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The existence of orthogonal, field aligned coordinate systems, i.e. coordinate systems with one coordinate direction always parallel to the local magnetic field, is critically reviewed. Such coordinates are widely used e.g. in wave propagation studies of magnetospheric plasmas and would as well be useful in theoretical studies of fusion plasma configurations, particularly of configurations without spatial symmetry. With the help of simple model magnetic configurations it is demonstrated that the presence of local magnetic shear precludes in general the existence of such coordinates.

1. Introduction

In the investigation of any problem of plasma physics, such as, for example, the propagation of waves in a magnetized plasma, a suitable coordinate system has to be chosen. Only in the simplest cases will Cartesian coordinates be adequate. Symmetries in the configuration considered may suggest the use of cylindrical coordinates, etc. If no symmetries are present, and if one wants to keep all options open, it is possible to work with a system of yet unspecified coordinates and to use covariant notation. For this freedom, however, one has to pay with the ubiquitous co- and contra-variant metric coefficients $g_{ij} = \partial_i \mathbf{r} \cdot \partial_j \mathbf{r}$ and $g^{ij} = \nabla r^i \cdot \nabla r^j$, respectively, where $\mathbf{r}$ is the coordinate vector with components r^1, r^2, r^3 and $\partial_i = \partial/\partial r^i$, for $i = 1, 2, 3$. Between these extremes of trivial and of general coordinates there exist several classes of coordinate systems which are adapted in some way to the physics and/or to the topology involved in the problem. In the case of magnetized plasmas such coordinate systems are commonly based on the magnetic field structure and may become rather elaborate [1].

In fusion plasma physics it is often assumed that the magnetic field lines are confined to closed nested toroidal surfaces, the so called "magnetic surfaces" or "flux surfaces", $\psi(\mathbf{r}) =$ const, which, in the magnetohydrodynamic (MHD) plasma model, are also surfaces of constant pressure. The variable $\psi(\mathbf{r})$ is then used as one of the coordinates. The two other coordinates are a poloidal angle-like coordinate and a toroidal angle-like coordinate on the flux surfaces, which can be chosen in such a way that the lines of the magnetic field become straight lines. Various options on these "flux coordinates" can in addition be satisfied, which give rise to specific coordinate systems such as the Boozer-Grad coordinates or the Hamada coordinates [1].

In the following we will focus our attention on field line based coordinates rather than on flux based coordinates. In field line coordinate systems, the intersections of two families of coordinate surfaces trace out magnetic field lines. Let the intersecting coordinate surfaces be $\alpha(\mathbf{r}) =$ const and $\beta(\mathbf{r}) =$ const. The magnetic field $\mathbf{B}$ then has the so-called Euler or Clebsch representation

$$\mathbf{B} = f(\alpha, \beta)\, [\nabla\alpha \times \nabla\beta], \tag{1}$$

where $f(\alpha, \beta)$ is a scalar function of α and β. The coordinates (α, β), in conjunction with a third coordinate $\gamma(\mathbf{r})$, which for example could be the distance along the field lines, are well known and documented in the literature [1], [2].

A class of field aligned coordinates that is particularly popular in studies of waves and resonances in magnetospheric plasmas is characterized by coordinate surfaces that are also orthogonal to each other, see e.g. [3] – [13]. Unfortunately, such orthogonal, field aligned coordinates, denoted here as OF coordinates, do not exist for arbitrary magnetic field configurations. Some authors do correctly indicate that the use of OF coordinates places restrictions on the admissible configurations [5]. Other authors [7], [12] consider a priori only those magnetic field geometries for which the existence of OF coordinates is self-evident. An example of the latter geometry class is an axisymmetric configuration with a purely meridional (poloidal) magnetic field. In most of the references cited above, no discussion on the existence of OF coordinates is included. It is clearly possible that the authors of these references had in mind magnetic configurations that are consistent with the constraints of OF coordinates, without explicitly noting this. In many of the papers based on OF coordinates the implicit restrictions on the acceptable magnetic configurations do not necessarily place severe limits on the attainable goals of the investigations. There are, however, papers where this is not so clear [3], [4], [5]. In these papers the effects due to nontrivial magnetic geometry are stated explicitly as part of the focus of the investigation [4] or as its main focus [3], [5], although in such general configurations OF coordinates likely do not exist.

Since OF coordinate systems are so widely used, and in view of the general significance of hypothetical OF coordinates, it seems appropriate to explore in some detail the problems related to their existence. This is the general aim of the present paper. We want to point out specifically that orthogonal, field aligned coordinates should not be used without a clear proof that they exist. A paper that is particlularly relevant for our purposes is Ref. [3] by Klimushkin, Leonovich and Mazur. These authors propose a method of construction of orthogonal, field aligned coordinates for magnetic configurations for which OF coordinates do exist. Their method of construction and their discussion of results for nontrivial magnetic field configurations could suggest that OF coordinates exist in fairly general situations. At first glance, it is not clearly evident where their method of construction might fail, once existence of surfaces normal to **B** is established. The authors reenforce this impression with the incorrect claim that Hamada coordinates for arbitrary MHD equilibria are OF coordinates provided that Eq. (4) in Sect. 2 is satisfied. In order to elucidate the reasons for possible nonexistence of OF coordinates, we will treat in this paper a very simple, helical magnetic field configuration for which practically all desired quantities and properties can be obtained analytically. With this example it will be demonstrated where and why the construction process presented in Ref. [3] can and usually does go wrong. The insight gained with this example is pertinent for general magnetic field configurations as well.

