16}$$

$$\left. - \tfrac{1}{2}(\mathcal{L}_X g)_{ij} + 2V\left(1 - \frac{\mathrm{tr}^g \pi}{n-1} \right)g_{ij} + 2V\pi_{ij} \right).$$

The following lemma is the analogue of [13, Lemma 20] in the asymptotically hyperbolic setting. The proof of the cited Lemma is similar to [9, Proposition 3.1].

Lemma 3.3. *If (M, g, π) is an asymptotically hyperbolic initial data set of type (α, τ) for $0 < \alpha < 1$ and $\tau > 0$ then the linear map $A : W_\delta^{2,p} \times W_\delta^{1,p} \to W_\delta^{0,p}$ defined by*

$$A(h, w) = D\Phi|_{(g,\pi)}(h, w) - (0, \tfrac{1}{2}h_j^l J_l)$$

is surjective for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$. In particular, $D\Phi|_{(g,\pi)} : W_\delta^{2,p} \times W_\delta^{1,p} \to W_\delta^{0,p}$ is surjective for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$.

Proof. The first step is to show that A has closed range. For this we compute

$$
\begin{aligned}
A(vg, \mathring{\mathcal{L}}_Z g) = \Big(&(n-1)(\Delta^g v - nv) + (\mathrm{Scal}^g + n(n-1))v \\
&- 2v(\mathrm{tr}^g \pi - \tfrac{1}{n-1}(\mathrm{tr}^g \pi)^2 + |\pi|_g^2) + 2\langle \pi, \mathring{\mathcal{L}}_Z g \rangle, \\
&(\Delta_L Z)_i - v(\mathrm{div}^g \pi)_i + (\tfrac{n}{2}-1)\pi_i^j \nabla_j v - \tfrac{1}{2}\mathrm{tr}^g \pi \nabla_i v - \tfrac{1}{2}v J_j \Big).
\end{aligned}
$$

Reasoning as in the proof of Lemma 3.2 we conclude that the operator

$$
(v, Z) \mapsto A(vg, \mathring{\mathcal{L}}_Z g)
$$

is a Fredholm operator $W_\delta^{2,p} \to W_\delta^{0,p}$ for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$. Its range is contained in the range of the operator A. Consequently, the range of the operator A has finite codimension in $W_\delta^{0,p}$, and hence it is closed.

Next we need to show that $\ker A^*$ is trivial. Let p^* be such that $\frac{1}{p} + \frac{1}{p^*} = 1$. Then $W_{-\delta}^{0,p^*}$ is dual to $W_\delta^{0,p}$ under the standard L^2 pairing, see [18, Chapter 3]. Note that we have $-1 < -\delta + \frac{n-1}{p^*} < n$ as a consequence of $-1 < \delta + \frac{n-1}{p} < n$. It follows from (16) that $\ker A^*$ consists of $(V, X) \in W_{-\delta}^{0,p^*}$ such that

$$
\begin{aligned}
(\Delta V)g_{ij} - \nabla_i \nabla_j V + V \mathrm{Ric}_{ij} = 2V &\left(1 - \frac{\mathrm{tr}^g \pi}{n-1}\right)\pi_{ij} + 2V \pi_{ik}\pi_j^k \\
&- \tfrac{1}{2}(\pi_{jk}\nabla_i X^k + \pi_{ik}\nabla_j X^k) \\
&+ \tfrac{1}{2}(\mathrm{div}\,\pi)_k X^k g_{ij} + \tfrac{1}{4}\langle \pi, \mathcal{L}_X g\rangle g_{ij} \\
&- \tfrac{1}{2}X^k \nabla_k \pi_{ij} - \tfrac{1}{2}(\mathrm{div}\,X)\pi_{ij} \\
&+ \tfrac{1}{4}(X_i J_j + X_j J_i),
\end{aligned}
\tag{17}
$$

$$
\mathcal{L}_X g = 4V\left(1 - \frac{\mathrm{tr}^g \pi}{n-1}\right)g + 4V\pi.
$$

As a consequence of the second equation we have $\mathcal{L}_X g \in W_{-\delta}^{0,p^*}$. Taking the trace of the first equation we have

$$
\Delta V - nV = V \star O^\alpha(e^{-\tau r}) + X \star O^\alpha(e^{-\tau r}) + \mathcal{L}_X g \star O^\alpha(e^{-\tau r}),
\tag{18}
$$

where $O^\alpha(e^{-\tau r})$ denotes a section T of some geometric tensor bundle of relevant type such that $T \in C_\tau^{0,\alpha}$, and $A \star B$ denotes a tensor which is obtained from $A \otimes B$ by raising and lowering indices, taking any number of contractions, and switching any number of components in the product. By standard elliptic regularity [18, Lemma 4.8 (a)] we conclude that $V \in W_{-\delta}^{2,p^*}$. As a consequence of the second equation in (17) we have

$$
\mathring{\mathcal{L}}_X g = 4V\left(\pi - \frac{\mathrm{tr}^g \pi}{n}g\right).
$$

Taking the divergence, we obtain

$$
\Delta_L X = V \star O^\alpha(e^{-\tau r}) + \nabla V \star O^\alpha(e^{-\tau r}).
\tag{19}
$$

Thus $X \in W_{-\delta}^{2,p^*}$, again by standard elliptic regularity. Since $(V, X) \in W_{-\delta}^{2,p^*}$ the right hand sides of equations (18) and (19) are both in $W_{-\delta+\tau}^{0,p^*}$. Using Proposition A.2, improved elliptic regularity [18, Proposition 6.5], and the continuity of the embedding $W_\varepsilon^{k,p^*} \hookrightarrow W_{\varepsilon'}^{k,p^*}$ for $\varepsilon > \varepsilon'$, we conclude that $(V, X) \in W_\gamma^{2,p^*}$ for

any γ such that $-1 < \gamma + \frac{n-1}{p^*} < n$. Therefore we may without loss of generality assume that $1 < \gamma < n - \frac{n-1}{p^*} = 1 + \frac{n-1}{p}$.

In fact, we can show that $(V, X) \in C^{2,\beta}_\gamma$ for some $0 < \beta < 1$. Indeed, if $p^* < n$ then $(V, X) \in W^{2, \frac{np^*}{n-p^*}}_\gamma$ by the Sobolev embedding theorem [18, Lemma 3.6 (c)] and standard elliptic regularity applied to equations (18) and (19). Repeating this argument we obtain that $(V, X) \in W^{2,q}_\gamma$ for some $q > n$ and thus $(V, X) \in C^{1,\beta}_\gamma$ for some $0 < \beta < 1$ by Sobolev embedding [18, Lemma 3.6 (c)]. Applying standard elliptic regularity to the equations (18) and (19) we conclude that $(V, X) \in C^{2,\beta}_\gamma$.

Next we show that (V, X) vanishes to infinite order at infinity. That is, $(V, X) = O(e^{-Nr})$ for any $N > 0$. As a consequence of (17) and Definition 2.2 we see that (V, X) is a solution to the system

$$\text{Hess}^b V - Vb = V \star O(e^{-\tau r}) + \nabla V \star O(e^{-\tau r}) + X \star O(e^{-\tau r}) + \nabla X \star O(e^{-\tau r}),$$
$$\mathcal{L}_X b = 4Vb + X \star O_1(e^{-\tau r}) + V \star O_1(e^{-\tau r}),$$

where $O_1(e^{-\tau r})$ denotes a section T of the relevant geometric tensor bundle such that $T \in C^1_\tau$. From the first equation and the fact that $(V, X) \in C^{2,\beta}_\gamma$ we conclude that V satisfies the ordinary differential equation

$$\partial^2_{rr} V - V = \widetilde{f}$$

along radial geodesic rays, where $\widetilde{f} = O(e^{-(\tau+\gamma)r})$. Since $\tau + \gamma > 1$, it follows that $V = O(e^{-(\tau+\gamma)r})$, see formula (42) for the explicit form of the solution. Then $V \in C^{2,\beta}_{\tau+\gamma}$ by standard elliptic regularity applied to (18). Combining this with the second equation, we see that $(\mathcal{L}_X b)_{rr} = O(e^{-(\tau+\gamma)r})$, which yields $\partial_r X_r = O(e^{-(\tau+\gamma)r})$. Integrating this relation from r to ∞, we obtain that $X_r = O(e^{-(\tau+\gamma)r})$. Differentiating the expression for X_r with respect to μ we obtain $\partial_\mu X_r = O(e^{-(\tau+\gamma)r})$. Here we work in polar coordinates for hyperbolic space, $(\mathbb{H}^n, b) = ((0, \infty) \times S^{n-1}, dr^2 + \sinh^2 r \, \sigma)$, the subscript r denotes the radial component and μ denotes components in a coordinate system on the sphere. It follows that

$$\partial_r X_\mu - 2 \coth r X_\mu = \bar{f},$$

where $\bar{f} = O(e^{-(\tau+\gamma-1)r})$, and hence $X_\mu = \sinh^2 r \int_r^\infty \frac{\bar{f}}{\sinh^2 s} \, ds = O(e^{-(\tau+\gamma-1)r})$. Thus $|X|_b = O(e^{-(\tau+\gamma)r})$, and hence $X \in C^{2,\beta}_{\tau+\gamma}$ by standard elliptic regularity applied to (19). We proceed by induction and deduce that $(V, X) = O(e^{-Nr})$ for any $N > 0$.

