IPP-Report

Max-Planck-Institut
für Plasmaphysik

H. Wobig, D. Pfirsch

# Guiding Center Orbits of Passing Particles in Toroidal Systems

IPP III/245

September1999

H. Wobig, D. Pfirsch

# Guiding Center Orbits of Passing Particles in Toroidal Systems



IPP III/245

September1999

# Guiding Center Orbits of Passing Particles in Toroidal Systems

H. Wobig, D. Pfirsch

Max-Planck Institut für Plasmaphysik, EURATOM Association
D-85740 Garching bei München, Germany

**Abstract:**

Passing particles in toroidal geometry are described in Hamiltonian formalism including time-dependent electric and magnetic fields. These particles are characterised by a non-vanishing toroidal velocity. The introduction of the toroidal angle as independent variable instead of the time allows one to derive a map of the poloidal plane onto itself, which is similar to the Poincaré map of magnetic field lines. In time-dependent fields the energy of the particles is not conserved leading to two coupled maps, which is characteristic for autonomous systems with three degrees of freedom. As a result Arnold diffusion occurs and KAM surfaces, which in case of energy conservation separate stochastic regions in phase space, can be bypassed leading to enhanced radial transport of particles. The mechanism of enhanced transport is resonance streaming along resonance lines, which build–up the complex Arnold web. The structure of this web depends on the drift rotational transform of drift orbits and the toroidal transit time of passing particles. Numerical examples of Arnold diffusion of test paticles will be given.

## 1. Introduction

In non-axisymmetric stellarator configurations absolute confinement of particle orbits does not exist, in general. Quasi-helical or quasi-axisymmetric configurations are exceptions; in real configurations, however, there are always small deviations from the symmetry and some particles may exhibit stochastic behaviour, which is the origin of enhanced losses.

The standard description of particle orbits in stellarator employs the guiding centre model in magnetic or Boozer co-ordinates. However, outside the last magnetic surface and in inner stochastic regions magnetic surfaces do not exist and one must return to a general co-ordinate system, which does not require magnetic surfaces as co-ordinate surfaces. Another case, where a proper choice of the co-ordinate system is recommended, is the problem of trapped particles under the influence of field errors or electrostatic or electromagnetic fluctuations. Here the transition to action-angle variables is useful, which allows one to study the long-term behaviour of the particles. In the present paper we shall outline the general method to describe particle orbits in a magnetic field and then focus on the orbits of circulating particles in toroidal geometry, which represent the majority of particles in a toroidal magnetic field. These particles do not change sign of the parallel velocity and circulate around the torus in one direction. They approximately follow magnetic field lines. If these particles stay on closed drift surfaces, their orbits exhibit the same features as magnetic surfaces: they also have a rotational transform, which in contrast to magnetic surfaces depends on the energy and the magnetic moment of the particles. Since magnetic surfaces and drift surfaces differ only by order of gyro radius, the rotational transform of particles differs only slightly from the magnetic transform. Closed drift surfaces can be destroyed under the influence of magnetic and electric fields, in particular, poloidal electric fields can lead to island formation of drift surfaces, while magnetic surfaces are not affected by these fields. Poloidal and parallel electric field exist in a neoclassical plasma and they are also an unavoidable by-product of plasma instabilities.

In the following we consider at first the drift surfaces of circulating particles in time-independent electromagnetic fields. Island formation and island overlap of drift surfaces is the origin of enhanced radial transport of circulating particles. In case of magnetic surfaces, islands are separated by KAM-surfaces, which form transport barriers, if the perturbations stay below a certain threshhold. The destruction of the last KAM surface opens the path for strong radial diffusion of field lines. The same effect has to be expected in case of drift surfaces and destruction by time-independent electromagnetic fields. Mathematically the analysis leads to an autonomous system with two degrees of freedom. In case of time-dependent fluctuations, which be also considered, the theory of circulating particles is described by an autonomous system with three degrees of freedom. In such a case resonance surfaces, on which islands can arise, are no longer separated by KAM-surfaces. The new feature in systems with three degrees of freedom is the Arnold diffusion, which allows particles to go around the KAM-surfaces.

The aim of the paper is to investigate how the properties of the unperturbed system, the rotational transform and the shear, affect the stochastisation of particle orbits. In the literature various descriptions of drift orbits exist. In the paper of Morozov and Solov'ev[1] the drift velocity of a guiding centre in a time-independent vacuum field is given by

$$\frac{d\boldsymbol{x}}{dt} = \frac{v_{\parallel}}{B}\nabla\times\left(\boldsymbol{A} + \rho_{\parallel}\boldsymbol{B}\right) \;;\quad \rho_{\parallel} = \sigma\frac{1}{\Omega}\sqrt{\frac{2}{m}\left(E - \mu B - q\Phi\right)} \qquad \text{Eq. 1.1}$$

[1] Morozov, Solov'ev, Rev. of Plasma Physics, Vol. 2, p.229, Consultants Bureau, New York 1966

$\rho_{\parallel}$ is the parallel gyro radius, A the vector potential of the magnetic field and $\mu$ the magnetic moment of the particle. $\sigma=\pm 1$ is the sign of the parallel velocity. The orbits of circulating particles which do not change sign are the „field lines" of the divergence free field

$$\boldsymbol{B}^{*} = \nabla \times \left(\boldsymbol{A} + \rho_{\parallel}\boldsymbol{B}\right) \qquad \text{Eq. 1.2}$$

The Poincaré plot of the intersection points in a poloidal plane is the image of a flux conserving map. Therefore the topology of particle orbits is the same as the topology of magnetic field lines, which means that islands and stochastic regions may occur. White, Boozer and Hay[2] have criticised the formulation of Morozov and Solov'ev as being not compatible with Liouville's theorem and they suggest a slightly different form of the guiding centre velocity

$$\frac{d\boldsymbol{x}}{dt} = \frac{\upsilon_{\parallel}}{B}\frac{1}{1+\rho_{\parallel}\boldsymbol{b}\bullet\left(\nabla\times\boldsymbol{b}\right)}\nabla\times\left(\boldsymbol{A}+\rho_{\parallel}\boldsymbol{B}\right) \;; \quad \boldsymbol{b} = \frac{\boldsymbol{B}}{B} \qquad \text{Eq. 1.3}$$

This relation can be obtained from a Lagrangian derived by Littlejohn[3]. The orbits of circulating particles, however, are described by the same equation 1.2 as in Morozov's form. In time-dependent fields the variation of energy has to be taken into account and the description of particle orbits and their general properties are best described in terms of Hamiltonian formalism corresponding to Littlejohn´s Lagrangian.
Passing particles under the effect of electrostatic drift waves have been analysed by Horton et al[4]. These authors employ a mapping technique to study the long-term behaviour of the particles. However the variation of particle energy by the waves has not been retained in this paper.
In the first part of the paper we consider circulating particles in time-independent fields and start from the drift velocity in eq. 1.3. An area-preserving map will be derived which by iteration yields the Poincaré plot of circulating particles. In the second part the theory is extended to time-dependent magnetic and electric fields. The main step is the reduction of the six-variable Lagrangian of the guiding center to a five-variable Lagrangian and a four-variable Hamiltonian. A Hamiltionian valid in time-dependent fields has been derived by Boozer[5], the reduction to a four variable Hamiltonian was made by neglecting the radial component of $\boldsymbol{B}$. Hazeltine and Meiss[6] succeeded to eliminate the radial component by constructing a proper co-ordinate system. In case of time-dependent fields this co-ordinate system is also time-dependent leading to an additional term in the Hamiltonian, which was neglected by Hazeltine and Meiss. This additional term will be rigorously included in the following paper. In practical cases, however, it turns out that this term is small. Starting from the canonical equations the mapping equations will be derived and some numerical examples of Arnold diffusion will be given.

---

[2] R.B. White, A.H. Boozer and R. Hay, *Phys. Fluids* **25**, (1982), 575

[3] R.J. Littlejohn, *Phys.Fluids* **24**, 1730 (1981)

[4] W. Horton, H.B. Park, JM Kwon, DI Choi , D. Strozzi and P.J. Morrison, Phys. of Plasmas 5 (11) 3910, (1998)

[5] A. Boozer , *Phys. Fluids* **27** (10) 2441 (1984)

[6] J.D. Meiss, R.D. Hazeltine *Phys. Fluids* B2, (1990) 2563

## 2. Time-independent Fields

In time-independent electromagnetic fields the energy of the particles is conserved, and the equation of motion can be reduced to two canonical equations. The orbits of circulating particles provide an area-preserving map of the poloidal plane onto itself. To illustrate this we start from the drift velocity of a guiding centre as given in eq.1.3. This equation holds under the condition $\boldsymbol{j} \bullet \boldsymbol{B} = 0$, which is satisfied in vacuum fields. The orbits of circulating particles which do not change sign are the „field lines“ of the divergence free field

$$\boldsymbol{B}^* = \nabla \times \left(\boldsymbol{A} + \rho_{\|}\boldsymbol{B}\right) \qquad \text{Eq. 2.1}$$

The Poincaré plot of the intersection points is the image of a flux-conserving map. Therefore the topology of particle orbits is the same as the topology of magnetic field lines, islands and stochastic regions may occur. In analogy to field lines of magnetic fields[7] one can obtain the orbit equations as Euler equations of the variational principle

$$\delta \int \boldsymbol{A}^* \bullet d\boldsymbol{x} = 0 \;\; ; \;\; \boldsymbol{A}^* = \boldsymbol{A} + \rho_{\|}\boldsymbol{B} \qquad \text{Eq. 2.2}$$

Let the transformation x,y,z —> r,θ,φ introduce a toroidal co-ordinate system with r(x,y,z) = const. being toroidally closed surfaces and θ,φ a poloidal and toroidal angle. Introducing an appropriate gauge the vector potential can be written as

$$\boldsymbol{A}^* = \left\{0, A_\theta^*, A_\varphi^*\right\} \qquad \text{Eq. 2.3}$$

where the radial covariant component of $\boldsymbol{A}$ has been eliminated by a gauge transformation. Explicitly the variational principle is

$$\delta \int A_\theta^* d\theta + A_\varphi^* d\varphi = 0 \qquad \text{Eq. 2.4}$$

Defining a variable $p_\theta$

$$p_\theta = A_\theta^*\left(r, \theta, \varphi\right) \qquad \text{Eq. 2.5}$$

and postulating that

$$\frac{\partial}{\partial r} A_\theta^* \neq 0 \qquad \text{Eq. 2.6}$$

allows one to invert equation 2.5 and to eliminate the radial variable r:

$$r = r\left(p_\theta, \theta, \varphi\right) \qquad \text{Eq. 2.7}$$

We introduce the function $K$ by

$$K\left(p_\theta, \theta, \varphi\right) = -A_\varphi^*\left(r\left(p_\theta, \theta, \varphi\right), \theta, \varphi\right) \qquad \text{Eq. 2.8}$$

and write the variational principle in the form

$$\delta \int p_\theta d\theta - K d\varphi = 0 \qquad \text{Eq. 2.9}$$

---

[7] J.R. Gary, R.G. Littlejohn, Annals of Physics (NY)151, 1 (1983)

$K$ is a Hamiltonian with φ replacing the time and the canonical orbit equations are

$$\frac{d\theta}{d\varphi}=\frac{\partial K}{\partial p_\theta} \quad ; \quad \frac{dp_\theta}{d\varphi}=-\frac{\partial K}{\partial \theta} \qquad \text{Eq. 2.10}$$

.Energy and magnetic moment are fixed parameters. Let $K_0(p_\theta)$ be the average over the angular variables.

$$K=K_0(p_\theta)+K_1(p_\theta,\theta,\varphi)=\int^{p_\theta}\iota(p)dp+K_1(p_\theta,\theta,\varphi) \qquad \text{Eq. 2.11}$$

$\iota(p_\theta,E,\mu)$ is the rotational transform of the unperturbed drift surface. The canonical equations are

$$\frac{d\theta}{d\varphi}=\iota(p_\theta)+\frac{\partial K_1}{\partial p_\theta} \quad ; \quad \frac{dp_\theta}{d\varphi}=-\frac{\partial K_1}{\partial \theta} \qquad \text{Eq. 2.12}$$

Note that the "perturbation" $K_1$ is not assumed to be small. Particle orbits of circulating particles provide an area-preserving map of the poloidal plane onto itself. This is the result of Liouville´s theorem, which says that the area

$$\oint p_\theta d\theta = const \qquad \text{Eq. 2.13}$$

along orbits in phase space. This is the analogue to the flux-conserving map of magnetic field lines.