In Sect. 2 a few basic properties of OF coordinates are reviewed. In Sect. 3 we provide a short sketch of the construction method of Ref. [3] that is based on Dupin's theorem [14]. We then introduce our example configuration and demonstrate the stated contradictions in the construction of orthogonal coordinates. In the literature the role of axisymmetry in regards to the existence of orthogonal, field aligned coordinates is sometimes presented, or rather the presentation is avoided, in such a way that incorrect interpretations could ensue [3], [9], [11], [13]. We therefore discuss some pertinent aspects of the general axisymmetric case in Sec. 4. A discussion and the conclusions are contained in Section 5. Three appendices are present. Appendix A contains a short derivation of the lines of minimal and maximal curvature of the helical magnetic field. Appendix B summarizes the lines of minimal and maximal curvature in two degenerate versions of the helical magnetic field. Finally, Appendix C contains a treatment and refutation of OF coordinates which is more formal than the presentation in Sect. 3. The counterexample employed in this appendix is, however, somewhat more general.

2. Some properties of orthogonal, field aligned coordinate systems

The representation (1) for the magnetic field, which automatically satisfies $\operatorname{div} \mathbf{B} = 0$, always exists, at least locally [1]. With a redefinition of α or β the coefficient $f(\alpha, \beta)$ can be transformed into unity. Sometimes α and β in the representation (1) are called "unmatched" Euler potentials, while they are called "matched" if $f = 1$. A consequence of Eq. (1) is

$$\mathbf{B} \cdot \nabla\alpha = \mathbf{B} \cdot \nabla\beta = 0. \tag{2}$$

Field lines are described by the intersection of the $\alpha =$ const and $\beta =$ const coordinate surfaces. If all three coordinate surfaces are assumed to be orthogonal to each other, the magnetic field must be in the direction of the normal to the third coordinate surface, $\gamma(\mathbf{r}) =$ const say. This implies

$$\mathbf{B} = g\nabla\gamma, \tag{3}$$

where $g = g(\mathbf{r})$ may also be a function of position. By taking the curl of Eq. (3) one obtains

$$\mathbf{B} \cdot \operatorname{curl} \mathbf{B} = 0. \tag{4}$$

With $\mathbf{j} = \operatorname{curl} \mathbf{B}$ this condition, $\mathbf{j} \cdot \mathbf{B} = 0$, implies that the current density $\mathbf{j}$ must not have a component parallel to the magnetic field. Equation (4) is also sufficient for $\mathbf{B}$ to have normal surfaces [15]. In the current-free case g is a pure function of γ and can be transformed into unity. Condition (4) is a severe restriction on the magnetic fields admissible for OF coordinates.

Once functions $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$ and $\gamma(\mathbf{r})$ for a magnetic field are known, it follows from Eqs. (1) and (3) that the orthogonality conditions

$$\nabla\alpha \cdot \nabla\gamma = \nabla\beta \cdot \nabla\gamma = 0 \tag{5}$$

are satisfied automatically. The third orthogonality condition

$$\nabla\alpha \cdot \nabla\beta = 0, \tag{6}$$

however, has to be imposed in order to make the coordinates orthogonal.

In magnetospheric literature, OF coordinates are used in two, slightly different, ways. Either Eqs. (1) and (3) are used explicitly, [5], [7] – [10], [12], or, without applying these two equations, orthogonal, curvilinear coordinates are postulated, and the magnetic field is assumed to have only one component, in the direction of one of the coordinate lines [3], [4], [6], [11], [13].

3. An example without orthogonal, field aligned coordinates

The construction of an orthogonal, field aligned coordinate system, as proposed by Klimushkin, Leonovich and Mazur [3], very roughly sketched, proceeds as follows. Consider a bundle of field lines, the bundle having sufficiently small but finite width. In the region of the bundle the desired coordinate system is to be constructed. For this purpose, construct the (continuum of) surfaces $\gamma(\mathbf{r}) = \text{const}$ which are orthogonal to the field lines. On these surfaces, construct the lines of curvature, i.e. the curves which at each point follow the directions of minimal or maximal normal curvature of the surface. It is well known [16] that through each point of a surface there are exactly two such curves and they cross each other at right angles (except for spheres and planes where there are infinitely many). In this way an orthogonal grid of lines covers each surface. The lines of minimal and maximal curvature, denoted here as LMMC, stacked along the bundle, form surfaces $\alpha =$ const and $\beta =$ const, respectively, say. From this procedure a tri-orthogonal family of surfaces results which can be used as mutually orthogonal coordinate surfaces. Since the construction is based on field lines everywhere the intersection of any two surfaces $\alpha =$ const and $\beta =$ const constitutes again a field line.

In order to look deeper into this method of construction, we use a very simple axisymmetric, helical magnetic field configuration. Cylindrical coordinates (r, θ, z) are employed throughout. It is assumed that the magnetic field $\mathbf{B}$ only depends on the radial coordinate r and that $\mathbf{B}$ has no radial component. Thus, the non-vanishing components are $B_\theta(r)$ and $B_z(r)$. The dependence on r, so far, is arbitrary. The symmetry assumed makes the field $\mathbf{B}(r)$ automatically divergence free, if the axis $r = 0$ is excluded.

