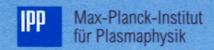Albert Salat

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Radial Dependence of Magnetohydrodynamic Continuum Modes in Axisymmetric Toroidal Geometry

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A characteristic feature of modes in the continua ("Alfvén" and "slow") of magnetohydrodynamics (MHD) is their singularity at a magnetic surface $\psi = \psi_0$. For axisymmetric toroidal configurations the dependence of continuum modes on the distance $\psi - \psi_0$ is reexamined. It is found that, in general, the normal component ξ_ψ of the plasma displacement has an oscillatory type of singularity: $\xi_\psi \sim (\psi - \psi_0)^\sigma \sim \sin(|\sigma| \ln |\psi - \psi_0| + \text{const})$, where σ is an imaginary constant. For special classes of MHD configurations σ vanishes. In this case the previously derived law $\xi_\psi \sim \ln |\psi - \psi_0|$, [Y. Pao, Nucl. Fusion **15**, 631 (1975)] remains valid. Configurations with up/down symmetry pertain to these classes.

1. Introduction

It is well known that the frequency spectrum of plasmas in the magnetohydrodynamic (MHD) description has both discrete and continuous parts [1], [2]. In axisymmetric geometry, to any frequency in the continuous part there corresponds a flux surface where the eigenmodes become singular [1], [3]. In three-dimensional geometry without symmetries the situation is far less clear [4], and comparably much less analytic work has been done. In the present investigation, however, we restrict our analysis completely to the axisymmetric case. Also, we consider finite toroidal mode numbers and thus exclude ballooning type modes.

The continuum plays a important role in linearized MHD. In addition to its fundamental aspect in the mathematical theory of MHD, it also has practical aspects. The localized accumulation of energy in the continuum modes can be utilized in heating schemes such as the so called Alfvén wave heating method [5]. From a more fundamental point of view, the continuum is of interest because it describes inherent plasma properties independent of external boundary conditions. Gaps in the continuous spectrum which open up in toroidal geometry are the preferred locus of global modes [2], [6] which can be dangerous for alpha-particle heating in nuclear fusion reactors. Also, global modes may experience "continuum damping" if they do collide with parts of the continuum. Observations of signals from the earth's magnetosphere hint at transfer of energy from compressional phenomena into shear Alfvén waves of the MHD continuum [7], [8]. Thus, the study of the MHD continuum is a worthwhile subject.

In axisymmetry, two classic papers on this topic have been published in the same year. One by Goedbloed [1] and a second by Pao [3]. In both papers a subset of linear differential equations, relevant for the properties of the "eigenmodes" *within* the magnetic surfaces, is derived. The derivatives in these equations are derivatives with respect to the path length, taken along the unperturbed magnetic field lines. The square of the mode frequency ω plays the role of an eigenvalue parameter. Periodicity conditions on the torus determine the eigenvalues ω_m^2, $m = 1, 2, \cdots$. The ω_m^2 depend continuously on the magnetic surface label. Both subsets, although written for different variables and with a different degree of elimination, are mutually equivalent.

Goedbloed [1] discusses in detail how toroidicity couples the Alfvén continuum and the slow continuum, which are decoupled in screw pinch or sheet pinch geometry, and he also discusses the occurrence of discrete modes. Furthermore, he shows that there exists a sequence of integrable functions in Hilbert space which satisfy the eigenmode equations better and better but which do not converge to an integrable function. He thus proves the existence of improper (hence the quotes) "eigenfunctions", which are the signature of a spectral continuum in the mathematical sense.

Pao [3] is interested not only in the spectrum and the properties within the resonant surface but also in the dependence of the continuum modes on the "radial" coordinate ψ, where the flux function ψ labels the pressure surfaces. He seeks approximate solutions, for *all* mode equations, valid in the proximity of the resonant surface at $\psi = \psi_0$, for frequencies ω taken from the continuum.

In *simple* geometries, such as the sheet pinch or the screw pinch, it is well known from the explicit solution of the equations in terms of Bessel functions, that the radial component of the displacement vector, to leading order, has a logarithmic dependence, while the component within the surface but orthogonal to the magnetic field, behaves as $1/(\psi - \psi_0)$. In Ref. [3] Pao claims that the logarithm and the inverse power law still describe the leading order behavior also in *arbitrary axisymmetric toroidal* geometry. The aim of the present paper is to show that this is not the case, in general. We find a radial dependence of the form $(\psi - \psi_0)^\sigma$ and $(\psi - \psi_0)^{\sigma-1}$, respectively, where σ is an imaginary constant that is generally non-zero. In special cases, with $\sigma = 0$, however, the results of Ref. [3] remain valid. The most important exceptions to $\sigma \neq 0$ are equilibria with up/down symmetry with respect to the "equatorial" mid-plane. Other exceptions are e.g. pressureless equilibria with purely poloidal magnetic fields. This case is relevant e.g. in magnetospheric physics, where it received and still receives much attention [9], [10]. Reference [3] is still one of the fundamental treatments of the subject and is quoted in many papers on Alfvén continua, see e.g. [8], [10], [11]. We therefore try to be as clear as possible in substantiating our claim.

In order to determine the leading order behavior of continuum modes, Hameiri devised a scheme [12] which differs from that applied in Refs. [1] and [3]. Also, in Ref. [12] it is mentioned in passing, without an explicit proof, that the result of Pao [3] (logarithmic and inverse dependence on the distance) can even be extended to the general three-dimensional equilibrium case. Not only in view of our present results but also in the light of the difficulties encountered in Refs. [4], with non-symmetric 3-D equilibria, we are skeptical about this remark.

The present paper is organized as follows. In the next section we present our general method, how to obtain the leading (and higher) order radial dependence. The method partly agrees with that used in Ref. [12]. A dimensionless exponent σ, see above, defined by averaging procedures around the torus, emerges as relevant for the radial dependence. In Sect. 3 this so far abstract quantity σ is expressed as much as possible in terms of equilibrium quantities. A discussion of special cases, those with $\sigma = 0$, follows in Sect. 4. Conclusions are given in Sect. 5.

Appendix A is a supplement to the case $\sigma = 0$. Problems that arise in derivations of the logarithm are pointed out and an improved derivation is presented. The insight

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gained here is also helpful in Appendix B, which is devoted to a discussion of Pao's work [3] in the light of the results of Sects. 2 and 3. In Appendix B, we attempt to determine the origin of the logarithmic solution that Pao found. In Appendix C the transition from a torus to a straight cylinder is investigated.