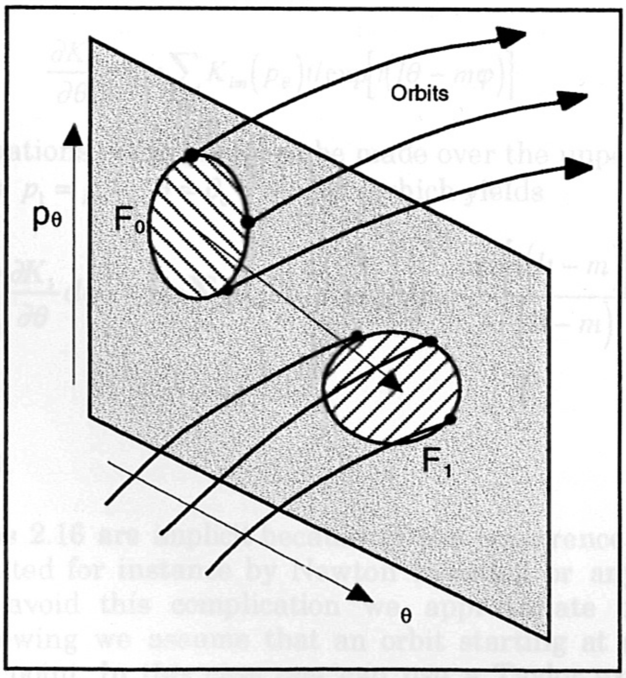


**Fig. 1:** Surface of section in the $p_\theta$–θ plane. In case of constant energy the orbits of circulating particles generate an area-preserving map of the $p_\theta$–θ plane onto itself.

We define the surface of section by $\varphi = \varphi_0$ and describe the map

$$T: \quad \theta_0,p_0,\varphi_0 \Rightarrow \theta_1,p_1,\varphi_0+1 \qquad \text{Eq. 2.14}$$

using the generating function

$$S = p_1\theta_0 + \int^{p_1} \iota(x)dx + S_1(p_1,\theta_0) \qquad \text{Eq. 2.15}$$

All details of the perturbing Hamiltonian $K_1$ are described by the function $S_1$. The mapping equations are

$$p_0 = p_1 + \frac{\partial S_1(p_1,\theta_0)}{\partial \theta_0} \quad ; \quad \theta_1 = \theta_0 + \iota(p_1) + \frac{\partial S_1(p_1,\theta_0)}{\partial p_1} \qquad \text{Eq. 2.16}$$

which is the general form of a twist map. In order to get an approximate expression of the generating function we formally integrate eq. 2.12 over the particle orbit and find

$$p_1 = p_0 - \int_0^1 \frac{\partial K_1}{\partial \theta} d\varphi \qquad \text{Eq. 2.17}$$

Comparing this with eq. 2.15 yields

$$S_1 = \int^{\theta_0} d\theta \left\{ \int_0^1 \frac{\partial K_1}{\partial \theta} d\varphi \right\} \qquad \text{Eq. 2.18}$$

Expanding $K_1$ in a Fourier series

$$K_1 = \sum_{lm} K_{lm}(p_\theta) \exp[i(l\theta - m\varphi)] \qquad \text{Eq. 2.19}$$

yields

$$\frac{\partial K_1}{\partial \theta} = \mathrm{Re} \sum_{lm} K_{lm}(p_\theta) il \exp[i(l\theta - m\varphi)] \qquad \text{Eq. 2.20}$$

In case of small pertubations integration can be made over the unperturbed orbits, which are described by $p_1 = p_0 \;;\; \theta = \theta_0 + \iota(p_\theta)\varphi$ , which yields

$$S_1 = \int^{\theta_0} d\theta \left\{ \int_0^1 \frac{\partial K_1}{\partial \theta} d\varphi \right\} = \mathrm{Re} \sum_{lm} K_{lm}(p_\theta) il \exp[il\theta_0] \frac{\exp[i(l\iota - m)] - 1}{(l\iota - m)} \qquad \text{Eq. 2.21}$$

## 2.1. Local Map

The mapping equations 2.16 are implicit because of the occurrence of $S_1$, the momentum $p_1$ must be computed for instance by Newton iteration or any other root finding routine. In order to avoid this complication we approximate the map by Taylor expansion. In the following we assume that an orbit starting at $p_0 = P$ stays in the neighbourhood of this point. In this case one can use a Taylor expansion of the map around this point.

$$S_1(p_1,\theta_0) = S_1(P,\theta_0) + S_2(P,\theta_0)(p_1 - P) \qquad \text{Eq. 2.22}$$

which leads to the mapping equations

$$y_0 = \left(1 + \frac{\partial S_2(P,\theta_0)}{\partial \theta_0}\right) y_1 + \frac{\partial S_1(P,\theta_0)}{\partial \theta_0} \quad ; \qquad \theta_1 = \theta_0 + \iota(P) + s y_1 + S_2(P,\theta_0) \qquad \text{Eq. 2.23}$$

Here we have introduced $y = p_1 - P$ as a new variable; $s$ is the shear defined by

$$s = \frac{d\iota}{dp_1} \qquad \text{Eq. 2.24}$$

Introduction of a new variable $Y = sy$ yields the equations

$$Y_0 = \left(1 + \frac{\partial S_2(P,\theta_0)}{\partial \theta_0}\right) Y_1 + s\frac{\partial S_1(P,\theta_0)}{\partial \theta_0} \quad ; \qquad \theta_1 = \theta_0 + \iota(P) + Y_1 + S_2(P,\theta_0) \qquad \text{Eq. 2.25}$$

If in $S_1$ the dependence on the variable $p$ is negligible ($S_2 = 0$) this reduces to a twist map

$$Y_1 = Y_0 - s\frac{\partial S_1(\theta_0)}{\partial \theta_0} \quad ; \qquad \theta_1 = \theta_0 + \iota(P) + Y_1 \qquad \text{Eq. 2.26}$$

This formulation shows the important result that shear enhances the effect of a perturbation. The excursions of the orbits in Y-direction grow with increasing shear.

## 2.2. The Role of Shear

In the literature of dynamical system[8] it is shown that on resonant surfaces, which are defined by

$$\iota(p_\theta) = \frac{m}{l} \qquad \text{Eq. 2.27}$$

primary islands exist. The size of these islands is given by

$$\delta p_\theta = 2\sqrt{\frac{K_{lm}}{s}} \quad ; \quad s = \frac{d\iota}{dp_\theta} \qquad \text{Eq. 2.28}$$

where $s$ is the shear on the resonant surface. The size of the islands grows if the shear decreases. If there are more than one resonant surface, these islands can overlap and the region between islands becomes stochastic[9]. The distance between two resonant surfaces is

$$\delta P_\theta = \frac{1}{s}\left(\frac{m_2}{l_2} - \frac{m_1}{l_1}\right) \qquad \text{Eq. 2.29}$$

which shows that the distance between islands decreases linearily with shear while the islands size decreases with the square root of the shear. Accordind to the overlap

[8] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion*, Springer Verlag 1983

[9] B.V. Chirikov, A universal instability of many-dimensional oscillator systems, Physics Reports 52, (1979) 263, North-Holland Publishing Company

criterion of Chirikov the KAM-surfaces between islands are destroyed if the size of islands is larger than the distance.

$$2\sqrt{K_{lm}s} + 2\sqrt{K_{nk}s} \geq \left( \frac{m}{l} - \frac{n}{k} \right) \qquad \text{Eq. 2.30}$$

The relevant parameter for island overlap is the product of shear and the coefficient of the resonant partubation.

As an example we consider a simple twist map with a linear iota profile

$$p_0 = p_1 - a\sin(4\pi\theta_0) - b\sin(6\pi\theta_0) \quad ; \quad \theta_1 = \theta_0 + \iota(p_1) \quad ; \quad \iota(p) = \iota_0 + sp \qquad \text{Eq. 2.31}$$