Let us look for a representation of $\mathbf{B}$ in the form of Eq. (3). The functions $\gamma(\mathbf{r})$ has to satisfy the (vector) differential equation

$$[\mathbf{B} \times \nabla\gamma] = 0. \tag{7}$$

From Eq. (7) one finds the two conditions

$$\partial_r \gamma = 0 \tag{8}$$

and

$$B_z \partial_\theta \gamma - r B_\theta \partial_z \gamma = 0. \tag{9}$$

There is a compatibility condition between Eqs. (8) and (9). Take the r-derivative of Eq. (9) in the form

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{B_z}{r B_\theta} = \frac{\partial}{\partial r} \frac{\partial_z \gamma}{\partial_\theta \gamma} = \frac{(\partial_\theta \gamma)(\partial_z \partial_r \gamma) - (\partial_z \gamma)(\partial_\theta \partial_r \gamma)}{(\partial_\theta \gamma)^2}. \tag{10}$$

From Eq. (8) it follows that the right-hand side of Eq. (10) is zero. The resulting equation for the left-hand side, $\mathrm{d}[B_z/(rB_\theta)]/\mathrm{d}z = 0$, is just the expected condition $\mathbf{B} \cdot \mathrm{curl}\, \mathbf{B} = 0$, Eq. (4), as is easily verified. This condition is satisfied for all configurations with

$$B_z(r) = c r B_\theta(r), \tag{11}$$

where $B_\theta(r)$ is an arbitrary function, and c is an arbitrary constant. This relation is assumed to hold, henceforth. It will become evident, however, that Eq. (4) is not enough, in general, to guarantee the existence of OF coordinates. The magnetic field, in Cartesian components, becomes

$$\mathbf{B} = B_\theta(r)\,(-\sin\theta, \cos\theta, cr). \tag{12}$$

With Eq. (11) the solution of Eqs. (8) and (9) simply is

$$\gamma(\theta + cz) = const, \tag{13}$$

where γ is an arbitrary function of its argument. These normal surfaces are also of helical shape. Since they do not close onto themselves if θ is increased by 2π, radial cuts have to be made in order to make them globally unique. For the present purpose, however, a local consideration with a finite but small region of θ values is sufficient.

The field lines $r(z)$, $\theta(z)$ are determined by

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{B_r}{B_z} = 0, \qquad \frac{r\,\mathrm{d}\theta}{\mathrm{d}z} = \frac{B_\theta}{B_z} = \frac{1}{cr}, \tag{14}$$

with the solution

$$r = r_0 = const, \qquad \theta = \theta_0 + \frac{z - z_0}{c r_0^2}. \tag{15}$$

Here, r_0 and z_0 are arbitrary constants, $\theta_0(r_0)$ is an arbitrary function, and $c \neq 0$ is assumed. The field lines are helices whose steepness varies with the inverse square of distance to the axis. $\theta_0 - z_0/(c r_0^2)$ could have been combined into a function $\theta_{00}(r_0)$, but the representation (15) emphasizes that the field line crosses the plane $z = z_0$ at $\theta = \theta_0$.

Next we look for surfaces $\alpha(\mathbf{r}) = \text{const}$ and $\beta(\mathbf{r}) = \text{const}$ which are tangent to field lines. They are defined by

$$\mathbf{B} \cdot \nabla\alpha = 0 \tag{16}$$

and the analogous equation for β. In order to avoid unnecessary duplication of equations, the pair α and β will be collectively denoted, in the following, either by α or by β. Where the distinction between the two functions is relevant, the original α and β will be replaced by α_1 and α_2 or by β_1 and β_2. We start with the notation α. The characteristic equation for (16) is

$$\mathrm{d}\theta = \frac{\mathrm{d}z}{cr^2}. \tag{17}$$

This can be integrated, with the result

$$\alpha(r, \theta, z) = \alpha(p, q), \qquad p \equiv \theta - \frac{z}{cr^2}, \quad q \equiv r, \tag{18}$$

where α is an arbitrary function of its two arguments p and q. Clearly, from Eqs. (15), α is a constant along each field line. At each point in space the surfaces $\gamma = \text{const}$ and $\alpha = \text{const}$ are orthogonal to each other, as is easily confirmed with the relations $\nabla r = (\cos\theta, \sin\theta, 0)$, $\nabla\theta = (-\sin\theta, \cos\theta, 0)/r$ and $\nabla z = (0, 0, 1)$, in Cartesian coordinates. Equations (5) are therefore satisfied identically.

The arbitrary functional dependence of the α on their arguments p and q is to be used to satisfy the orthogonality condition (6), i.e.

$$\nabla\alpha_1 \cdot \nabla\alpha_2 = 0, \tag{19}$$

everywhere. According to Klimushkin et al. this can be done with the help of the curvature lines, the LMMC, which are orthogonal to each other. In appendix A the LMMC are derived. They can be put in the form

$$\theta_j(r) = \theta_{j0} + \ln\frac{Q_j(r)}{Q_{j0}}, \qquad z_j(r) = z_{j0} - \frac{1}{c}\ln\frac{Q_j(r)}{Q_{j0}}, \qquad \text{for } j = 1,\, 2, \tag{20}$$

where $\sigma_1 = 1$, $\sigma_2 = -1$ and $Q_j(r) = \sigma_j c r + \sqrt{1 + c^2r^2}$. Also, $Q_{j0} \equiv Q_j(r_{j0})$. Each of the two curvature lines $\mathbf{r}_j(s)$ passes through an arbitrary point $\mathbf{r} = \mathbf{r}_{j0}$, defined by the triplets $(r_{j0}, \theta_{j0}, z_{j0})$, with $j = 1,\, 2$. If the surface which embeds the LMMC is described by $\gamma(\mathbf{r}) = \gamma_0 = \text{const}$, the quantities θ_{j0}, z_{j0} and γ_0 are constrained by the relations

$$\gamma(\theta_{j0} + cz_{j0}) = \gamma_0. \tag{21}$$

In the limit $r \to \infty$ $Q_1(r)$ diverges while $Q_2(r)$ goes to zero. This implies that the two curves are spirals, one turning clockwise around the origin and the other counterclockwise.