2. Radial dependence: general considerations

The linearized MHD equations [13] can be written in the form

$$-\omega^2 \rho \boldsymbol{\xi} = [\mathrm{curl}\,\mathbf{B} \times \mathbf{b}] + [\mathrm{curl}\,\mathbf{b} \times \mathbf{B}] - \nabla p \;, \tag{1}$$

$$\mathbf{b} = \mathrm{curl}\,[\boldsymbol{\xi} \times \mathbf{B}] \;, \tag{2}$$

$$p = -\boldsymbol{\xi} \cdot \nabla P - \gamma P \,\mathrm{div}\,\boldsymbol{\xi} \;, \tag{3}$$

where $\boldsymbol{\xi}$, $\mathbf{b}$ and p represent, respectively, the displacement vector, the magnetic field and the fluctuating pressure of the modes, and γ designates the ratio of specific heats. Capital letters denote equilibrium values. In addition, a harmonic time dependence of the form $\exp(i\omega t)$ with frequency ω has been assumed. The vacuum permeability μ_0 has been set to unity in order to stay with the convention adopted in Ref. [1].

Inspection of Eqs. (1) – (3) readily shows that some but not all components of the modes are differentiated in radial direction, out of the magnetic surfaces. We have in mind the axisymmetric case, where the flux surface label ψ is used as one of the coordinates, the toroidal angle ϕ is another, and a poloidal variable χ, which for the moment need not be specified, is the third coordinate. The equations (1) – (3), therefore, can be written in matrix form in the following way

$$\mathsf{A}\,\mathbf{v} = \mathsf{B}\,\mathbf{w}, \tag{4}$$

$$\frac{\partial}{\partial \psi}\,\mathbf{w} = \mathsf{C}\,\mathbf{v} + \mathsf{D}\,\mathbf{w}. \tag{5}$$

Here, A, B, C, D are matrix operators acting on the column vectors $\mathbf{v}$ and $\mathbf{w}$. The operators contain derivatives only within the magnetic surfaces. The perturbation p and the vector components of the perturbations $\mathbf{b}$ and $\boldsymbol{\xi}$ are subdivided into the two vectors $\mathbf{w}$ and $\mathbf{v}$, depending on whether they are differentiated with respect to ψ or not. With respect to the ϕ-dependence, the ansatz $\sim e^{in\phi}$ is made. As a result of this ansatz, $\mathbf{v}$ and $\mathbf{w}$ depend on ψ and χ alone, and the only differentiation operation that is left in A – D is with respect to χ.

For the present purposes, it is sufficient to know this basic structure of the governing wave equations. It is also advantageous to have this general framework in order to compare effectively our analysis with the analysis in Ref. [3], see Appendix B. In the next two sections this framework will be completed with explicit details.

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We anticipate that the singularity of "eigenmodes" in the continuum, at a resonant surface $\psi = \psi_0$, manifests itself by steep gradients of the modes in the vicinity of ψ_0. Let ϵ be a small number, $\epsilon \ll 1$, and let $y \equiv (\psi - \psi_0)/\epsilon$. In the following, y is chosen as the new radial variable in place of ψ, and the region in y considered is of order one. With $\partial w/\partial \psi = \epsilon^{-1} \partial w/\partial y$ a factor ϵ^{-1} is picked up on the left-hand side of Eq. (5). As a consequence, this equation can be satisfied non-trivially to lowest order, only provided w is smaller than v, order of magnitude wise, by a factor ϵ. This suggests a series ansatz for the modes in the form

$$\mathbf{v} \equiv \mathbf{v}^{(0)} + \epsilon \mathbf{v}^{(1)} + \epsilon^2 \mathbf{v}^{(2)} + \cdots = \mu_0(y)\mathbf{v}_0 + \epsilon \mu_1(y)\mathbf{v}_1 + \epsilon^2 \mu_2(y)\mathbf{v}_2 + \cdots, \tag{6}$$

$$\mathbf{w} \equiv \epsilon \left[\mathbf{w}^{(0)} + \epsilon \mathbf{w}^{(1)} + \epsilon^2 \mathbf{w}^{(2)} + \cdots \right] = \epsilon \left[\nu_0(y)\mathbf{w}_0 + \epsilon \nu_1(y)\mathbf{w}_1 + \epsilon^2 \nu_2(y)\mathbf{w}_2 + \cdots \right], \tag{7}$$

where the $\mathbf{v}_i$ and $\mathbf{w}_i$ are functions of χ alone. Finally, all equilibrium quantities, A say, are Taylor expanded with respect to ψ which implies an expansion in y of the form

$$A = A_0 + \epsilon y A_1 + \epsilon^2 y^2 A_2 + \cdots. \tag{8}$$

This notation is also transferred to the matrix operators A – D, where is is supposed to apply to each matrix element. The expansions (6) – (8) transform Eqs. (4) and (5) into

$$(\mathsf{A}_0 + \epsilon y \mathsf{A}_1)(\mathbf{v}^{(0)} + \epsilon \mathbf{v}^{(1)}) = (\mathsf{B}_0 + \epsilon y \mathsf{B}_1)\epsilon(\mathbf{w}^{(0)} + \epsilon \mathbf{w}^{(1)}) + \cdots, \tag{9}$$

$$\frac{\partial}{\partial y}(\mathbf{w}^{(0)} + \epsilon \mathbf{w}^{(1)}) = (\mathsf{C}_0 + \epsilon y \mathsf{C}_1)(\mathbf{v}^{(0)} + \epsilon \mathbf{v}^{(1)}) + (\mathsf{D}_0 + \epsilon y \mathsf{D}_1)\epsilon(\mathbf{w}^{(0)} + \epsilon \mathbf{w}^{(1)}) + \cdots. \tag{10}$$

To the lowest order in ϵ this implies

$$\mu_0 \mathsf{A}_0 \mathbf{v}_0 = 0, \tag{11}$$

$$\nu_0' \mathbf{w}_0 = \mu_0 \mathsf{C}_0 \mathbf{v}_0, \tag{12}$$

where the prime denotes differentiation with respect to y. To first order there results

$$\mu_1 \mathsf{A}_0 \mathbf{v}_1 + y \mu_0 \mathsf{A}_1 \mathbf{v}_0 = \nu_0 \mathsf{B}_0 \mathbf{w}_0, \tag{13}$$

$$\nu_1' \mathbf{w}_1 = \mu_1 \mathsf{C}_0 \mathbf{v}_1 + y \mu_0 \mathsf{C}_1 \mathbf{v}_0 + \nu_0 \mathsf{D}_0 \mathbf{w}_0. \tag{14}$$

From Eqs. (11) and (12) we find

$$\mathsf{A}_0 \mathbf{v}_0 = 0, \tag{15}$$

$$\nu_0' = \mu_0, \tag{16}$$

$$\mathbf{w}_0 = \mathsf{C}_0 \mathbf{v}_0. \tag{17}$$

Equation (15), together with periodic boundary conditions in the poloidal variable χ, determines the continuous part of the spectrum. Since Eq. (15) has nontrivial solutions,

