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Abstract

A statistical theory of nonlinear-nonequilibrium plasma state with strongly developed turbulence and with strong inhomogeneity of the system has been developed. A unified theory for both the thermally excited fluctuations and the strongly turbulent fluctuations is presented. With respect to the turbulent fluctuations, the coherent part to a certain test mode is renormalized as the drag to the test mode, and the rest, the incoherent part, is considered to be a random noise. The renormalized operator includes the effect of nonlinear destabilization as well as the decorrelation by turbulent fluctuations. Formulation is presented by deriving an Fokker-Planck equation for the probability distribution function. Equilibrium distribution function of fluctuations is obtained. Transition from the thermal fluctuations, that is governed by the Boltzmann distribution, to the turbulent fluctuation is clarified. The distribution function for the turbulent fluctuation has tail componenet and the width of which is in the same order as the mean fluctuation level itself. The Lyapunov function is constructed for the strongly turbulent plasma, and it is shown that an approach to a certain equilibrium distribution is assured. The result for the most probable state is expressed in terms of 'minimum renormalized dissipation rate', which is given by the ratio of the nonlinear decorrelation rate of fluctuation energy and the random excitation rate which includes both the thermal noise and turbulent self-noise effects. Application is made for example to the current-diffusive interchange mode turbulence in inhomogeneous plasmas. The applicability of this method covers plasma turbulences in much wider circumstance as well as neutral fluid turbulence. This method of analyzing strong turbulence has successfully extended the principles of statistical physics, i.e., Kubo-formula, Prigogine's principle of minimum entropy production rate. The condition for the turbulence transition is analogous to the Maxwell's construction in the phase transition physics in thermodynamical equilibrium. The method provides the extension of the nonequilibrium statistical physics to the far-nonequilibrium states.

§1. Introduction

Strong turbulence in high temperature plasmas is one of the most challenging problems of statistical physics for systems far from thermodynamic equilibrium. Understanding of such systems is far from satisfactory, in contrast to those near thermodynamic equilibrium in which the principles that govern fluctuations (i.e., equipartition of energy, Einstein relation, fluctuation-dissipation (FD) theorem, etc.) are established [1-3].

Statistical physics picture for turbulence has been developed for homogeneous turbulence in neutral fluid. Based on the conventional DIA (direct interaction approximation) method [4], various closed sets of equations have been formulated by use of two-time correlation functions of fields and the averaged quantities of response functions, including extensions to the two-scale DIA for nonequilibrium situations [5,6]. In another method, a Fokker-Planck equation was formulated for the truncated nonlinearlities, by introducing the concepts of turbulence viscosity as a drag and the turbulent diffusion in a functional space [7,8]. To resolve the infrared divergence, efforts have been made to utilize the Lagrangean statistical quantities [9]. In these approaches, the role of strong instabilities has not been clearly identified. For the study of strong temporal-spatial change of flow, an independent approach, namely, the "rapid distortion theory" (RDT), has been used [10]. In this type of approach, strong nonuniformity of the system could be incorporated; however, the statistical feature of the nonlinear turbulence is not well treated.

Plasma turbulence is often known to be inhomogeneous and the method for a homogeneous turbulence is insufficient. For strongly unstable plasmas, nonlinear theory has been developed, based upon the methodology of clump and two-point correlation functions [11] or on the method of dressed-test mode [12]. The anomalous transport and the improved confinement in toroidal plasmas, so long as their averaged properties are concerned, have been successfully explained [13]. In particular, the roles of pressure gradient and the gradient of radial electric field have been considerably clarified [e.g., 12-19] (see [20] for a review). In the study of statistical nature of plasma turbulence, an effort has been made to formulate a Langevin equation for weak turbulence [21]. Based on the DIA method, a Langevin equation for turbulence is derived for the strong turbulence [22-25]. This approach has been investigated for the case of linearly-unstable drift waves.

Recently, a statistical description and the extended analyses have been developed for a self-sustained strong turbulence which is caused by the subcritically excited interchange mode. Report were made [26-28], which we call I, II and III, respectively, in this article. A Langevin equation for a dressed test mode is formulated: In this theoretical framework (i) the nonlinear interactions are divided into the drag term (coherent interactions) and the random noise term (incoherent ones), and (ii) thermal excitation is interpreted as the collisional drag term and the thermal noise term. Random coupling model (RCM)[29] has been used to model the self-noise term. Imposing ansatz (1) of a large number of degrees of freedom in the turbulence (extensiveness) and (2) of the randomness of self-noise, the turbulent level and decorrelation rate of turbulence and the auto- and cross-correlation functions have been solved. The extended FD-theorem (Einstein relation) has been explicitly described by the nonequilibrium-parameter (the gradient) of the system [27, 28]. The transition from thermal fluctuations to turbulent fluctuations has been described. In the space of the temperature and the gradient, the phase diagram of fluctuations has been obtained.

In this article, we further extend the analysis to formulate the Fokker-Planck equations for the probability distribution function, including the effects of nonlinear instability, nonlinear self-noise as well as thermal fluctuations. In the treatment of thermal fluctuations, their coherent interactions with the plasma collective mode (e.g., interchange mode or current-diffusive-interchange mode, CDIM [12]) are represented by the collisional drags, and their incoherent interactions are considered to be a random noise being characterized by plasma temperature T. Taking a certain test mode from turbulence modes, the coherent interactions to the mode are represented by the renormalized turbulent drags, and the incoherent interactions are considered to be a random self-noise. From a Langevin equation, a Fokker-Planck equation is formulated in the presence of both the thermal excitations and the inhomogeneous turbulent excitations. The probability distribution function of fluctuation amplitude is solved. Distribution function of fluctuation for the case of coexisting thermal fluctuation and turbulent fluctuation is obtained. The characteristic features, e.g., the double peaks and their widths are illustrated. Transition from thermal fluctuation to the turbulent fluctuation occurs at a certain critical gradient. It is also shown that the probability distribution function has a power-law tail. This is caused by the self-sustaining mechanism of turbulence through nonlinearlities. The index to the power is obtained. Formulation is also developed for coarse-grained quantities. The probability distribution function of macro variable (say volume averaged fluctuation energy) is obtained. The result shows that the most probable state of turbulence is dictated by the principle of minimum renormalized dissipation rate of fluctuation energy. This minimum principle reduces to, if the self-noise disappears, the Prigogine's minimum entropy production rate which was derived for the case of nonequilibrium but near thermodynamical equilibrium. Thus the thermodynamical law is extended to the system with strong turbulence, being a theoretical deduction based on statistical description. The application is made to CDIM turbulence. However, the method developed in this article is applicable to more general circumstances. Application to the other example would lead to a quantitative difference. However, the qualitative conclusions like the existence of power law tail or the minimum principle hold in general. These results of turbulence transitions and the power law distribution are characteristic features of strong plasma turbulence, which is far from thermal equilibrium.

§2. Basic Equation and Statistical Approach

2.1 Plasma model and basic equation

We consider a slab plasma which is inhomogeneous in the x-direction and is immersed in an inhomogeneous and sheared magnetic field. The magnetic field is given as $\boldsymbol{B} = B_0(0, sx, 1)$ with $B_0(x) = \left(1 + \Omega' x + \cdots\right)B_0$. In this system, a collective mode, interchange mode, can be subcritically excited due to the turbulent current diffusivity.[12,

[30] (It is also called the current-diffusive interchange mode, CDIM.) In the dynamics of its mode, the electron viscosity prohibits the free-motion of electrons along the magnetic field line, and makes the system be nonlinearly unstable. This dissipative instability system in the presence of thermal fluctuations is of our interest. The reduced set of equations for the electrostatic potential ϕ, current J and pressure p is employed to describe the system.[31]

The dynamics of micro fluctuations are studied in the presence of global inhomogeneity of plasma pressure. Quantities that are averaged over the (y, z)-plane are denoted by suffix 0, as p_0 and ϕ_0; We set $\phi = \phi_0 + \tilde{\phi}$, $J = J_0 + \tilde{J}$ and $p = p_0 + \tilde{p}$. The pressure and electrostatic potential could be inhomogeneous (i.e., inhomogeneous in the $\hat{x}$-direction) in the global scale. Parameters ∇p_0 and $\nabla_\perp^2 \phi_0$ together with Ω' represent the inhomogeneity of the system. The scale separation is introduced, in this article, between the dynamics of the micro fluctuations and macroscopic structures: $|p_0^{-1}\, \partial p_0/\partial t| << |\tilde{p}^{-1}\, \partial \tilde{p}/\partial t|$, and $|p_0^{-1} \nabla p_0| << |\tilde{p}^{-1} \nabla \tilde{p}|$. The symbol ~ which denotes the fluctuating field components is suppressed for the simplicity of expression.

This system has a strong instability source due to the presence of inhomogeneities, and the product of pressure gradient and magnetic field inhomogeneity,

$$G_0 = \Omega' p_0', \tag{1}$$

denotes the driving parameter, being fixed in time in this article.

With the help of assumption of space-time scale separation, the dynamical equations of fluctuation fields are given as

$$\frac{\partial}{\partial t} f + \mathcal{L}^{(0)} f = \mathcal{N}(f) + \tilde{S}_{th}, \tag{2}$$

where $\mathcal{L}^{(0)}$ denotes the linear operator

$$\mathcal{L}^{(0)} = \begin{pmatrix} -\mu_c \nabla_\perp^2 & -\nabla_\perp^{-2}\nabla_{\parallel} & -\nabla_\perp^{-2}\Omega' \frac{\partial}{\partial y} \\ \xi \nabla_{\parallel} & -\mu_{ec}\nabla_\perp^2 & 0 \\ -\frac{dp_0}{dx}\frac{\partial}{\partial y} & 0 & -\chi_c \nabla_\perp^2 \end{pmatrix}, \tag{3}$$

$\boldsymbol{f}$ denotes the fluctuating field,

$$\boldsymbol{f} = \begin{pmatrix} \phi \\ J \\ p \end{pmatrix}. \tag{4}$$

and $\mathcal{N}(\boldsymbol{f})$ stands for the nonlinear terms

$$\mathcal{N}(\boldsymbol{f}) = -\begin{pmatrix} \nabla_\perp^{-2}[\phi, \nabla_\perp^2 \phi] \\ [\phi, J] \\ [\phi, p] \end{pmatrix}. \tag{5}$$

The bracket $[f, g]$ denotes the Poisson bracket,

$$[f, g] = (\nabla f \times \nabla g) \cdot \boldsymbol{b},$$

($\boldsymbol{b} = \boldsymbol{B}_0/B_0$), $\Delta_\perp = \nabla_\perp^2$, Ω' is the average curvature of the magnetic field, Ψ is the vector potential, and $1/\xi$ denotes the finite electron inertia, $1/\xi = (\delta/a)^2$, δ being the collisionless skin depth. Length, time, static potential and pressure are normalized to the global plasma size a, the Alfven transit time $\tau_{Ap} = a/v_{Ap}$, $av_{Ap}B_0$ and $B_0^2 R/2a\mu_0$, respectively (a and R are minor and major radii of torus, $v_{Ap} = B_0(2\mu_0 m_i n_i)^{-1/2} aRq^{-1}$, m_i is the ion mass, and n_i is the ion density; see ref.30 for details). (It is also noted that the study of the response to $\mathcal{L}^{(0)}$ corresponds to the conventional application of RDT in neutral fluid.)

The electron inertia effect should be kept, because this effect is amplified by the nonlinear shielding effect of the turbulence.[32, 33] The classical resistivity is neglected for the simplicity of the argument. Three field equations in the presence of thermal excitations are now employed: i.e., the equation of motion, the Ohm's law and the energy balance equation. The electrostatic approximation is used, i.e., the inductive electric field in the Ohm's law and the nonlinear terms of the form $[\Psi, \cdots]$ are neglected. (See refs. 34 and 35 for high-β cases.) The interchange mode (CDIM) has a quasi-2 dimensional nature, $|\nabla_{\parallel}^2| << |\nabla_{\perp}^2|$; nevertheless, the existence of small but finite $\nabla_{\parallel}$ is essential.

We consider the thermal fluctuations in the range of plasma frequency ω_p and the time scale between microscopic mode considered here (CDIM) and the thermal fluctuation is well separated. In the thermal fluctuations, coherent parts to the microscopic CDIM are given by the collisional transport coefficients μ_c, μ_{ec} and χ_c (the ion viscosity, electron viscosity and thermal diffusivity, respectively). Incoherent parts are considered to be a random noise and expressed as $\tilde{S}_{th}$.[3)]

2.2 Langevin equation

The system which has a large number of degrees of freedom and has many positive Lyapunov exponents is considered. A part of the Lagrangean nonlinearity is considered to cause the drag to this collective mode (CDIM) and this part is renormalized to the eddy-viscosity type nonlinear transfer rate γ_j. The other part is regarded as a random noise, which has a faster decorrelation time than γ_j according to RCM.[29)] As has been discussed in I-III, a projection operator $\mathcal{P}$ is introduced to divide the nonlinear interactions into the drag and others. The nonlinear drag term is written in an apparent linear term as

$$\mathcal{P}\mathcal{N}(f) = \begin{pmatrix} \mu_N \nabla_\perp^2 f_1 \\ \mu_{Ne} \nabla_\perp^2 f_2 \\ \chi_N \nabla_\perp^2 f_3 \end{pmatrix} = - \begin{pmatrix} \gamma_1 f_1 \\ \gamma_2 f_2 \\ \gamma_3 f_3 \end{pmatrix} \tag{6}$$

and the rest part is rewritten as

$$\tilde{S} = (1 - \mathcal{P})\mathcal{N}(f) \tag{7}$$

Then a Langevin equation is derived as[23, 26-29, 36)]

$$\frac{\partial}{\partial t} f + \mathcal{L} f = \tilde{S} + \tilde{S}_{th} \tag{8}$$

with

$$\mathcal{L}_{ij} = \mathcal{L}_{ij}^{(0)} + \gamma_i \delta_{ij} \tag{9}$$

(δ_{ij} is the Kronecker's delta) and

$$\tilde{S} = \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \end{pmatrix}. \tag{10}$$

Notation here follows the convention in ref.36. In this article, suffix $i,\ j = 1,2,3$ denotes the i-th or j-th field. In the following, Fourier transformation is used, and $k,\ p,\ q$ describes the wave number of Fourier components. Suffix $k,\ p,\ q$ is often omitted unless confusion is caused.