The variation of the parameters θ_{10} and θ_{20} generates a dense set of LMMC on $\gamma = \gamma_0$. They constitute a dense, orthogonal grid of coordinate lines. The additional variation of

r_{10} and r_{20} merely shifts the "bookkeeping" position of the LMMC to different radii and adds no extra freedom. At any fixed point on $\gamma = \gamma_0$ the directions $\mathbf{r}_1'(s)$ and $\mathbf{r}_2'(s)$ of the LMMC, passing through it, are given by Eqs. (A.5) and (A.6). It is easily confirmed that the orthogonality relations

$$\mathbf{r}_1' \cdot \mathbf{r}_2' = 0, \qquad \mathbf{B} \cdot \mathbf{r}_j' = 0, \quad \text{for } j = 1, 2 \tag{22}$$

are satisfied identically not only on $\gamma = \gamma_0$ but at every point in space (within the bundle). Actually, as can easily be seen, $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are combinations of the normal and the binormal unit vectors $\mathbf{n}$ and $\mathbf{b}$ of the field line passing through $\mathbf{r}$, according to $\mathbf{r}_1' = (\mathbf{b} + \mathbf{n})/\sqrt{2}$, $\mathbf{r}_2' = (\mathbf{b} - \mathbf{n})/\sqrt{2}$.

After these preparations, it remains to extend the local orthogonality relations (22) to the surfaces $\alpha_j(\mathbf{r}) = \text{const}$, see Eq. (19). The intersection of the surfaces $\alpha_j(p, q) = \alpha_{j0} = \text{const}$ with the surface $\gamma = \gamma_0$ defines two families of curves, $\tilde{\mathbf{r}}_1(s)$ and $\tilde{\mathbf{r}}_2(s)$, say. As will be presently shown, they can be made to coincide with the set of LMMC $\mathbf{r}_1(s)$ and $\mathbf{r}_2(s)$ from Eq. (20). For this purpose, the first argument, $p = \theta - z/(cr^2)$, of α is evaluated along both sets of LMMC. The result,

$$p_j = \theta_{j0} - \frac{z_{j0}}{cr^2} + \frac{w^2(r)}{(cr)^2} \ln \frac{Q_j(r)}{Q_{j0}}, \tag{23}$$

is a function of r only. By choosing a suitable dependence on the second argument $q = r$ it is thus possible to compensate this r -dependence completely and to achieve $\alpha_j = \text{const}$ along all the LMMC on the chosen surface $\gamma = \gamma_0$. One thus obtains

$$\alpha_j(r, \theta, z) = \alpha_j \left(\theta - \theta_{j0} - \frac{z - z_{j0}}{cr^2} - \frac{w^2(r)}{(cr)^2} \ln \frac{Q_j(r)}{Q_{j0}} \right), \tag{24}$$

where the α_j are now two arbitrary functions of their single argument, respectively. It is evident that the parameter θ_{j0} is equivalent to the parameter α_{j0} as regards the selection of surfaces $\alpha(\mathbf{r}) = \alpha_{j0}$. The intersecting curve, $\widetilde{\theta}(z)$, $\widetilde{r}(z)$, say, between any two surfaces $\alpha_1(\mathbf{r}) = \alpha_{10} = \text{const}$ and $\alpha_2(\mathbf{r}) = \alpha_{20} = \text{const}$ is again a field line. This is most easily seen if, without restriction of generality, the two arbitrary points $\mathbf{r}_{j0}$ on $\gamma(\mathbf{r}) = \gamma_0$ are chosen to coincide, $\mathbf{r}_{j0} = \mathbf{r}_0$, say, and the constants α_{10} and α_{20} are both set equal to zero. From the two equations in Eq. (24) one obtains, first by subtraction and then directly,

$$\widetilde{r} = r_0 = \text{ const}, \qquad \widetilde{\theta} = \theta_0 + \frac{z - z_0}{cr_0^2}. \tag{25}$$

This is simply the field line which passes through $\mathbf{r} = \mathbf{r}_0$ on $\gamma = \gamma_0$.

We now address the crucial question: given an arbitrary but then fixed "reference surface" $\gamma(\mathbf{r}) = \gamma_0$, is the orthogonality relation (19) satisfied identically everywhere along the intersecting field line of the two surfaces $\alpha_j(\mathbf{r}) = 0$. For $\nabla\alpha_1 \cdot \nabla\alpha_2$ there results

$$\nabla\alpha_1 \cdot \nabla\alpha_2 = \alpha_1'\alpha_2' \frac{2}{c^2 r^6} \left[2\Delta z_1 \, \Delta z_2 + (\Delta z_1 - \Delta z_2) r w \right], \tag{26}$$

where the prime here denotes the derivative with respect to the argument, w is defined in Eq. (A.3), and

$$\Delta z_j \equiv z - z_0 + \frac{1}{c}\ln\frac{Q_j(r)}{Q_{j0}}. \tag{27}$$

Since $r = \widetilde{r} = r_0$, along the intersection, the last term in Δz_j vanishes, and only $\Delta z_j = z - z_0$, for $j = 1$, 2, remains. Hence, Eq. (26) simplifies to

$$\nabla\alpha_1 \cdot \nabla\alpha_2 = \alpha_1' \alpha_2' \frac{4}{c^2 r^6}\,(z - z_0)^2. \tag{28}$$