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the inhomogeneous Eq. (13) for $\mathbf{v}_1(\chi)$ can have solutions only if a solvability condition is satisfied. The product of $\hat{\mathbf{v}}_0$, in some properly defined function space, see below, with the inhomogeneous terms must vanish, where $\hat{\mathbf{v}}_0$ is a solution of the adjoint equation $A_0^\dagger \hat{\mathbf{v}}_0 = 0$. If we denote the product with acute brackets and keep in mind that the operator A_0 is self-adjoint [1], [3], the solvability condition becomes

$$y\mu_0 < \mathbf{v}_0^\dagger A_1 \mathbf{v}_0 > = \nu_0 < \mathbf{v}_0^\dagger B_0 \mathbf{w}_0 > . \tag{18}$$

Here $\hat{\mathbf{v}}_0 = \mathbf{v}_0^*$ was taken, and $\mathbf{v}_0^\dagger$ is the transpose of $\mathbf{v}_0^*$. With Eq. (17) and the definition

$$\sigma \equiv \frac{< \mathbf{v}_0^\dagger B_0 C_0 \mathbf{v}_0 >}{< \mathbf{v}_0^\dagger A_1 \mathbf{v}_0 >} \tag{19}$$

there results

$$y\mu_0 = \sigma \, \nu_0. \tag{20}$$

From here onwards our treatment differs from that in Ref. [12]. If Eq. (20) is differentiated and ν_0' is taken from Eq. (16) there results

$$(y\mu_0)' = \sigma\mu_0. \tag{21}$$

Its solution is

$$\mu_0(y) = c_0 \, y^{\sigma-1}, \tag{22}$$

where c_0 is an arbitrary constant. From Eq. (16) there results

$$\nu_0(y) = c_{00} + \frac{c_0}{\sigma} y^\sigma, \qquad \text{for } \sigma \neq 0, \tag{23}$$

where c_{00} is another arbitrary constant, and

$$\nu_0(y) = c_{00} + c_0 \ln |y|, \qquad \text{for } \sigma = 0. \tag{24}$$

A remark concerning the case $\sigma = 0$ is appropriate here. In this case, Eq. (20) becomes

$$y\mu_0 = 0, \tag{25}$$

This condition is not satisfied by the solution (22), except for the trivial case $c_0 = 0$. Therefore, strictly speaking, the logarithm is not an acceptable solution for this expansion scheme. In Ref. [12] this problem was circumvented by an elegant *ad hoc* recipe. In Appendix A we discuss this point further and put the logarithm on a firm basis. In Appendix B we put Pao's treatment of the singularities [3] in parallel to the treatment above. We try to shed light on his at times somewhat unclear procedure.

In agreement with Ref. [12] the conclusion is reached that a logarithmic dependence on $\psi - \psi_0$ of some variables, together with a $(\psi - \psi_0)^{-1}$-dependence of others occurs only

provided the quantity σ, defined in Eq. (19), vanishes. The determination of σ requires an explicit representation of the operators $\mathsf{A} - \mathsf{C}$ and a consideration of the eigenvalue equation (15). This task is undertaken in the next two sections. As mentioned before, we find $\sigma \neq 0$, in general, and $\sigma = 0$ in special cases.

In the case $\sigma \neq 0$ the series ansatz (7), (8) for $\mathbf{v}$ and $\mathbf{w}$ can easily be solved to higher orders as well. It may suffice here to indicate the first steps. If Eq. (13) is differentiated with respect to y, and μ_0, ν_0 from Eqs. (22), (23) are inserted there results

$$\mu_1' \mathsf{A}_0 \mathbf{v}_1 + c_0 \sigma y^{\sigma-1} \mathsf{A}_1 \mathbf{v}_0 = c_0 y^{\sigma-1} \mathsf{B}_0 \mathbf{w}_0. \tag{26}$$

This implies

$$\mu_1(y) = c_{10} + \frac{1}{\sigma} c_0 y^\sigma, \tag{27}$$

where c_{10} is a constant, and

$$\mathbf{v}_1 = \mathsf{A}_0^{-1} \left\{ -\sigma \mathsf{A}_1 \mathbf{v}_0 + \mathsf{B}_0 \mathbf{w}_0 \right\}. \tag{28}$$

The inverse operator A_0^{-1} exists, since the solvability condition to Eq. (13) is satisfied. Equation (13), in its original form, shows that $c_{00} = c_{10} = 0$. The function $\mu_1(y)$ therefore, is given by $y\mu_0(y)$ (up to a constant factor, which may be absorbed in the χ-dependent part of $\mathbf{v}^{(1)}$. It can be shown that $\mu_i = y\mu_{i-1}$ holds in any order i, and analogously with ν_i.). Equation (14) for $\nu_1(y)$ and $\mathbf{w}_1(\chi)$ can also be solved in a straightforward manner. Thus, for $\sigma \neq 0$, the ansatz (7), (8) for $\mathbf{v}$ and $\mathbf{w}$ leads to a formal power series solution of the "eigenmode" equations, in the vicinity of the resonant surface.

3.   Evaluation of the exponent σ

In the previous section it was found that the dependence of the continuum modes on the radial variable is governed by a quantity σ which is defined in a rather abstract way, see Eq. (19). The present section is devoted to a more detailed evaluation of this quantity.

For the present task a definite coordinate system and a definite choice of the dependent quantities to work with is necessary. We adopt the treatment of the continuum in Ref. [1] for this purpose and we heavily rely on relations derived there. Nevertheless, a few remarks on the equilibrium, the coordinates and the representation of the modes are repeated here for convenience.

The coordinates (ψ, χ, ϕ) are chosen as in Sect. 2 above, except that now ψ denotes the poloidal magnetic flux, and the poloidal coordinate χ is specified by the orthogonality condition $\nabla\psi \cdot \nabla\chi = 0$. This turns the unit vectors $\mathbf{e}_\psi$, $\mathbf{e}_\chi$, $\mathbf{e}_\phi$ into a mutually orthogonal triplet, where $\mathbf{e}_i$ is defined by $\mathbf{e}_i = (\partial\mathbf{r}/\partial i)/|\partial\mathbf{r}/\partial i|$, $i = \psi, \chi, \phi$, and $\mathbf{r}$ is the coordinate vector.