The operator to the k-th component, $\mathcal{L}_k$,

$$\mathcal{L}_k f_k = \mathcal{L}_{0,\,k} f_k - \mathcal{P}_k \mathcal{N}_k(f), \tag{11}$$

is the renormalized operator, which includes the renormalized transfer rates of

$$\gamma_{i,\,k} = -\sum_{\Delta} M_{i,\,kpq} M_{i,\,qkp}^{*} \theta_{qkp}^{*} \left| \tilde{f}_{1,p}^{2} \right| . \tag{12}$$

The self-noise has a much shorter correlation time as is discussed in I-III, and is approximated to be given by the Gaussian white noise term $w(t)$ as

$$\tilde{S}_{i,\,k} = w(t)\tilde{g}_{i,\,k} \equiv w(t)\sum_{\Delta} M_{i,\,kpq}\sqrt{\theta_{kpq}}\,\zeta_{1,\,p}\zeta_{i,\,q}\;. \tag{13}$$

In these expressions, summation Δ indicates the constraint $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$. The explicit form of the nonlinear interaction matrix is given as, e.g., $M_{1,\,kpq} = ((\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{b})(p_\perp^2 - q_\perp^2)k_\perp^{-2}$, or $M_{(2,\,3),\,kpq} = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{b}$, and the propagator satisfies the relation $(\partial/\partial t + \mathcal{L}(k) + c.p.)\theta_{kpq} = 1$, where c.p. indicates the counter part, i.e., $\mathcal{L}(p) + \mathcal{L}(q)$.[36]

The term $\zeta_{j,\,p}$ in a random noise represents the j-th field of q-component in the nonlinear term $\mathcal{N}$, and their correlation functions satisfy the average relations of the mode, which we call an Ansatz of equivalence in correlation in the following, as

$$\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle \tag{14}$$

with

$$\langle \zeta_{i,\,p} \zeta_{j,\,q} \rangle \propto \delta_{pq} \tag{15}$$

where the bracket $\langle\;\rangle$ indicates the statistical average.

The thermal excitation is also assumed to be a Gaussian white noise,

$$\tilde{S}_{th,\,i} = w(t)\tilde{g}_{th,\,i} \tag{16}$$

The statistical independence between the incoherent parts of thermal and turbulent fluctuations is also imposed, that is,

$$\langle \tilde{S}_{th,\,i} \tilde{S}_j \rangle = 0 \tag{17}$$

2.3 One branch approximation and Fokker-Planck equation

The basic set of equation, which describes interchange mode (or CDIM), also contains two others branches of plasma mode. These additional two branches are much more stable, and are considered to be excited up to much smaller amplitude. Based on this approximation, the Langevin equation is reduced to that for only one branch. The detailed procedure of decomposition is described in the previous articles II and III.

The matrix $\exp[-\mathcal{L}(t-\tau)]$, which provides the solution of the Langevin equation, was explicitly expressed as

$$\exp[-\mathcal{L}(t-\tau)] = \boldsymbol{A}\exp(-\lambda_1(t-\tau)) + \boldsymbol{A}^{(2)}\exp(-\lambda_2(t-\tau)) + \boldsymbol{A}^{(3)}\exp(-\lambda_3(t-\tau)) \quad (18)$$

where the elements of matrix $\boldsymbol{A}$ are given as

$$\boldsymbol{A} = \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\begin{pmatrix} (\bar{\gamma}_e-\lambda_1)(\bar{\gamma}_p-\lambda_1) & \dfrac{-ik_{\parallel}(\bar{\gamma}_p-\lambda_1)}{k_\perp^2} & \dfrac{-ik_y\Omega'(\bar{\gamma}_e-\lambda_1)}{k_\perp^2} \\ -ik_{\parallel}\xi(\bar{\gamma}_p-\lambda_1) & \dfrac{\xi k_{\parallel}^2}{k_\perp^2}\dfrac{(\lambda_1-\bar{\gamma}_p)}{(\lambda_1-\bar{\gamma}_e)} & \dfrac{-\xi k_{\parallel}k_y\Omega'}{k_\perp^2} \\ ip_0'k_y(\bar{\gamma}_e-\lambda_1) & \dfrac{k_{\parallel}k_y p_0'}{k_\perp^2} & \dfrac{G_0 k_y^2}{k_\perp^2}\dfrac{(\bar{\gamma}_e-\lambda_1)}{(\lambda_1-\bar{\gamma}_p)} \end{pmatrix} \quad (19)$$

where $-\lambda_m$ ($m = 1,2,3$ and $\lambda_1 < \lambda_2 < \lambda_3$) represents the eigenvalue of the non-normal matrix $\mathcal{L}$, which gives the homogeneous solution of eq.(8) if $\mathcal{L}$ is constant. The eigenvalue is determined by:

$$\det(\lambda \boldsymbol{I} + \mathcal{L}) = 0 \quad (20)$$

and $\boldsymbol{I}$ is a unit tensor. The eigenvector with the eigenvalue $-\lambda_1$ corresponds to the least stable branch, the decay time of which is the longest. Others with $(-\lambda_2, -\lambda_3)$ denote highly-stable branches, which decay much faster. (Elements $A_{ij}^{(2,3)}$ are also obtained in a similar way, and are given in II, being not repeated here.)

In the one branch approximation, only the pole of $(s + \lambda_1)^{-1}$ is kept. Then, the Langevin equation is deduced to that of one field, e.g., $f_1 = \phi$, as is discussed in II,

$$\frac{\partial}{\partial t}\phi + \lambda_1 \phi = \tilde{s} \tag{21}$$

with the source of

$$\tilde{s}_k = w(t)\left(g_k + g_{th,\,k}\right) \tag{22}$$

In this expression, both the contributions of the turbulent self noise and thermal noise are retained. The magnitude of the noise source is given by use of the matrix $\boldsymbol{A}$ as

$$g_k = \Re e\left(\sum_{j=1}^{3} A_{1j} g_{j,\,k}\right), \tag{23}$$

and

$$g_{th,\,k} = \sum_{j=1}^{3} A_{1j} g_{th,\,j,\,k} \tag{24}$$

By retaining the real part in eq.(23), the possible problem of complex quantity of $g_{i,\,k}$ is eliminated, and the diffusion process is assured in the Fokker-Planck equation. The coefficient g_k is statistically independent for each k-component, $\langle g_k g_{k'} \rangle = \left\langle\left| g_k^2 \right|\right\rangle \delta_{k,\,k'}$, $\langle g_{th,\,k} g_{th,\,k'} \rangle = \left\langle\left| g_{th,\,k}^2 \right|\right\rangle \delta_{k,\,k'}$ and $\langle g_k g_{th,\,k'} \rangle = 0$. Then the Liouville equation is reduced to a Fokker-Planck equation [37]. For the probability distribution function $P\left(\{\phi_k\}\right)$, one has a Fokker-Planck equation as

$$\frac{\partial}{\partial t}P = \sum_k \frac{\partial}{\partial \phi_k}\left(\lambda_{1,\,k}\phi_k + \frac{1}{2}\hat{g}_k \frac{\partial}{\partial \phi_k}\hat{g}_k\right)P\,. \tag{25}$$

The diffusion coefficient, $\hat{g}_k$, contains two statistically independent noises and the form

$$\hat{g}_k^2 = g_k^2 + g_{th,\, k}^2 \tag{26}$$

is employed.

From the fluctuation-dissipation (FD) theorem for the thermodynamical equilibrium, the thermal excitation rate is expressed in terms of the temperature as[28]

$$\sum_{j,\, j'} A_{1j} A_{1j'}^* \tilde{S}_{th,\, j} \tilde{S}_{th,\, j'} = 2\mu_{vc} \hat{T}. \tag{27}$$

where the normalized temperature (with an additional dimension of volume) is introduced as

$$\hat{T} = \frac{2\mu_0}{B_p^2} k_{\mathrm{B}} T\ . \tag{28}$$

The thermal contribution to the diffusion term is rewritten as

$$g_{th,\, k}^2 = 2\mu_{vc} \hat{T} \tag{29}$$

This term is independent of the choice of test mode number k, so long as collisional viscosity is independent of the scale size.

§3 Steady State Solution of Fokker-Planck Equation and Characteristics

3.1 Probability distribution function

Steady state probability distribution function P_{eq} satisfies the equation

$$\sum_k \frac{\partial}{\partial \phi_k} \left(\lambda_{1,k} \phi_k + \frac{1}{2} \hat{g}_k \frac{\partial}{\partial \phi_k} \hat{g}_k \right) P_{eq} = 0 \tag{30}$$

An equilibrium distribution function satisfies the detailed balance,

$$\left(2\lambda_{1,k}\phi_k + \hat{g}_k \frac{\partial}{\partial \phi_k} \hat{g}_k\right) P_{eq} = 0 \qquad \text{(for all } k\text{)} \tag{31}$$

The constant of integral (right hand side) is chosen to be zero, based on the boundary condition that the relation $\phi_k P_{eq} = \partial P_{eq}/\partial \phi_k = 0$ holds as $\phi_k \to \infty$. (The condition that P_{eq} vanishes much faster than ϕ_k^{-1} as $\phi_k \to \infty$ is a necessary condition for the requirement that an integral of P_{eq} could be normalized to unity.)

The detailed-balance equation suggests the form of equilibrium distribution function as

$$P_{eq}\left(\left\{\phi_k\right\}\right) = \bar{P} \prod_k \frac{1}{\hat{g}_k} \exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k \, \hat{g}_k^{-2} \, d\phi_k\right\} \tag{32}$$

where $\bar{P}$ is a normalization constant. This equation is an exact solution if g_k is independent of $\phi_{k'}$ for $k' \neq k$. The solution is obtained by the help of ansatz of a large number of degrees of freedom. Although $\phi_{k'}$ makes an influence on the value of g_k, the influence from one particular $\phi_{k'}$ on g_k is weak. This is because many numbers of fluctuating components statistically contribute to g_k through γ_v, γ_j and γ_p, so that the most sensitive dependence of ϕ_k on P_{eq} appears through the term of $\exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k \, \hat{g}_k^{-2} \, d\phi_k\right\}$. Based on this fact, an approximate solution of the equilibrium distribution function is given as eq.(32).

3.2 Overview of distribution function

Let us discuss the qualitative characteristics of the equilibrium distribution function P_{eq}, taking an example analysis on current-diffusive interchange mode (CDIM). The basic concepts of this method apply to other modes of plasma turbulence. The analysis has been done by solving the Langevin equation .[27] By employing a one-pole approximation, where only the least stable branch is considered, the eigenvalue has been obtained [27], and the interpolation formula to fit the eigenvalue λ_1 is given as

$$-\lambda_1 = \frac{G_0^{3/5}}{s^{2/5}} \left(\frac{\delta}{a}\right)^{2/5} \bar{\mu}_e^{1/5} k_\perp^{4/5} - \bar{\mu}_v k_\perp^2 , \tag{33}$$

where $\bar{\mu}_v = \mu_N + \mu_c$ and $\bar{\mu}_e = \mu_{Ne} + \mu_{ec}$.[38] The first term in the right hand side is destabilization by the current diffusivity (turbulent one as well as collisional one), the formula for which was derived for the small dissipation limit near marginal point. The second term in the right hand side represents the damping by the effective viscosity. The nonlinear marginal stability condition, $\lambda_1 = 0$, which is deduced from this interpolation formula, provides qualitatively correct result in comparison with the direct solution of original equation. The decorrelation rate λ_1 is plotted in Fig.1 against the eddy damping rate γ_v (both being normalized to γ_{vc}) for various values of G_0 that represents the plasma gradient. The fluctuation level $I_k = \phi_k^2$ and the eddy damping rate γ_v are related to each other. The fluctuation spectrum is a rapidly decaying function of k, and an approximate relation has been derived as[38, 39]

$$\gamma_v(\gamma_v + \gamma_{v,c}) \cong k_\perp^6 I_k \tag{34}$$

In the presence of both the thermal and turbulent excitations of CDIM, the relations between the fluctuation level of this branch I_k and the nonequilibrium parameter G_0 and the temperature has been obtained in II. One example with the hysteresis characteristics between I_k and G_0 is shown in Fig.2 for a fixed temperature of $\bar{T} = 0.003$. (In Fig.2, normarived parameters for gradient fluctuation level and temparature are used following the convention of III as $\hat{G}_0 = \left(\delta k_\perp\right)^{2/3} s^{-2/3} \gamma_{vc}^{-4/3} G_0$, $\hat{I} = k_\perp^6 \gamma_{vc}^{-2} I_k$ and $\bar{T} \equiv k_\perp^4 \gamma_{vc}^{-2} \hat{T}$).[28] This system is subject to subcritical excitation of turbulence.[39] In Fig.3, the statistically-averaged values (being normalized) of eddy damping time γ_v and decorrelation rate λ_1 are shown as functions of the gradient G_0. In the branch (left) which is connected to the thermodynamical equilibrium limit

$(G_0 \to 0)$, the decorrelation rate λ_1 decreases as the gradient parameter G_0 increases. This decrease corresponds to an access to the linearly unstable regime, showing that the correlation time becomes longer. As G_0 becomes large, λ_1 does not converge to zero, but starts to increase again owing to the bifurcation to the branch of strong turbulence (right).