To conclude the proof, note that, as a consequence of (17), $(V, X) \in C^2$ satisfies a differential inequality

$$|\Delta(V, X)| \le C \left(|(V, X)| + |\nabla(V, X)| \right),$$

where $\Delta = \nabla^* \nabla$ is the rough Laplacian. Since (V, X) vanishes to infinite order at infinity, a standard unique continuation argument, see Appendix C, implies that (V, X) vanishes identically. $\qquad\square$

We use the subscript c on a function space to denote the subspace of sections with compact support.

Lemma 3.4. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type (α, τ), where $0 < \alpha < 1$ and $\frac{n}{2} < \tau < n$. Then for any $f \in C_\tau^{0,\alpha}$ there exist $(v, Z) \in C_\tau^{2,\alpha}$ and symmetric 2-tensors $(h, w) \in C_c^{3,\alpha}$ so that*

$$DT|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(h, w) = (f, \tfrac{1}{2} h_j^l J_l). \tag{20}$$

If in addition $f \in C_{n+\tau_0}^{0,\alpha}$ for some $\tau_0 > 0$ then $(v, Z) \in C_n^{2,\alpha}$.

Proof. For some $p > n$ we choose $\gamma > 0$ so that $-1 < \gamma + \frac{n-1}{p} < \tau$. In this case $C_\tau^{k,\alpha} \hookrightarrow W_\gamma^{k,p}$ for $k = 0, 1, 2$. Further, by Lemma 3.2 the operator $DT|_{(1,0)} : W_\gamma^{2,p} \to W_\gamma^{0,p}$ is Fredholm with index zero. Since the linear map $A : W_\gamma^{2,p} \times W_\gamma^{1,p} \to W_\gamma^{0,p}$ defined in Lemma 3.3 is surjective, we can find symmetric 2-tensors $(h_k, w_k) \in W_\gamma^{2,p} \times W_\gamma^{1,p}$, $k = 1, \dots, N$, such that their images $A(h_k, w_k)$ span a subspace that complements $DT|_{(1,0)}(W_\gamma^{2,p})$ in $W_\gamma^{0,p}$. Note that by the density of compactly supported sections, [18, Lemma 3.9], together with the continuity of A we may assume that $(h_k, w_k) \in C_c^{3,\alpha}$. Consequently, since $f \in W_\gamma^{0,p}$ we can find $(v, Z) \in W_\gamma^{2,p}$ and $(h, w) \in C_c^{3,\alpha}$ such that (20) holds. By Sobolev embedding $(v, Z) \in C_\gamma^{1,\alpha}$. Since $\gamma > 0$ and $f \in C_\tau^{0,\alpha}$ it follows from (15) that $(\Delta v - nv, \Delta_L Z) \in C_\tau^{0,\alpha}$. From [18, Proposition 6.5] we conclude that $(v, Z) \in C_\tau^{2,\alpha}$.

To prove the second claim note that outside a sufficiently large compact set $(v, Z) \in C_\tau^{2,\alpha}$ satisfies $(\Delta v - nv, \Delta_L Z) \in C_{n+\varepsilon}^{0,\alpha}$ for some $\varepsilon > 0$. This is an immediate consequence of (15) and the fact that $\tau > \frac{n}{2}$. The claim follows from Proposition B.2. $\qquad \square$

Proof of Theorem 3.1. With the above lemmas at hand, the proof differs very little from that of [13, Theorem 22]. We choose a positive $C^{3,\alpha}$ function f such that

$$f = e^{-(n+\min\{1, \tau_0\})r}$$

near infinity, and let $(v, Z) \in C_n^{2,\alpha}$ and $(h, w) \in C_c^{3,\alpha}$ be a solution of the system

$$DT|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(h, w) = (-f, \tfrac{1}{2} h_j^l J_l),$$

which exists by Lemma 3.4. We will show that for a sufficiently small $t > 0$

$$\bar{g} = (1 + tv)^\kappa (g + th) \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2}(\pi + t\mathring{\mathcal{L}}_Z g + tw)$$

is an initial data set whose existence is asserted in the theorem. Note that $\|g - \bar{g}\|_{C_\tau^{2,\alpha}} \leq \varepsilon$, $\|\pi - \bar{\pi}\|_{C_\tau^{1,\alpha}} \leq \varepsilon$ provided that t is sufficiently small.

We will verify that $\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$ for some $\gamma > 0$ depending on t. Set $u = 1 + tv$ and define

$$\Phi_1(1 + tv, tZ, th, tw) = (-2u^\kappa \bar{\mu}, u^{\kappa/2} \bar{J}).$$

Linearizing we have

$$\Phi_1(1 + tv, tZ, th, tw) = \Phi_1(1, 0, 0, 0) + tD\Phi_1|_{(1,0,0,0)}(v, Z, h, w) + \mathcal{R}$$
$$= (-2\mu, J) + tDT|_{(1,0)}(v, Z) + tD\Phi|_{(g,\pi)}(h, w) + \mathcal{R} \tag{21}$$
$$= (-2\mu, J) + t(-f, \tfrac{1}{2} h_i^k J_k) + \mathcal{R},$$

where the remainder term $\mathcal{R} = \mathcal{R}(t, v, Z, h, w)$ can be written as

$$\mathcal{R}(t, v, Z, h, w) = \Phi_1(1 + tv, tZ, th, tw) - \Phi_1(1, 0, 0, 0) - tD\Phi_1|_{(1,0,0,0)}(v, Z, h, w)$$
$$= t \int_0^1 \left[D\Phi_1|_{(1+\theta tv, \theta tZ, \theta th, \theta tw)} - D\Phi_1|_{(1,0,0,0)} \right] (v, Z, h, w) \, d\theta,$$

by the mean value theorem.

We first prove that
$$|\mathcal{R}| \le Ct^2 e^{-2nr} = O(t^2 f), \tag{22}$$
where the constant $C > 0$ does not depend on t and is uniform for all points. For this it suffices to estimate $\mathcal{R}$ outside the support of (h, w) where it takes the form
$$\mathcal{R}(t, v, Z) = t \int_0^1 \left[DT|_{(1+\theta tv, \theta tZ)} - DT|_{(1,0)} \right] (v, Z) \, d\theta.$$
Using (14) we compute
$$DT|_{(u,Y)}(v, Z) = \Big(\tfrac{4(n-1)}{n-2}(-u^{-2}v\Delta^g u + u^{-1}\Delta^g v) - n(n-1)\kappa u^{\kappa-1}v$$
$$+ \kappa u^{\frac{\kappa}{2}-1} v \operatorname{tr}^g \pi + 2\langle \pi, \mathring{\mathcal{L}}_Z g\rangle + 2\langle \mathring{\mathcal{L}}_Z g, \mathring{\mathcal{L}}_Y g\rangle,$$
$$(\Delta_L Z)_j + \tfrac{2(n-1)}{n-2}\nabla_k(vu^{-1})(\pi + \mathring{\mathcal{L}}_Y g)^k_j$$
$$+ \tfrac{2(n-1)}{n-2}u^{-1}\nabla_k u(\mathring{\mathcal{L}}_Z g)^k_j - \tfrac{2}{n-2}\nabla_j(vu^{-1})\operatorname{tr}^g \pi \Big).$$
Then it is not complicated to check that
$$\left| DT|_{(1+\theta tv, \theta tZ)}(v, Z) - DT|_{(1,0)}(v, Z) \right| \le \theta t \mathcal{Q}(v, Z)$$
where $\mathcal{Q}$ is a quadratic function of v, its first and second order covariant derivatives, and $\mathring{\mathcal{L}}_Z g$, which is uniformly bounded in $\theta \in [0, 1]$. Hence (22) holds.