The Poincaré plots are shown in the next figures

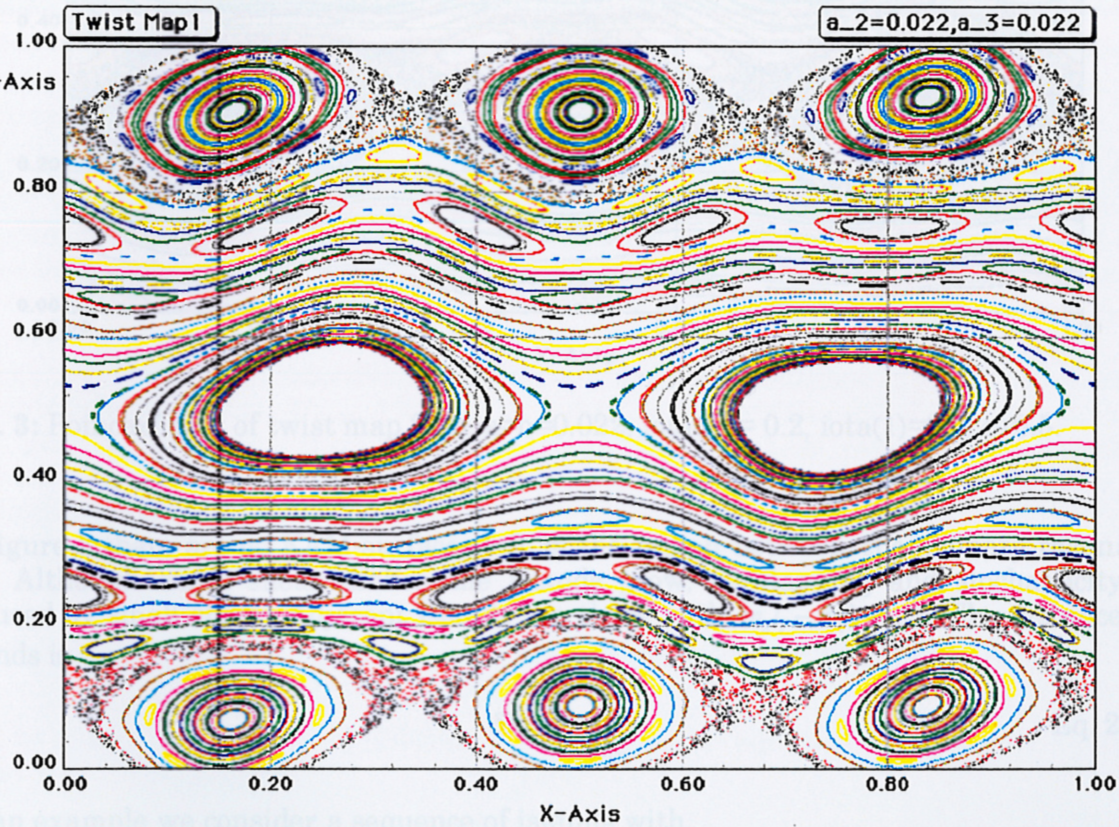


**Fig. 2:** Poincaré plot of twist map eq. 2.31. a=b=0.022, iota(0) = 0.3, iota(1)=0.7, s=0.4.

The resonant islands in Fig. 2 are the primary islands caused by the two harmonics in Eq. 2.30. The large islands arise at iota =1/3, 1/2, 2/3. Between these islands KAM-surfaces exist. Secondary islands are separated by KAM-surfaces. Increasing the shear to s = 0.5 makes the islands shrink, however the distance will shrink faster as can be seen in the next figure.

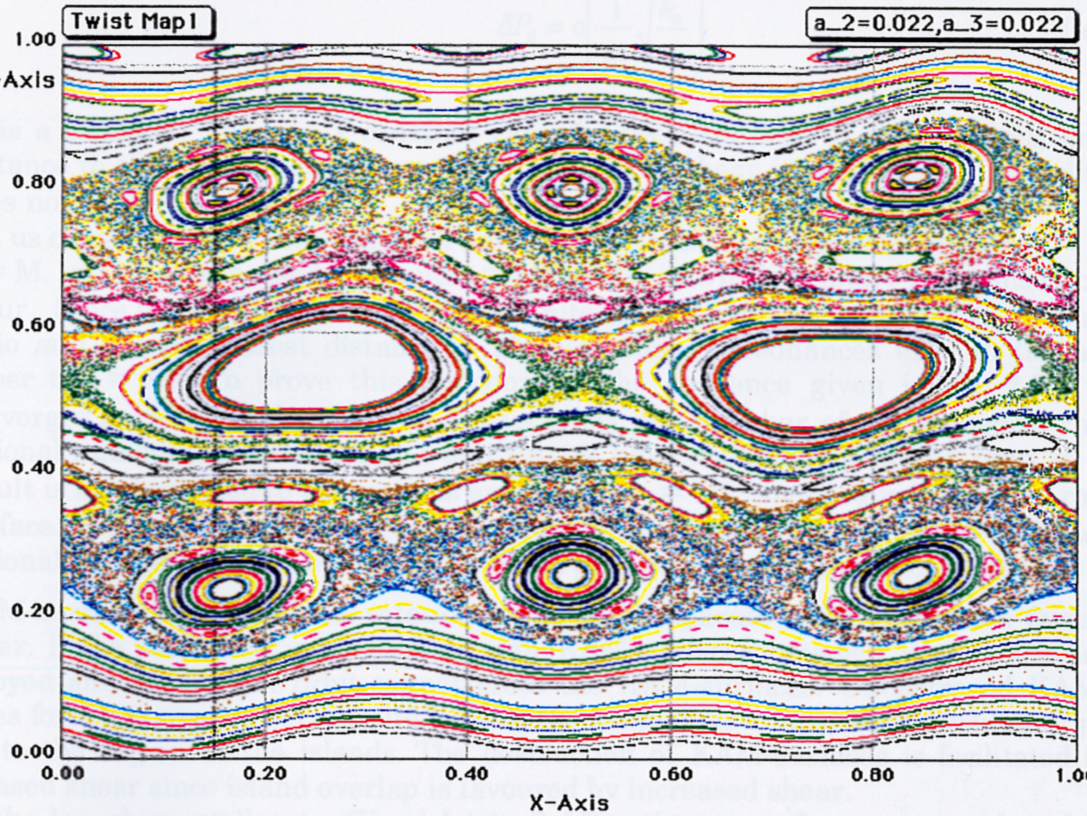


**Fig. 3:** Poincaré plot of twist map 2.30. a=b=0.022, iota(0) = 0.2, iota(1)=0.8, s=0.5.

In figure 3 there are no KAM-surfaces left in the region between the primary resonances. Although small shear makes the islands grow, overlapping and stochasticity is reduced by small shear. Adjacent islands are those with $m_2 l_1 - m_1 l_2 = \pm 1$. The distance of islands is

$$\delta P_\theta = \pm \frac{1}{s l_1 l_2} \qquad \text{Eq. 2.32}$$

As an example we consider a sequence of islands with

$$\iota_m = \frac{m}{2m-1} \quad ; \quad m = 1,2,3... \qquad \text{Eq. 2.33}$$

This sequence converges to 1/2 and all islands of this sequence are adjacent. The distance between these islands and the resonant surfaces ι=1/2 is

$$\delta P_\theta = \frac{1}{s2(2m-1)} = o\left(\frac{1}{sm}\right) \qquad \text{Eq. 2.34}$$

Since the Fourier series Eq. 3.18 converges the coefficients $K_{lm}$ decrease faster or equal to $1/ml = 1/m(2m\text{-}1)$ and for large $m$ the size of the island is

$$\delta P_\theta = o\left(\frac{1}{m}\sqrt{\frac{k_0}{s}}\right) \qquad \text{Eq. 2.35}$$

$k_0$ is a constant. There is a constant ratio between the size of these islands and the distance to the rational surfaces with $\iota = 1/2$. If the shear is small enough, overlapping does not occur.
Let us consider a case with a finite Fourier series which is truncated at some $l$ = L and $m$ = M. In this case a region around $\iota = 1/2$ exists, in which primary resonances cannot occur. Since the number of harmonics is finite there exists one harmonic $K_{lm}$ where the ratio $m/l$ has the closest distance to 1/2. No primary resonances exist in this region closer to $\iota = 1/2$. To prove this we consider the sequence given in Eq. 2.30, which converges towards 1/2 from above. Let us select any member of the sequence, then all rational numbers between this $\iota_m$ and $\iota = 1/2$ have numerators higher than $m$. This result is not restricted to the rational surface with $\iota = 1/2$, they are valid on any rational surface. The result can be formulated as follows: In case the Fourier series is finite, any rational surface has a finite vicinity in which primary resonances do not exist.

KAM-surfaces with irrational rotational transform separate rational surfaces from each other. If the perturbation is small enough KAM-surfaces between islands are not destroyed and the particle orbit cannot cross the KAM-surfaces. Undestroyed KAM-surfaces form a transport barrier and localise the anomalous transport of circulating particles to the region of the islands. The destruction of KAM-surfaces is facilitated by increased shear since island overlap is favoured by increased shear.
In the low-shear stellarator Wendelstein 7-A[10] optimum confinement was found in the neighbourhood of iota = 1/2 and iota = 1/3. A qualitative explanation of this phenomenon was given by the observation that magnetic surfaces in these regions are particularly robust against magnetic perturbations[11]. The same effect has been observed in the stellarator Wendelstein 7-AS[12].

## 3. Circulating Particles in Time-dependent Fields

Let us consider a toroidal equilibrium described by the equation of ideal MHD-equilibrium

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla p \qquad \text{Eq. 3.1}$$

There may be islands and stochastic regions and a last magnetic surface surrounded by a stochastic sea. Flux surfaces do not exist everywhere and therefore the use of magnetic co-ordinates or Boozer co-ordinates may not be applicable everywhere. In this case the so-called canonical co-ordinate system proposed by Hazeltine and Meiss (ref. 6) is more appropriate to describe particle orbits and to explore the general feature of these. This co-ordinate system has closed toroidal surfaces $r$ = const., which are nearly tangential to magnetic field lines and they may coincide with magnetic surfaces where these exist. On these toroidal surfaces labelled by $r$, two angular co-ordinates are

---

10 H. Wobig, H. Maassberg, H. Renner et al. 11th IAEA-Conf. Kyoto 1986, Vol. 2, p. 369

11 H. Wobig, Z. Naturforsch. 42a, 1054 – 1066 (1987)

12 H. Renner et al. Proc. 16th EPS-Conf. Venice 1989, 1579-1596

chosen such as to eliminate the radial covariant components of the magnetic field and its vector potential. The co-ordinate system is defined by $r,\theta,\varphi$ with

$$\boldsymbol{A}=\left(0,A_\theta,A_\varphi\right) \quad ; \quad \boldsymbol{B}=\left(0,B_\theta,B_\varphi\right) \qquad \text{Eq. 3.2}$$

It is always possible to eliminate the radial component of $\boldsymbol{A}$ by a gauge transformation. Eliminating the radial component of $\boldsymbol{B}$, however, requires a special choice of either the poloidal or the toroidal angular co-ordinate. The existence of such angular co-ordinates is discussed in ref. 6.