A bird's eye view of statistical average of fluctuation amplitude, the relation $\langle \hat{I} \rangle \left[\hat{G}_0, \hat{T} \right]$, is shown in Fig.4 (conceptual). The cusp type manifold structure is seen. In the low temperature limit, the function $\langle \hat{I} \rangle \left[\hat{G}_0; \mathit{fixed}\ \hat{T} \right]$ has a hysteresis. The upper plane represents the turbulent fluctuation level, and the lower plane corresponds to the one excited by the thermal fluctuation. The middle one satisfies the neutral condition, but is an unstable branch: a small deviation from this branch leads to convergence to either upper or lower branch.

These features of the fluctuation level and eigenvalue λ_1 illustrate an important role of gradient parameter G_0. Dependencies on the global plasma parameters have been obtained.[32] Typical values of the wave number, k_0, turbulent visocsity, $\mu_{N,eq}$, and the eddy damping rate, $\gamma_{k,eq}$, and fluctuation amplitude $\phi_{k,eq}^2$, are given in a strong turbulence limit as[40, 41]

$$k_0 \simeq G_0^{-1/2} s a \delta^{-1} \tag{35a}$$

$$\mu_{N,eq} \simeq G_0^{-3/2} s^{-2} \delta^2 a^{-2} \tag{35b}$$

$$\gamma_{k,eq} = \mu_{N,eq} k_0^2 \simeq G_0^{1/2} \tag{35c}$$

and

$$\phi_{k,eq}^2 = \phi_{eq,0}^2 \left(k_\perp / k_0 \right)^{-6} \simeq G_0 k_\perp^{-6} . \qquad (k_\perp > k_0) \tag{35d}$$

By use of these expressions, order-of-magnitude estimate of various terms are available.

The eigenvalues λ_1 (in the limit of $T \to 0$) for cases (a), (b) and (c) in Fig.1 correspond to those along the lines (a), (b) and (c) in Fig.2. The solid line in Fig.2 is the statistical average of spectrum, being given by the solution of the Langevin equation. The distribution function along lines (a)-(c) are discussed as typical examples below.

3.2.1 Nonequilibrium but near thermodynamical equilibrium (case A)

If the turbulent driving force is small as in the case of (a) in Fig.2, the mode is mainly excited by thermal excitations. The noise term of eq.(26) which causes the diffusion is determined by $g_{th,k}^2$ and the damping rate λ_1 approaches to $\gamma_{vc} = \mu_{vc} k_\perp^2$ in the thermodynamical equilibrium limit. Substitution of these estimates into eq.(32) gives a limiting solution

$$P_{eq}\left(\{\phi_k\}\right) = \bar{P} \prod_k \frac{1}{g_{th,k}} \exp\left\{ - \int^{\phi_k} \frac{\lambda_{1,k}\left(\phi_k^2 \to 0\right)}{g_{th,k}^2} \, \mathrm{d}\phi_k^2 \right\} \tag{36}$$

Noting the relation $\lambda_1 \to \mu_{vc} k_\perp^2$ and eq.(29) in this limit, eq.(36) is rewritten as

$$P_{eq}\left(\{\phi_k\}\right) \to \bar{P} \prod_k \frac{1}{\sqrt{2\mu_{vc}\hat{T}}} \exp\left(- \frac{k_\perp^2 \phi_k^2}{2\hat{T}} \right) \tag{37a}$$

By using the kinetic energy of fluctuations,

$$\mathcal{E}_k \equiv \frac{1}{2} k_\perp^2 \phi_k^2 ,$$

eq.(37a) is expressed as

$$P_{eq}\left(\left\{\mathcal{E}_k\right\}\right) \propto \frac{1}{\sqrt{\hat{T}}} \exp\left\{-\sum_k \frac{\mathcal{E}_k}{\hat{T}}\right\} \tag{37b}$$

i.e., the Boltzmann relation. The gaussian distribution near thermodynamical equilibrium in refs.42-45 is confirmed, and it is clear from eq.(37b) that the equipartition law holds for fluctuation energy.

3.2.2 Regimes of nonlinear excitation and linear excitation (case (b) and (c))

As the turbulent excitation becomes stronger, i.e., G_0 becomes larger, the turbulence is excited nonlinearly and is sustained. In the case of (b) in Fig.2, both the thermal fluctuation and turbulent fluctuation are possible to exist, and determins the distribution function P_{eq} as an average. Two paths of integration are considered in evaluating the equilibrium distribution function. On one path, ϕ_k is varied and others are fixed

$$\phi_k = \ell\ \phi_{k,0} \ \text{ but } \left\{\phi_{k'}\right\} = \textit{fixed} \ \ (\text{all } k',\ k' \neq k\,) \ \ \text{path I.} \tag{38a}$$

On another path, all the fluctuation components change as

$$\{\phi_k\} = \{\ell\ \phi_{k,0}\}, \qquad (\text{all } k\,) \qquad \text{path II} \tag{38b}$$

In the latter case, the term g_k^2 in eq.(26) is in proportion to $|\phi|_k^3$ in a strong turbulence limit. The dependence of λ_1 on the fluctuation amplitude is also deduced from eqs.(33) and (34). As a result, the schematic drawings of equilibrium probability distribution functions are shown in Fig.5(a), (b) and (c). Figure 5(a) corresponds to the case near the thermodynamical equilibrium, showing that the distribution function is close to the Gaussian. Figure 5 (b) and (c) represent cases of (b) and (c) in Fig.2. In the case of (b), λ_1 vanishes at two values of the fluctuation amplitude as is shown in Fig.1, and accordingly a pair of local minimum and local maximum are observed in Fig.5(b). Two peaks are observed; the one near

$$|\phi_k| \simeq 0 \tag{39}$$

comes from the thermal excitation, and the other peak around

$$\lambda_1 \simeq 0 \tag{40}$$

is contributed from the nonlinearly excited turbulence fluctuation. Both peak values and widths vary depending on the plasma parameters. The case (c) corresponds to the strong turbulent limit, where the thermal excitation is unimportant. In this case, g_k^2 and λ_1 have asymptotic dependencies $g_k^2 \propto |\phi_k|^3$ and $\lambda_k \propto |\phi_k|$, respectively, along the path II. Substitution of these estimates into eq.(32) shows that the asymptotic relation

$$\lambda_{1,k}\phi_k \hat{g}_k^{-2} \propto \phi_k^{-1}$$

holds along the path II. The integral has a logarithmic dependence as

$$\int^{\phi_k} 2\lambda_{1,k}\phi_k \hat{g}_k^{-2} \, d\phi_k \propto \ln\left(|\phi|\right) \tag{41}$$

Namely, the equilibrium distribution has the tail component in a large $|\phi_k|$ regime as

$$P_{eq}\left(\left\{\phi_k \to \infty\right\}\right) \propto \frac{1}{|\phi|^{3/2}} \frac{1}{|\phi|^{\alpha}} \tag{42}$$

where α is a numerical factor, and is discussed later. Equation (42) shows the power-law dependence of equilibrium distribution function.

3.3 Characteristics of distribution function

Several characteristic features are drawn from the equilibrium distribution function of probability.

The average

An integral $\langle \phi_k^2 \rangle = \int d\phi \, P_{eq} \phi_k^2$ yields the average $\langle \phi_k^2 \rangle$, where ϕ represents a set of $\{\phi_k\}$ for all the k-th component and the notation $d\phi = \prod_k d\phi_k$ is used. Multiplying ϕ_k to the relation eq.(31) and integrating the second term by part, one has

$$\int d\phi \, P_{eq} \phi_k^2 = \int d\phi \, \hat{g}_k P_{eq} \frac{\partial}{\partial \phi_k} \left(\frac{\hat{g}_k \phi_k}{2\lambda_{1,k}} \right) . \tag{43}$$

The ansatz of a large number of degrees of freedom is used to approximate g_k and $\lambda_{1,\,k}$ to be slowly varying functions of ϕ_k, and the derivative in the right hand side of eq.(43) is evaluated by $\partial\left(\hat{g}_k \lambda_{1,k}^{-1} \phi_k\right) / \partial \phi_k \simeq \hat{g}_k \lambda_{1,k}^{-1}$. Substituting this evaluation into the right hand side of eq.(43), one obtains the relation between two moments as

$$\int d\phi \, \phi_k^2 P_{eq} = \int d\phi \, \frac{\hat{g}_k^2}{2\lambda_{1,k}} P_{eq} . \tag{44}$$

This relation, $\langle \phi_k^2 \rangle = \langle \hat{g}_k^2 / 2\lambda_{1,k} \rangle$, is considered as an extended FD Theorem of the second kind. If an Ansatz of equivalence of correlation, $\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle$ and the decomposition approximation like $\langle \phi\phi\phi\phi \rangle \sim \langle \phi\phi \rangle \langle \phi\phi \rangle$ are imposed to estimate $\langle \hat{g}_k^2 \rangle$ as in I-III, the relation equation by use of the Langevin equation approach is obtained.

The peak

The peak of the equilibrium probability distribution is given by $\frac{\partial}{\partial \phi_k} P_{eq} = 0$, which satisfies the condition

$$2\lambda_{1,k} \phi_k + \hat{g}_k \frac{\partial}{\partial \phi_k} \hat{g}_k = 0 \qquad (\text{at } \phi_k = \phi_{k,\,p}) . \tag{45}$$

If the random source term is independent of the fluctuation field (*e.g.*, the case of an external random forcing), the relation $\partial \hat{g}_k^2 / \partial \phi_k^2 = 0$ holds. Then the condition, $\lambda_1 = 0$, provides the peak of the equilibrium probability distribution function. This estimate has been used in refs 38-41, 46, 47 to obtain the $k-$ and $\omega-$ spectrum of fluctuations.

In the strong turbulence limit, $\lambda_{1,k}$ and $\hat{g}_k^2$ in eq.(45) are modelled along the path II of eq.(38) as

$$\lambda_1 \simeq \lambda_0 + \lambda_1' |\phi_k| \text{ and } g_k^2 = g_{k0}^2 |\phi_k|^3 . \tag{46}$$

Using this approximate fitting form, one can obtain the peak position as

$$|\phi_k|_{peak} \sim \frac{-\lambda_0}{\lambda_1' + 3 g_{k0}^2/4} \tag{47}$$

instead of $|\phi_k|_{peak} \sim -\lambda_0/\lambda_1'$ which is deduced from the condition $\lambda_1 = 0$.

The width

The denominator $\hat{g}_k^{-2}$ in the integrand in the term $\exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k \hat{g}_k^{-2}\, d\phi_k\right\}$ dictates the width of the distribution function.

Near the thermodynamical equilibrium limit, like the case of (a) in Fig.5(a), the width is well approximated by the plasma temperature. In the case of (b), the width of the peak near $\phi \simeq 0$ becomes broader and is modified to have a power dependence due to the turbulent excitation. In the strong turbulent limit, cases of Fig.5(c) as well as the second peak in Fig.5(b), the width of the distribution is given comparable to the averaged value, i.e., $\left\langle \left(\phi_k - \sqrt{\langle \phi_k^2 \rangle}\right)^2 \right\rangle \sim \langle \phi_k^2 \rangle$. Namely, the variance is comparable to the average value itself.

The tail component

The fact that the random noise level, g_k, depends on the turbulence level leads to a possibility of power-law in the tail of probability distribution. This situation is different from the case of thermal fluctuation, where the random noise level is ultimately determined by the temperature.

The rough estimate of the tail component of eq.(42) is done by use of the approximation eq.(46) with $\lambda_0 = 0$. This gives the relation between the power index of α in eq.(42) and the coefficients λ_1' and g_{k0}^2 in eq.(46) as $\alpha = 2\lambda_1' g_{k0}^{-2}$.