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The magnetic field is represented in the form

$$\mathbf{B} = \frac{1}{R}(-\nabla\psi \times \mathbf{e}_\phi + I\mathbf{e}_\phi), \tag{29}$$

where $R = R(\psi, \chi)$ is the distance away from the axis of symmetry, $I \equiv RB_\phi = I(\psi)$ is an arbitrary flux function, and ψ, in cylindrical coordinates R, z, ϕ, is determined by

$$\left(R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2}\right)\psi(R, z) = -R^2 P'(\psi) - II'(\psi). \tag{30}$$

$P = P(\psi)$, the equilibrium pressure, is also an arbitrary flux function. In the new coordinates Eq. (30) is a relation between the Jacobian $J = 1/(\nabla\psi \cdot [\nabla\chi \times \nabla\phi])$, the poloidal magnetic field B_χ and the flux functions P and I,

$$P' + \frac{1}{J}(JB_\chi^2)' + \frac{1}{R^2}II' = 0. \tag{31}$$

The prime denotes (partial) differentiation with respect to ψ.

The linearized mode equations (1) – (3) are condensed in Ref. [1] into the matrix equation

$$\sum_j \mathcal{F}_{ij} X_j = -\rho\omega^2 \alpha_i X_i, \tag{32}$$

where $\mathbf{X} = (X, Y, Z)$ is defined by

$$\mathbf{X} = \left(JRB_\chi \xi_\psi, \; i\frac{B_\phi}{B_\chi}\frac{B_\phi\xi_\chi - B_\chi\xi_\phi}{B^2}, \; i\frac{B_\chi\xi_\chi + B_\phi\xi_\phi}{B^2}\right), \tag{33}$$

the matrix $\mathcal{F}_{ij}$ is given in the form

$$\mathcal{F}_{ij} = \begin{pmatrix} D\tilde{B}^2 D + a_{11} & DG\tilde{B}^2 + a_{12} & D\gamma PF \\ -\tilde{B}^2 GD + a_{21} & -G\gamma PG - B^2 G\frac{1}{B^2}GB^2 + a_{22} & -G\gamma PF \\ -F\gamma PD & -F\gamma PG & -F\gamma PF \end{pmatrix}, \tag{34}$$

where

$$\tilde{B}^2 = \gamma P + B^2, \tag{35}$$

and

$$a_{11} = -\frac{1}{J}F\frac{1}{R^2 B_\chi^2}F\frac{1}{J} - \frac{2}{J^2}(\tau' + q\lambda'), \qquad a_{12} = -\frac{2}{IJ}\left(\tau n + \lambda i\frac{\partial}{\partial\chi}\right)B^2, \tag{36}$$

$$a_{21} = -\frac{2}{IJ}B^2\left(\tau n + \frac{\partial}{\partial\chi}\lambda i\right), \qquad a_{22} = -B^2 F\frac{B_\chi^2}{B_\phi^2 B^2}FB^2, \tag{37}$$

and

$$\alpha_1 = \frac{1}{J^2 R^2 B_\chi^2}, \qquad \alpha_2 = \frac{B_\chi^2 B^2}{B_\phi^2}, \qquad \alpha_3 = B^2, \tag{38}$$

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$$\tau = \frac{B_\chi}{J}(JB_\chi)', \qquad \lambda = \frac{B_\phi}{J}R'. \tag{39}$$

Furthermore, D is defined by $D = \frac{1}{J}\frac{\partial}{\partial\psi}$, and the definitions of F and G are

$$F = -\frac{i}{J}\frac{\partial}{\partial\chi} + \frac{nq}{J}, \qquad G = -\frac{i}{J}\frac{\partial}{\partial\chi} - \frac{nB_\chi^2}{I}. \tag{40}$$

The, locally defined, safety factor q is

$$q = \frac{JB_\phi}{R}. \tag{41}$$

In order to obtain a first-order system with respect to the ψ-derivatives it is advantageous to introduce a function $S(\psi,\chi)$ by the definition

$$S \equiv \tilde{B}^2 DX + G\tilde{B}^2 Y + \gamma PFZ. \tag{42}$$

Equations (32) – (34) can then be written in the standard form of Eqs. (4), (5),

$$\mathsf{A}\begin{pmatrix} Y \\ Z \end{pmatrix} = \mathsf{B}\begin{pmatrix} X \\ S \end{pmatrix} \tag{43}$$

$$\frac{\partial}{\partial\psi}\begin{pmatrix} X \\ S \end{pmatrix} = \mathsf{C}\begin{pmatrix} Y \\ Z \end{pmatrix} + \mathsf{D}\begin{pmatrix} X \\ S \end{pmatrix}, \tag{44}$$

where

$$\mathsf{A} = \begin{pmatrix} A_{11} & -\tilde{B}^2 G\dfrac{\gamma P}{\tilde{B}^2}F + G\gamma PF \\[2ex] -F\dfrac{\gamma P}{\tilde{B}^2}G\tilde{B}^2 + F\gamma PG & -F\gamma P\dfrac{\gamma P}{\tilde{B}^2}F + F\gamma PF - \rho\omega^2\alpha_3 \end{pmatrix}, \tag{45}$$

with

$$A_{11} = -\tilde{B}^2 G\frac{1}{\tilde{B}^2}G\tilde{B}^2 + B^2 G\frac{1}{B^2}GB^2 + G\gamma PG - a_{22} - \rho\omega^2\alpha_2, \tag{46}$$

$$\mathsf{B} = \begin{pmatrix} a_{21} & -\tilde{B}^2 G\dfrac{1}{\tilde{B}^2} \\[2ex] 0 & -F\dfrac{\gamma P}{\tilde{B}^2} \end{pmatrix}, \qquad \mathsf{C} = J\begin{pmatrix} \dfrac{-1}{\tilde{B}^2}G\tilde{B}^2 & \dfrac{-\gamma P}{\tilde{B}^2}F \\[2ex] -a_{12} & 0 \end{pmatrix}, \tag{47}$$

$$\mathsf{D} = J\begin{pmatrix} 0 & \dfrac{1}{\tilde{B}^2} \\[2ex] -a_{11} - \rho\omega^2\alpha_1 & 0 \end{pmatrix}. \tag{48}$$

Equations (43) and (44) define explicitly the elements of the column vectors $\mathbf{v} = (Y, Z)^T$ and $\mathbf{w} = (X, S)^T$. Note that in all equations derivatives with respect to χ operate on all quantities to their right, except where indicated below.