Time-dependent magnetic fields and electric fields can strongly disturb particle orbits and enhanced radial transport is the result of this perturbation. If the magnetic field is time-dependent, the co-ordinate system, which eliminates the $B_r$-component, is also time-dependent[13]. The guiding centre Lagrangian of charged particles is [14]

$$L=\frac{m}{2}\left(\frac{dx}{dt}\bullet b\right)^2+q\frac{dx}{dt}\bullet A-\mu B-q\Phi \quad ; \quad b=\frac{\boldsymbol{B}}{B} \qquad \text{Eq. 3.3}$$

In this form it holds in every co-ordinate system. It is also valid in time-dependent electromagnetic fields as long as the frequency is much smaller than the gyro frequency. In a general co-ordinate system the Lagrangean is

$$L=\frac{m}{2B^2}\left(v^rB_r+v^\theta B_\theta+v^\varphi B_\varphi\right)^2+q\left(v^rA_r+v^\theta A_\theta+v^\varphi A_\varphi\right)-\mu B-q\Phi \quad ; \quad \boldsymbol{b}=\frac{\boldsymbol{B}}{B} \qquad \text{Eq. 3.4}$$

$\mu$ is the magnetic moment and q the charge of the particle. $\mathbf{v}$ is the velocity of the particle. $\Phi$ is the electric potential. In order to anihilate the radial component of the magnetic field we introduce a time-dependent transformation of the poloidal angle

$$\theta\rightarrow\eta=\eta(x,t) \qquad \text{Eq. 3.5}$$

In this new coordinate system the magnetic field is

$$\boldsymbol{B}=B_r\nabla r+B_\eta\nabla\eta+B_\varphi\nabla\varphi \qquad \text{Eq. 3.6}$$

The condition $B_r = 0$ requires

$$\left(\boldsymbol{B}\times\nabla\varphi\right)\bullet\nabla\eta=0 \qquad \text{Eq. 3.7}$$

which is a differential equation for $\eta(r,\theta,\varphi,t)$, where $\varphi,t$ are parameters. If the magnetic field does not depend on time, this transformation is also time-independent. In the following we consider magnetic fields with periodic time-dependent terms

$$\boldsymbol{B}=\boldsymbol{B}_0(x)+\varepsilon\boldsymbol{B}_1(x,t) \qquad \text{Eq. 3.8}$$

The smallness parameter $\varepsilon$ is introduced to indicate that $\boldsymbol{B}_1$ is a small time-dependent perturbation. This implies that the time derivative of the co-ordinate transformation $\eta$ is of the order $\varepsilon$. Normalising to the period $\tau$ of the oscillation we introduce a dimensionless time variable t which ranges between 0 and $2\pi$. In the following the time has the same dimension and the same range as the angular variables. The Lagrangian in these new co-ordinates is

---

[13] In Ref. 3 this transformation has been made for time-independent magnetic fields

[14] R.J. Littlejohn, *Phys.Fluids* **24**, 1730 (1981)

$$L=\frac{m}{2B^2}\left(v^\eta B_\eta+v^\varphi B_\varphi\right)^2+q\left(v^\eta A_\eta+v^\varphi A_\varphi\right)-\mu B-q\Phi \qquad \text{Eq. 3.9}$$

In writing this Lagrangian in terms of the time derivatives of the co-ordinates we must take into account that the relations

$$\frac{d\varphi}{dt}=v\bullet\nabla\varphi \quad ; \quad \frac{d\eta}{dt}=v\bullet\nabla\eta+\frac{\partial\eta}{\partial t} \qquad \text{Eq. 3.10}$$

hold and the Lagrangian can be written as

$$L=L\left(\frac{d\eta}{dt},\frac{d\varphi}{dt},r,\eta,\varphi,t\right)$$

$$L=\frac{m}{2B^2}\left(\frac{d\eta}{dt}B_\eta-\frac{\partial\eta}{\partial t}B_\eta+\frac{d\varphi}{dt}B_\varphi\right)^2+q\left(\frac{d\eta}{dt}A_\eta-\frac{\partial\eta}{\partial t}A_\eta+\frac{d\varphi}{dt}A_\varphi\right)-\mu B-q\Phi \qquad \text{Eq. 3.11}$$

In the paper by Meiss and Hazeltine the last term in eq. 3.10 has been neglected, hence their results are not valid in time-dependent magnetic fields, strictly speaking. The canonical momenta derived from the Lagrangian are

$$p_r=\frac{\partial L}{\partial v^r}=0 \qquad \text{Eq. 3.12}$$

$$p_\eta=\frac{muB_\eta}{B}+qA_\eta \qquad \text{Eq. 3.13}$$

$$p_\varphi=\frac{muB_\varphi}{B}+qA_\varphi \qquad \text{Eq. 3.14}$$

The parallel velocity is

$$u=\frac{1}{B}\left(v^\eta B_\eta+v^\varphi B_\varphi\right)=\frac{1}{B}\left(\frac{d\eta}{dt}B_\eta-\frac{\partial\eta}{\partial t}B_\eta+\frac{d\varphi}{dt}B_\varphi\right) \qquad \text{Eq. 3.15}$$

The Hamiltonian is

$$H=p_\eta\frac{d\eta}{dt}+p_\varphi\frac{d\varphi}{dt}-L \qquad \text{Eq. 3.16}$$

Inserting eqs. 3.11, 3.13 and 3.14 into eq. 3.16 yields

$$H=\frac{m}{2}u^2+\mu B+q\Phi+p_\eta g \quad ; \quad g(r,\eta,\varphi,t)=\frac{\partial\eta}{\partial t} \qquad \text{Eq. 3.17}$$

The last term is a new term, which arises due to the time-dependent co-ordinate transformation. The Hamiltonian is not yet written in terms of canonical variables. For this purpose the equations 3.13 and 3.14 must be inverted which yields

$$u=u\left(p_\eta,p_\varphi,\eta,\varphi,t\right) \quad ; \quad r=r\left(p_\eta,p_\varphi,\eta,\varphi,t\right) \qquad \text{Eq. 3.18}$$

This inversion is possible provided the Jacobian of the transformation 3.13 and 3.14 is non-zero. In ref. 6 this condition is mentioned, however, no example is given, when this condition can be satisfied. In the following we assume, that the toroidal component $B_\varphi$ is much larger than the other ones and $B_\varphi \neq 0$. We can write eq. 3.14 as

$$u = \frac{B}{mB_\varphi}\left(p_\varphi - qA_\varphi\right) \qquad \text{Eq. 3.19}$$

The radial co-ordinate $r$ follows from

$$p_\eta B_\varphi - p_\varphi B_\eta = q\left(A_\eta B_\varphi - A_\varphi B_\eta\right) \qquad \text{Eq. 3.20}$$

which in lowest order in $B_\eta$ yields

$$p_\eta = qA_\eta\left(r, \eta, \varphi, t\right) \qquad \text{Eq. 3.21}$$

For inverting this equations it is necessary that

$$B^\varphi = \frac{\partial A_\eta}{\partial r} \neq 0 \qquad \text{Eq. 3.22}$$

The result can be formulated as follows: If $B_\varphi \neq 0$ and $B_r << B_\varphi, B_\eta << B_\varphi$ there exists a unique solution of eq. 3.21.

$$r = r\left(p_\eta, p_\varphi, \eta, \varphi, t\right) \qquad \text{Eq. 3.23}$$

After eliminating the parallel velocity and the radial co-ordinate $r$ the Hamiltonian is

$$H = \frac{m}{2}u^2\left(p_\eta, p_\varphi, \eta, \varphi, t\right) + \mu B\left(p_\eta, p_\varphi, \eta, \varphi, t\right) + q\Phi\left(p_\eta, p_\varphi, \eta, \varphi, t\right) + p_\eta g \qquad \text{Eq. 3.24}$$

or

$$H = \frac{B^2}{2mB_\varphi^2}\left(p_\varphi - qA_\varphi\right)^2 + \mu B\left(p_\eta, p_\varphi, \eta, \varphi, t\right) + q\Phi\left(p_\eta, p_\varphi, \eta, \varphi, t\right) + p_\eta g \qquad \text{Eq. 3.25}$$

The Hamiltonian has only four variables, the two angles and their conjugate momenta; the radial conjugate momentum does not occur in the Hamiltonian. In a paper by A. Boozer[15], it was only argued that the radial $B_r$ –component can be neglected since it is small. Hamilton's equations are

$$\frac{dp_\eta}{dt} = -\frac{\partial H}{\partial \eta} = -mu\frac{\partial u}{\partial \eta} - \frac{\partial\left(\mu B + q\Phi\right)}{\partial \eta} - p_\eta \frac{\partial g}{\partial \eta} \qquad \text{Eq. 3.26}$$

$$\frac{dp_\varphi}{dt} = -\frac{\partial H}{\partial \varphi} = -mu\frac{\partial u}{\partial \varphi} - \frac{\partial\left(\mu B + q\Phi\right)}{\partial \varphi} - p_\eta \frac{\partial g}{\partial \varphi} \qquad \text{Eq. 3.27}$$

$$\frac{d\eta}{dt} = \frac{\partial H}{\partial p_\eta} = mu\frac{\partial u}{\partial p_\eta} + \frac{\partial\left(\mu B + q\Phi\right)}{\partial p_\eta} + g + p_\eta \frac{\partial g}{\partial p_\eta} \qquad \text{Eq. 3.28}$$

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial p_\varphi} = mu\frac{\partial u}{\partial p_\varphi} + \frac{\partial\left(\mu B + q\Phi\right)}{\partial p_\varphi} + p_\eta \frac{\partial g}{\partial p_\varphi} \qquad \text{Eq. 3.29}$$

These equations are the Euler equations of the variational principle

---

15 A. H.Boozer, Phys. Fluids 27, 2441 (1984)

$$\delta \int L dt = 0 \quad \Rightarrow \quad \delta \int p_\eta d\eta + p_\varphi d\varphi - H dt = 0 \qquad \text{Eq. 3.30}$$

In the limit of m -> 0 the Hamiltonian is zero and we get the field line equation[16]

$$\delta \int A_\eta d\eta + A_\varphi d\varphi = 0 \qquad \text{Eq. 3.31}$$

## 4. Passing Particles

In the following we consider particles circulating around the torus without changing the sign of the parallel velocity. Basically these particles follow the field lines, magnetic drift and electric drift, however, cause some deviation from the field lines. Particle orbits even can be stochastic, if field lines are regular. In order to obtain a twist map for circulating particles the toroidal angle as independent variable is introduced instead of the time. To do so we invert the Hamiltonian and write the toroidal momentum $p_\varphi$ as function of energy E and the other variables

$$p_\varphi = qA_\varphi + \sigma \sqrt{\frac{2mB_\varphi^2}{B^2}\left(E - \mu B - q\Phi - p_\eta g\right)} =: -K\left(E, t, p_\eta, \eta, \varphi\right) \qquad \text{Eq. 4.1}$$

which defines a new Hamiltonian $K$. σ is the sign of the circulating particle and it is either 1 or –1. Equation 4.1 is only a formal solution of this inversion problem since the magnetic field and the electric field in 4.1 are still functions of $p_\varphi$. However, in case of circulating particles this inversion is possible since for these we have

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial p_\varphi} \neq 0 \qquad \text{Eq. 4.2}$$

The new variational principle is now

$$\delta \int p_\eta d\eta - K d\varphi - E dt = 0 \qquad \text{Eq. 4.3}$$

and the new canonical equations are

$$\frac{dp_\eta}{d\varphi} = -\frac{\partial K}{\partial \eta} \quad ; \quad \frac{d\eta}{d\varphi} = \frac{\partial K}{\partial p_\eta} \qquad \text{Eq. 4.4}$$

$$\frac{dE}{d\varphi} = \frac{\partial K}{\partial t} \quad ; \quad \frac{dt}{d\varphi} = -\frac{\partial K}{\partial E} \qquad \text{Eq. 4.5}$$

In this formulation conservation of energy is obvious if the fields are time-independent. Only the first two equations 4.4 are relevant in this case and the problem has been reduced to a problem with two canonical variables. The phase space is the $p_\eta$,η-space and according to basic theorems of mechanics the area

$$J = \oint p_\eta d\eta \qquad \text{Eq. 4.6}$$

is conserved around the torus. This property is the equivalent to the flux conservation of magnetic field lines. It implies that all properties of magnetic field lines also apply to

[16] J.R. Gary, R.G. Littlejohn, Annals of Physics (NY)151, 1 (1983)

circulating particles in a torus. In particular stochastisation and island overlap may lead to enhanced losses if the self-consistent electric field is properly retained. In axisymmetric tokamaks there is no dependence on the toroidal angle and eqs. 3.27 says that the toroidal canonical momentum is conserved. In non-axisymmetric stellarators this is no longer true, however the conservation of J is still valid (eq. 4.6).