By (21) we have
$$u^\kappa \bar{\mu} = \mu + \frac{t}{2}f + O(t^2 f) \qquad \text{and} \qquad u^{\kappa/2}\bar{J}_i = J_i + \frac{t}{2}h_i^k J_k + O(t^2 f).$$
In particular, for sufficiently small $t > 0$, we have
$$u^\kappa \bar{\mu} > \mu + \frac{t}{3}f. \tag{23}$$
Recall that h is compactly supported, hence we may write
$$\bar{g}^{ij} = u^{-\kappa}(g^{ij} - tg^{ik}h_k^j + O(t^2 f)).$$
Since f is positive, we obtain
$$(u^\kappa|\bar{J}|_{\bar{g}})^2 = u^{2\kappa}\bar{g}^{ij}\bar{J}_i\bar{J}_j$$
$$= (g^{ij} - tg^{ik}h_k^j + O(t^2 f))(J_i + \tfrac{t}{2}h_i^l J_l + O(t^2 f))(J_j + \tfrac{t}{2}h_j^m J_m + O(t^2 f))$$
$$= |J|_g^2 + O(t^2|J|_g f + t^3 f^2)$$
$$= \left(|J|_g + \frac{tf}{4}\right)^2 - \frac{tf}{2}|J|_g - \frac{t^2 f^2}{16} + O\left(t\left(\frac{tf}{2}|J|_g + \frac{t^2 f^2}{16}\right)\right)$$
$$< \left(|J|_g + \frac{tf}{4}\right)^2$$
for $t > 0$ small enough, so we find that
$$u^\kappa|\bar{J}|_{\bar{g}} < |J|_g + \frac{t}{4}f \tag{24}$$
for such t.

Fix $t > 0$ such that (23) and (24) hold. Note that our choice of f implies that $\sup_M (|J|_g/f) < \infty$. Therefore for any $x \in M$ such that $|\bar{J}|_{\bar{g}}(x) \ne 0$ we have
$$\frac{\bar{\mu}}{|\bar{J}|_{\bar{g}}} = \frac{u^\kappa \bar{\mu}}{u^\kappa|\bar{J}|_{\bar{g}}} > \frac{\mu + tf/3}{|J|_g + tf/4} \ge \frac{|J|_g + tf/3}{|J|_g + tf/4} = 1 + \frac{t}{12(|J|_g/f) + 3t} \ge 1 + \gamma,$$

for

$$\gamma := \frac{t}{12 \sup_M \left(|J|_g/f \right) + 3t}.$$

In points where $|\bar{J}|_{\bar{g}}(x) = 0$ we have $\bar{\mu} > 0$ by (23). Consequently, we have $\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$ everywhere on M as desired.

Note also that $(\bar{\mu}, \bar{J}) \in C^{0,\alpha}_{n+\tau'_0}$ for $\tau'_0 = \min\{1, \tau_0\}$ by (21), the asymptotics of u, and the properties of $\mathcal{R}$. In particular $\|(\mu, J) - (\bar{\mu}, \bar{J})\|_{C^0_{n+\tau'_0}}$ can be made arbitrarily small for a sufficiently small t. Thus Proposition 2.8 guarantees that $\left| \mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V) \right| < \varepsilon$ holds. $\qquad \square$

4. Perturbation to conformally hyperbolic asymptotics

In this section we prove the following result.

Theorem 4.1. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type (α, τ, τ_0) for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$. Assume that the dominant energy condition $\mu \geq |J|_g$ holds. Then for every $\tau' < \tau$ and $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$ which has conformally hyperbolic asymptotics with respect to the same chart, and is such that*

$$\|g - \bar{g}\|_{C^{2,\alpha}_{\tau'}} < \varepsilon, \qquad \|\pi - \bar{\pi}\|_{C^{2,\alpha}_{\tau'}} < \varepsilon,$$

the strict dominant energy condition

$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

holds, and

$$|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)| < \varepsilon$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

In [10, Appendix B] a similar result was proven in the simpler case when $\pi = 0$. The proof of Theorem 4.1 is very similar to [13, Proof of Theorem 18]. Its main ingredients are Theorem 3.1 and the following lemma.

Lemma 4.2. *Let (g, π) be an asymptotically hyperbolic initial data set of type (α, τ) for $0 < \alpha < 1$ and $\frac{n}{2} < \tau < n$ and suppose that $\tau' < \tau$. Then there are positive constants C_0 and δ_0 such that for any $(\bar{\mu}, \bar{J}) \in C^{0,\alpha}_\tau$ with $\|(\mu - \bar{\mu}, J - \bar{J})\|_{C^{0,\alpha}_\tau} \leq \delta \leq \delta_0$, there exists initial data $(\bar{g}, \bar{\pi})$ of type (α, τ) with the following properties.*

- *The energy and momentum densities of $(\bar{g}, \bar{\pi})$ are $\bar{\mu}$ and $\bar{J}$.*
- *Outside of a compact set $(\bar{g}, \bar{\pi})$ is of the form*

$$\bar{g} = u^\kappa b, \qquad \bar{\pi} = u^{\kappa/2} \mathring{\mathcal{L}}_Y b$$

  for $(u - 1, Y) \in C^{2,\alpha}_\tau$.
- *The initial data $(\bar{g}, \bar{\pi})$ is close to (g, π) in the sense that*

$$\|g - \bar{g}\|_{C^{2,\alpha}_{\tau'}} \leq C_0 \delta, \qquad \|\pi - \bar{\pi}\|_{C^{1,\alpha}_{\tau'}} \leq C_0 \delta.$$

Proof. The proof uses the construction introduced by Corvino and Schoen in [9, Proof of Theorem 1], which is similar to the one that was used in the proof of Theorem 3.1. Given (g, π) as in the statement of the theorem and $(u - 1, Y) \in C^{2,\alpha}_\tau$, we define the map

$$T_{(g,\pi)}(u, Y) = \Phi(u^\kappa g, u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g)).$$

It follows from (14) that the components of $T_{(g,\pi)}$ are given by

$$-2\widetilde{\mu} = \frac{4(n-1)}{n-2}u^{-\kappa-1}\Delta^g u - u^{-\kappa}\mathrm{Scal}^g - n(n-1) + 2u^{-\frac{\kappa}{2}}\mathrm{tr}^g\,\pi$$

$$\qquad - \frac{1}{n-1}u^{-\kappa}(\mathrm{tr}^g\,\pi)^2 + u^{-\kappa}\left(|\pi|_g^2 + 2\langle\pi,\mathring{\mathcal{L}}_Y g\rangle + |\mathring{\mathcal{L}}_Y g|_g^2\right),$$

$$\widetilde{J}_j = u^{-\frac{\kappa}{2}}(\Delta_L Y + \mathrm{div}^g\,\pi)_j + \frac{2(n-1)}{n-2}u^{-\frac{\kappa}{2}-1}(\pi + \mathring{\mathcal{L}}_Y g)_j^k\nabla_k u \tag{25}$$

$$\qquad - \frac{2}{n-2}u^{-\frac{\kappa}{2}-1}\,\mathrm{tr}^g\,\pi\nabla_j u,$$

for $j = 1, 2, \ldots, n$. From this formula it is straightforward to compute the linearization

$$DT_{(g,\pi)}|_{(1,0)}(v, Z) = \Big(\frac{4(n-1)}{n-2}\Delta^g v + \frac{4}{n-2}\mathrm{Scal}^g v - \frac{4}{n-2}(\mathrm{tr}^g\,\pi)v$$

$$\qquad + \frac{\kappa}{n-1}(\mathrm{tr}^g\,\pi)^2 v - \frac{4}{n-2}|\pi|_g^2 v + 2\langle\pi, \mathring{\mathcal{L}}_Z g\rangle,$$

$$\qquad (\Delta_L Z)_j - \frac{2}{n-2}(\mathrm{div}^g\,\pi)_j v + \frac{2(n-1)}{n-2}\pi_j^k\nabla_k v - \frac{2}{n-2}(\mathrm{tr}^g\,\pi)\nabla_j v\Big).$$