3.4 Accessibility

Accessibility to the equilibrium distribution function, P_{eq}, is shown by constructing a Lyapunov function

$$\mathcal{H}(t) = \int d\phi \, P(\phi;\, t) \, \ell n\left(\frac{P(\phi;\, t)}{P_{eq}(\phi)}\right) \tag{48}$$

where ϕ represents a set of $\{\phi_k\}$ for all the *k*-th components. Taking the time derivative and assuming the Markov process [1], one has

$$\frac{d}{dt}\mathcal{H}(t) = \int d\phi \left[\left\{\frac{d}{dt}P(\phi;\, t)\right\} \ell n\left(\frac{P(\phi;\, t)}{P_{eq}(\phi)}\right)\right] \tag{49}$$

Substituting dP/dt of eq.(25) into eq.(49) and performing a partial integration, one obtains

$$\frac{d}{dt}\mathcal{H}(t) = -\int d\phi \left[\sum_k \left(\lambda_{1,k}\phi_k P + \frac{1}{2}\hat{g}_k \frac{\partial}{\partial \phi_k}\hat{g}_k P\right)\left\{\frac{\partial}{\partial \phi_k} \ell n\left(\frac{P(\phi;\, t)}{P_{eq}(\phi)}\right)\right\}\right] \tag{50}$$

The terms in the right hand side of eq.(50) are calculated. By use of the relation $\hat{g}_k \frac{\partial}{\partial \phi_k}\left(\hat{g}_k P\right) = \hat{g}_k^2 P \frac{\partial}{\partial \phi_k} \ell n\left(\hat{g}_k P\right)$, eq.(50) is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = -\int \mathrm{d}\phi \left[\sum_k \frac{1}{2}\hat{g}_k^2 P \left(\frac{2\lambda_{1,k}\phi_k}{\hat{g}_k^2} + \frac{\partial}{\partial\phi_k} \ell n\left(\hat{g}_k P\right) \right) \left\{ \frac{\partial}{\partial\phi_k} \ell n\left(\frac{P(\phi;t)}{P_{eq}(\phi)} \right) \right\} \right] . \quad (51)$$

Noting identities of eq.(31) together with $\hat{g}_k\, \partial\left(\hat{g}_k P_{eq}\right)/\partial\phi_k = \hat{g}_k^2 P_{eq}\, \partial\, \ell n\left(\hat{g}_k P_{eq}\right)/\partial\phi_k$, one has the relation

$$2\lambda_{1,k}\phi_k \hat{g}_k^{-2} = -\partial\, \ell n\left(\hat{g}_k P_{eq}\right)/\partial\phi_k \, . \quad (52)$$

Substituting this expression into eq.(51), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = -\int \mathrm{d}\phi \left[\sum_k \frac{g_k^2 P}{2} \left\{ \frac{\partial}{\partial\phi_k} \ell n\left(\frac{P(\phi;t)}{P_{eq}(\phi)} \right) \right\}^2 \right] \quad (53)$$

The integrand is positive or zero, and an inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \le 0 \quad (54)$$

holds. The condition $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = 0$ is satisfied if the probability distribution function is equal to the equilibrium distribution function $P(\phi;t) = P_{eq}(\phi)$. The fact that the construction of the Lyapunov function is possible means that one of steady state distributions is guaranteed in a time-asymptotic behaviour.

§4 Extension to thermodynamical quantities

4.1 Fokker-Planck equation for macro variable (coarse-grained quantity)

In order to study the statistical property of thermodynamical quantities, we employ the dynamical equations of macro variables which are quantities integrated over some finite-size volume. The total fluctuating energy in this voulme

$$\mathcal{E} \equiv \frac{1}{2} \sum_k k_\perp^2 \phi_k^2 \tag{55}$$

and the average potential

$$\Phi^2 \equiv \sum_k \phi_k^2 \tag{56}$$

are taken as examples.

4.1.1 Langevin equation

From eq.(21), one has

$$\frac{1}{2} \frac{\partial}{\partial t} \sum_k k_\perp^2 \phi_k^2 + \sum_k \lambda_{1,k} k_\perp^2 \phi_k^2 = \sum_k k_\perp^2 \tilde{s}_k \phi_k \tag{57}$$

By introducing a time constant

$$\Lambda \equiv \frac{\sum_k \lambda_{1,k} k_\perp^2 \phi_k^2}{\sum_k k_\perp^2 \phi_k^2} , \tag{58}$$

the Langevin equation for the total fluctuating energy is given as

$$\frac{\partial}{\partial t} \mathcal{E} + 2\Lambda \mathcal{E} = \sum_k k_\perp^2 \tilde{s}_k \phi_k \tag{59}$$

In a similar manner, one obtains from eq.(21) as

$$\frac{1}{2} \frac{\partial}{\partial t} \sum_k \phi_k^2 + \sum_k \lambda_{1,k} \phi_k^2 = \sum_k \tilde{s}_k \phi_k \tag{60}$$

The equation for the average potential is derived as

$$\frac{\partial}{\partial t} \Phi^2 + 2\Lambda_\phi \Phi^2 = 2 \sum_k \tilde{s}_k \phi_k \tag{61}$$

with another time constant

$$\Lambda_\phi \equiv \frac{\sum_k \lambda_{1,k}\,\phi_k^2}{\sum_k \phi_k^2} . \tag{62}$$

4.1.2 Fokker-Planck equation

Following the similar procedure of §3, the probability distribution function of the coarse grained quantity, $\mathcal{E} \equiv \frac{1}{2}\sum_k k_\perp^2 \phi_k^2$ or Φ^2, is described by the Fokker-Planck equation. In the Langevin equation for the average energy $\mathcal{E}$, the magnitude of the statistical source term is written as $g^2 = \sum_k \tilde{s}_k^2 k_\perp^4 \phi_k^2$, i.e.,

$$g^2 = \sum_k 2\mu_{vc}\hat{T}\, k_\perp^4 \phi_k^2 + \sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \tag{63}$$

where the cross terms between the thermal noise and self noise vanish based on the assumption of mutual statistical independence. The cross terms between different *k*-components also vanish, owing to the independence property of the source term. In deriving this expression, eq.(29) is used to rewrite the contribution of thermal excitations. It is also useful to introduce an average of the classical decorrelation rate

$$\gamma_m \equiv \frac{\sum_k \gamma_{vc} \mathcal{E}_k}{\mathcal{E}} \tag{64}$$

($\gamma_{vc} = \mu_{vc} k_\perp^2$). When the fluctuation level changes, this coefficient might deviate form the value in the limit of thermodynamical equilibrium. The amplitude of the source term is rewritten as

$$g^2 = 4\hat{T}\gamma_m \mathcal{E} + \sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \tag{65}$$

The Fokker-Planck equation for the probability distribution function $P(\mathcal{E})$ is given as

$$\frac{\partial}{\partial t}P(\mathcal{E}) = \frac{\partial}{\partial \mathcal{E}}\left(2\Lambda\mathcal{E} + \frac{1}{2}\, g\, \frac{\partial}{\partial \mathcal{E}}\, g\right)P(\mathcal{E}) \tag{66}$$

4.1.3 Amplitude dependencies of the drag term and noise source term

4.1.3(a) Thermodynamical equilibrium limit

In the thermodynamical equilibrium limit, $\phi_k \to 0$, the self-noise term vanishes, and the relation

$$\gamma_m \mathcal{E} \to \sum_k \gamma_{vc} \mathcal{E}_{k,\, th} \tag{67}$$

holds, where $\mathcal{E}_{k,\, th}$ is the amplitude in the thermodynamical equilibrium. One has

$$g^2 = 4\hat{T} \sum_k \gamma_{vc} \mathcal{E}_{k,\, th} \quad . \tag{68}$$

In the limit of zero-amplitude fluctuation, the eigenvalue λ_1 approaches to γ_{vc} . As a result, the average damping rate Λ converges as

$$\Lambda \to \gamma_m \qquad \text{or} \qquad \Lambda\mathcal{E} \to \sum_k \gamma_{vc} \mathcal{E}_{k,\, th} \, . \tag{69}$$

4.1.3(b) Strong turbulent with large amplitude limit

It is generally shown in a strong turbulent limit [II] that the relation

$$\left(\sum_{j=1}^{3} A_{1j} g_{j,\, k}\right)^2 \propto |\phi|^3 \tag{70}$$

holds. Based on this property, one has a dependence

$$\sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \propto |\phi|^5 . \tag{71}$$

Noting the relation $\mathcal{E} \propto |\phi|^2$, we write the asymptotic form of the second term of eq.(63) as

$$\sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 = \bar{\bar{g}}_0^2 \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{5/2} \tag{72}$$

where $\bar{\bar{g}}_0^2$ is a coefficient and $\mathcal{E}_{eq}$ is the most probable value of the average fluctuation energy $\mathcal{E}$. Combining with the thermal excitation term, the amplitude of the noise source term is written as

$$g^2 = 4\hat{T}\gamma_m \mathcal{E} + \bar{\bar{g}}_0^2 \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{5/2} \tag{73}$$

The coefficient $\bar{\bar{g}}_0^2$ is calculated as

$$\bar{\bar{g}}_0^2 = \lim_{\mathcal{E} \to \infty} \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{-5/2} \sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 , \tag{74}$$

and an order of magnitude estimate is given as

$$\bar{\bar{g}}_0^2 \simeq \left[\sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \right]_{\mathcal{E} = \mathcal{E}_{eq}} . \tag{75}$$

Based on the general dimensional analysis, one has the relation $\Lambda \propto |\phi|$, i.e., $\Lambda \propto \mathcal{E}^{1/2}$, in a strong turbulence limit. We rewrite as

$$\Lambda = \bar{\bar{\Lambda}} \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{1/2} \tag{76}$$

where the coefficient $\bar{\bar{\Lambda}}$ is given as

$$\bar{\bar{\Lambda}} = \lim_{\mathcal{E} \to \infty} \Lambda \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{-1/2} \tag{77}$$

Order-of-magnitude estimates of coefficients $\bar{\bar{\Lambda}}$ and $\bar{\bar{g}}_0^2$ are available by use of eq.(35). One has an estimate $\bar{\bar{\Lambda}} \sim \max(\lambda_1)$, because the coefficient Λ is an average of λ_1 over a spectrum and the spectrum is peaked near $k_\perp \simeq k_0$. The maximum value of λ_1 is of the order of $\gamma_{k,eq}$. Therefore, one has an estimate

$$\bar{\bar{\Lambda}} \sim \gamma_{k,eq} \simeq G_0^{1/2} \tag{78}$$

The coefficient $\bar{\bar{g}}_0^2$ and average energy $\mathcal{E}_{eq}$ depend on the volume of integration of concern, because they contain the sum over the k-space. This is discussed later in detail, but a dimensional argument is available. From a dimensional analysis, we have estimates

$$\bar{\bar{g}}_0^2 \propto \phi_{eq,0}^4 k_0^6 \gamma_{k,eq} \tag{79a}$$

and

$$\mathcal{E}_{eq} \propto \phi_{eq,0}^2 k_0^4 \tag{79b}$$

apart from a proportionality coefficient which depends on the volume of the coarse-grained average. This provides parameter dependences

$$\bar{\bar{g}}_0^2 \propto G_0^{11/2} s^{-6} \delta^6 a^{-6} \tag{80a}$$

and

$$\mathcal{E}_{eq} \propto G_0^2 (\delta / sa)^2 . \tag{80b}$$

Dependencies (78) and (80) illustrate the importance of plasma gradient G_0 for the evolution of the probability distribution function of the turbulent fluctuations.

4.2 Probability distribution function

The equilibrium probability distribution function for the average energy P_{eq} is obtained by putting $\partial P / \partial t = 0$ in eq.(66). It is expressed as

$$P_{eq}(\mathcal{E}) = \bar{P} \frac{1}{g} \exp\left(- \int_0^{\mathcal{E}} \frac{4\Lambda}{g^2} \mathcal{E} \mathrm{d}\mathcal{E} \right) \tag{81}$$

where $\bar{P}$ is a normalization constant.

An exponential function $\exp\left(- \int_0^{\mathcal{E}} 4\Lambda g^{-2} \mathcal{E} \mathrm{d}\mathcal{E} \right)$ dictates the distribution function. When Λ is positive, it is a decreasing function of $\mathcal{E}$. On the contrary, when Λ is negative, this function is an increasing function of $\mathcal{E}$. At zeros of Λ, the function takes the extremum values. As is illustrated in Fig.1, Λ is always positive when the gradient, which is parameterized by G_0, is very small. The probability distribution function is peaked near zero $\mathcal{E}$, as is the case of thermodynamical equilibrium. In the large gradient limit, Λ is negative at smaller value of $\mathcal{E}$ (representing the existence of linear/nonlinear instability) but becomes positive at larger value of $\mathcal{E}$. In this case, the probability distribution function mainly peaks at the finite and large value of $\mathcal{E}$. In the region of intermediate values of the gradient, Λ might vanish at two values of $\mathcal{E}$. This is the case of subcritical excitation of fluctuations. In such a case, the probability distribution has two peaks. These properties are summarized in Fig.5.

4.2.1 Thermodynamical equilibrium limit

In this case, limiting formula hold as eqs.(68) and (69). The integrand in the equilibrium probability distribution function, eq.(81), converges to

$$\frac{4\Lambda\mathcal{E}}{g^2} \to \frac{1}{\hat{T}} \tag{82}$$

Substituting this relation into eq.(81), the thermodynamical equilibrium limit is obtained as

$$P_{eq}(\mathcal{E}) \to \bar{P}\,\frac{1}{\sqrt{4\hat{T}\sum_k \gamma_{vc}\mathcal{E}_{k,\,th}}}\exp\left(-\int_0^{\mathcal{E}}\frac{\mathrm{d}\mathcal{E}}{\hat{T}}\right). \tag{83}$$

The equilibrium distribution function satisfies the relation

$$P_{eq}(\mathcal{E}) \propto \frac{1}{\sqrt{\hat{T}}}\exp\left(-\frac{\mathcal{E}}{\hat{T}}\right) \tag{84}$$

i.e., the probability distribution function of the fluctuation amplitude satisfies the Boltzmann relation. This result agrees with eq.(37b).

4.2.2 Large amplitude limit and tail of distribution function

One important feature of the strong turbulence limit is the tail component in the distribution function as is discussed in §3.