The matrix A may be simplified by applying the definitions (35) – (40). The result is

$$\mathsf{A} = \begin{pmatrix} B^2 F \dfrac{B_\chi^2}{B_\phi^2 B^2} FB^2 + \dfrac{\gamma P}{\tilde{B}^2}\left(\dfrac{\partial_\chi B^2}{JB}\right)^2 - \rho\omega^2\alpha_2 & -\dfrac{i}{J}\dfrac{\gamma P}{\tilde{B}^2}(\partial_\chi B^2)F \\[3mm] F\dfrac{\gamma P}{\tilde{B}^2}\dfrac{i}{J}(\partial_\chi B^2) & F\dfrac{\gamma P}{\tilde{B}^2}B^2 F - \rho\omega^2\alpha_3 \end{pmatrix}. \tag{49}$$

Here, the χ-derivatives in front of B^2 and enclosed in brackets do not extend beyond the brackets. In our derivation, the continuous spectrum is determined by Eq. (15), $\mathsf{A}_0\mathbf{v}_0 = 0$. If $\mathbf{v}_0$, see the ansatz (7), is written in components, the continuum equation becomes

$$\mathsf{A}_0 \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} = 0. \tag{50}$$

Equation (40) in Ref. [1] serves the same purpose there. A comparison with the operator A, above, shows complete agreement (up to a typing error in Ref. [1]).

The fact that the operator A, for periodic boundary conditions, is self-adjoint can be checked easily, by partial integrations, if the operator $<>$ in function space is defined by

$$< a > \equiv \oint a J\, d\chi. \tag{51}$$

The index σ, Eq. (19), which determines the radial dependence of the modes has the form $\sigma = \beta/\alpha$, with

$$\beta = < \mathbf{v}_0^\dagger \mathsf{B}_0 \mathsf{C}_0 \mathbf{v}_0 >, \qquad \alpha = < \mathbf{v}_0^\dagger \mathsf{A}_1 \mathbf{v}_0 > . \tag{52}$$

β uses the product BC,

$$\mathsf{BC} = \begin{pmatrix} -a_{21}\dfrac{J}{\tilde{B}^2}G\tilde{B}^2 + \tilde{B}^2 G\dfrac{1}{\tilde{B}^2}Ja_{12} & -a_{21}J\dfrac{\gamma P}{\tilde{B}^2}F \\[3mm] F\dfrac{\gamma P}{\tilde{B}^2}Ja_{12} & 0 \end{pmatrix}. \tag{53}$$

With repeated partial integrations it is straightforward, albeit tedious, to bring β into a reasonably condensed form:

$$\begin{aligned}
\beta &= 2inI < \frac{1}{J B^4}\left\{\tilde{B}^4\partial_\chi\left(\frac{B^4}{\tilde{B}^4}\frac{B_\chi}{B^2 R}\kappa_p\right) - B^4\partial_\chi\left(\frac{B_\chi}{B^2 R}\kappa_t\right)\right\}|\hat{Y}_0|^2 > \\
&\quad + 4i\gamma PI^2\,\Im\, < \frac{1}{J\tilde{B}^2}\frac{1}{JRB_\chi}\frac{(\partial_\chi B^2)}{B^4}\kappa_t\,(\partial_\chi\hat{Y}_0^*)\hat{Y}_0 > \\
&\quad - 4i\gamma PI\,\Im\, < \frac{1}{J\tilde{B}^2}\frac{1}{JRB_\chi}\kappa_t\,(\partial_\chi\hat{Y}_0^*)\,(\partial_\chi + inq)\,Z_0 > \\
&\quad - 4in\gamma P\,\Re\, < \frac{1}{J\tilde{B}^2}\frac{B_\chi}{R}\kappa_p\,\hat{Y}_0^*\,(\partial_\chi + inq)\,Z_0 >,
\end{aligned} \tag{54}$$

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where $\Re$ and $\Im$ denote the real and imaginary parts, $\partial_\chi = \partial/\partial\chi$ and

$$\widehat{Y}_0 \equiv \frac{B_0^2}{I_0} Y_0. \tag{55}$$

While the factor B^2, in this definition, is for convenience, the denominator I ensures that $\widehat{Y}_0$ stays finite in the limit of purely poloidal fields, i.e. for $B_\phi = I/R \to 0$, see the second component of $\mathbf{X}$ in Eq. (33). For equilibrium quantities, the index zero has been omitted for convenience of notation. The poloidal and toroidal curvatures, $\kappa_p = \mathbf{e}_\chi \cdot \nabla \mathbf{e}_\psi$ and $\kappa_t = \mathbf{e}_\phi \cdot \nabla \mathbf{e}_\psi$, are given by [1]

$$\kappa_p = \frac{R}{J}(JB_\chi)' \, , \qquad \kappa_t = B_\chi R'. \tag{56}$$

In Eq. (54), the χ-derivatives end after the nearest closing bracket, except in the last two terms, where they still operate on Z_0.

It is instructive to consider a few special cases. In the pressureless case, $P = 0$, the first term in β simplifies, owing to $\widetilde{B}^2 = B^2$. while the remaining terms vanish. The result is

$$\beta(P = 0) \;=\; 2inI < \frac{1}{J} |\widehat{Y}_0|^2 \, \partial_\chi \left[\frac{B_\chi}{B^2 R}(\kappa_p - \kappa_t) \right] > . \tag{57}$$

If the magnetic field is purely poloidal, $I = 0$, as e.g. approximately realized in the earth's magnetosphere, it holds that

$$\beta(I = 0) \;=\; -4in\gamma P \, \Re < \frac{1}{J\widetilde{B}^2} \frac{B_\chi}{R} \kappa_p \, \widehat{Y}_0^* \partial_\chi Z_0 > . \tag{58}$$

The denominator α of σ is nonzero, in general [3]. It is obtained from

$$\alpha = \left. \frac{\partial A}{\partial \psi} \right|_{\psi = \psi_0}, \tag{59}$$

where $A(\psi) \equiv < \mathbf{v}^\dagger \mathbf{A} \mathbf{v} >$. This follows from the definition of $\mathbf{A}_1$, and the fact that $\mathbf{A}\mathbf{v}_0 = 0$ at $\psi = \psi_0$. It is straightforward, using partial integration and periodicity conditions for A and $\mathbf{v}$, to put A into the form

$$A = < \frac{B_\chi^2 R^2}{J^2 B^2} \left| (\partial_\chi + inq)\widehat{Y} \right|^2 + \frac{1}{J^2 B^2} \frac{\gamma P}{\widetilde{B}^2} \frac{1}{B^4} \left| I(\partial_\chi B^2)\widehat{Y} - B^4(\partial_\chi + inq)Z \right|^2 >$$
$$- \omega^2 < \rho \left[\frac{B_\chi^2 R^2}{B^2} |\widehat{Y}|^2 + B^2 |Z|^2 \right] >, \tag{60}$$

where $\widehat{Y} = B^2 Y/I$. Manifestly, $A(\psi)$ is real. So then is α. Note that consequently σ is purely imaginary, if it does not vanish.

Equation (60) also proves that α stays finite in the poloidal limit, $I \to 0$, when expressed in terms of $\widehat{Y}_0$ and Z_0.