Since $\mathrm{Scal}^g + n(n-1) \in C_\tau^{0,\alpha}$, we may argue as in the proof of Lemma 3.2 and show that $DT_{(g,\pi)}|_{(1,0)}$ is Fredholm of index zero as an operator $C_\delta^{2,\alpha} \to C_\delta^{0,\alpha}$ for $-1 < \delta < n$ and as an operator $W_\gamma^{2,p} \to W_\gamma^{0,p}$ for $-1 < \gamma + \frac{n-1}{p} < n$. Fix $\tau' \in (\frac{n}{2}, \tau)$ and let $U_{\tau'}$ denote a complementing space for the kernel of $DT|_{(1,0)}$ in $C_{\tau'}^{2,\alpha}$. Arguing as in the proof of Lemma 3.4 we conclude that there exist finitely many pairs of compactly supported symmetric 2-tensors (h_k, w_k), $k = 1, \ldots, N$, such that their images $D\Phi|_{(g,\pi)}(h_k, w_k)$ form a basis for a subspace which complements $DT_{(g,\pi)}|_{(1,0)}(C_{\tau'}^{2,\alpha})$ in $C_{\tau'}^{0,\alpha}$. Set $V_{\tau'} := \mathrm{span}\{(h_k, w_k)\}_{k=1,\ldots,N}$. We define the map $\Xi_{(g,\pi)} : U_{\tau'} \times V_{\tau'} \to C_{\tau'}^{0,\alpha}$ by

$$\Xi_{(g,\pi)} : (u, Y, h, w) \mapsto \Phi(u^\kappa g + h, u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g) + w). \tag{26}$$

Then the linearization $D\Xi_{(g,\pi)}|_{(1,0,0,0)} : U_{\tau'} \times V_{\tau'} \to C_{\tau'}^{0,\alpha}$ is given by

$$D\Xi_{(g,\pi)}|_{(1,0,0,0)} : (v, Z, \eta, \omega) \mapsto DT_{(g,\pi)}|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(\eta, \omega)$$

and is an isomorphism by construction.

Using the chart at infinity $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus B_{R_0}$, we define the cut-off function $\chi_\lambda(x) = \chi(r(x)/\lambda)$, where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. For a sufficiently large $\lambda > 0$, the cut-off initial data (g_λ, π_λ) is given by

$$g_\lambda = \chi_\lambda g + (1 - \chi_\lambda)\Psi^* b, \qquad \pi_\lambda = \chi_\lambda \pi.$$

Since $\tau' < \tau$ we have

$$\|g - g_\lambda\|_{C_{\tau'}^{2,\alpha}} \to 0 \qquad \text{and} \qquad \|\pi - \pi_\lambda\|_{C_{\tau'}^{1,\alpha}} \to 0 \tag{27}$$

as $\lambda \to \infty$. Hence

$$\|\Phi(g, \pi) - \Phi(g_\lambda, \pi_\lambda)\|_{C_{\tau'}^{0,\alpha}} \to 0 \tag{28}$$

as $\lambda \to \infty$.

Similarly to (26), we define the map $\Xi_{(g_\lambda, \pi_\lambda)} : U_{\tau'} \times V_{\tau'} \to C_{\tau'}^{0,\alpha}$ by

$$\Xi_{(g_\lambda, \pi_\lambda)} : (u, Y, h, w) \mapsto \Phi(u^\kappa g_\lambda + h, u^{\kappa/2}(\pi_\lambda + \mathring{\mathcal{L}}_Y g_\lambda) + w).$$

The linearization $D\Xi_{(g_\lambda, \pi_\lambda)}|_{(1,0,0,0)} : U_{\tau'} \times V_{\tau'} \to C_{\tau'}^{0,\alpha}$ is given by

$$D\Xi_{(g_\lambda, \pi_\lambda)}|_{(1,0,0,0)} : (v, Z, \eta, \omega) \mapsto DT_{(g_\lambda, \pi_\lambda)}|_{(1,0)}(v, Z) + D\Phi|_{(g_\lambda, \pi_\lambda)}(\eta, \omega).$$

As a consequence of (27), the operators $D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)}$ converge to the isomorphism $D\Xi_{(g,\pi)}|_{(1,0,0,0)}$ as $\lambda \to \infty$ in the uniform operator topology. It follows that there exists a positive λ_0 such that for any $\lambda \geq \lambda_0$ the linearization $D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)}$ is an isomorphism. Note that $\Xi_{(g_\lambda,\pi_\lambda)}(1,0,0,0) = \Phi(g_\lambda,\pi_\lambda) = (-2\mu_\lambda, J_\lambda)$. Applying the Inverse Function Theorem, see for example [15, Theorem 4.2 and Remark 4.3]), it is not complicated to check that there exists $\rho_0 > 0$ depending only on (g,π) such that $\Xi_{(g_\lambda,\pi_\lambda)} : B_{\rho_0}(1,0,0,0) \to \Xi_{(g_\lambda,\pi_\lambda)}(B_{\rho_0}(1,0,0,0))$ is a diffeomorphism for any $\lambda \geq \lambda_0$. Furthermore, there exists a constant $C > 0$ depending only on (g,π) such that

$$C\|(u,Y,h,w) - (1,0,0,0)\|_{C_{\tau'}^{2,\alpha}} \leq \|\Xi_{(g_\lambda,\pi_\lambda)}(u,Y,h,w) - (-2\mu_\lambda, J_\lambda)\|_{C_{\tau'}^{0,\alpha}} \quad (29)$$

holds for any $(u,Y,h,w) \in B_{\rho_0}(1,0,0,0)$, and such that

$$B_{C\rho_0}(-2\mu_\lambda, J_\lambda) \subset \Xi_{(g_\lambda,\pi_\lambda)}(B_{\rho_0}(1,0,0,0)).$$

Now set $\delta_0 = C\rho_0/2$ and suppose that $(\bar{\mu}, \bar{J})$ is such that $\|(2(\mu - \bar{\mu}), J - \bar{J})\|_{C_\tau^{0,\alpha}} \leq \delta \leq \delta_0$. By (28) we may assume that $\lambda \geq \lambda_0$ is such that $\|(2(\mu - \mu_\lambda), J - J_\lambda)\|_{C_{\tau'}^{0,\alpha}} \leq \delta$, hence $(-2\bar{\mu}, \bar{J}) \in B_{C\rho_0}(-2\mu_\lambda, J_\lambda)$. Then it follows from the above discussion that there exists $(u,Y,h,w) \in B_{\rho_0}(1,0,0,0)$ such that $\Xi_{(g_\lambda,\pi_\lambda)}(u,Y,h,w) = (-2\bar{\mu}, \bar{J})$. As a consequence of (29) we have

$$\|(u-1,Y,h,w)\|_{C_{\tau'}^{2,\alpha}} \leq 2C^{-1}\delta.$$

Further, it follows from (25) that outside of a compact set (u,Y) satisfies

$$\begin{aligned}
-2u^{\kappa+1}\bar{\mu} &= \tfrac{4(n-1)}{n-2}\Delta^b u + n(n-1)(u - u^{\kappa+1}) + u|\mathring{\mathcal{L}}_Y b|_b^2, \\
u^{\kappa/2}\bar{J}_j &= (\Delta_L Y)_j + \tfrac{2(n-1)}{n-2}u^{-1}(\mathring{\mathcal{L}}_Y b)_j^k \nabla_k u.
\end{aligned} \quad (30)$$

This implies that $v = u - 1$ and Y are such that $(v,Y) \in C_{\tau'}^{2,\alpha}$ and

$$(\Delta^b v - nv, \Delta_L Y) \in C_\tau^{0,\alpha},$$

hence $(v,Y) \in C_\tau^{2,\alpha}$ by the improved elliptic regularity [18, Proposition 6.5].