It is evident that $\nabla\alpha_1 \cdot \nabla\alpha_2$ is different from zero, in general. The only place where it vanishes is on the reference surface $\gamma = \gamma_0$ with $z = z_0$, where indeed, by construction, it has to. We thus obtain the result: two coordinate surfaces, $\alpha_1(\mathbf{r}) = \text{const}$ and $\alpha_2(\mathbf{r}) = \text{const}$, which are made up of field lines and which are orthogonal to each other on an orthogonal reference surface $\gamma(\mathbf{r}) = \gamma_0$, do not remain orthogonal in general along the field line that they have in common. This conflicts with the construction in Ref. [3] where the orthogonality property is tacitly assumed to persist along the field line.

Our negative result is a consequence of the local magnetic shear, which is present in the configuration. Field lines at different radial positions have different helical pitches. On the reference surface $\gamma = \gamma_0$ consider the foot-point of the common field line and the foot-points of two arbitrary field lines on $\alpha_1(\mathbf{r}) = \text{const}$ and $\alpha_2(\mathbf{r}) = \text{const}$. By construction, they form a right angle. Since the common field line rotates around the axis at a different angular "speed" compared to that of the other two field lines, the right angle is distorted.

Thus, the construction of an orthogonal system of surfaces does not work, in general, if the field lines are to be coordinate lines. A complementary part of the construction procedure in Ref. [3] which leads to the same conclusion highlights another aspect. Instead of a reference surface consider a single, arbitrary but fixed reference field line, F say, which is embedded within a bundle of neighboring field lines. One may ask first whether there is a coordinate system $(\beta_1, \beta_2, \gamma)$, say, whose surfaces $\gamma(\mathbf{r}) = \text{const}$ are orthogonal to the field lines, as before, and whose surfaces $\beta_1(\mathbf{r}) = \text{const}$, $\beta_2(\mathbf{r}) = \text{const}$ are orthogonal to each other and to $\gamma(\mathbf{r}) = \text{const}$ *everywhere along* F. A posteriori it is tested whether field lines are coordinate lines.

One can proceed as follows. Let F be given by

$$\widehat{r}(z) = \widehat{r}_0 = const, \qquad \widehat{\theta}(z) = \frac{z - \widehat{z}_0}{c\widehat{r}_0^{\,2}} + \widehat{\theta}_0, \tag{29}$$

where $\widehat{r}_0$, $\widehat{\theta}_0$ and $\widehat{z}_0$ are arbitrary constants. Along this field line plus bundle there exists the continuum of orthogonal surfaces $\gamma(\mathbf{r}) = \text{const}$, as given by Eq. (13). On each of them the curvature lines, the LMMC, are known, see Eqs. (20). In particular, on each surface, we focus on those two LMMC which meet (orthogonally) at the point where the field line

F crosses the surface. The totality of all these pairs of LMMC along F form the two surfaces $\beta_1(\mathbf{r}) = \text{const}$ and $\beta_2(\mathbf{r}) = \text{const}$. They are easily derived. In Eqs. (20) the two arbitrary points $(r_{j0},\ \theta_{j0},\ z_{j0})$ have to be replaced by the common values $(\widehat{r}(\widehat{z}),\ \widehat{\theta}(\widehat{z}),\ \widehat{z})$, where $\widehat{z}$ is written for the continuous z-value along F. With Eqs. (29) this parameter $\widehat{z}$ can be eliminated. Altogether, with Eqs. (20), a relation between r, θ and z is obtained which defines the two desired surfaces. They may be written in the form $\beta_j(\mathbf{r}) = 0$, with

$$\beta_j(r, \theta, z) \equiv \beta_j\left(\theta - \widehat{\theta}_0 - \frac{z - \widehat{z}_0}{c\widehat{r}_0^2} - \frac{w^2(\widehat{r}_0)}{(c\widehat{r}_0)^2} \ln \frac{Q_j(r)}{\widehat{Q}_{j0}}\right), \tag{30}$$

where the functions β_j may depend arbitrarily on their argument, but such that $\beta_j(u) = 0$ for $u = 0$, and $\widehat{Q}_{j0} \equiv Q_j(\widehat{r}_0)$. Note the similarities and the differences between $\alpha_j(\mathbf{r})$, Eq. (24), and $\beta_j(\mathbf{r})$. It is easily confirmed that $\nabla\beta_1 \cdot \nabla\beta_2 = 0$ along F, as it must be. With Eq. (12), however, it also follows that

$$\mathbf{B} \cdot \nabla\beta_j = \beta_j' \frac{B_\theta}{c^2 r \widehat{r}_0^2}(\widehat{r}_0^2 - r^2). \tag{31}$$

This vanishes only along the field line F, at $r = \widehat{r}_0$, (and, owing to the symmetry of the configuration, also on all field lines with the same radial distance to the axis). Consequently, except for F, field lines are not simutaneously coordinate lines of the system $\beta_1(\mathbf{r})$, $\beta_2(\mathbf{r})$, $\gamma(\mathbf{r})$, which along F is orthogonal.