Let us first consider briefly the time independent case in a stellarator and write the Hamiltonian $K$ in the form

$$K = K_0(p_\eta, E) + K_1(p_\eta, E, \eta, \varphi) \qquad \text{Eq. 4.7}$$

where $K_0$ is the average of $K$ over the two angular variables as in Eq. 2.11 and the energy $E$ is constant. The canonical equations are

$$\frac{dp_\eta}{d\varphi} = -\frac{\partial K_1}{\partial \eta} \quad ; \quad \frac{d\eta}{d\varphi} = \iota(p_\eta) + \frac{\partial K_1}{\partial p_\eta} \quad ; \quad \iota(p_\eta) = \frac{\partial K_0}{\partial p_\eta} \qquad \text{Eq. 4.8}$$

which are the same equations as in 2.12.

Next, we return to the time-dependent case, where the electromagnetic oscillations introduce a small correction term $K_2$.

$$K = K_0(p_\eta, E) + K_1(p_\eta, E, \eta, \varphi, t) \qquad \text{Eq. 4.9}$$

The unperturbed Hamiltonian $K_0$ describes unperturbed drift surfaces characterised by the two "frequencies"

$$\iota(p_\eta, E) = \frac{\partial K_0}{\partial p_\eta} \quad ; \quad T(p_\eta, E) = \frac{\partial K_0}{\partial E} \qquad \text{Eq. 4.10}$$

The term $K_1$ describes time-dependent perturbations, which can disturb the orbits as described in section 2. It also destroys the energy conservation. $T$ is the toroidal transit time of circulating particles normalised to the period $\tau$ of the oscillations. This is a decreasing function of the particle energy since the transit time of fast particles is shorter than the transit time of slow particles. Particles circulating around the torus with initial values $\eta_0, t_0, p_0, E_0$ will change these values to $\eta_1, t_1, p_1, E_1$ after one toroidal transit. The map describing this transformation has the generating function

$$S = S(p_1, \eta_0, E_1, t_0) = p_1\eta_0 + E_1 t_0 + K_0(E_1, p_1) + S_1(p_1, \eta_0, E_1, t_0) \qquad \text{Eq. 4.11}$$

The four the mapping equations are

$$E_0 = E_1 + \frac{\partial S_1}{\partial t_0} \quad ; \quad t_1 = t_0 + T(E_1 p_1) + \frac{\partial S_1}{\partial E_1} \qquad \text{Eq. 4.12}$$

and

$$p_0 = p_1 + \frac{\partial S_1}{\partial \eta_0} \quad ; \quad \eta_1 = \eta_0 + \iota(p_1, E_1) + \frac{\partial S_1}{\partial p_1} \qquad \text{Eq. 4.13}$$

The perturbation $S_1$ is uniquely determined by the perturbation of the Hamiltonian $K_1$. An approximation to this function $S_1$ will be given in the next section.

## 5. Approximation of the Map

In the analysis above no assumption about the smallness of $K_1$ or $S_1$ has been made. In the following we assume that magnetic surfaces exist and the magnetic field can be written as

$$\boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{x}) + \boldsymbol{B}_1(\boldsymbol{x},t) \qquad \text{Eq. 5.1}$$

with nested magnetic surfaces of the lowest order field. The Hamiltonian is

$$p_\varphi = qA_\varphi^0(p_\eta) + A_\varphi^1(p_\eta, \eta, E, t, \varphi) + \sigma\sqrt{\frac{2mB_\varphi^2}{B^2}(E - \mu B - q\Phi - p_\eta g)} \qquad \text{Eq. 5.2}$$

Averaging the second and the third term over the angles and the time (which is periodic) yields

$$K_0(p_\eta, E) = qA_\varphi^0(p_\eta) + \sigma\left\langle\sqrt{\frac{2mB_\varphi^2}{B^2}(E - \mu B - q\Phi - p_\eta g)}\right\rangle \qquad \text{Eq. 5.3}$$

and

$$K_1 = K - K_0 \qquad \text{Eq. 5.4}$$

The derivative with respect to $p_\eta$ is the rotational transform of the unperturbed drift surface. Next, we consider $K_1$ as a small perturbation and integrate equation 4.8 over the unperturbed orbits. The unperturbed orbits are written as

$$p_1 = p_0 \;\;;\;\; E_1 = E_0 \;\;;\;\; \eta_1 = \eta_0 + \iota(E_1, p_1)\varphi \;\;;\;\; t = t_0 + \frac{\partial K_0}{\partial E}\varphi \qquad \text{Eq. 5.5}$$

The formal integration of the canonical equation 4.4 over the toroidal angle φ and along the orbit yields

$$p_1 = p_0 - \int_0^1 \frac{\partial K_1}{\partial \eta_0}(E_1, p_1, \eta_0 + \iota\varphi, \varphi, t(\varphi))d\varphi \qquad \text{Eq. 5.6}$$

Because of 4.13 we obtain the generating function

$$S_1 = \int^{\eta_0}\int_0^1 \frac{\partial K_1}{\partial \eta}(E_1, p_1, \eta + \iota\varphi, \varphi, t(\varphi))d\varphi d\eta \qquad \text{Eq. 5.7}$$

Expanding the Hamiltonian $K_1$ in a Fourier series

$$K_1 = \sum_{nml} K_{mnl}(E_1, p_1)\exp(i[l\eta - m\varphi + n\omega t]) \qquad \text{Eq. 5.8}$$

yields

$$S_1 = \mathrm{Re}\sum_{lmn} K_{lmn}(E_1, p_1) il\ \exp(il\eta_0 + in\omega t_0)\frac{(1 - \exp(i[l\iota - m + n\omega T]))}{i[l\iota - m + n\omega T]} \qquad \text{Eq. 5.9}$$

## 6. Extended Phase Space

Our canonical equations 4.4 and 4.5 describe a non-autonomous system with two degrees of freedom. The Hamiltonian $K$ depends on 4 phase space variables $p_\eta,\eta,E,t$ and the independent variable $\varphi$. Considering $-K$ and $\varphi$ as new conjugate variables and $\lambda$ as a new independent variable leads to an autonomous system with 6 phase space variables (or three degrees of freedom)[17]. The new Hamiltonian is

$$K \Rightarrow K^*(p_\eta,\eta,E,t,\varphi,K) = K(p_\eta,\eta,E,t,\varphi) - k \qquad \text{Eq. 6.1}$$

The new independent variable $\lambda$ does not occur in the Hamiltonian $K^*$ which implies that $K^*$ is conserved. The canonical equations are now

$$\frac{dp_\eta}{d\lambda} = -\frac{\partial K^*}{\partial \eta} \quad ; \quad \frac{d\eta}{d\lambda} = \frac{\partial K^*}{\partial p_\eta} \qquad \text{Eq. 6.2}$$

$$\frac{dE}{d\lambda} = \frac{\partial K^*}{\partial t} \quad ; \quad \frac{dt}{d\lambda} = -\frac{\partial K^*}{\partial E} \qquad \text{Eq. 6.3}$$

$$\frac{dk}{d\lambda} = \frac{\partial K^*}{\partial \varphi} \quad ; \quad \frac{d\varphi}{d\lambda} = -\frac{\partial K^*}{\partial k} = 1 \qquad \text{Eq. 6.4}$$

The new Hamiltonian $K^*$ is constant and defines a five-dimensional plane in the six-dimensional phase space. The Hamiltonian can be written as

$$K^* = K_0(p_\eta,E) - k + K_1(p_\eta,\eta,E,t,\varphi) \qquad \text{Eq. 6.5}$$

The unperturbed Hamiltonian is

$$K_0^*(p_\eta,E,k) = K_0(p_\eta,E) - k \qquad \text{Eq. 6.6}$$

and the resonance surfaces are defined by

$$l\iota(p_\eta,E) + n\omega T(p_\eta,E) - m = 0 \qquad \text{Eq. 6.7}$$

On these resonance surfaces islands can arise if the Fourier spectrum of perturbations contains resonant components or by non-linear coupling of non-resonant harmonics. In contrast to time-independent case discussed above resonant surfaces can intersect. They are no longer separated by KAM surfaces. This property is the origin of the Arnold diffusion[18].
The resonance surfaces according to eq. 6.7 are independent of the co-ordinate k, the projections onto the $k$-plane describe curves in the $p_\eta$-$E$-plane.

[17] See Lichtenberg and Lieberman, *Regular and Stochastic Motion*, Chapter 6.
[18] V.I.Arnold, Mathematical Methods of Classical Mechanics, Springer -Verlag

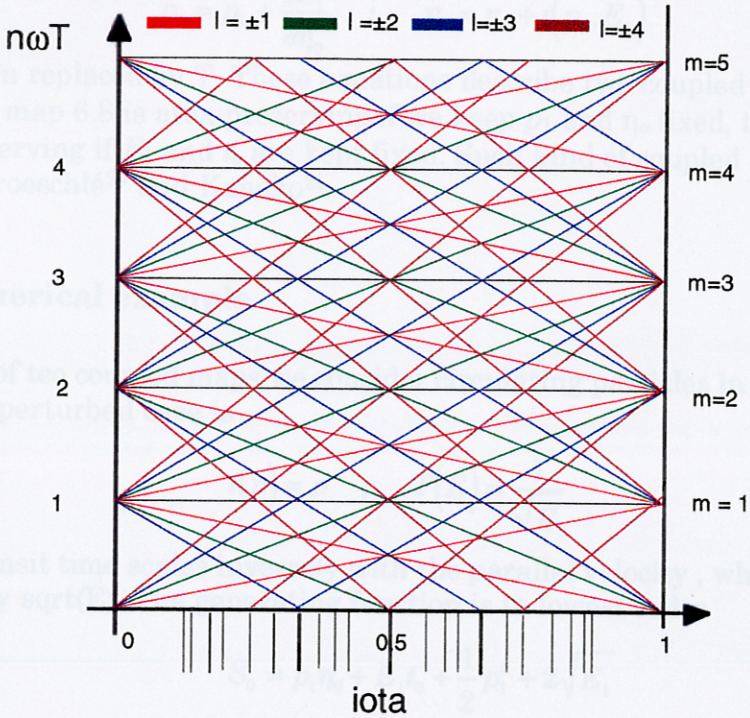


**Fig. 4:** Arnold web in the nωT-iota plane, m=1, ...5, l=±1,... ,±4.

In a time-independent system this resonance condition 6.7 reduces to $\iota$-$m = 0$ and the resonant lines do not intersect for constant energy $E$. In general, the surfaces defined by eq. 6.7 do intersect. These intersection points are indicated by vertical lines on the x-axis in Fig 4. Particles stay on surfaces of constant K* and by slow diffusion along the resonance layers – the Arnold diffusion – they can diffuse from one resonance layer to another. The effect is similar to collisions, which change the energy of the particles in a random fashion. By changing the energy the particle can jump from one drift surface onto another one. Without islands the radial step width of these random processes is of the order gyro radius and leads to classical diffusion or Pfirsch-Schlüter diffusion. If islands in drift surfaces exist, the KAM surfaces between islands can be “tunneled” by collisions leading to enhanced radial transport. The transport barrier formed by KAM surfaces can be “torn down” by collisions. Arnold diffusion in action space has the same effect.