It is now straightforward to check that there exists a constant $C_0 > 0$ such that the initial data

$$\bar{g} = u^\kappa g_\lambda + h, \qquad \bar{\pi} = u^{\kappa/2}(\pi_\lambda + \mathring{\mathcal{L}}_Y g_\lambda) + w$$

has all the required properties. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following result is not complicated to prove.

Proposition 4.3. *Suppose that (M,g,π) is an asymptotically hyperbolic initial data set of type (α,τ,τ_0) for $\tau_0 \geq 1$ such that*

$$g = u^\kappa b, \qquad \pi = u^{\kappa/2}\mathring{\mathcal{L}}_Y b$$

outside of a compact set for some $(u-1,Y) \in C_\tau^{2,\alpha}$. Then (g,π) has conformally hyperbolic asymptotics in the sense of Definition 2.3.

Proof. In this case it follows from (30) that outside of a compact set $v = u - 1$ and Y satisfy

$$(\Delta^b v - nv, \Delta_L Y) \in C_{n+\varepsilon}^{0,\alpha}$$

for some $\varepsilon > 0$. Then $(v,Y) \in C_n^{2,\alpha}$ by Proposition B.2. More specifically, from the proof of Proposition B.2 we see that (v,Y) is of the form (5), where (v_0,Y_0) does not depend on r, and $(v_1,Y_1) \in C_{n+\varepsilon}^{2,\alpha}$.

Inserting $u = v + 1$ and Y back in (30) we conclude that $(v_1, Y_1) \in C^{2,\alpha}_{n+\varepsilon}$ satisfies

$$(\Delta^b v_1 - nv_1, \Delta_L Y_1) \in C^{0,\alpha}_{n+1}.$$

Thus $(v_1, Y_1) \in C^{2,\alpha}_{n+1}$ by Proposition B.2.   □

Proof of Theorem 4.1. By Theorem 3.1 we may without loss of generality assume that $\mu > (1 + \gamma)|J|_g$ for some $\gamma > 0$. Let ξ be a smooth function such that $\xi(x) = e^{-r(x)}$ outside a compact set. For χ_λ as in the proof of Lemma 4.2 set $\xi_\lambda := \chi_\lambda + (1 - \chi_\lambda)\xi$. Then $(\xi_\lambda \mu, \xi_\lambda J) \in C^{0,\alpha}_{n+1+\tau_0}$ and

$$\|(\mu, J) - (\xi_\lambda \mu, \xi_\lambda J)\|_{C^{0,\alpha}_{n+\tau'_0}} \to 0 \tag{31}$$

as $\lambda \to \infty$ for any $\tau'_0 < \tau_0$. By Lemma 4.2 and Proposition 4.3 we may construct initial data sets (g_λ, π_λ) with conformally hyperbolic asymptotics, whose energy and momentum densities are $(\xi_\lambda \mu, \xi_\lambda J)$, and such that

$$\|g - g_\lambda\|_{C^{2,\alpha}_{\tau'}} \to 0, \qquad \|\pi - \pi_\lambda\|_{C^{1,\alpha}_{\tau'}} \to 0 \tag{32}$$

for any $\tau' < \tau$ as $\lambda \to \infty$. In particular, $\|g - g_\lambda\|_{C^0} \to 0$ as $\lambda \to \infty$ thus $|J|^2_{g_\lambda} = |J|^2_g(1 + o(1))$ as $\lambda \to \infty$. It follows that

$$\xi_\lambda \mu > \xi_\lambda(1 + \gamma)|J|_g \geq \left(1 + \frac{\gamma}{2}\right)|\xi_\lambda J|_{g_\lambda}$$

for λ sufficiently large, hence (g_λ, π_λ) satisfies the strict dominant energy condition. Further, since (31) and (32) hold for any $0 < \tau'_0 < \tau_0$ and $\frac{n}{2} < \tau' < \tau$, it follows by Proposition 2.8 that

$$\mathcal{M}_{(g_\lambda, \pi_\lambda)}(V) \to \mathcal{M}_{(g,\pi)}(V)$$

for all $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ as $\lambda \to \infty$.   □

5. Initial data with Wang's asymptotics

In this section we show that the results of Section 3 extend to yet another important class of asymptotically hyperbolic initial data.

Definition 5.1. Let (M, g, π) be asymptotically hyperbolic initial data of type (α, n) for $0 \leq \alpha < 1$. We say that (M, g, π) has *Wang's asymptotics* if there exists $R_0 > 0$ and a diffeomorphism

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus B_{R_0}$$

with the following properties.

1. The pushforward of the metric g under Ψ satisfies

$$\Psi_* g = dr^2 + \sinh^2 r g_r, \tag{33}$$

where

$$g_r = \sigma + me^{-nr} + O(e^{-(n+1)r})$$

is an r-dependent family of symmetric 2-tensor on S^{n-1}, and m is a symmetric 2-tensor on S^{n-1}.

2. The pushforward of the 2-tensor π under Ψ satisfies

$$\begin{aligned}
(\Psi_* \pi)_{rr} &= p_{rr}e^{-nr} + O(e^{-(n+1)r}), \\
(\Psi_* \pi)_{r\mu} &= p_{r\mu}e^{-(n-1)r} + O(e^{-nr}), \\
(\Psi_* \pi)_{\mu\nu} &= p_{\mu\nu}e^{-(n-2)r} + O(e^{-(n+1)r}),
\end{aligned} \tag{34}$$

where p does not depend on r and μ, ν denote components in a coordinate system on the sphere.

3. The asymptotic expansion (33) is twice differentiable and the asymptotic expansion (34) is once differentiable.

Asymptotically hyperbolic metrics g with asymptotics (33) were considered by Wang in [25] and have been studied in various contexts, see for example [1], [22], and [23].

The assumption that the metric has an asymptotic expansion (33) is not as restrictive as it might seem. Any sufficiently regular conformally compactifiable metric with the round sphere as the boundary at infinity and deviating from the hyperbolic metric at the "critical" order $|g - b|_b = O(e^{-nr})$ can be written in the form (33) in appropriate coordinates. See, for example, [3, Section IV], [1, Section 3], or [8].

In the case when (M, g, π) is an initial data set with Wang's asymptotics a direct computation shows that the mass functional is given by

$$\mathcal{M}(V_{(0)}) = \tfrac{1}{2(n-1)\omega_{n-1}} \int_{S^{n-1}} (n \operatorname{tr}_\sigma m - 2p_{rr}) \, d\mu^\sigma,$$

and

$$\mathcal{M}(V_{(i)}) = \tfrac{1}{2(n-1)\omega_{n-1}} \int_{S^{n-1}} x^i (n \operatorname{tr}_\sigma m - 2p_{rr}) \, d\mu^\sigma,$$

for $i = 1, \ldots, n$, where m and p_{rr} are as in Definition 5.1. Further, we have the following result.

Theorem 5.2. *Suppose that (M, g, π) is an initial data set of type (α, n) for $0 < \alpha < 1$ such that it has Wang's asymptotics and satisfies the dominant energy condition $\mu \geq |J|_g$. Then, for any $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$ of the same type with Wang's asymptotics such that*

$$\|g - \bar{g}\|_{C_n^{2,\alpha}} < \varepsilon, \qquad \text{and} \qquad \|\pi - \bar{\pi}\|_{C_n^{1,\alpha}} < \varepsilon, \tag{35}$$

the strict dominant energy condition

$$\bar{\mu} > |\bar{J}|_{\bar{g}} \tag{36}$$

holds, and

$$|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)| < \varepsilon \tag{37}$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

Proof. Let $\varepsilon > 0$ be fixed. Since $C_n^{k,\alpha} \hookrightarrow C_\tau^{k,\alpha}$ for $n > \tau$ and $k = 0, 1, \ldots$, we can argue as in the proof of Theorem 3.1 to find $(v, Z) \in C_n^{2,\alpha}$, and $(h, w) \in C_c^{3,\alpha}$ such that for some sufficiently small $t > 0$ the perturbed initial data

$$\bar{g} = (1 + tv)^\kappa (g + th) \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2} (\pi + t\mathring{\mathcal{L}}_Z g + tw)$$

satisfies (35) and (36). In particular, the positive function f used in this construction should be chosen so that $f = O(e^{-(n+1)r})$ in order to ensure that $(v, Z) = (v_0, Z_0)e^{-nr} + (v_1, Z_1)$ for $(v_0, Z_0) \in C_{loc}^{2,\alpha}$ independent of r and $(v_1, Z_1) \in C_{n+1}^{2,\alpha}$ (compare the proof of Proposition 4.3). Since in this case f might decay faster than $J = O(e^{-nr})$, it is not clear that there is a $\gamma > 0$ such that $\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$. However, this is not important in the current setting since we do not intend to make a further perturbation of $(\bar{g}, \bar{\pi})$.