4. Further comments on OF coordinates in the axisymmetric case

First, a trivial observation regarding orthogonal, field aligned coordinates in axisymmetric configurations is made. In the magnetospheric literature it is often stated that in such cases the angle of symmetry, ϕ, can be used as one of the coordinates α or β in Eq. (1), see e.g. Refs. [3], [4], [9], [11], [13]. This identification implies $\mathbf{B} \cdot \nabla\phi = 0$, i.e. the absence of a toroidal component of the magnetic field. This is a further, severe restriction of the admissible configurations, which however is seldom mentioned by the authors. For example, if the *only* comment on restrictions to the admissible configurations is something like, "The only restriction of the magnetospheric plasma ... is its axial symmetry, implying that all equilibrium quantities — plasma density ρ_0, pressure P_0, and magnetic field $\mathbf{B}$ — are independent of the azimuthal coordinate" [13], then this is rather misleading. Neither models of the magnetosphere which would retain axisymmetry but would be more realistic with respect to a local azimuthal field component nor axisymmetric fusion devices with finite rotational transform can be treated with the coordinate system and the identification of ϕ just mentioned.

A less trivial result on OF coordinates in axisymmetric configurations follows from theorems of differential geometry. Let ψ be an index which labels the circularly symmetric magnetic shells or surfaces. Since $\mathbf{B} \cdot \nabla\psi = 0$ is satisfied by definition, ψ can also be identified with α or β (without a specification of the other member). Dupin's theorem [14] states that in a system of three mutually orthogonal surfaces these surfaces meet in curvature lines of the surfaces. This implies in particular that field lines must be curvature lines of the $\psi(\mathbf{r}) = \text{const}$ surfaces. Curvature lines of surfaces of revolution, however, are the meridians and the circles of constant latitude only [17]. In configurations whose field lines do not belong to this particular, degenerate class (purely poloidal or purely toroidal), therefore, the magnetic surfaces cannot play the role of the coordinate surfaces α or β. If OF coordinates exist at all, their surfaces $\alpha = \text{const}$ and $\beta = \text{const}$ must be tilted with respect to the ψ surfaces.

We end with a comment on field aligned but not fully orthogonal coordinate systems. If Eq. (1) for the magnetic field is kept, but Eq. (3) is not retained, the restriction (4) on the current density and the orthogonality relations (5) are removed. One could however still ask whether coordinates α, β exist such that their mutual orthogonality condition (6) is satisfied, and such that α corresponds to the axisymmetric magnetic surfaces ψ. Also, let β be a coordinate, more general than ϕ, but such that the partial derivative of equilibrium quantities with respect to β vanishes. Without proof we state our result, namely that such, potentially desirable, coordinates are again not possible for general magnetic field configurations.

5. Discussion and conclusions

A simple, helical magnetic field configuration was used to study explicitly the existence of orthogonal, field aligned coordinate systems for which a construction method is given in Ref. [3]. It was demonstrated why this construction goes wrong in general: the demand for a mutually orthogonal system of coordinate surfaces and the demand that field lines should also be coordinate lines are incompatible in general.

The problem does not lie in the demand for mutually orthogonal coordinate surfaces. Locally, they always exist. There are two methods to construct them which are related with the two aspects of the problem discussed in Sect. 2. The first method is based on Darboux's embedding theorem [14]. It states that if on an arbitrary surface $\gamma(\mathbf{r}) = \gamma_0$, say, a grid of curvature lines is given, this surface can always be embedded, in the local neighborhood, in a mutually orthogonal system of coordinate surfaces. The second method has been advocated by Mercier and others [18], [19]. It singles out an arbitrary suitable curve, $\mathbf{r}_c(\ell)$ say, which may also be a field line, as an "axis" of the configuration.

The planes spanned by the normal and the binormal directions, $\mathbf{n}(\ell)$ and $\mathbf{b}(\ell)$, to $\mathbf{r}_c(\ell)$ are considered. In these planes, the distance ρ to the central curve and a suitably defined angle ω, together with the coordinate ℓ can always be forged into an orthogonal coordinate system (in the local neighborhood of the central curve). Mercier coordinates have their merits and they are used up to recently [20]. In the two types of orthogonal coordinate systems just mentioned, however, field lines are not simultaneously coordinate lines, in general. In Sect. 2 it was made plausible that local magnetic shear is the culprit which prevents the coincidence of the two items. This suggestion is corroborated by an example for OF coordinates, mentioned in Ref. [19]. In a solenoid with an axis twisted away from the equatorial plane the magnetic field lines in the vicinity of the axis are also coordinate lines of an orthogonal (Mercier-) coordinate system. This configuration has no shear in the neighborhood of its axis.

The claim that for a rather general class of MHD equilibria OF coordinates exist [3] still needs to be discussed. MHD equilibria with plasma pressure $P(\mathbf{r})$ satisfy the equation $[\mathbf{j} \times \mathbf{B}] = \nabla P$. For $\mathbf{j} \cdot \mathbf{B} = 0$ the normal to the pressure surfaces, the current density lines and the field lines are indeed mutually orthogonal. If OF coordinates exist, surfaces normal to all three families of coordinate lines must also exist. The condition for surfaces normal to the current density lines $\mathbf{j}$ to exist is the analogue to Eq. (4) for the field lines, namely

$$\mathbf{j} \cdot \operatorname{curl} \mathbf{j} = 0. \tag{32}$$

This can be tested with our example configuration of Sect. 2 which is easily extended into an MHD equilibrium. The components of $\mathbf{j}$ are

$$j_r = 0, \qquad j_\theta = -B_z', \qquad j_z = \frac{B_z'}{cr}, \tag{33}$$

where $B_z(r) = crB_\theta(r)$, and the prime denotes derivation with respect to r. The MHD equation is satisfied provided $P = P(r)$, with $P'(r) = -w^2(r) B_z B_z'/(c^2 r^2)$. From Eqs. (33) one obtains

$$\mathbf{j} \cdot \operatorname{curl} \mathbf{j} = -\frac{2B_z'(r)}{cr^2}. \tag{34}$$

This, however, shows that condition (32) is not satisfied in general, contrary to the claims in Ref. [3]. (Our analysis is valid with or without the use of Hamada coordinates). The only acceptable case, in our example, is the current free case, with $B_\theta(r) \sim 1/r$ and $B_z = \text{const}$. This result underlines our word of caution that orthogonal, field aligned coordinates only exist under very restricted circumstances.