A classical example exhibiting Arnold diffusion is the billiard ball problem with a ball bouncing between a plane wall and a rippled wall[19]. The resulting four mapping equations have the same structure as eq. 4.12 and eq. 4.13.

Let us assume that the function $S_1$ is independent of the energy $E$ and $p$. The mapping equations are

$$E_0 = E_1 + \frac{\partial S_1}{\partial t_0} \quad ; \quad t_1 = t_0 + T\left(E_1, p_1\right) \qquad \text{Eq. 6.8}$$

and

[19] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion*, Springer Verlag 1983

$$p_0 = p_1 + \frac{\partial S_1}{\partial \eta_0} \quad ; \quad \eta_1 = \eta_0 + \iota(p_1, E_1) \qquad \text{Eq. 6.9}$$

Her ωT has been replaced by T. These equations describe two coupled area-preserving maps. The first map 6.8 is area-preserving if we keep $p_1$ and $\eta_o$ fixed, the second map 6.9 is area-preserving if $E_1$ and $t_o$ are kept fixed. Such kind of coupled maps have been studied by C. Froeschlé[20] and Kaneko[21].

## 6.1. Numerical example

As an example of teo coupled maps we consider circulating particles in a torus. The model of the unperturbed case is

$$\iota(p) = p \quad ; \quad T(E) = \frac{1}{\sqrt{E}} \qquad \text{Eq. 6.10}$$

The toroidal transit time scales inversely with the parallel velocity , which has been approximated by sqrt(E). The generating function is in lowest order

$$S_0 = p_1\eta_0 + E_1 t_0 + \frac{1}{2}p_1^2 + 2\sqrt{E_1} \qquad \text{Eq. 6.11}$$

and the perturbation is

$$S_1 = \frac{a}{2\pi\sqrt{E_1}}\cos(2\pi\eta_0) + \frac{b}{2\pi}\cos(2\pi(\eta_0 - t_0)) \qquad \text{Eq. 6.12}$$

which leads to the mapping equations

$$E_0 = E_1 + b\sin(2\pi(\eta_0 - t_0)) \quad ; \quad t_1 = t_0 + T(E_1) - \frac{a}{2E_1^{3/2}}\cos(2\pi\eta_0) \qquad \text{Eq. 6.13}$$

$$p_0 = p_1 - \frac{a}{\sqrt{E_1}}\sin(2\pi\eta_0) - b\sin(2\pi(\eta_0 - t_0)) \quad ; \quad \eta_1 = \eta_0 + p_1 \qquad \text{Eq. 6.14}$$

Setting $b$ to 0 makes the energy constant and the remaining two equations describe the standard map. This is shown in Fig. 6. The time-independent part of $S_1$ describes a magnetic drift or electric drift, which causes the drift orbits to deviate from magnetic surfaces. This difference scales inversely with the parallel velocity of the particles. The first equations 6.13 yield an area preserving map if η is a fixed parameter. This map is shown in Fig. 7. There are two resonances, the upper one is defined by

$$T(E_1) - \frac{a}{2E_1^{3/2}}\cos(2\pi\eta_0) = 1 \qquad \text{Eq. 6.15}$$

and the lower one by

$$T(E_1) - \frac{a}{2E_1^{3/2}}\cos(2\pi\eta_0) = 2 \qquad \text{Eq. 6.16}$$

[20] C. Froeschle, Astron. Astrophys. 16 (1972) 172

[21] R. Kaneko, R.J. Bagley, Physics Letters 110 A, 435-40 (1985)

A third (small) resonance occurs at 1.5.

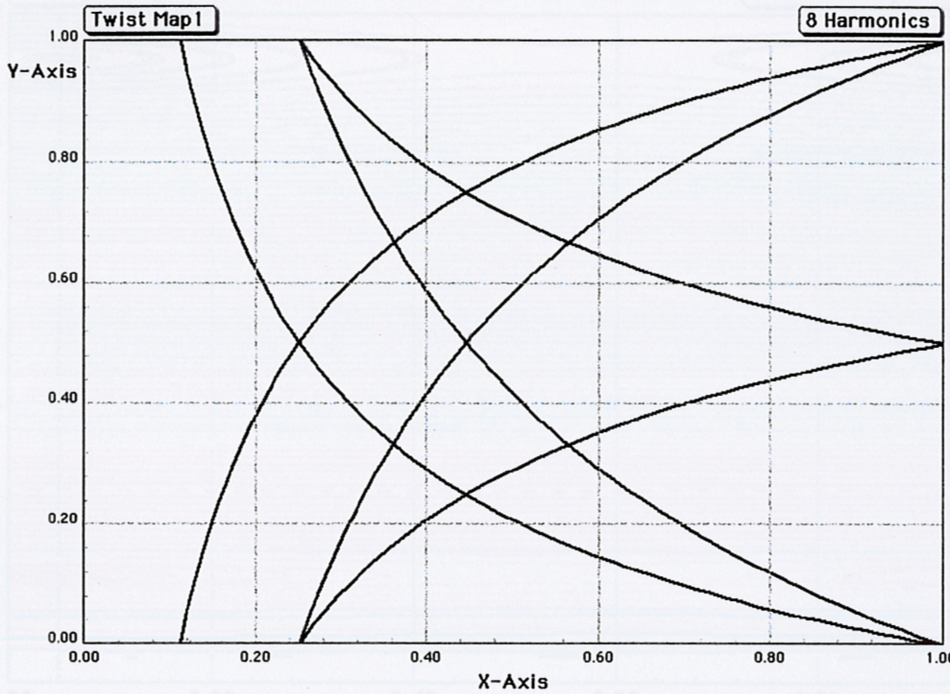


**Fig. 5:** Arnold web in the p-E-plane. Horizontal axis: energy, vertical axis: poloidal momentum (radial co-ordinate).

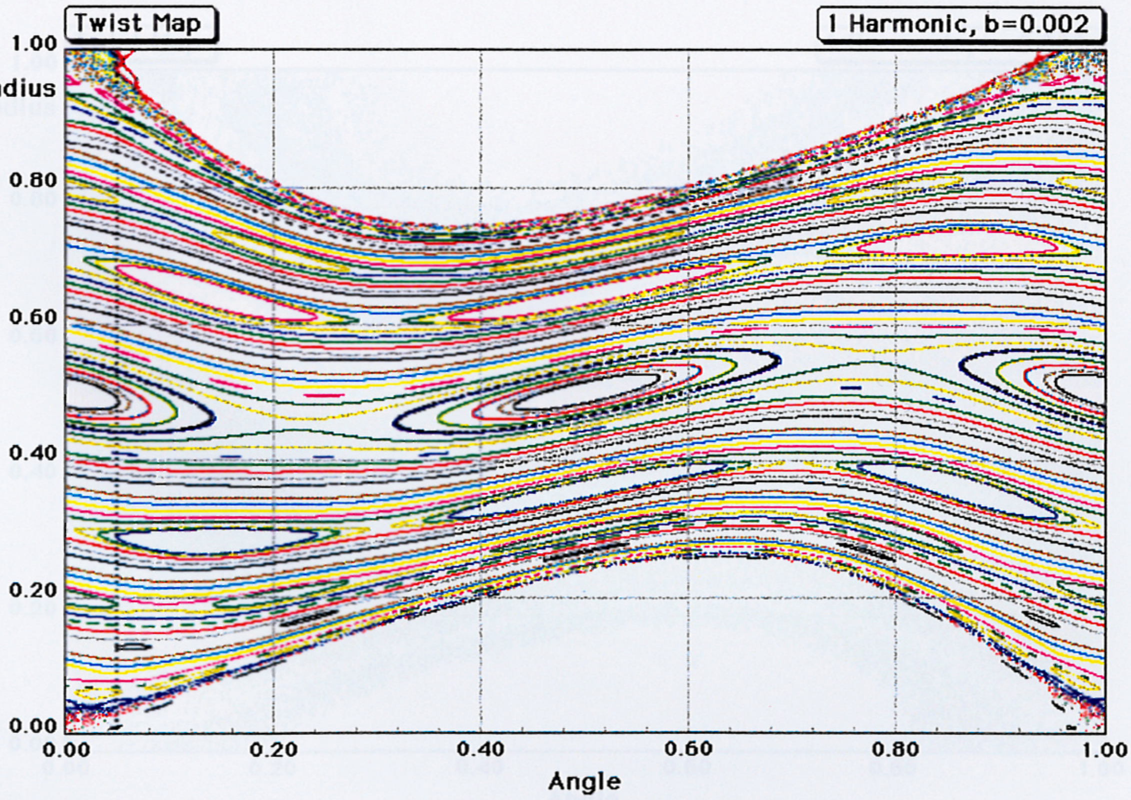


**Fig. 6:** Standard map with a = 0.066, E = 0.444. Solid black line: Start line of particles. The particles start in the island with iota = 0.5. b = 0.002, E,t fixed.