Substituting the power law relations of the large amplitude limit, i.e., eqs.(73) and (76), into eq.(81), one has the formula in the large amplitude limit as

$$P_{eq}(\mathcal{E}) = \bar{P}\,\frac{1}{\sqrt{4\hat{T}\gamma_m\mathcal{E} + \bar{\bar{g}}_0^2\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{5/2}}}\exp\left(-\int^{\mathcal{E}}\frac{4\bar{\bar{\Lambda}}\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{1/2}}{4\hat{T}\gamma_m\mathcal{E} + \bar{\bar{g}}_0^2\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{5/2}}\mathcal{E}\,\mathrm{d}\mathcal{E}\right). \tag{85}$$

In a strong turbulent limit,

$$\bar{\bar{g}}_0^2\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{5/2} >> 4\hat{T}\gamma_m \mathcal{E} , \tag{86}$$

one has an asymptotic relation

$$P_{eq}(\mathcal{E}) \sim \bar{P}\, \bar{\bar{g}}_0^{-1}\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{-5/4} \exp\left(-\int^{\mathcal{E}} 4\, \bar{\bar{\Lambda}}\, \bar{\bar{g}}_0^{-2}\, \mathcal{E}_{eq}^2 \mathcal{E}^{-1} \mathrm{d}\mathcal{E}\right). \tag{87}$$

This equation provides a power law

$$P_{eq}(\mathcal{E}) \sim \bar{P}\, \bar{\bar{g}}_0^{-1}\left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{-5/4 - 4\, \bar{\bar{\Lambda}}\, \bar{\bar{g}}_0^{-2}\, \mathcal{E}_{eq}^2} \tag{88}$$

for the tail of the distribution function.

4.2.3 Multiple peaks of distribution function

The probability distribution function $P_{eq}(\mathcal{E})$ can have multiple peaks, associated with the hysteresis in the average level versus the pressure gradient (Fig.2). This is the case (b) in Fig.2. The study of distribution function in such a case is important in order to understand the transition of turbulence state.

In order to have an analytic insight of the problem, we model the form of Λ of case (b) of Fig.1 as a function of 'fluctuation velocity' υ ,

$$\mathcal{E} = \upsilon^2 \tag{89}$$

as is shown in Fig.6:

$$\Lambda(\upsilon) = \Lambda_0 - \Lambda_0' \upsilon \qquad (0 < \upsilon < \upsilon_c) \tag{90a}$$

$$\Lambda(v) = \Lambda_1' \, v - \bar{\Lambda}_0 \qquad (v > v_c) \tag{90b}$$

This model form of Λ vanishes at two values of v

$$v_{*1} = \frac{\Lambda_0}{\Lambda_0'} \qquad \text{and} \qquad v_{*2} = \frac{\bar{\Lambda}_0}{\Lambda_1'} \tag{91}$$

The function Λ is a decreasing function of v in the region $0 < v < v_c$, representing the subcritical excitation by nonlinear instability mechanism. In the region of $v > v_{*2}$, the function Λ is positive and is an increasing function of v. This dependence models the fact that decorrelation rate increases as the fluctuation level becomes larger in the large amplitude limit. The asymptotic form of this model Λ satisfies the scaling relation eq.(76), i.e., $\Lambda \propto \sqrt{\mathcal{E}}$, in the large amplitude limit. By use of eq.(73), the equilibrium probability distribution function eq.(81) is rewritten as

$$P_{eq}(v) = \bar{P} \, \frac{1}{g} \exp\left(- \int^{\mathcal{E}} \frac{2\,\Lambda(v)}{\hat{T}\gamma_m + \bar{g}_0^2 \mathcal{E}_{eq}^{-5/2} v^3} \, v dv \right). \tag{92}$$

Substituting the model form of Λ, eq.(90), into eq.(92), an analytic form of $P_{eq}(v)$ is obtained.

Performing the integral one finds

$$P_{eq}(v) = \bar{P}_1 \frac{\left(v^3 + d^3\right)^{a_1}}{\left(v + d\right)^{3b_1}} \exp\left(-2\sqrt{3}\, b_1 \arctan\left(\frac{2v - d}{\sqrt{3}\, d} \right) \right) \qquad (0 < v < v_c) \tag{93a}$$

$$P_{eq}(v) = \bar{P}_2 \frac{\left(v^3 + d^3\right)^{a_2}}{\left(v + d\right)^{3b_2}} \exp\left(2\sqrt{3}\, b_2 \arctan\left(\frac{2v - d}{\sqrt{3}\, d} \right) \right) \qquad (v > v_c) \tag{93b}$$

In these expressions the parameter d is defined as

$$d^3 = \hat{T}\gamma_m \bar{\bar{g}}_0^{-2} \mathcal{E}_{eq}^{5/2} \tag{94}$$

and other numerical coefficients are introduced as

$$a_1 = -\frac{\Lambda_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2}} \left(\frac{1}{3d} - \frac{2\Lambda_0'}{\Lambda_0} \right) - \frac{1}{2} \qquad b_1 = \frac{\Lambda_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2}} \frac{1}{3d} \tag{95a}$$

and

$$a_2 = \frac{\bar{\Lambda}_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2}} \left(\frac{1}{3d} - \frac{2\Lambda_1'}{\bar{\Lambda}_0} \right) - \frac{1}{2} \qquad b_2 = \frac{\bar{\Lambda}_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2}} \frac{1}{3d} \ . \tag{95b}$$

From the result of eq.(93), several features of the equilibrium distribution function are observed. In the low amplitude region the distribution function deviates from the Boltzmann distribution, owing to the reduction of Λ by the finite-value effect of fluctuation amplitude. By this effect, although the probability distribution is a decreasing function of v in the region $0 < v < v_c$, it deviates from an exponential function and has a tail element. The exponential part has an argument $-2\sqrt{3}\, b_1 \arctan\left(\frac{2v - d}{\sqrt{3}d} \right)$; therefore, the characteristic value of v for the variation of $P_{eq}(v)$ is estimated by the values of $\left| d/b_1 \right|$ and $a_1 - b_1$.

This distribution function has the second peak near

$$v = v_{*2} \tag{96}$$

which represents the most probable turbulent state. The variation of the distribution in this region is dictated by the parameters $\left| d/b_2 \right|$ and $a_2 - b_2$. If one expands the integrand of eq.(92) in the vicinity of v_{*2} as,

$$\frac{2\,\Lambda(v)v}{\hat{T}\gamma_m + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{5/2} v^3} \simeq \frac{2\,v_{*2}\Lambda_0'}{\hat{T}\gamma_m + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{5/2} v_{*2}^3}\left(v_{*2} - v\right) \tag{97}$$

the exponential part of eq.(92) is given as

$$\exp\left(-\frac{v_{*2}\Lambda_0'}{\hat{T}\gamma_m + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{5/2} v_{*2}^3}\left(v - v_{*2}\right)^2\right) \tag{98}$$

The half width of the peak Δv is approximately given as

$$\Delta v = \sqrt{\frac{\hat{T}\gamma_m + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{5/2} v_{*2}^3}{v_{*2}\Lambda_0'}} \ . \tag{99}$$

In a strong turbulence limit where the thermal excitation is neglected, one has

$$\Delta v = \sqrt{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{3/2} \Lambda_0'^{-1}} \ . \tag{100}$$

Finally, this distribution function has tail as $P_{eq}(v) \propto v^{3a_2 - 3b_2 - 1}$. This recovers the asymptotic form in a strong turbulence limit. Evaluation of the width of peak and power index is discussed in the next subsection.

4.3 Evaluation of the width of peak and power index

4.3.1 Evaluation of self-noise term

To estimate the width of the peak, Δv , or the power index, η , the evaluation of the magnitude of self noise $\bar{\bar{g}}_0^2$, eq.(75), is necessary. We approximate the magnitude $\bar{\bar{g}}_0^2$ by the average of the noise amplitude which was obtained by the substitution of a statistical average $\langle \phi_k^2 \rangle$ of Langevin equation [II,III]. By use of this procedure, one has an estimation as

$$\bar{\bar{g}}_0^2 \simeq C_0 \sum_k k_\perp^4 \gamma_{k,\,eq} \phi_{k,\,eq}^4 , \tag{101}$$

where C_0 is of the order of unity, and $\gamma_{k,\,eq}$ is the decorrelation rate at the most probable fluctuation amplitude. Detailed argument together with the estimation for the coefficient C_0 is discussed in the appendix A.

Statistical characteristics vary depending on the choice of volume of average. In order to clarify the dependence on the volume of observations, we consider the average over the flux tube, the cross-section of which is given as L^2 . Fourier sum is performed, noting the fact that we are interested in the average in the volume with cross-section L^2 . Since all the length (including k) is normalized to a , the Fourier mode number in this averaging volume takes an integer multiplied by a/L . Therefore, the summation over k is estimated by

$$\sum_k \cdot\cdot\cdot = \int_{k_0} \cdot\cdot\cdot \left(\frac{L}{a}\right)^2 k_\perp \, dk_\perp \tag{102}$$

The power-law spectrum was obtained,[27, 40] which is given as a statistical average of the turbulent state as $\phi_{k,\,eq}^2 = \phi_{eq,\,0}^2 \left(k_0/k_\perp\right)^6$. For this spectrum, the coefficient $\bar{\bar{g}}_0^2$ and $\mathcal{E}_{eq}$ are calculated (see Appendix A for detail) as

$$\bar{\bar{g}}_0^2 \simeq \frac{C_0}{6} k_0^6 \gamma_{k,\,eq} \phi_{eq,\,0}^4 \left(\frac{L}{a}\right)^2 \tag{103}$$

and

$$\mathcal{E}_{eq} = \frac{1}{4}\, \phi_{eq,\,0}^2 k_0^4 \left(\frac{L}{a}\right)^2 \ . \tag{104}$$

If one sums up all the fluctuation energy in the volume of plasma, $L \to a$, one has $\mathcal{E}_{eq} \to G_0^2 \left(\delta/sa\right)^2/4$. This recovers the previous estimate of fluctuation energy.

By use of the estimate of characteristic values of fluctuations, eq.(35), results eqs.(103) and (104) are expressed in terms of the global plasma parameters including the gradient explicitly. Substituting eq.(35) into eqs.(103) and (104), we have

$$\bar{\bar{g}}_0^2 = \frac{C_0}{6} G_0^{11/2} \left(\frac{\delta}{s\,a}\right)^6 \left(\frac{L}{a}\right)^2 , \tag{105}$$

and

$$\mathcal{E}_{eq} = \frac{1}{4} G_0^2 \left(\frac{\delta}{s\,a}\right)^2 \left(\frac{L}{a}\right)^2 . \tag{106}$$

4.3.2 Width of the peak of strong turbulence

With the help of the order estimate

$$\Lambda_0' \simeq \frac{\gamma_{k,\,eq}}{\mathcal{E}_{eq}^{1/2}} \tag{107}$$

one has the expression

$$\Delta v \simeq \sqrt{\frac{\bar{\bar{g}}_0^2}{\mathcal{E}_{eq}\gamma_{eq}}} . \tag{108}$$

Substituting eqs.(35c), (105) and (106) into eq.(108), one has

$$\Delta v \simeq \sqrt{\frac{2}{3} C_0 \, G_0^{3/2}} \left(\frac{\delta}{s\,a}\right)^2 \tag{109}$$

It is compared to the average velocity, $v_{*2} = \sqrt{\mathcal{E}_{eq}}$, which is given as

$$v_{*2} \simeq \frac{1}{2} G_0 \frac{\delta}{s\,a} \frac{L}{a} \tag{110}$$

by the help of eq.(106). One has

$$\frac{\Delta v}{v_{*2}} \simeq \sqrt{\frac{8}{3} C_0 \, G_0^{1/2}} \, \frac{\delta}{s\,a} \frac{a}{L} \tag{111}$$

or

$$\frac{\Delta v}{v_{*2}} \simeq \sqrt{\frac{8}{3}C_0}\left(\frac{k_0 L}{a}\right)^{-1} \tag{112}$$

As is discussed in the end of §3.3, the width of distribution is given comparable to the average value if the local value is discussed.

4.3.3 Power index

The analysis in the large amplitude limit has shown the existence of a tail component as is shown by eq.(88), i.e.,

$$P_{eq}(\mathcal{E}) \propto \left(\frac{\mathcal{E}}{\mathcal{E}_{eq}}\right)^{-\eta}, \tag{113}$$

with

$$\eta = 5/4 + 4\,\overline{\overline{\Lambda}}\,\bar{\bar{g}}_0^{-2}\,\mathcal{E}_{eq}^2\,. \tag{114}$$

Substituting eqs.(105) and (106) into eq.(114), we have the relation for the power index as

$$\eta = \frac{5}{4} + \frac{3}{2C_0}\frac{\overline{\overline{\Lambda}}}{\gamma_{k,\,eq}}\,s\left(\frac{L}{\delta}\right)^2 G_0^{-1}\ , \tag{115}$$

or

$$\eta = \frac{5}{4} + \frac{3}{2C_0}\frac{\overline{\overline{\Lambda}}}{\gamma_{k,\,eq}}\left(k_0\frac{L}{a}\right)^2 . \tag{116}$$

4.3.4 Impact of plasma gradient

These results clearly illustrate the important role of plasma gradient.