Cases where OF coordinates trivially exist include axisymmetric configurations with a purely meridional (poloidal) magnetic field. This is an often assumed model field in wave and resonance studies of magnetospheric plasmas. (For the helical magnetic field studied here such degenerate cases are discussed in Appendix B.) Unfortunately, purely poloidal (or purely toroidal) magnetic fields are unsuitable in fusion plasma physics where the same wave phenomena, e.g. resonances in the Alfvén continuum, are of interest. The extension of their analysis in the 2D, axisymmetric case [21] to the 3D case without symmetry, which is still an outstanding task, would have profitted from the existence of OF coordinates.

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Appendix A: derivation of the curvature lines

It suffices to quote just a few steps of the derivation of the lines of minimal and maximal curvature, LMMC, since the method can be found in many textbooks on differential geometry [16].

On the surfaces $\gamma =$ const the first and the second metric forms, $g_{ik} = \partial_i \mathbf{r} \cdot \partial_k \mathbf{r}$ and $L_{ik} = \mathbf{N} \cdot \partial^2_{ik}\mathbf{r}$, for $i,\ k = r,\ z$, are required. $\mathbf{N} = \mathbf{B}/B$ is the unit normal vector to the surface, and r and z are used as coordinate pair on the surface. The main curvatures $\lambda = \kappa_1,\ \kappa_2$ follow from

$$\det\{L_{ik} - \lambda g_{ik}\} = 0. \tag{A.1}$$

With

$$g_{rr} = 1, \quad g_{rz} = 0, \quad g_{zz} = w^2, \qquad L_{rr} = 0, \quad L_{rz} = -z/w, \quad L_{zz} = 0, \tag{A.2}$$

where

$$w = w(r) \equiv \sqrt{1 + c^2 r^2}, \tag{A.3}$$

it is found that $\kappa_1 = c/w^2$ and $\kappa_2 = -c/w^2$. The two main curvature directions $\mathbf{r}'_j(s)$, $j = 1,\ 2$, where the prime denotes differentiation with respect to the arc length s along the LMMC, can be decomposed into components ξ^i_j according to $\mathbf{r}'_j = \xi^i_j \partial_i \mathbf{r}$, with summation over $i = 1,\ 2$. The ξ^i_j are determined from the coupled equations

$$\kappa_j = L_{ik}\xi^i_j\xi^k_j, \qquad 1 = g_{ik}\xi^i_j\xi^k_j. \tag{A.4}$$

After a bit of algebra the main curvature directions, in Cartesian representation, are obtained as

$$\mathbf{r}'_1 = \frac{1}{\sqrt{2}w}(w\cos\theta - cr\sin\theta,\ w\sin\theta + cr\cos\theta,\ -1), \tag{A.5}$$

$$\mathbf{r}'_2 = \frac{1}{\sqrt{2}w}(-w\cos\theta - cr\sin\theta,\ -w\sin\theta + cr\cos\theta,\ -1). \tag{A.6}$$

In order to obtain the curvature lines one still has to integrate the differential equations with respect to s,

$$\mathbf{r}'_j = (r\cos\theta, r\sin\theta, z)'_j = (r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta, z')_j, \qquad (A.7)$$

where $\mathbf{r}'_1$ and $\mathbf{r}'_2$, on the left-hand side, are given in Eqs. (A.5), (A.6). Since the desired curves $\mathbf{r}_j(s)$, with components $r_j(s)$, $\theta_j(s)$, $z_j(s)$, are on $\gamma(\theta + cz) = \text{const}$, it holds that $\theta'_j = -cz'_j$. From Eqs. (A.5), (A.6) and (A.7) the additional relations $r'_1(s) = -r'_2(s) = 1/\sqrt{2}w$ and $z'_1(s) = z'_2(s) = -1/(\sqrt{2}w)$ are obtained. Their integration is straightforward. If the parameter s is eliminated in favor of r one finally obtains for the two LMMC

$$\theta_j(r) = d_j + \ln Q_j(r), \qquad z_j(r) = b_j - \frac{1}{c}\ln Q_j(r), \qquad j = 1, 2, \qquad (A.8)$$

where $Q_j \equiv \sigma_j cr + w(r)$ and $\sigma_1 = 1$, $\sigma_2 = -1$. b_j and d_j are four arbitrary integration constants, except that $d_1 + cb_1 = d_2 + cb_2$ which follows from Eq. (13).