There is no evidence whatsoever, in the general case, that relations originating from the equilibrium conditions or from $A_0 v_0 = 0$ would bring σ *identically* to zero. For particular types of configurations, however, $\sigma = 0$ indeed holds. This is discussed in the next section.

4. Configurations with $\sigma = 0$

From Eqs. (57) and (58) it is obvious that the exponent σ vanishes identically under particular conditions on the equilibrium configurations. This is so for pressureless configurations ($P = 0$), provided the magnetic field is purely poloidal ($I = 0$) or provided the perturbation is axisymmetric (toroidal mode number $n = 0$). The case of a pressureless, purely poloidal magnetic field configuration has been treated carefully also in Ref. [10]. In agreement with our result $\sigma = 0$ a logarithmic dependence was obtained. For the modes, an extended series solution, analogous to Eqs. (A.1), (A.2), was derived.

Equation (58) shows that, for finite pressure, a purely poloidal field with an axisymmetric perturbation also brings about $\sigma = 0$.

There exists another class of configurations with $\sigma = 0$ which might be more relevant with respect to applications. It is the class of up/down symmetric configurations, i.e. configurations which are symmetric with respect to the "equatorial" plane, at $\chi = 0$, say.

The proof is based on the fact that the numerator β of σ is proportional to an integral around the poloidal cross section and on symmetry properties of Y and Z that can be inferred from Eqs. (49) and (50). It is easily seen that the operator $A^*(\chi)$, acting on $v_0^*(\chi) = (Y_0^*, Z_0^*)^T$, is the same as the operator $A(-\chi)$, acting on $v_0(-\chi)$. As a consequence, it holds that

$$v_0(-\chi) = f v_0^*(\chi), \tag{61}$$

where f is a complex constant. From the continuity of v_0 at $\chi = 0$ it follows that

$$|f|^2 = 1. \tag{62}$$

Equation (61) holds componentwise for Y_0 and Z_0. From these relations it is straightforward, with a bit of algebra, to see that the integrand in all four terms of Eq. (54) is antisymmetric with respect to χ. This proves $\beta = \sigma = 0$, and hence the logarithmic dependence of ξ_ψ on the distance $\psi - \psi_0$.

One might ask whether σ goes to zero for *arbitrary* equilibrium configurations if the torus is opened up into a straight cylinder, i.e. in the limit $R_0 \to \infty$, where R_0 is the average major radius. It can be shown that, in general, this is not the case, see Appendix C. The exponent σ vanishes only in configurations which are the axial analogues of the

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toroidal equilibria discussed above in this Section. The only difference is that $\sigma = 0$ holds in arbitrary cylindrical equilibria provided the modes are constant along the cylinder axis (n=0). The fact that continuum modes in cylindrical plasmas with non symmetric cross section behave differently from those in symmetric configurations, does not seem to be widely known.

5. Conclusions

It was shown in the previous sections that in axisymmetric toroidal MHD equilibria the radial dependence of modes in the spectral continuum is, generically, of the form $(\psi - \psi_0)^\sigma$ or $(\psi - \psi_0)^{\sigma-1}$, where σ is a purely imaginary number. Expressed in real form, this corresponds to $\sin[|\sigma| \ln|\psi - \psi_0| + \text{const}]$ or $\sin[|\sigma| \ln|\psi - \psi_0| + \text{const}]/(\psi - \psi_0)$. This describes an *oscillatory* behavior whose period shrinks more and more, in coming closer to the singular magnetic surface $\psi = \psi_0$. This radial dependence does not agree with the previously obtained monotonic form $\ln|\psi - \psi_0|$ or $1/(\psi - \psi_0)$, [3], which is (implicitly) claimed to hold for *arbitrary* axisymmetric MHD configurations. We think that this claim [3] is not substantiated enough, see Appendix B.

The impact of this difference on practical applications may be small, in particular, in view of commonly encountered cases where Pao's result remains true. The qualitative difference, however, concerning a fundamental aspect of the MHD continuum, is substantial.

Under particular circumstances, our result, Eq. (54), leads to the special value $\sigma = 0$. In these cases the results of Ref. [3], with the logarithmic space dependence of the normal component of the displacement ξ remain valid. The most prominent such cases are configurations with up/down symmetry. Similarly, $\sigma = 0$ holds provided any two out of the following three conditions are satisfied simultaneously: (I) The magnetic field has no toroidal component. (II) The MHD equilibrium is pressureless. (III) The plasma perturbation is axisymmetric.

The transition from the oscillatory behavior to the logarithmic behavior, when a configuration with $\sigma \neq 0$ is deformed into a configuration with vanishing σ is a gradual one. The radial zone of oscillations shrinks more and more towards the singular surface.

Acknowledgment

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Appendix A: series expansion in the case $\sigma = 0$

It has been realized repeatedly [10], [14], [15] that in the presence of logarithmic terms "eigenfunctions" require a more general ansatz than Eqs. (7), (8) in order to allow a self-consistent solution. Such an ansatz which takes into account Eq. (16) and the relations between μ_i, ν_i and μ_{i-1}, ν_{i-1} from Sect. 2 is

$$\mathbf{v} = \nu'(y)[\mathbf{v}_0 + \epsilon y \mathbf{v}_1 + \cdots] + \epsilon \nu(y)[\mathbf{k}_0 + \epsilon y \mathbf{k}_1 + \cdots], \tag{A.1}$$

$$\mathbf{w} = \epsilon \nu(y)[\mathbf{w}_0 + \epsilon y \mathbf{w}_1 + \cdots] + \nu'(y)[\epsilon y \mathbf{l}_1 + \epsilon^2 y^2 \mathbf{l}_2 + \cdots]. \tag{A.2}$$

Here, the functions $\mathbf{v}_i$, $\mathbf{w}_i$, $\mathbf{k}_i$ and $\mathbf{l}_i$ are functions of χ, and the first part in Eqs. (A.1) and (A.2), respectively, corresponds to the ansatz (7) and (8). To lowest order in ϵ one obtains again Eq. (15), and the relation

$$\nu' \mathbf{w}_0 + (y\nu')' \mathbf{l}_1 = \nu' \mathsf{C}_0 \mathbf{v}_0. \tag{A.3}$$