Next we estimate the difference between the masses of the original data (g, π) and the perturbed data $(\bar{g}, \bar{\pi})$. Outside a compact set we have

$$\bar{g} = (1 + tv)^{\kappa} g \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2} (\pi + t\mathring{\mathcal{L}}_Z g).$$

Set $U = (1 + tv)^{\kappa} - 1$, then $\bar{g} - g = Ug$, $\bar{\pi} - \pi = U\pi + t(1 + tv)^{\kappa/2}\mathring{\mathcal{L}}_Z g$ where $U = O_1(tv) = O_1(e^{-nr})$. A straightforward computation shows that for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ we have

$$\mathcal{M}_{(\bar{g}, \bar{\pi})}(V) - \mathcal{M}_{(g, \pi)}(V)$$
$$= \frac{1}{2(n-1)\omega_{n-1}} \lim_{R \to \infty} \int_{S_R} \left((n-1)(U dV - V dU)(\nu) - 2t\mathring{\mathcal{L}}_Z b(\nabla^b V, \nu) \right) d\mu^b.$$

Consequently, we have

$$|\mathcal{M}_{(\bar{g}, \bar{\pi})}(V) - \mathcal{M}_{(g, \pi)}(V)| \leq Ct(|v|_{C_n^1} + |Z|_{C_n^1})$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ and (37) follows, after decreasing t if necessary.

Finally, recall that g has asymptotic expansion (33) with respect to a certain chart at infinity. With respect to this chart, $\bar{g}$ has the expansion

$$\bar{g} = (1 + tv)^{\kappa} dr^2 + \sinh^2 r \left(\sigma + (m + t\kappa v_0 \sigma)e^{-nr} + O(e^{-(n+1)r}) \right).$$

This expansion is not of the form (33). Nevertheless, we may argue as in [3, Section IV] or [8] and introduce a new radial coordinate $\bar{r} = r - \frac{t\kappa}{2n} v_0 e^{-nr} + O(e^{-(n+1)r})$ so that $(\bar{g}, \bar{\pi})$ has Wang's asymptotics in the new coordinates. In particular,

$$\bar{g} = d\bar{r}^2 + \sinh^2 \bar{r} \left(\sigma + (m + \frac{t\kappa(n+1)}{n} v_0 \sigma)e^{-n\bar{r}} + O(e^{-(n+1)\bar{r}}) \right).$$

Using the asymptotic expansions of (g, π) and (v, Z) it is straightforward to check that this change of coordinates does not affect the mass. $\qquad \square$

Using the computations performed in the proof of Theorem 5.2 it is straightforward to see that initial data sets with conformally hyperbolic asymptotics can be interpreted as initial data sets with Wang's asymptotics after a change of the radial coordinate. This gives us the following equivalent form of Theorem 4.1.

Theorem 5.3. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type (α, τ, τ_0) for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$. Assume that the dominant energy condition $\mu \geq |J|_g$ holds. Then for every $\tau' < \tau$ and $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$ with Wang's asymptotics such that*

$$\|g - \bar{g}\|_{C_{\tau'}^{2,\alpha}} < \varepsilon, \qquad \|\pi - \bar{\pi}\|_{C_{\tau'}^{2,\alpha}} < \varepsilon,$$

the strict dominant energy condition

$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

holds, and

$$|\mathcal{M}_{(g, \pi)}(V) - \mathcal{M}_{(\bar{g}, \bar{\pi})}(V)| < \varepsilon$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

6. CONCLUDING REMARKS

In this paper we have focused on the charge integrals $\mathbb{Q}_{(V,-dV)}$, where $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$. These charge integrals are associated (as described in Section 2.2) to the Killing vectors $\partial_t, \partial_{x^1}, \ldots, \partial_{x^n}$ of Minkowski spacetime which generate infinitesimal translations in time and space. One may ask if the analogue of Theorem 4.1 can be proven for the remaining charges associated with the Killing vectors $x^i\partial_t + t\partial_{x^i}$, $1 \le i \le n$, and $x^i\partial_{x^j} - x^j\partial_{x^i}$, $1 \le i < j \le n$, which generate respectively infinitesimal boosts and rotations. In fact, using the general theory by Michel [20, Section IV.B] it is straightforward to check that these charges are well-defined and continuous under the assumptions of Proposition 2.4 and Proposition 2.8. As a consequence, we expect that the perturbation results of this paper can be extended to apply to these charges as well. Note that this is quite different from the situation in the asymptotically Euclidean setting, where the charges associated with boosts and rotations are determined by terms of lower order in the asymptotic expansion of the initial data than the charges associated with translations in time and space. For this reason a given asymptotically Euclidean initial data set can be perturbed slightly to achieve any value of angular momentum and center of mass in such a way that the mass and linear momentum do not change, see Huang, Schoen, and Wang [16]. (In particular, this shows that the mass and angular momentum inequality will in general not hold for asymptotically Euclidean initial data sets without the assumption of axial symmetry.) One does not expect such a result to hold in the asymptotically hyperbolic setting.

It is also clear that we could have assumed higher regularity for the initial data (g, π), say $g \in C_\tau^{k,\alpha}$ and $\pi \in C_\tau^{k-1,\alpha}$ for $k > 2$, $0 < \alpha < 1$ and $\frac{n}{2} < \tau < n$, in which case the perturbed data $(\bar{g}, \bar{\pi})$ would have the same regularity throughout the article, as a consequence of standard elliptic regularity.

Regarding the extension of the results to the case of weighted Sobolev spaces, note that in this case it is not possible to rely on the beautiful theory of Lee [18]. Instead, methods for operators *asymptotic to geometric operators on hyperbolic space* (compare Bartnik [4, Definition 1.5]) can be used, see for example the proof of Lemma 3.2.

APPENDIX A. FREDHOLM OPERATORS ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS: CHART-DEPENDENT APPROACH

Theorem C in Lee's monograph [18] proves the Fredholm property of geometric elliptic operators acting on weighted Sobolev and Hölder spaces on conformally compact manifolds. In this appendix we will show that the same result holds for asymptotically hyperbolic manifolds in the sense of Definition 2.1. We use the same definition of geometric tensor bundles and geometric elliptic partial differential operators as in the cited monograph.

Let (M, g) be a $C_\tau^{l,\beta}$-asymptotically hyperbolic n-manifold in the sense of Definition 2.1 for $n \ge 2$, $l \ge 2$, $0 \le \beta < 1$, and $\tau > 0$. Let $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0}$ be the chart at infinity. Given a geometric elliptic partial differential operator $P : C^\infty(M; E) \to C^\infty(M; E)$ of order $m \le l$ we define the *indicial map* $I_s(P) : E|_{S^{n-1}} \to E|_{S^{n-1}}$ by setting

$$I_s(P)\overline{u} := \lim_{r \to \infty} e^{sr}P(e^{-sr}\overline{u}).$$

Following [18, Section 4], we call $s \in \mathbb{C}$ a *characteristic exponent* at $p \in S^{n-1}$ if $I_s(P)$ is singular at p. Using the fact that $|\Psi_* g - b|_b = O(e^{-\tau r})$ it is not complicated to check that the characteristic exponents of P are constant on S^{n-1}. Further, if P is self-adjoint one may verify that the set of characteristic exponents is symmetric about the line $\text{Re } s = \frac{n-1}{2} - k$, where $k = k_1 - k_2$ is the rank of the geometric tensor bundle $E \subset T^{k_1}_{k_2} M$, see [18, Proposition 4.4]. Similarly to the conformally compact case, we define the *indicial radius* of P as the smallest non-negative number R such that P has a characteristic exponent whose real part is $\frac{n-1}{2} - k + R$.