From the result, we have the following; First, the plasma gradient G_0 is essential for determining the distribution function. For instance, the location of the peak, v_{*2}, shifts to the higher values, the width Δv becomes broader and the power index η becomes smaller, as the gradient becomes steeper, for a fixed value of integration size, L^2. This is understood by noticing the dependencies

$$v_{*2} \propto G_0 \tag{117a}$$

$$\Delta v \propto G_0^{3/2} \tag{1176}$$

and

$$(\eta - 5/4) \propto G_0^{-1} \tag{117c}$$

Then the volume of observation is also important. When one studies the probability distribution of the energy which is averaged over the global scale length (whole plasma volume),

$$L \simeq a \ , \tag{118}$$

the relative width $\Delta v / v_{*2}$ is of the order of k_0^{-1} (i.e., δ / a), and the power index η is of the order of k_0^2 (i.e., a^2/δ^2). The width is narrow, and the index η is much greater than unity, and the distribution function is close to the delta-function. There is almost no probability to observe the tail distribution in the total-volume-averaged fluctuation energy. In contrast, if one is intersted in the average over the correlation scale length,

$$L \simeq k_0^{-1} \tag{119}$$

the relative width and the power index are given as

$$\frac{\Delta v}{v_{*2}} \simeq \sqrt{\frac{8}{3}C_0} \ , \tag{120a}$$

and

$$\eta \simeq \frac{5}{4} + \frac{3}{2C_0}\frac{\overline{\overline{\Lambda}}}{\gamma_{k,\,eq}} \ . \tag{120b}$$

Noting the evaluations that

$$C_0 \simeq O(1) \ , \ \frac{\overline{\overline{\Lambda}}}{\gamma_{k,\,eq}} < 1 \ , \tag{121}$$

the relative width and the index are given to be of the order of unity as

$$\frac{\Delta v}{v_{*2}} \simeq O(1) \tag{122a}$$

and

$$\eta = \frac{5}{4} + O(1) \ . \tag{122b}$$

Noticeable width and power-law tail are predicted in the distribution function.

§5 Extension of Thermodynamical Relations

5.1 Principle of minimum renormalized dissipation

5.1.1 Minimum renormalized dissipation

The probability distribution function of the turbulent state is an extension of the Boltzmann distribution in a thermodynamical equilibrium. The equilibrium distribution function has the form of

$$P_{eq} \sim \exp\left(-S\right) \tag{123}$$

with the exponential part as

$$S = \int_0^{\mathcal{E}} \frac{4\,\Lambda \mathcal{E}}{g^2}\, \mathrm{d}\mathcal{E}\ . \tag{124}$$

The integrand is a ratio of the dissipation rate (or action) $\Lambda \mathcal{E}$ to the magnitude of random sources. The probability becomes large when the functional S is small. When S takes a local minimum value, the distribution function takes a local maximum. An example is given in Fig.7.

This result shows the fact that the most probable solution of turbulent state (being subject to the thermal excitations) satisfies that the integral of renormalized dissipation rate

$$S \text{ is minimum.} \tag{125}$$

5.1.2 Critical condition for turbulent transition

There are two minima of S, one corresponds to the thermodynamical equilibrium and the other is in the turbulent state. In such a case, one out of two is the dominant solution. A critical condition for turbulent transition can be obtained in terms of the probabilities of states. The selection rule is drawn from the condition

$$P_{eq}(0) \sim P_{eq}(v_{*2}) \tag{126}$$

This is simplified as

$$S(v_{*2}) \simeq 0 \tag{127}$$

assuming that g^2 is a slowly-varying function of the fluctuation level. If

$$S(v_{*2}) > 0 \tag{128}$$

holds, the thermal fluctuation near thermodynamical equilibrium is a dominant solution. If the condition

$$S(v_{*2}) < 0 \tag{129}$$

is satisfied, the turbulent state becomes the dominant state.

This rule, eq.(127), is analogous to the rule of "Maxwell's construction"[48] for thermodynamical equilibrium, and is an extended version to the turbulent state.

5.2 Comparison with near-thermodynamical equilibrium

5.2.1 Thermodynamical equilibrium limit

The principle of minimum renormalized dissipation rate, eq.(125), is compared to the thermodynamical equilibrium limit. In this limit, the probability distribution reduces the Boltzmann distribution as is shown in eq.(84). The Boltzmann distribution in a thermodynamical equilibrium could be reformulated as

$$P(X) \propto \exp\left(\frac{S_{ent}(X)}{k_B}\right) \tag{130}$$

where X is a thermodynamical variable and $S_{ent}(X)$ is the entropy.[49] One can reform it via a relation $S_{ent}(X)/k_B = -\left(\mathcal{E}/\hat{T} + ½\ln(\hat{T})\right)$. In stead of the entropy, which is difficult to be defined in the turbulent state being far-from-thermal equilibrium, the integral of renormalized dissipation rate S plays the role in controlling the probability distribution function.

5.2.2 Nonequilibrium but near-thermodynamical equilibrium

It should also be noted that our result is also a natural extension of the 'principle of minimum dissipation rate' in the near equilibrium. Prigogine has proposed a rule for the near thermodynamical equilibrium that the 'minimum principle of the entropy production (dissipation) rate'.[49, 50] This principle is valid if the random noise is governed by the thermodynamical equilibrium temperature. The result of this article can be reduced to such a situation. When the self-noise is absent, we have $g^2 = 4\hat{T}\gamma_m\mathcal{E}$ as is given in eq.(82), and the coefficient γ_m is independent of the fluctuation energy. Therefore, we have the limiting formula

$$S = \int_0^{\mathcal{E}} \frac{4\Lambda\mathcal{E}}{g^2} \mathrm{d}\mathcal{E} \rightarrow \frac{1}{\hat{T}\gamma_m} \int_0^{\mathcal{E}} \Lambda \mathrm{d}\mathcal{E} \tag{131}$$

in this case. The integral in eq.(131)

$$\int_0^{\mathcal{E}} \Lambda \mathrm{d}\mathcal{E} \tag{132}$$

is the dissipation rate of the fluctuating energy. One can define the entropy production rate as

$$\left.\frac{\partial S_{ent}}{\partial t}\right|_{irr} = \frac{1}{T} \int_0^{\mathcal{E}} \Lambda \mathrm{d}\mathcal{E} \,. \tag{133}$$

If one uses this definition, one finds that the relation

$$S \propto \left.\frac{\partial S_{ent}}{\partial t}\right|_{irr} \tag{134}$$

holds near the thermodynamical equilibrium, where γ_m is the damping rate close to thermodynamical equilibrium. Therefore in this limit, the minimum principle of S in this article reduces to the principle of the minimum dissipation rate of nonequilibrium but near the thermodynamical equilibrium. Our result corresponds to an extension of the Prigogine's formula.

§6. Summary and Discussion

A statistical theory of nonlinear-nonequilibrium plasma state with strongly developed turbulence and with strong inhomogeneity of the system has been further developed to include the thermal fluctuations. A unified theory for both the thermally excited fluctuations and the strongly turbulent fluctuations is presented. With respect to the turbulent fluctuations, the coherent part to a certain test mode is renormalized as the drag to the test mode, and the rest, the incoherent part, is considered to be a random noise. In the thermal fluctuation analysis, the coherent part to some mode is denoted by the collisional drag, and the rest (incoherent part) is considered to be the thermal noise of the temperature T. Combining both the contributions, we make a bridge between thermal fluctuations and turbulent fluctuations. The turbulent fluctuations are treated with the similar line of thought of the statistical physics approach.

In this article, formulation is presented by deriving a Fokker-Planck equation for the probability distribution function. By use of the one-branch approximation, the Liouville equation is approximated to a form of Fokker-Planck equation. The 'drag' and 'diffusion' coefficients in the Fokker-Planck equation are modelled based upon the renormalization of the turbulent effects. On the basis of this equation, equilibrium distribution function of turbulence level is derived. The Lyapunov function is constructed for the strongly turbulent plasma. The time derivative of this functional is shown to be negative definite, which shows that an approach to a certain equilibrium distribution is assured.

The equilibrium probability distribution function is obtained and the characteristic features are discussed. The difference from the thermal excitation is that the turbulent level is determined not by the heat-bath temperature but by the internal balance of turbulence-driven drag and the turbulent self-noise. The width of the distribution function for the turbulent fluctuation is in the same order as the mean value itself. This is because the excitation by the self-noise becomes larger as the mean fluctuation level increases.

Transition from the thermodynamical fluctuations, that is governed by the Boltzmann distribution, to the turbulent fluctuation is clarified. The subcritical excitation is possible, because the renormalized operator includes both the effects of nonlinear destabilization due to turbulent drag as well as the decorrelation by turbulent fluctuations. It is shown that the cusp-catastrophe is constructed in the average amplitude of spectrum on the plane of gradient and temperature. In the case that the average amplitude has a hysteresis, the probability distribution function has multiple peaks. The criterion is obtained for the transition from one state to the other, each of which is characterized by a peak of the distribution function.

The obtained equilibrium distribution function is found to be associated with a small but finite tail component. It has a power law distribution in its large amplitude limit. This power law dependence is caused from the fact that the random noise is the self-noise: namely, the enhancement of the fluctuation level simultaneously increases the noise pumping, establishing a self-sustained tail distribution. This result is obtained by keeping the self-noise term in the denominator of the integral which appears in the probability function. In a similar way of physics picture, the distribution function was discussed in the neutral fluid theory [8]. In this work, the self-noise term is treated to be small and is expanded. Therefore, no tail distribution was derived there.

Furthermore, coarse-grained volume-integrated quantities, like a total fluctuating energy, the total dissipation rate and the associated time constant, are introduced and the Fokker-Planck equation is reformulated. The equilibrium distribution function is obtained and is expressed in terms of the macro variables.

These results show that our theory corresponds to an extension from the previous theory of 'nonequilibrium but near thermodynamical equilibrium' into the one of 'far nonequilibrium' systems. According to the distance from the thermodynamic equilibrium, the probability distribution function deviates from the Boltzmann distribution function. The power-index of the tail distribution is also expressed in terms of global parameters. (It depends on the volume over which the fluctuation average is made.) The distinction between the 'near' and 'far' nonequilibrium systems (the distance from the thermodynamical equilibrium) is made by use of the nonequilibrium parameter of gradient. For instance, the critical gradient G_{0*} in Fig.2, divides the system near thermal equilibrium ($G_0 < G_{0*}$) and the one far from thermal equilibrium ($G_0 > G_{0*}$).

The most probable state is inferred from the equilibrium probability distribution function of fluctuating energy. The most probable state is expressed in terms of 'minimum renormalized dissipation rate', which is given by the ratio of the nonlinear decorrelation rate of fluctuation energy to the random excitation rate which includes both the thermal noise of T and the self-noise effects of turbulence. If one takes the limit of the thermodynamical equilibrium, this minimum principle reduces to the "minimum entropy". For nonequilibrium but near equilibrium state, the fluctuation source is given by the thermal excitation; then this renormalized dissipation rate is in proportion to the entropy production rate. In this way, our result recovers, in the limit near the thermodynamical equilibrium, the Prigogine's principle of minimum entropy-production rate.

In the presence of hysteresis in the average spectrum, the probability distribution function can have multiple peaks. The critical condition is derived, which governs the transition from thermal fluctuation to the turbulent fluctuation. This rule has a similarity to the 'Maxwell's construction' of conventional transition theory.

Application is made for example to the current-diffusive interchange mode turbulence in inhomogeneous plasmas. The applicability of this method is not restricted to this case, but could cover plasma turbulence in much wider circumstance as well as neutral fluid turbulence. The method of the analysis in this article could also be developed even in the presence of oscillating modes, by use of the model of self-noise which satisfies the realizability.[22-25] The analysis which utilizes the realizable models is discussed in the Appendix B. We would therefore conclude that the methodology of this article has a wider applicability than the example which is explicitly shown in this article.

Based on the results in I-III and this article, this method of analyzing strong turbulence has succeeded to extend principles of statistical physics, i.e., Kubo-formula and Prigogine's principle of minimum entropy production rate. Also presented is a criterion that is analogous to the Maxwell's construction. These considerations show that the method and result in this article provide the extension from the nonequilibrium statistical physics to the far-nonequilibrium one.

It is noted that the theory in this article is developed based on the assumption of scale separation between the microscopic fluctuations and macroscopic structure. In more general cases, this assumption does not always hold. If this scale separation does not hold, the system of fluctuating elements might not satisfy the extensiveness, so that the Gibbs-Boltzmann distribution is not valid even in the thermodynamical equilibrium state. This direction of extension in the statistical physics has also attracted attentions, and Tsallis has developed a statistical physics of nonextensive systems.[51)] Extension of the present theory of far-nonequilibrium systems to the case of nonextensive systems is left for future study.

The transition probability between the two peaks in the probability distribution function can be calculated by extending the method in this article. This subject is essential in the understanding of turbulence-turbulence transition,[52)] and shall be discussed in a forthcoming article.

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Dedication

This article is dedicated to the memory of Prof. R. Kubo.