Since the dependence on y has to be the same in all terms it follows that

$$(y\nu')' = c\nu' \tag{A.4}$$

with an arbitrary constant c. There are two types of solution. For $c = 0$ there results

$$\nu(y) = \ln |y|, \tag{A.5}$$

while for $c \neq 0$, the solution is

$$\nu(y) = \frac{1}{c} y^c. \tag{A.6}$$

The case with $c = 0$ is the interesting one. In this case $(y\nu')' = 0$, and the term with $\mathbf{l}_1$ drops out of Eq. (A.3). It thus goes over into Eq. (17). Note that an equation $y\nu' = 0$, analogous to Eq. (25), does not occur here. This is a consequence of the series ansatz made in Eqs. (A.1), (A.2), which for $\nu(y) = \ln |y|$ is more general than Eqs. (7) and (8). To the next order one obtains

$$\mathsf{A}_0(y\nu' \mathbf{v}_1 + \nu \mathbf{k}_0) + y\mathsf{A}_1 \nu' \mathbf{v}_0 = \mathsf{B}_0(\nu \mathbf{w}_0 + y\nu' \mathbf{l}_1). \tag{A.7}$$

The terms with $y\nu'$ and those with ν have to vanish separately. There results

$$\mathsf{A}_0 \mathbf{v}_1 + \mathsf{A}_1 \mathbf{v}_0 = \mathsf{B}_0 \mathbf{l}_1, \tag{A.8}$$

$$\mathsf{A}_0 \mathbf{k}_0 = \mathsf{B}_0 \mathbf{w}_0. \tag{A.9}$$

To both equations a solvability condition has to be considered. With Eq. (15) there results

$$< \mathbf{v}_0^\dagger \mathsf{A}_1 \mathbf{v}_0 > = < \mathbf{v}_0^\dagger \mathsf{B}_0 \mathbf{l}_1 >, \tag{A.10}$$

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$$0 = < \mathbf{v}_0^\dagger \mathsf{B}_0 \mathbf{w}_0 > = < \mathbf{v}_0^\dagger \mathsf{B}_0 \mathsf{C}_0 \mathbf{v}_0 > . \tag{A.11}$$

The first equation merely fixes a weighted mean value of the function l_1. Equation (A.11), however, is equivalent to the condition $\sigma = 0$, see Eq. (19). Thus the case $\sigma = 0$ is indeed compatible with a logarithmic solution for $\mathbf{v}$, as we wanted to show.

$\mathbf{v}_1$ and $\mathbf{k}_0$ can now be obtained from Eqs. (A.8) and (A.9). The higher order terms follow similarly. If, however, the power law solution (A.6) with $c \neq 0$, instead of the logarithmic one, is inserted into Eq. (A.7), it turns out that $\sigma = 0$ leads to a contradiction. This proves that $\sigma = 0$ is both, necessary and sufficient for a logarithmic dependence to occur (within the ansatzes made).

Appendix B: comparison with Pao

The aim of this Appendix is to compare the treatment of Pao in Ref. [3] with our analysis. We begin by associating the key equations in Ref. [3], which we mark here with a capital P, with ours. In so doing, we will condense the component-wise notation of Ref. [3] into a convenient vector notation. Pao's equations (P11) – (P14) and (P15) – (P16) are represented by our Eqs. (4) and (5), respectively. The components of the vector $\mathbf{v}$ are $(u_\theta, u_\chi, b_\theta, b_\chi)$, while those of $\mathbf{w}$ are (ξ, τ). In order to distinguish variables in Ref. [3] from the corresponding variables used here, Pao's variables will be labeled henceforth with a subscript P. His vector $\mathbf{g}$ with components (g_1, g_2, g_3, g_4) that Eqs. (P19) – (P22) specify is given by

$$\mathbf{g} \equiv \mathsf{B}\mathbf{w}_P. \tag{B.1}$$

Equations (P23) – (P26), which define the eigenfrequencies ω and the associated singular surface $\psi_0(\omega)$, are represented by the vector equation

$$\mathsf{A}_0 \mathbf{v}_P = 0. \tag{B.2}$$

Solutions of this equation are designated $\mathbf{v}_{P0}$ analogously to the notation adopted in Ref. [3]. Equation (4) is an inhomogeneous partial differential equation that relates $\mathbf{v}$ to $\mathbf{w}$. A solution for $\mathbf{v}$ exists only if a solvability condition is satisfied. The solvability condition Eq. (P31) that Pao introduces is,

$$< \mathbf{v}_{P0}^\dagger \mathbf{g} > \equiv < \mathbf{v}_{P0}^\dagger \mathsf{B}\mathbf{w}_P > = 0. \tag{B.3}$$

Equation (B.3), however, lacks consistency as it stands because $\mathbf{v}_{P0}$ is defined only on the singular surface $\psi = \psi_0$, whereas Eq. (4) is defined on a general magnetic surface labeled by ψ.

Pao next makes an ansatz for the forms of $\mathbf{v}_P$ and $\mathbf{w}_P$ in the vicinity of the singular magnetic surface with Eqs. (P35) – (P38),

$$\mathbf{v}_P = \lambda'(y)[\mathbf{v}_{P0} + \epsilon y \mathbf{v}_{P1} + \cdots], \tag{B.4}$$

$$\mathbf{w}_P = \lambda(y)[\mathbf{w}_{P0} + \epsilon y \mathbf{w}_{P1} + \cdots], \tag{B.5}$$

which are power series in the radial coordinate $y = \psi - \psi_0$ multiplied by the factors λ and λ'. The leading coefficient of Eq. (B.4), $\mathbf{v}_{P0}$, is a solution of Eq. (B.2). These expansions are equivalent to the first parts of Eqs. (A.1) and (A.2), respectively, and also to Eqs. (7) and (8). It should be noted that the role of $\nu(y)$ is played by $\lambda(y)$, and that the expansion factor ϵ in Eq. (A.2) has been omitted. However, the relation $\lambda/\lambda' \to 0$ as $y \to 0$ is assumed to be valid.

The expansion coefficients are derived by substituting Eqs. (B.4) and (B.5) in Eqs. (4) and (5). To lowest order, Pao finds

$$A_0 \mathbf{v}_{P0} = 0, \tag{B.6}$$

which is Eq. (B.2). To the next order, Pao obtains his equations (P39) – (P42), which in our notation are represented by the vector equation

$$y\lambda' A_0 \mathbf{v}_{P1} = -y\lambda' A_1 \mathbf{v}_{P0} + B_0 \lambda \mathbf{w}_{P0}. \tag{B.7}$$

where it is recalled that $\mu_1 = y\mu_0$ and $\mu_0 = \nu_0'$. Equation (B.7) is equivalent to Eq. (13). Pao actually expresses Eq. (B.7) in a slightly different form. Specifically, he writes

$$y\lambda' A_0 \mathbf{v}_{P1} = y\lambda' \Lambda + \mathbf{g}, \tag{B.8}$$

where $\Lambda(\mathbf{v}_{P0}) \equiv -A_1 \mathbf{v}_{P0}$, and $\mathbf{g}$ is defined by Eq. (B.1). At this point, Pao introduces a critical step. Without a clear explanation, he differentiates Eq. (B.7) with respect to y, and then adopts the resulting inhomogeneous equation as the governing equation for $\mathbf{v}_{P1}$. The solvability condition that results from Eq. (B.6) is