Appendix B: degenerate cases

In the cases $c = 0$ ("poloidal" field) or $c = \infty$ ("toroidal" field) the field lines degenerate into circles in the $z = \text{const}$ planes or into straight lines in z direction, respectively. In both cases, the normal surfaces $\gamma(\mathbf{r}) = \text{const}$ are planes, either with $\theta = \text{const}$ or with $z = \text{const}$, respectively. The curvature lines, the LMMC, from Eqs. (20) become straight lines in the case $c = 0$,

$$\theta_j(r) = \theta_{j0}, \qquad z_j(r) = z_{j0} - \sigma_j(r - r_{j0}), \qquad (B.1)$$

or logarithmic spirals for $c = \infty$,

$$\theta_j(r) = \theta_{j0} + \ln\left(\frac{r}{r_{j0}}\right)^{\sigma_j}, \qquad z_j(r) = z_{j0}, \quad \text{for } j = 1, 2, \qquad (B.2)$$

where $\sigma_1 = 1$ and $\sigma_2 = -1$. LMMC in planes, however, are not unique. In the poloidal case the lines r, $z = \text{const}$ are also mutually orthogonal LMMC, as are the lines r, $\theta = \text{const}$ in the toroidal case. The continuation of these LMMC along the field lines gives rise to a trivial system of mutually orthogonal coordinate surfaces.

Appendix C: more general counterexample

As in Sect. 3 we consider an axisymmetric magnetic field which depends on the radial coordinate r only and which has no radial component. The components $B_\theta(r)$ and $B_z(r)$, however, are arbitrary. In general, therefore Eq. (11) does not hold and there exist no surfaces with normal direction parallel to the field lines. In this appendix the field is represented in the form of Eq. (1), and it is investigated whether the orthogonality relation $\nabla\alpha \cdot \nabla\beta = 0$ can be satisfied everywhere along a field line.

The functions $\alpha(\mathbf{r}) \equiv \alpha_1(\mathbf{r})$ and $\beta(\mathbf{r}) \equiv \alpha_2(\mathbf{r})$ are defined by Eq. (16). Its characteristic equation is

$$\frac{r\,\mathrm{d}\theta}{B_\theta(r)} = \frac{\mathrm{d}z}{B_z(r)}, \tag{C.1}$$

which is easily integrated. The solution for α_j, for $j = 1$, 2, is

$$\alpha_j(r, \theta, z) = \alpha_j(p, q), \tag{C.2}$$

where the dependence on p and q,

$$p \equiv \theta - u(r)z, \quad q \equiv r, \tag{C.3}$$

is arbitrary, and $u(r) \equiv B_\theta(r)/(rB_z(r))$. For $\nabla\alpha_1 \cdot \nabla\alpha_2$ one obtains

$$\nabla\alpha_1 \cdot \nabla\alpha_2 = \frac{\partial\alpha_1}{\partial p}\frac{\partial\alpha_2}{\partial p}\left[(u'z)^2 + 1 + u^2\right] - \left(\frac{\partial\alpha_1}{\partial p}\frac{\partial\alpha_2}{\partial q} + \frac{\partial\alpha_1}{\partial q}\frac{\partial\alpha_2}{\partial p}\right)u'z + \frac{\partial\alpha_1}{\partial q}\frac{\partial\alpha_2}{\partial q}, \tag{C.4}$$

where $u' \equiv \mathrm{d}u(r)/\mathrm{d}r$. If the orthogonality condition $\nabla\alpha_1 \cdot \nabla\alpha_2 = 0$ is satified, the right-hand side of Eq. (C.4) must vanish all along a field line. The intersection between two different surfaces $\alpha_j(p, q) = \text{const}$ defines a field line in the form

$$p = \text{const}, \quad q = \text{const}. \tag{C.5}$$

For finite and regular $u(r)$ one can always transform from the coordinate triple (r, θ, z) to the triple (p, q, z). In Eq. (C.4) this transformation is trivial and affects only the term $u' = \mathrm{d}u(q)/\mathrm{d}q$. At fixed p and q the coordinate z corresponds to a coordinate along the field line. Hence, the right-hand side of Eq. (C.4) must vanish for all values of z, at fixed p and q. Consequently, the coefficients of z^n, with $z^n = 0$, 1, 2, have to vanish separately. This implies

$$\frac{\partial\alpha_1}{\partial p}\frac{\partial\alpha_2}{\partial p} = 0, \quad \frac{\partial\alpha_1}{\partial q}\frac{\partial\alpha_2}{\partial q} = 0 \tag{C.6}$$

and

$$\frac{\partial\alpha_1}{\partial p}\frac{\partial\alpha_2}{\partial q} + \frac{\partial\alpha_1}{\partial q}\frac{\partial\alpha_2}{\partial p} = 0. \tag{C.7}$$

From Eqs. (C.6) it follows that either

$$\frac{\partial\alpha_1}{\partial p} = \frac{\partial\alpha_1}{\partial q} = 0, \tag{C.8}$$

or

$$\frac{\partial\alpha_1}{\partial p} = 0, \quad \frac{\partial\alpha_2}{\partial q} = 0, \tag{C.9}$$

or the same relations with the indices 1 and 2 interchanged. A function $\alpha_j(p,q)$ which does not depend on both of its arguments does not define a surface. Hence, only the case (C.9) is acceptable. From Eq. (C.7) there remains

$$\frac{\partial \alpha_1}{\partial q}\frac{\partial \alpha_2}{\partial p} = 0. \tag{C.10}$$

Together with Eqs. (C.9), however, it again follows that at least one of the two functions $\alpha_j(p,q)$ is independent of both arguments. This proves that $\nabla\alpha_1 \cdot \nabla\alpha_2 = 0$ cannot be satisfied everywhere along a field line, for the configuration considered.

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