Appendix A: Evaluation of the power index

In the analysis of the statistical average based on the Langevin equation, the decomposition approximation was used as

$$\left\langle g_{1,k}^2 \right\rangle = 2\sum_q M_{1,\,kpq}^2 \theta_{kpq} \left\langle \zeta_{1,p}^2 \right\rangle \left\langle \zeta_{1,q}^2 \right\rangle \tag{A1}$$

where ζ is an independent statistical variable which has the common averaged-magnitude as the fluctuating quantities. That is, the relations $\left\langle \zeta_{1,p}^2 \right\rangle = \phi_{p,\,eq}^2$ and $\left\langle \zeta_{1,q}^2 \right\rangle = \phi_{q,\,eq}^2$ are imposed, and one has

$$\bar{\bar{g}}_0^2 \simeq \sum_{k,\,p} 2k_\perp^4 M_{1,\,kpq}^2 \theta_{kpq} \phi_{p,\,eq}^2 \phi_{q,\,eq}^2 \phi_{k,\,eq}^2 . \tag{A2}$$

When one further uses the diagonalized approximation,[27, 28]

$$\left(1 + \delta_{1j}\right)\sum_q M_{j,\,kpq} M_{j,\,kpq} \theta_{kpq} I_1(p) I_j(q) = C_0 \gamma_v I_j(k), \tag{A3}$$

one has an estimation as

$$\bar{\bar{g}}_0^2 \simeq C_0 \sum_k k_\perp^4 \gamma_{k,\,eq} \phi_{k,\,eq}^4 \tag{A4}$$

where C_0 is a numerical coefficient and $\gamma_{k,\,eq}$ is the decorrelation rate at the most probable fluctuation spectrum. The coefficient in the diagonalized approximation eq.(A3) was calculated for the spectrum that is given as a statistical average of the solution of Langevin equation. When the fluctuation spectrum $I_1(k) = \left\langle \phi_k^2 \right\rangle$ and eddy damping rate γ_v obey the power laws with respect to k, i.e.,

$$I_1(k) \simeq I_0 (k_\perp / k_0)^{-\alpha} \tag{A5}$$

and

$$\gamma_v(k) \simeq \gamma_{v0}(k_\perp/k_0)^\beta, \tag{A6}$$

the coefficient C_0 is calculated by the integral as

$$C_0 \simeq \frac{-\displaystyle\int_{>k} \frac{\left(\left(\boldsymbol{p}\times\boldsymbol{q}\right)\cdot\boldsymbol{b}\right)^2}{k_\perp^4}\left(\frac{p_\perp^2-q_\perp^2}{k_\perp^2}\right)^2\left(\frac{p_\perp}{k_\perp}\right)^{-\beta}\left(\frac{p_\perp q_\perp}{k_\perp^2}\right)^{-\alpha}\frac{\mathrm{d}\boldsymbol{p}_\perp}{k_\perp^2}}{\displaystyle\int_{>k} \frac{\left(\left(\boldsymbol{p}\times\boldsymbol{q}\right)\cdot\boldsymbol{b}\right)}{k_\perp^2}\left(\frac{p_\perp^2-q_\perp^2}{k_\perp^2}\right)\frac{\left(\left(\boldsymbol{k}\times\boldsymbol{p}\right)\cdot\boldsymbol{b}\right)}{k_\perp^2}\left(\frac{k_\perp^2-p_\perp^2}{q_\perp^2}\right)\left(\frac{p_\perp}{k_\perp}\right)^{-\alpha-\beta}\frac{\mathrm{d}\boldsymbol{p}_\perp}{k_\perp^2}}, \tag{A7}$$

which shows that C_0 is order unity. In performing the integral, $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the relation $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$.

If one employs the form of spectrum

$$\phi_{k,\,eq}^2 \propto k_\perp^{-6} \quad \text{and} \quad \gamma_{k,\,eq} \propto k_\perp^0, \tag{A8}$$

the approximate estimate for $\bar{\bar{g}}_0^2$ is made. Substituting the form

$$\phi_{k,\,eq}^2 = \phi_{eq,\,0}^2\left(k_0/k_\perp\right)^6 \tag{A9}$$

into eq.(A4), one has

$$\bar{\bar{g}}_0^2 \simeq C_0\sum_k k_\perp^4\gamma_{k,\,eq}\phi_{k,\,eq}^4 = C_0\int_{k_0} k_\perp^4\gamma_{k,\,eq}\phi_{eq,\,0}^4\left(k_0/k_\perp\right)^{12}\left(\frac{L}{a}\right)^2 k_\perp\,dk_\perp\,. \tag{A10}$$

Performing the integral, we have an estimate as

$$\bar{\bar{g}}_0^2 \simeq \frac{C_0}{6}k_0^6\gamma_{k,\,eq}\phi_{eq,\,0}^4\left(\frac{L}{a}\right)^2 \tag{A11}$$

The energy $\mathcal{E}_{eq} = \frac{1}{2} \sum_k k_\perp^2 \phi_{k,\,eq}^2$ is evaluated by use of eq.(A9) as

$$\mathcal{E}_{eq} = \frac{1}{2} \sum_k k_\perp^2 \phi_{k,\,eq}^2 = \frac{1}{2} \int_{k_0} k_\perp^2 \, \phi_{eq,\,0}^2 \left(k_0 / k_\perp\right)^6 \left(\frac{L}{a}\right)^2 k_\perp \, dk_\perp \quad . \tag{A12}$$

The expression for energy is obatined as

$$\mathcal{E}_{eq} = \frac{1}{4} \, \phi_{eq,\,0}^2 k_0^4 \left(\frac{L}{a}\right)^2 \quad . \tag{A13}$$

Combining results of $\bar{\bar{g}}_0^2$ and $\mathcal{E}_{eq}$, eqs.(A12) and (A13), we have the relation

$$\bar{\bar{g}}_0^2 \simeq \frac{8C_0}{3} \, k_0^{-2} \gamma_{k,\,eq} \mathcal{E}_{eq}^2 \left(\frac{L}{a}\right)^{-2} \quad . \tag{A14}$$

From these estimates, one obtains an evaluation as

$$4 \, \bar{\bar{\Lambda}} \, \bar{\bar{g}}_0^{-2} \, \mathcal{E}_{eq}^2 \sim \frac{3}{2C_0} \frac{\bar{\bar{\Lambda}}}{\gamma_{k,\,eq}} \, k_0^2 \left(\frac{L}{a}\right)^2 \quad . \tag{A15}$$

Appendix B: Other Models of Self-Noise Term

It has been shown that the model of self-noise term in eq.(13) might not be realizable when the excited modes have real frequency.[22-25] Applicability of eq.(13) might be limited to fluctuations without real frequency (e.g., interchange mode). In order to resolve this difficulty, several models have been proposed. By use of an alternative model, similar analysis is possible based on the Fokker-Planck equation.

Krommes has proposed a generalization for the self-noise term which is symbolically written as[23)]

$$\frac{\partial}{\partial t}\psi + \mathcal{L}\psi = \tilde{S}^K = M(\zeta - \psi_0)(\zeta - \psi_0) \tag{B1}$$

where ψ is a fluctuating field variable, $\mathcal{L}$ is a renormalized operator, M is the coefficient of nonlinear interaction, ζ is a random variable, and ψ_0 is a newly introduced (non-Gaussian) variable which is statistically dependent on ζ. Normalization is employed as $\langle \psi_0 \psi_0 \rangle = \langle \psi\psi \rangle$. The superfix K in $\tilde{S}^K$ represents the model by Krommes.

If one uses this model of random source term $\tilde{S}^K$, similar analysis is developed by use of the Fokker-Planck equation. The assumption of short correlation time is employed, $\tilde{S}^K = w(t) g^K$, and the result of the probability distribution function is recovered by replacing the magnitude of self-noise term by $|g^K|^2$. By this procedure, the important conclusion on the probability distribution function of the turbulence is not altered qualitatively. For instance, the derivation of the tail distribution of the strong turbulence is unchanged. This is because the scaling dependencies of g_0^2 and $|g^K|^2$ on the fluctuation amplitude are the same. (This can be understood from the fact that the dimensional dependence in the right hand side is equals to the one in eq.(13)). Therefore the logarithmic dependence on the fluctuating amplitude appears in the argument of exponential function of the distribution function. The tail distribution with power law is recovered. It is noted that the estimation of the power index depends on the coefficient C_0 which is given by an evaluation of integrals. This coefficient could be modified by the introduction of the new random variable.

Another type of realizable model was proposed by Bowman et al., which is given in a moment equation[24-25]

$$\frac{\partial}{\partial t}\Psi_k + 2\Re\gamma_k\Psi_k = 2F_k^B \tag{B2}$$

where $\Psi_k = \langle\psi\psi^*\rangle$ is the spectrum of fluctuating field, $\Re\gamma_k$ is a real part of γ_k and F_k^B denotes the magnitude of the self-noise term. The superfix B in the random source term denotes the model of Bowman et al. The magnitude of the self-noise term and the eddy-damping rate are expressed as

$$F_k^B = \sum_\Delta \left| M_{kpq} \right|^2 \Re\Theta_{kpq}\Psi_p^{1/2}\Psi_q^{1/2}, \tag{B3}$$

$$\gamma_k = \gamma_{cv} + \gamma_k^B, \tag{B4}$$

and

$$\gamma_k^B = -\sum_\Delta M_{kpq}M_{qkp}^*\Re\Theta_{pqk}^*\Psi_p^{1/2}\Psi_k^{-1/2}, \tag{B5}$$

where a new propagator Θ_{kpq} is introduced as

$$\frac{\partial}{\partial t}\Theta_{kpq} + \left[\gamma_k + \mathcal{P}(\gamma_p) + \mathcal{P}(\gamma_q)\right]\Theta_{kpq} = \Psi_p^{1/2}\Psi_q^{1/2}. \tag{B6}$$

In this equation, the operator $\mathcal{P}(\gamma_k) = \Re\gamma_k H(\Re\gamma_k) + i\,\Im\gamma_k$ is used where $H(\Re\gamma_k)$ is a Heaviside function and $\Im\gamma_k$ denotes the imaginary part of γ_k. By this modification, the realizability of the spectrum Ψ_k is recovered even in the case of fluctuation with real frequency.

The structure of the analysis in this article is not modified even if this model of the self-noise term is employed. Firstly, the eigenvalue of the renormalized operator $\mathcal{L}$ is obtained by the same procedure, by replacing $\gamma_{i,k}$ of eq.(9) by γ_k^B. The derivation of eigenvalues from eq.(20) and the decomposition of the matrix

$$\exp\left(\mathcal{L}t\right)$$

in terms of matrices $\boldsymbol{A}$, $\boldsymbol{A}^{(2)}$ and $\boldsymbol{A}^{(3)}$ are straightforward and unaltered. Therefore the nature of the subcritical excitation and self-sustaining process are deduced into the same form. The term Λ is defined accordingly. The dependencies of Λ on the gradient parameter as well as on the fluctuation amplitude is not modified. Fokker-Planck equation is constructed, in the same manner, by replacing the self-noise term. The background of this equation (B2) is that the random source in the Langevin equation is expressed as

$$\left\langle \tilde{S}_k^B(t)\tilde{S}_k^B(t')\right\rangle = \delta(t-t')F_k^B \,. \tag{B7}$$

A similar procedure can be developed in analyzing the distribution function. In the Fokker-Planck equation, the variable is changed to Ψ_k (or a coarse grained quantity $\Psi = \sum_k \Psi_k$), and the magnitude of the noise g_0^2 is replaced by F_k^B. As in the case of (B1), the qualitatively same conclusion is drawn.

For instance, the argument of the exponential function in the probability distribution function is expressed as

$$\int_0^v \frac{\Lambda\, v}{g_0^2}\, dv \rightarrow \int_0^u \frac{\Lambda\, u}{F^B}\, du \tag{B8}$$

where u is 'velocity' of perturbation field, $u = \sqrt{\Psi}$, and F^B is a weighted sum of F_k^B. In a large amplitude limit, the dependencies hold as

$$\Theta_{kpq} \propto \Psi_k^{1/2} \,, \tag{B9}$$

$$\Lambda \propto \Psi_k^{1/2} \,, \qquad \text{i.e.,}\ \Lambda \propto u \,, \tag{B10}$$

$$F_k^B \propto \Psi_k^{3/2} \,, \qquad \text{i.e.,}\ F^B \propto u^3 \,. \tag{B11}$$

From these relations, one recovers a dependence

$$\frac{\Lambda u}{F^B} \propto \frac{1}{u} \tag{B12}$$

i.e.,

$$P(u) \propto \frac{1}{\sqrt{F^B}} \exp\left(\int_0^u \frac{\Lambda u}{F^B}\, du \right) \propto u^{-\eta'} \tag{B13}$$

in the limit of strong turbulence. The power index η' can also be calculated if one employs the diagonalization approximation of eq.(A3),

$$\frac{F_k^B}{\gamma_k^B \Psi_k} = \frac{\sum_\Delta \left| M_{kpq} \right|^2 \Re \Theta_{kpq} \Psi_p^{1/2} \Psi_q^{1/2}}{-\sum_\Delta M_{kpq} M_{qkp}^* \Re \Theta_{pqk}^* \Psi_p^{1/2} \Psi_k^{1/2}} \simeq C_0' \tag{B14}$$

From these considerations, we see that the method of the analysis in this article could also be developed based on the model which satisfies the realizability even in the presence of oscillating modes. We would therefore conclude that the methodology of this article has a wider applicability than the example which is explicitly shown in this article.

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Figure Captions

Fig.1 Nonlinear decorrelation rate as a function of the fluctuation level. Fluctuation level is represented by the turbulent eddy damping rate γ_N . Cases (a)-(c) show various values of pressure gradient. In (a), $\hat{G}_0$ is small, so that λ_1 is always positive. If $\hat{G}_0$ increases, (b), λ_1 is positive in the zero-amplitude limit, but can be negative for finite values of fluctuation level. This represents the nonlinear instability mechanism. When $\hat{G}_0$ becomes large enough, λ_1 becomes negative in the zero-amplitude limit. This corresponds to the linear instability. In the large amplitude limit, λ_1 is positive and an increasing function of the fluctuation level.

Fig.2 Statistical average of fluctuation level as a function of the pressure gradient. The case of subcritical excitation is shown. In the case of low gradient, only thermal fluctuations are realized. Hysteresis is shown, and (a)-(c) correspond to those in Fig.1, respectively.

Fig.3 Rate of nonlinear decorrelation λ_1 and the eddy damping rate γ_N as a function of the pressure gradient parameter $\hat{G}_0$. In the zero-amplitude limit, γ_N vanishes and λ_1 converges to the damping rate owing to the molecular viscosity.