Theorem A.1. *Let (M, g) be a connected asymptotically hyperbolic n-manifold of class $C^{l,\beta}_\tau$, with $n \geq 2$, $l \geq 2$, $0 \leq \beta < 1$, and $\tau > 0$ and let $E \to M$ be a geometric tensor bundle over M. Suppose that $P : C^\infty(M; E) \to C^\infty(M; E)$ is an elliptic, formally self-adjoint, geometric partial differential operator of order m, $0 < m \leq l$, and assume that there exists a compact set $K \subset M$ and a positive constant C such that*

$$\|u\|_{L^2} \leq C\|Pu\|_{L^2} \tag{38}$$

for all $u \in C^\infty_c(M \setminus K; E)$. Let R be the indicial radius of P.

- *If $1 < p < \infty$ and $m \leq k \leq l$ then the natural extension*

$$P : W^{k,p}_\delta(M; E) \to W^{k-m,p}_\delta(M; E)$$

  is Fredholm for $|\delta + \frac{n-1}{p} - \frac{n-1}{2}| < R$. In that case, its index is zero, and its kernel is equal to the L^2 kernel of P.

- *If $0 < \alpha < 1$ and $m < k + \alpha \leq l + \beta$ then the natural extension*

$$P : C^{k,\alpha}_\delta(M; E) \to C^{k-m,\alpha}_\delta(M; E)$$

  is Fredholm for $|\delta - \frac{n-1}{2}| < R$. In that case, its index is zero, and its kernel is equal to the L^2 kernel of P.

Proof. The proof goes as in [18, Chapter 6], except for the steps which explicitly use coordinates at infinity. We verify that these steps can be carried out in our setting, that is for asymptotically hyperbolic manifolds as in Definition 2.1. Specifically, we need to adapt the construction of a parametrix given in Proposition 6.2 and Corollary 6.3 of [18]. In fact, the construction of a parametrix turns out to be simpler than in the conformally compact case, since instead of countably many Möbius charts covering a neighbourhood of infinity we have a single chart, so there is no need to patch the coordinate charts together using a partition of unity.

Let $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus B_{R_0}$ be a chart at infinity as in Definition 2.1. As in [14, Appendix A] we use this chart to construct a bundle $\breve{E} \to \mathbb{H}^n$ which is defined using the same $O(n)$-representation as the one which defines E, and an isomorphism $\Upsilon : \breve{E}|_{\mathbb{H}^n \setminus B_{R_0}} \to E|_{M \setminus K_0}$. The isomorphism Υ, its inverse and their first l derivatives all have uniformly bounded norms on $\mathbb{H}^n \setminus B_{R_0}$, respectively $M \setminus K_0$. Let $\breve{P} : C^\infty(\mathbb{H}^n; \breve{E}) \to C^\infty(\mathbb{H}^n; \breve{E})$ be the operator on hyperbolic space with the same local coordinate expression as P. We define $P' : C^\infty(\mathbb{H}^n \setminus B_{R_0}; \breve{E}) \to C^\infty(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$ by

$$P'u = \Upsilon^{-1} P \Upsilon u.$$

Let $R_1 \geq R_0$. Since P is a geometric operator and g is $C^{l,\beta}_\tau$-asymptotically hyperbolic, we conclude that for each $\delta \in \mathbb{R}$, $0 < \alpha < 1$, $1 < p < \infty$, and k such that

$m \leq k \leq l$ and $m < k + \alpha \leq l + \beta$ there exists a positive constant C independent of R_1 such that

$$\|P'u - \breve{P}u\|_{C^{k-m,\alpha}_\delta(\mathbb{H}^n \setminus B_{R_1}; \breve{E})} < Ce^{-\tau R_1}\|u\|_{C^{k,\alpha}_\delta(\mathbb{H}^n \setminus B_{R_1}; \breve{E})} \tag{39}$$

holds for all $u \in C^{k,\alpha}_\delta(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$, and

$$\|P'u - \breve{P}u\|_{W^{k-m,p}_\delta(\mathbb{H}^n \setminus B_{R_1}; \breve{E})} < Ce^{-\tau R_1}\|u\|_{W^{k,p}_\delta(\mathbb{H}^n \setminus B_{R_1}; \breve{E})} \tag{40}$$

holds for all $u \in W^{k,p}_\delta(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$.

Now suppose that P satisfies (38). Then, by the properties of Υ, the operator P' also satisfies (38). Consequently, if R_1 is sufficiently large, it follows by (40) and standard elliptic regularity that $\breve{P}$ satisfies $\|u\|_{L^2} \leq C\|\breve{P}u\|_{L^2}$ for all $u \in C^\infty_c(\mathbb{H}^n \setminus B_{R_1}; \breve{E})$, possibly with a larger value of C than in (38). By [18, Theorems 5.7 and 5.9] we conclude that $\breve{P}$ is invertible as an operator $W^{k,p}_\delta(\mathbb{H}^n; \breve{E}) \to W^{k-m}_\delta(\mathbb{H}^n; \breve{E})$ for $|\delta + \frac{n-1}{p} - \frac{n-1}{2}| < R$ and as an operator $C^{k,\alpha}_\delta(\mathbb{H}^n; \breve{E}) \to C^{k-m,\alpha}_\delta(\mathbb{H}^n; \breve{E})$ for $|\delta - \frac{n-1}{2}| < R$.

Assume that $R_1 \geq R_0$ is sufficiently large and let K_1 be such that $M \setminus K_1 = \Psi^{-1}(\mathbb{H}^n \setminus B_{R_1})$. We proceed by defining the operators $Q, S : C^\infty_c(M \setminus K_1, E) \to C^\infty_c(M \setminus K_1, E)$ by

$$Qu = \Upsilon \breve{P}^{-1} \Upsilon^{-1} u,$$

$$Su = \Upsilon \breve{P}^{-1}(P' - \breve{P})\Upsilon^{-1}u.$$

Then it is straightforward to check that

$$QPu = u + Su.$$

It follows from the above discussion that Q extends to bounded maps

$$Q : W^{0,p}_\delta(M \setminus K_0; E) \to W^{m,p}_\delta(M \setminus K_0; E) \quad \text{for} \quad |\delta + \tfrac{n-1}{p} - \tfrac{n-1}{2}| < R,$$

and

$$Q : C^{0,\alpha}_\delta(M \setminus K_0; E) \to C^{m,\alpha}_\delta(M \setminus K_0; E) \quad \text{for} \quad |\delta - \tfrac{n-1}{2}| < R.$$

Similarly, as a consequence of (39) and (40), S extends to bounded maps

$$S : W^{m,p}_\delta(M \setminus K_0; E) \to W^{m,p}_\delta(M \setminus K_0; E) \quad \text{for} \quad |\delta + \tfrac{n-1}{p} - \tfrac{n-1}{2}| < R,$$

and

$$S : C^{m,\alpha}_\delta(M \setminus K_0; E) \to C^{m,\alpha}_\delta(M \setminus K_0; E) \quad \text{for} \quad |\delta - \tfrac{n-1}{2}| < R,$$

such that if u is supported in $M \setminus K_1$, then

$$\|Su\|_{W^{m,p}_\delta} \leq Ce^{-\tau R_1}\|u\|_{W^{m,p}_\delta}, \quad \|Su\|_{C^{m,\alpha}_\delta} \leq Ce^{-\tau R_1}\|u\|_{C^{m,\alpha}_\delta}$$

for some constant C independent of R_1 and u. Without loss of generality we may assume that $Ce^{-\tau R_1} < \frac{1}{2}$, and it follows that the operators

$$\mathrm{Id} + S : W^{m,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E),$$

$$\mathrm{Id} + S : C^{m,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E)$$

have bounded inverses. This implies that, whenever u has support in $M \setminus K_1$, we have

$$u = \widetilde{Q}Pu,$$

where $\widetilde{Q} = (\mathrm{Id} + S)^{-1} \circ Q$ is bounded as an operator $W^{0,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E)$ and $C^{0,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E)$, where δ is in the same range

as above. As a consequence of this parametrix construction, improved elliptic regularity results [18, Proposition 6.5] hold for asymptotically hyperbolic manifolds as in Definition 2.1.