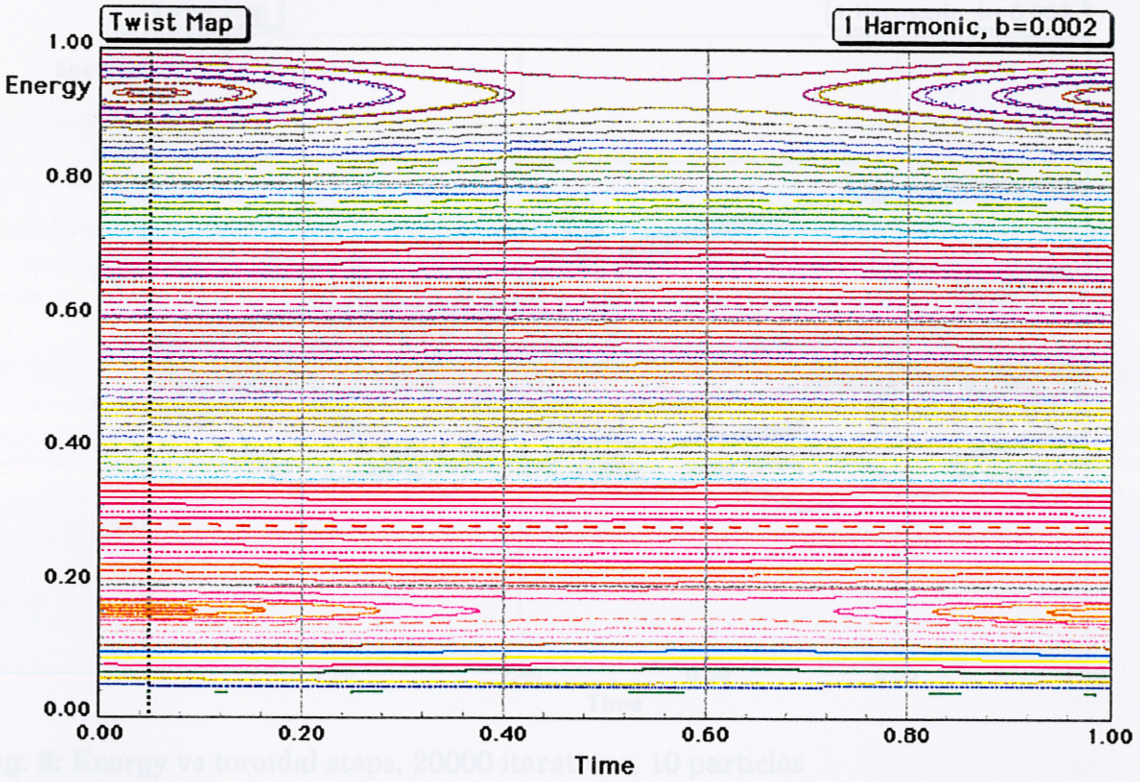


**Fig. 7:** E-t Map. b = 0.002, $\eta$ is fixed .

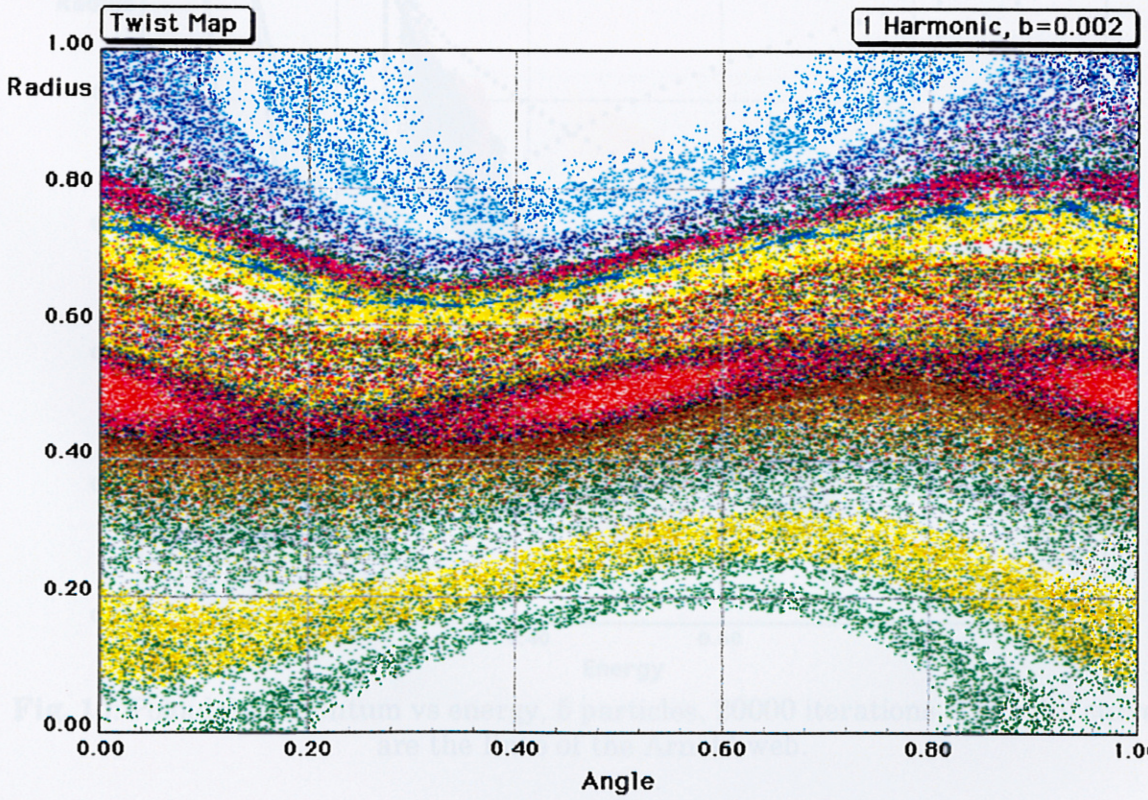


**Fig. 8:** Arnold diffusion of 10 particles, 20000 transits, b = 0.002, $E_0 = 0.444$. $x_o$ = 0.05.The particles start in the iota = 0.5 island.

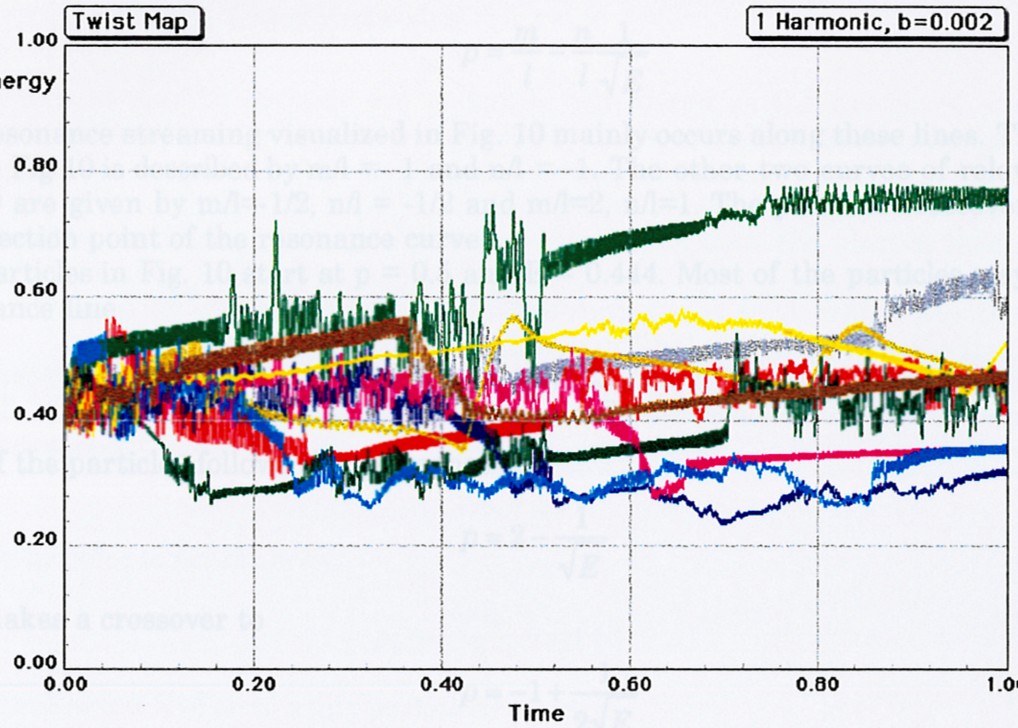


**Fig. 9:** Energy vs toroidal steps, 20000 iterations, 10 particles

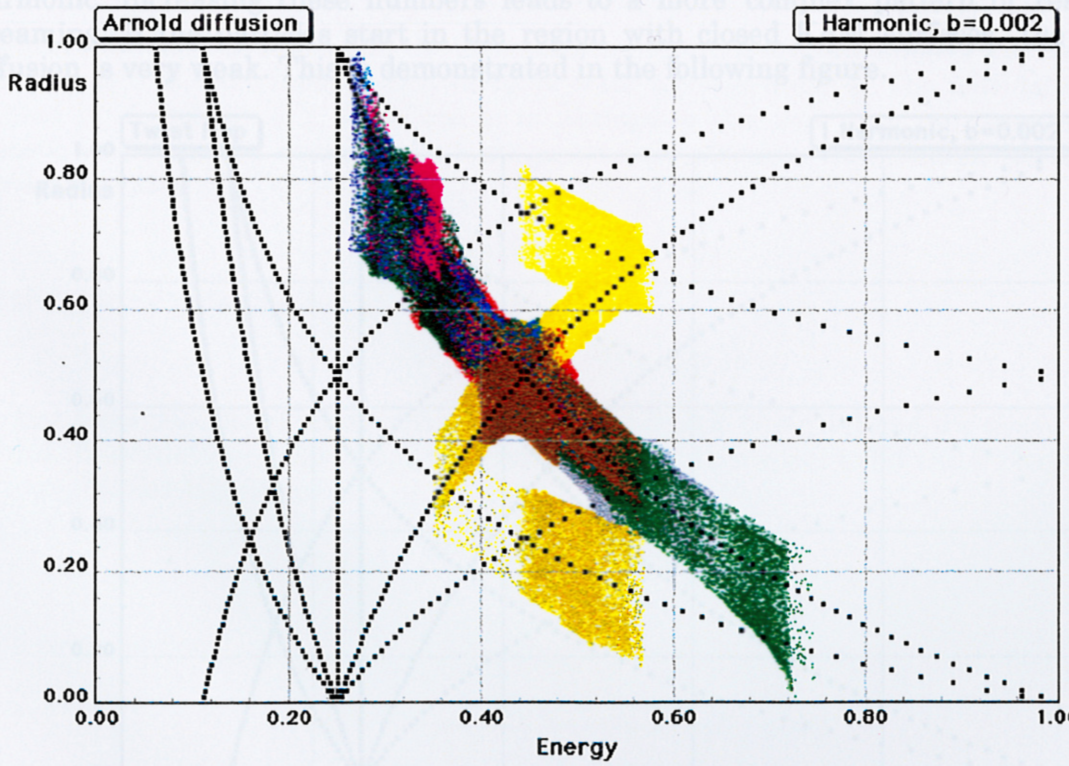


**Fig. 10**: Poloidal momentum vs energy, 5 particles, 20000 iterations. The solid curves are the lines of the Arnold web.

Fig 9 shows the energy variation of the particles, the chaotic behaviour of the energy is evident. The resonance condition 6.7 is written as

$$p = \frac{m}{l} - \frac{n}{l}\frac{1}{\sqrt{E}} \qquad \text{Eq. 6.17}$$

The resonance streaming visualized in Fig. 10 mainly occurs along these lines. The main line in Fig 10 is described by m/l = -1 and n/l = -1. The other two curves of relevance in Fig 10 are given by m/l=-1/2, n/l = -1/2 and m/l=2, n/l=1. The particles crossover at the intersection point of the resonance curves.

The particles in Fig. 10 start at p = 0.5 and E = 0.444. Most of the particles stay on the resonance line

$$p = -1 + \frac{1}{\sqrt{E}} \qquad \text{Eq. 6.18}$$

One of the particles follows the resonance line

$$p = 2 - \frac{1}{\sqrt{E}} \qquad \text{Eq. 6.19}$$

and makes a crossover to

$$p = -1 + \frac{1}{2\sqrt{E}} \qquad \text{Eq. 6.20}$$

The example given here employs only one harmonic in η and one time-dependent Harmonic. Increasing these numbers leads to a more complex pattern of resonance streaming. If the particles start in the region with closed KAM surfaces, the Arnold diffusion is very weak. This is demonstrated in the following figure.