$$\lambda' < \mathbf{v}_{P0}^\dagger B_0 \mathbf{w}_{P0} >= (y\lambda')' < \mathbf{v}_{P0}^\dagger A_1 \mathbf{v}_{P0} >, \tag{B.9}$$

or, in the notation of Pao,

$$< \mathbf{v}_{P0}^\dagger \mathbf{g}' >= (y\lambda')' G, \tag{B.10}$$

where

$$G \equiv < \mathbf{v}_{P0}^\dagger A_1 \mathbf{v}_{P0} > . \tag{B.11}$$

The left-hand side of Eq. (B.10), to lowest order, is evaluated with Eqs. (P15) and (P16), which correspond to Eq. (5). This leads to Eqs. (16) and (17). Translated into Pao's notation, these equations imply

$$\mathbf{w}_{P0} = C_0 \mathbf{v}_{P0}. \tag{B.12}$$

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After substitution of Eq. (B.12), Eq. (B.9) becomes

$$\lambda' < \mathbf{v}_{P0}^\dagger \mathbf{B}_0 \mathbf{C}_0 \mathbf{v}_{P0} >= (y\lambda')' < \mathbf{v}_{P0}^\dagger \mathbf{A}_1 \mathbf{v}_{P0} > . \qquad (B.13)$$

If $\lambda'(y)$ is replaced by $\mu_0(y)$, it becomes evident that Eq. (B.13) is identical to Eq. (21), with σ defined in Eq. (19). Pao does not present an analogue to Eq. (B.13). He simply states, "It turns out that, to this order, the integral on the left-hand side of ([P]43) vanishes, yielding the result $[y\lambda'(y)]'G = 0$." In other words, he claims without direct proof that σ (in our notation) vanishes for *arbitrary* axisymmetric MHD configurations, and that $\lambda(y) \sim \ln|y|$. Pao provides no explanation why Eq. (B.7) has to be differentiated. Had he not differentiated, he would have obtained the following solvability condition instead of Eq. (B.9),

$$\lambda < \mathbf{v}_{P0}^\dagger \mathbf{B}_0 \mathbf{w}_{P0} >= (y\lambda') < \mathbf{v}_{P0}^\dagger \mathbf{A}_1 \mathbf{v}_{P0} > . \qquad (B.14)$$

If the left-hand side of Eq. (B.14) were to vanish, as Pao claims it does, the following condition would emerge,

$$(y\lambda') < \mathbf{v}_{P0}^\dagger \mathbf{A}_1 \mathbf{v}_{P0} >= 0. \qquad (B.15)$$

Equation (B.15) yields $y\lambda' = 0$, which is analogous to Eq. (25). The implications are that $\lambda'(y)$ is proportional to the delta function $\delta(y)$, and therefore that $\lambda(y)$ is proportional to the step function. Equation $y\lambda' = 0$ and the ensuing appearance of distributional solutions is related to the form of the series ansatz made in Eqs. (B.4) and (B.5), as discussed in Appendix A.

Appendix C: transition to straight cylinder

Let R_0 be the average major radius of the toroidal configuration. The aim here is to discuss the behavior of the exponent σ in the limit $R_0 \to \infty$. It is assumed that during this transition the local equilibrium quantities should not change qualitatively. The poloidal magnetic flux *per unit length* along the toroidal circumference, for example, should not change its order of magnitude, for $R_0 \to \infty$. Similarly, the number of zeros per unit length in toroidal direction, of the modes should stay constant. The magnetic field and the displacement $\boldsymbol{\xi}$ should also retain their order of magnitude.

With these assumptions, with Eqs. (29) – (31), and Eqs. (41), (56), it follows that the asymptotic scaling with R_0 is as follows

$$\psi \sim R_0, \quad I \sim R_0, \quad J \sim 1, \quad q \sim R_0^{-1}, \quad n \sim R_0, \quad \kappa_p \sim 1, \quad \kappa_t \sim R_0^{-1}. \qquad (C.1)$$

The product nq scales ~ 1. For further relevant quantities and operators there results, see Eqs. (37), (38) and (40),

$$F \sim G \sim 1, \quad \alpha_2 \sim \alpha_3 \sim 1, \quad \mathbf{A} \sim 1. \qquad (C.2)$$

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The modes obey, see Eqs. (33) and (55),

$$X \sim R_0, \quad Y \sim Z \sim 1, \quad \hat{Y} \sim R_0^{-1}. \tag{C.3}$$

With these relations, it turns out that the numerator, β, of σ, Eq. (54), scales as R_0^{-1} as R_0 tends towards infinity. The effect of the decreasing toroidal curvature κ_t, in comparison to κ_p, is that all terms in β with a factor κ_t can be neglected. Only the first part of the first term, and the fourth term, in Eq. (54) survive. From Eqs. (59) and (60) it follows that the denominator α, owing to the ψ-derivative, also scales as R_0^{-1}. Hence, σ scales as ~ 1. A vanishing value of σ, therefore, relies again on special properties of the configuration which cut down the value of the poloidal integrals, or on situations with $nI = nP = 0$. As in the toroidal case, symmetry with respect to a mid-plane is sufficient for $\sigma = 0$.

We note in passing an exception to the rule $\sigma = 0$ for symmetric straight equilibria. This is an incompressible plasma, i.e. one with $\gamma = \infty$, with a circular cross section. In this limit the relation $\mathbf{A}_0 \mathbf{v}_0 = 0$ for a straight plasma degenerates into the equations $A_{11} Y_0 = 0$ and $A_{11} Z_0 = 0$. Thus, Y_0 and Z_0 are determined by the *same* equation. This permits Y_0 and Z_0 to be non-zero simultaneously. In analogy to Eq. (61) the symmetry properties are

$$Y_0(-\chi) = f_y Y_0^*(\chi), \qquad Z_0(-\chi) = f_z Z_0^*(\chi), \tag{C.4}$$

where f_y and f_z are arbitrary constants that satisfy the constraints $|f_y|^2 = |f_z|^2 = 1$. This permits the choice $f_y f_z^* = -1$. As a consequence, the fourth term of β in Eq. (54) is nonzero since with this choice the integrand is now *symmetric* with respect to the angle χ. The result, $\sigma \neq 0$, agrees with a straightforward analysis of the modes in an incompressible circular plasma cylinder. The radial displacement X_r obeys an ordinary second-order differential equation with respect to the radial variable r. In the vicinity of a singular surface $r = r_0$ a power law of the form $(r - r_0)^\sigma$ with an imaginary exponent σ is found [16].

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