The rest of the proof does not use coordinates at infinity, and the reader is referred to [18] for details. □

Proposition A.2. *The operator $\Delta - n$ and the vector Laplacian Δ_L satisfy the conditions of Theorem A.1 with $R = \frac{n+1}{2}$.*

Proof. It is not complicated to check that the L^2-estimate at infinity (38) holds for $\Delta - n$. For Δ_L the L^2-estimate at infinity can be proven by standard methods, see for example Appendix B in [14]. The critical exponents of both operators can be computed using the explicit expressions for their components, see the proof of Proposition B.2 below. □

APPENDIX B. SOLUTIONS OF CRITICAL ORDER

Suppose that $P : C^\infty(M; E) \to C^\infty(M; E)$ is a formally self-adjoint geometric elliptic operator of order m satisfying the conditions of Theorem A.1 and let $\delta_- < \delta_+$ be its critical exponents. Roughly speaking, if $u = O(e^{-\delta r})$ for some $\delta \in (\delta_-, \delta_+)$ then $Pu = O(e^{-\kappa r})$ for some $\kappa \in (\delta_-, \delta_+)$ implies that $u = O(e^{-\kappa r})$, see [18, Proposition 6.5]. At the same time, $Pu = O(e^{-\delta_+ r})$ does not necessarily imply $u = O(e^{-\delta_+ r})$. An extensive study of the asymptotic behaviour of solutions outside of the Fredholm interval can be found in [2, Chapter 4] in the case of conformally compact metrics. Analogous results can be proven for asymptotically hyperbolic manifolds as in Definition 2.1 using the following simple lemma.

Lemma B.1. *Consider the ordinary differential equation*
$$u'' + Au' + Bu = f. \tag{41}$$
Assume that $A^2 - 4B > 0$ so that the characteristic equation $\lambda^2 - A\lambda + B = 0$ has two distinct real roots $\delta_- < \delta_+$. Suppose that (41) holds for $u = u(r) = O(e^{-\delta r})$ and $f = f(r) = O(e^{-\kappa r})$ for some $\kappa > \delta_+$. Then
- *$\delta > \delta_-$ implies that $u = O(e^{-\delta_+ r})$, and*
- *$\delta > \delta_+$ implies that $u = O(e^{-\kappa r})$.*

Note that we use a possibly non-standard characteristic equation which results from substituting $u = e^{-\lambda r}$ rather than $u = e^{\lambda r}$ into (41).

Proof. This is a consequence of the explicit formula
$$
\begin{aligned}
u = &\Lambda_- e^{-\delta_- r} + \Lambda_+ e^{-\delta_+ r} \\
&- \frac{1}{\delta_+ - \delta_-} \left(e^{-\delta_- r} \int_r^\infty e^{\delta_- s} f(s)\, ds - e^{-\delta_+ r} \int_r^\infty e^{\delta_+ s} f(s)\, ds \right)
\end{aligned} \tag{42}
$$
for the solutions of (41). Note that Λ_- and Λ_+ do not depend on r. □

In this paper we use the following result.

Proposition B.2. *Let (M, g) be a connected asymptotically hyperbolic n-manifold of class $C_\tau^{l,\beta}$, with $n \geq 2$, $l \geq 2$, $0 \leq \beta < 1$, and $\tau > 0$.*
- *Assume that $v \in C_\delta^{0,0}$ is such that $\Delta v - nv \in C_{n+\varepsilon}^{0,\alpha}$ for $\varepsilon > 0$, $0 < \alpha < 1$, and $2 + \alpha \leq l + \beta$. If $\delta > -1$ then $v \in C_n^{2,\alpha}$. If $\delta > n$, then $v \in C_{n+\varepsilon}^{2,\alpha}$.*

- *Assume that $Z \in C_\delta^{0,0}$ is such that $\Delta_L Z \in C_{n+\varepsilon}^{0,\alpha}$ for $\varepsilon > 0$, $0 < \alpha < 1$, and $2 + \alpha \leq l + \beta$. If $\delta > -1$ then $Z \in C_n^{2,\alpha}$. If $\delta > n$, then $Z \in C_{n+\varepsilon}^{2,\alpha}$.*

Proof. We first prove the second claim. A straightforward computation shows that if $Z \in C_{\delta'}^{2,\alpha}$ then

$$(\Delta_L Z)_r = \frac{2(n-1)}{n}(\partial_{rr}^2 Z_r + (n-1)\partial_r Z_r - nZ_r) + O(e^{-(\delta'+\gamma)r}),$$

$$(\Delta_L Z)_\psi = \partial_{rr}^2 Z_\psi + (n-3)\partial_r Z_\psi - 2(n-1)Z_\psi + O(e^{-(\delta'+\gamma-1)r})$$

for $\gamma = \min\{1, \tau\}$. Note that in our case $Z \in C_{\delta'}^{2,\alpha}$ for any $\delta' \in (-1, n)$ as a consequence of improved elliptic regularity [18, Proposition 6.5] so we may assume that $\delta' + \gamma > n$. Hence the components of Z satisfy

$$\partial_{rr}^2 Z_r + (n-1)\partial_r Z_r - nZ_r = O(e^{-\kappa r}),$$

$$\partial_{rr}^2 Z_\psi + (n-3)\partial_r Z_\psi - 2(n-1)Z_\psi = O(e^{-(\kappa-1)r})$$

for $\kappa = \min\{\delta' + \gamma, n + \varepsilon\}$. From Lemma B.1 it follows that $Z_r = O(e^{-nr})$, and $Z_\psi = O(e^{-(n-1)r})$, hence $Z \in C_n^{2,\alpha}$ by standard elliptic regularity. Similarly, if $\delta > n$ it follows that $Z \in C_{n+\varepsilon}^{2,\alpha}$, possibly after repeating this argument finitely many times in order to ensure that $\kappa = n + \varepsilon$.

The first claim is proven similarly using Lemma B.1 and the fact that

$$\Delta v - nv = \partial_{rr}^2 v + (n-1)\partial_r v - nv + O(e^{-(\delta'+\gamma)r})$$

for $\gamma = \min\{1, \tau\}$ when $v \in C_{\delta'}^{2,\alpha}$. $\qquad\square$

Appendix C. On the unique continuation property

The following result is a straightforward consequence of the unique continuation results by Mazzeo [19, Theorem 7] and Kazdan [17, Theorem 1.8].

Proposition C.1. *Let (M, g) be a $C_\tau^{2,\beta}$-asymptotically hyperbolic manifold for $\tau > 0$ and $0 \leq \beta < 1$, and let E be a geometric tensor bundle over M. Suppose that $u \in C^2(M; E)$ satisfies the differential inequality*

$$|\Delta u| \leq C(|u| + |\nabla u|), \tag{43}$$

where $\Delta = -\nabla^*\nabla$ is the rough Laplacian. If u vanishes to infinite order at infinity, that is $|u| = O(e^{-Nr})$ for any $N > 0$, then $u = 0$ on M.

Proof. The hyperbolic metric $b = dr^2 + \sinh^2 r\,\sigma$ clearly satisfies the conditions (4)–(6) in [19]. We may therefore combine Theorem 7 in this reference with the fact that $g - b \in C_\tau^{2,\beta}$ to conclude that for any $z \in C^2(M; E)$ vanishing on $\{r \leq r_0\}$ and to infinite order at infinity we have

$$t^3 \int_M e^{2tr}|z|^2\,d\mu^g + t \int_M e^{2tr}|\nabla z|^2\,d\mu^g \leq C_0 \int_M e^{2tr}|\Delta z|^2\,d\mu^g. \tag{44}$$

Here it is assumed that t and r_0 are sufficiently large, and that C_0 does not depend on t. We now argue as in [19, Corollary 11] and set $z = \phi u$ where ϕ vanishes on $\{r \leq r_0\}$, and is equal to 1 on $\{r \geq r_0 + 1\}$. As a consequence of (44) combined with (43) we obtain

$$(t^3 - 2C_0 C^2) \int_{r_0+1}^\infty e^{2tr}|u|^2\,d\mu^g \leq C_0 \int_{r_0}^{r_0+1} e^{2tr}|\Delta z|^2\,d\mu^g.$$

When $t \to \infty$ the left hand side is at least of order $O(t^3 e^{2(r_0+1)t})$, whereas the right hand side has order $O(e^{2(r_0+1)t})$. Hence $u = 0$ on $\{r \geq r_0 + 1\}$. To conclude the proof, it suffices to note that u satisfies the conditions of the strong unique continuation theorem [17, Theorem 1.8], thus $u = 0$ on M. $\square$

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