Fig.4 Schematic illustration of the statistical average of fluctuation level $\langle I \rangle$ as a function of the global gradient and temperature. (See [II] for derivations.)

Fig.5 Schematic drawings of the equilibrium distribution function for cases (a), (b) and (c) of Figs.1 and 2. In (a), P_{eq} is an monotonic decreasing function of the fluctuation level. In (b), it has two peaks, i.e., one for thermal fluctuations and the other for self-sustained turbulence. In the case (c), distribution shows that of strong turbulence.

Fig.6 Model of the decorrelation rate Λ as a function of fluctuation velocity v. In the low amplitude region, Λ is a decreasing function of v, representing the subcritical excitation owing to nonlinear instability. In the large amplitude limit, Λ is an increasing function of v, and asymptotic dependence $\Lambda \propto v$ holds: A model form of $\Lambda(v) = \Lambda_1' v - \overline{\Lambda}_0$ is chosen here. Λ vanishes at two values of v, v_{*1} and v_{*2}.

Fig.7 Renormalized dissipation rate $S(v)$ .as a function of the fluctuation amplitude. The case of (b) in Fig.1 is schematically shown. Owing to the presence of zeros of Λ, $S(v)$ takes extremum at $v = v_{*1}$ and $v = v_{*2}$. $S(v)$ has local minima at $v = 0$ and $v = v_{*2}$.

Fig.1

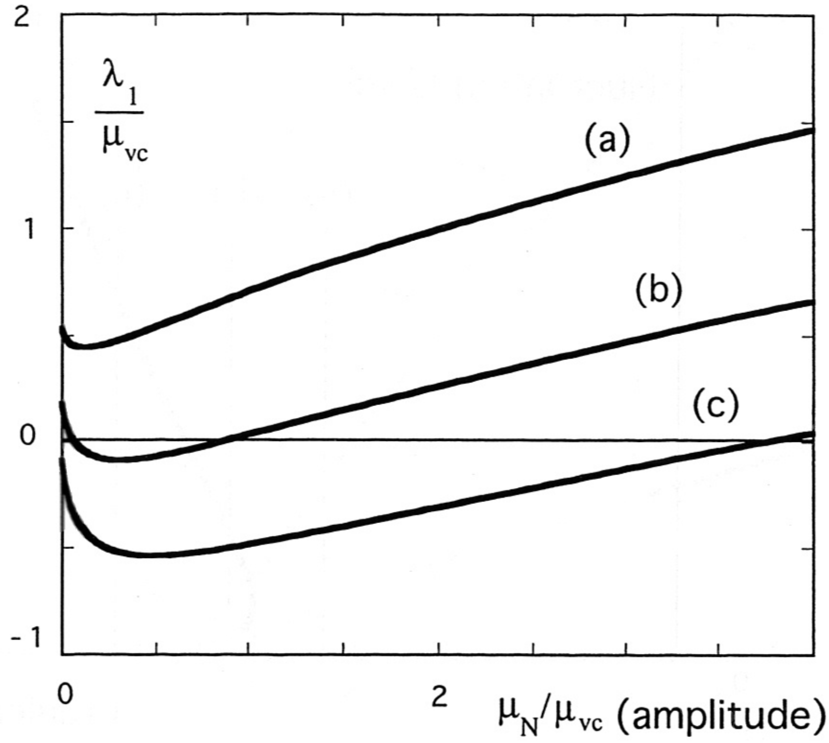


Fig.2

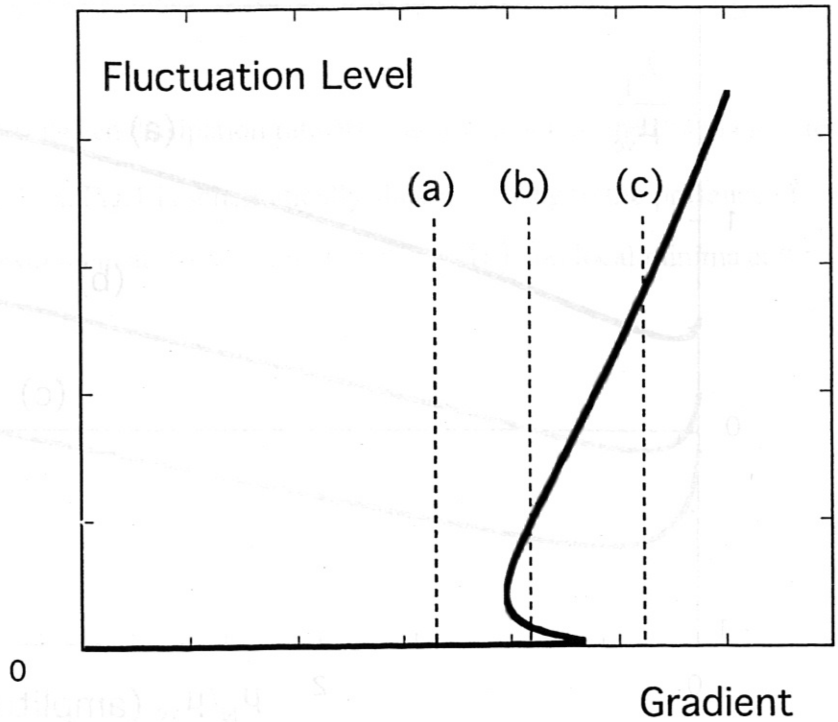
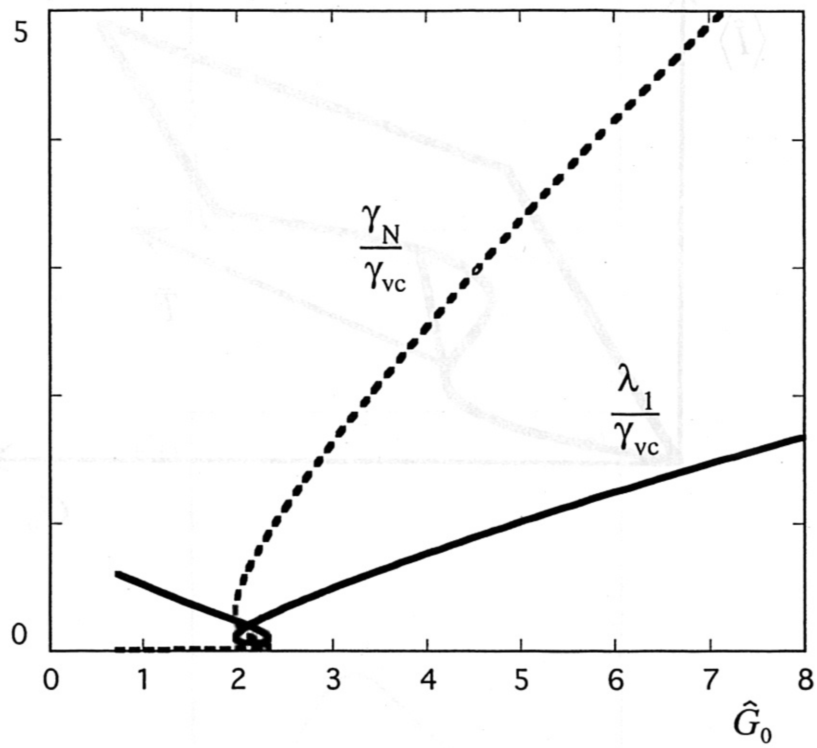


Fig.3



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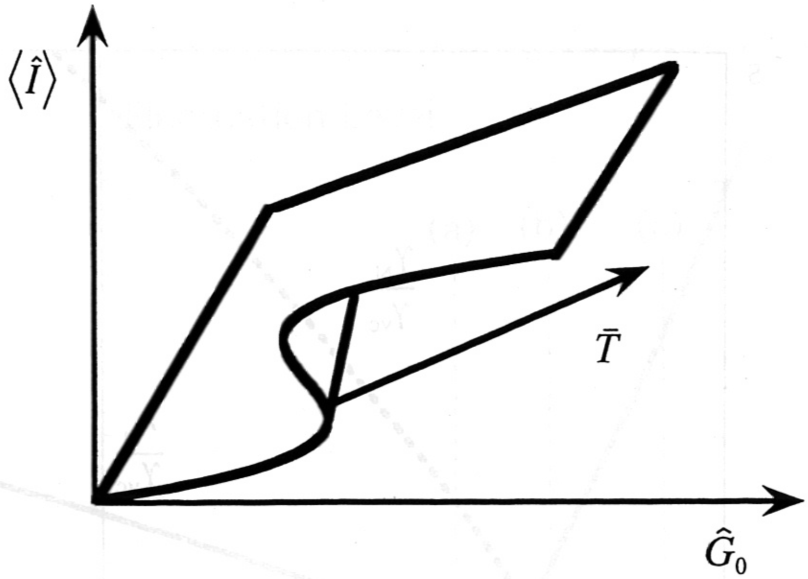
8
$\hat{G}_0$
$\frac{\gamma_N}{\gamma_{vc}}$
$\frac{\lambda_1}{\gamma_{vc}}$

Fig.4



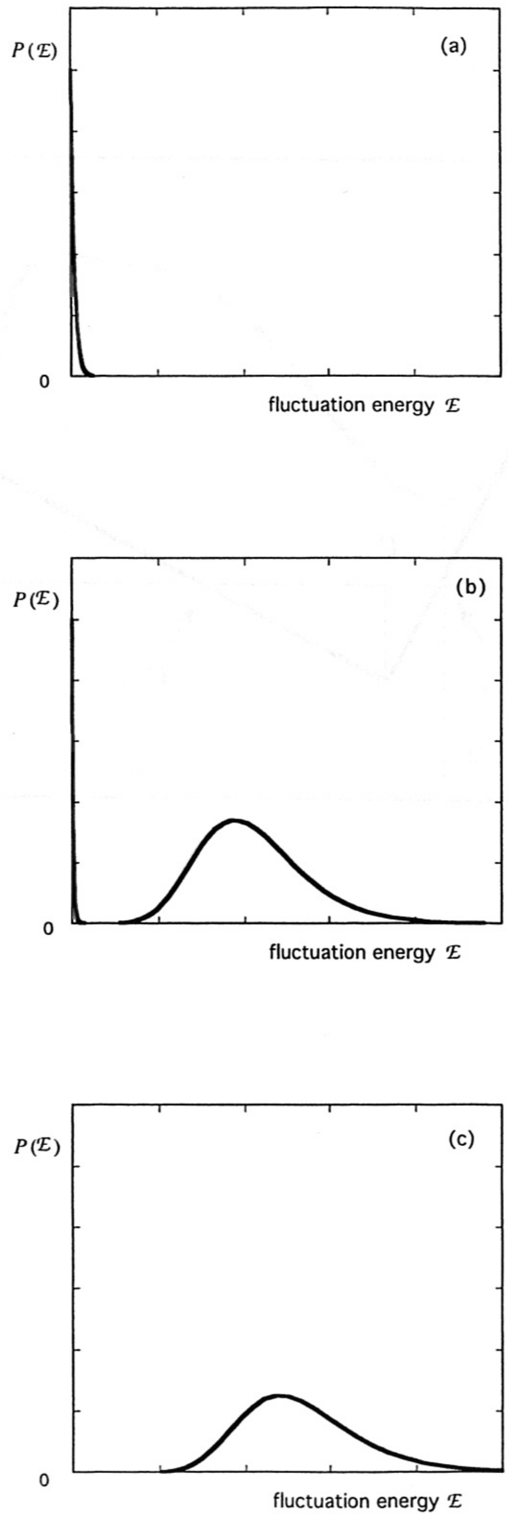

P(ℰ)
(a)
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Fig.6

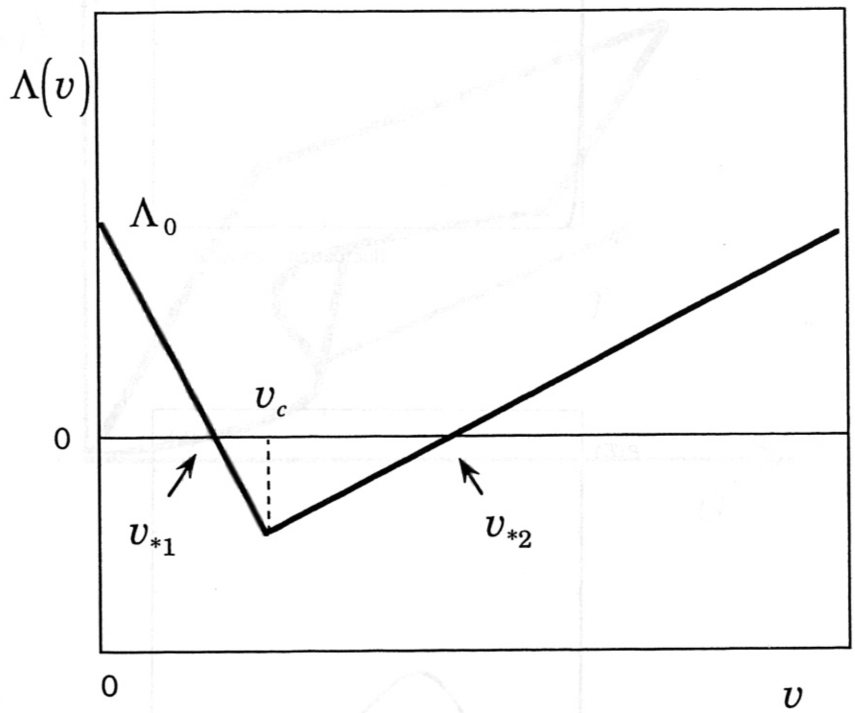

Fig.7