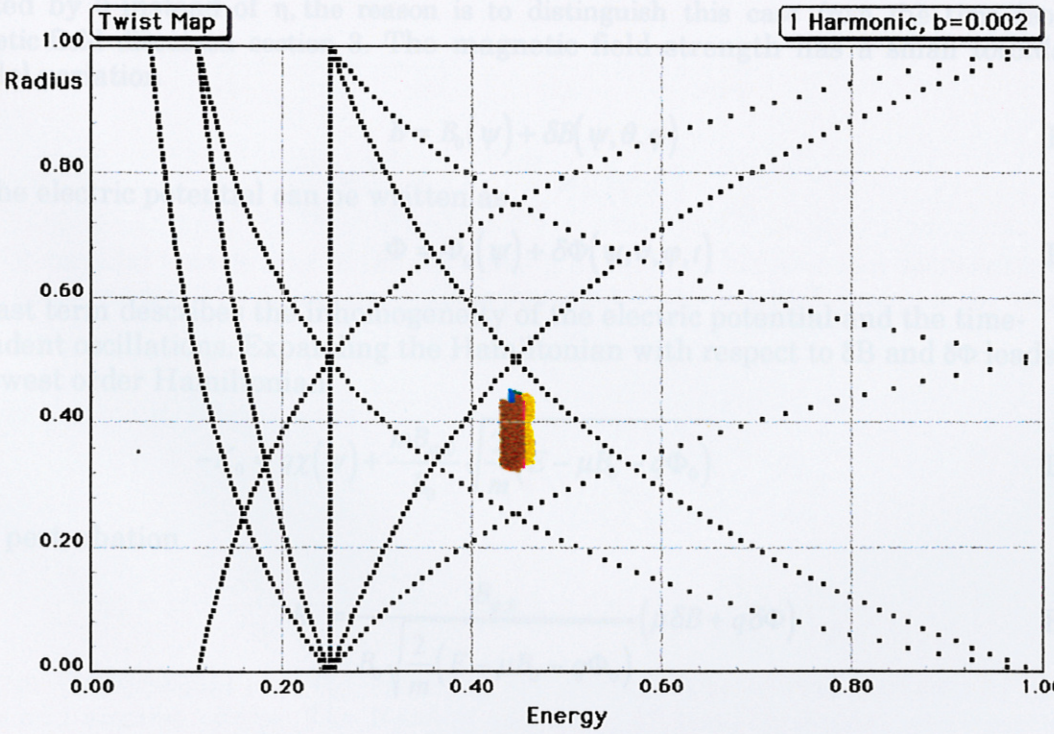


**Fig. 11:** Arnold diffusion of 5 particles, 20000 transits, b = 0.002, $E_0$ = 0.444. $x_o$ = 0.05.The particles start on the line 39 (iota =0.413).

# 7. Electrostatic Oscillations

Drift waves in toroidal systems are generally considered as the reason of anomalous transport. In lowest order approximation drift waves are electrostatic oscillations, the variation of the magnetic field is neglected. Particle orbits under the effect of drift waves have been studied by Horton et al.[22] employing the mapping technique. In this article, however, the variation of energy due to the interaction with the electric field has not been retained.
In the following we consider a time-independent magnetic field, which has closed flux surfaces. The flux surfaces are used as radial co-ordinates and by introducing flux co-ordinates the magnetic field lines become straight lines. Since the poloidal magnetic field is small compared with the toroidal one we may approximate the poloidal canonical momentum by

$$p_\eta = q\psi \qquad \text{Eq. 7.1}$$

and the toroidal momentum by

$$p_\varphi = \frac{muB_\varphi}{B} + q\chi(\psi) \qquad \text{Eq. 7.2}$$

The Hamiltonian $K$ of circulating particles becomes

$$-K = q\chi(\psi) + \frac{mB_\varphi}{B}\sqrt{\frac{2}{m}\left(E - \mu B - q\Phi\right)} \qquad \text{Eq. 7.3}$$

The magnetic field strength and the electrostatic potential are functions of the canonical variables $q\psi,\theta,-E,t$, the independent variable is the toroidal angle $\varphi$. The poloidal angle is denoted by $\theta$ instead of $\eta$, the reason is to distinguish this case from the time-dependent magnetic field discussed section 3. The magnetic field strength has a small toroidal and poloidal variation

$$B = B_0(\psi) + \delta B(\psi,\theta,\varphi) \qquad \text{Eq. 7.4}$$

and the electric potential can be written as

$$\Phi = \Phi_0(\psi) + \delta\Phi(\psi,\theta,\varphi,t) \qquad \text{Eq. 7.5}$$

The last term describes the inhomogeneity of the electric potential and the time-dependent oscillations. Expanding the Hamiltonian with respect to $\delta B$ and $\delta\Phi$ leads to the lowest order Hamiltonian

$$-K_0 = q\chi(\psi) + \frac{mB_{\varphi,0}}{B_0}\sqrt{\frac{2}{m}\left(E - \mu B_0 - q\Phi_0\right)} \qquad \text{Eq. 7.6}$$

and a perturbation

$$K_1 = \frac{B_{\varphi,0}}{B_0\sqrt{\frac{2}{m}\left(E - \mu B_0 - q\Phi_0\right)}}\left(\mu\delta B + q\delta\Phi\right) \qquad \text{Eq. 7.7}$$

---

[22] ref 4

The variation of $B_\varphi/B$ has been neglected. The perturbation is the sum of a time-independent and a time-dependent part.

$$K_{1,1} = \frac{B_{\varphi,0}\mu\delta B}{B_0\sqrt{\frac{2}{m}\left(E-\mu B_0 - q\Phi_0\right)}} \quad ; \quad K_{1,2} = \frac{B_{\varphi,0}q\delta\Phi}{B_0\sqrt{\frac{2}{m}\left(E-\mu B_0 - q\Phi_0\right)}} \qquad \text{Eq. 7.8}$$

The lowest order Hamiltonian describes the motion in magnetic surfaces

$$q\frac{d\psi}{d\varphi} = -\frac{\partial K_0}{\partial\theta} = 0 \quad ; \quad \frac{d\theta}{d\varphi} = \frac{\partial K_0}{q\partial\psi} = \iota(\psi) \qquad \text{Eq. 7.9}$$

The rotational transform of the particle is the sum of two terms: The first term is the rotational transform of the magnetic field $\iota_B(\psi)$ and the second term is the result of magnetic drift and electric drift. Especially a lowest order electric field will modify the drift rotational transform.

$$\iota = \frac{d\chi(\psi)}{d\psi} + \frac{\partial}{q\partial\psi}\left(\frac{mB_{\varphi,0}}{B_0}\sqrt{\frac{2}{m}\left(E-\mu B_0 - q\Phi_0\right)}\right) \qquad \text{Eq. 7.10}$$

If the electric field is large enough the drift rotational transform can be approximated by

$$\iota = \frac{d\chi(\psi)}{d\psi} + \frac{B_{\varphi,0}}{B_0}\frac{1}{u}\frac{\partial\Phi_0}{\partial\psi} \quad ; \quad u = \sqrt{\frac{2}{m}\left(E-\mu B_0 - q\Phi_0\right)} \qquad \text{Eq. 7.11}$$

Taking into account the finite perturbation leads to the following set of canonical equations

$$\frac{qd\psi}{d\varphi} = -\frac{\partial K_{1,1}}{\partial\theta} - \frac{\partial K_{1,2}}{\partial\theta} \quad ; \quad \frac{d\theta}{d\varphi} = \iota(\psi) + \frac{\partial K_1}{q\partial\psi} \qquad \text{Eq. 7.12}$$

$$\frac{dE}{d\varphi} = \frac{\partial K_{1,2}}{\partial t} \quad ; \quad \frac{dt}{d\varphi} = T(\psi,E) - \frac{\partial K_1}{\partial E} \qquad \text{Eq. 7.13}$$

T is the toroidal transit time of the circulating particles. This time is a decreasing function of the energy. The action generating function of the toroidal map has the form

$$S = S(\psi_1,\eta_0,E_1,t_0) = \psi_1\eta_0 + E_1 t_0 + K_0(E_1,\psi_1) + S_{1,1}(\psi_1,\eta_0,E_1) + S_{1,2}(\psi_1,\eta_0,E_1,t_0) \qquad \text{Eq. 7.14}$$

and the mapping equations are

$$E_0 = E_1 + \frac{\partial S_{1,2}}{\partial t_0} \quad ; \quad t_1 = t_0 + T(E_1,\psi_1) + \frac{\partial S_{1,1}}{\partial E_1} + \frac{\partial S_{1,2}}{\partial E_1} \qquad \text{Eq. 7.15}$$

$$\psi_0 = \psi_1 + \frac{\partial S_{1,1}}{\partial\eta_0} + \frac{\partial S_{1,2}}{\partial\eta_0} \quad ; \quad \theta_1 = \theta_0 + \iota(\psi_1,E_1) + \frac{\partial S_{1,1}}{\partial\psi_1} + \frac{\partial S_{1,2}}{\partial\psi_1} \qquad \text{Eq. 7.16}$$

The time-independent part $S_{1,1}$ describes the perturbation of drift surfaces by magnetic drifts and electric drifts. The Fourier spectrum of time-dependent electrostatic drift waves in general is very rich and has many frequencies, which are not always multiples of each other. The resonance condition (eq. 6.7) becomes

$$l\iota(p_\eta, E) + \omega_l T(p_\eta, E) - m = 0 \qquad \text{Eq. 7.17}$$

## 8. Summary and Conclusions

The Hamiltonian theory of guiding centre orbits in time-dependent electromagnetic fields has been revisited. By using a time-dependent co-ordinate system the radial covariant component of the magnetic field in the guiding centre Lagrangian can be eliminated. Similar to the procedure described by Hazeltine and Meiss (ref 6) the resulting Hamiltonian has two canonical momenta and two conjugate co-ordinates. An additional time-dependent term in the Hamiltonian is the result of the time-dependent co-ordinate transformation, this term has not been retained in the theory of Hazeltine and Meiss. However, this term is small, in general. In describing the effect of electrostatic oscillations on particle orbits, a time-independent transformation as in ref.6 can be used and the additional term does not occur.
In describing circulating particles in toroidal geometry the toroidal angle has been introduced as the independent variable instead of the time. Energy and time have been introduced as conjugate variables. The 4 canonical equations can be replaced by 4 mapping equations, which can be reduced to an area-preserving 2-dimensional map if the fields are time-independent and energy is conserved. This system of 4 phase space variables and the toroidal angle as independent variable is equivalent to an autonomous system with three degrees of freedom. Arnold diffusion is predicted in systems with more than two degrees of freedom and the diffusion mainly occurs in the neighbourhood of the Arnold web, which is defined by the resonance condition. A simple model has been established which describes the basic features of passing particles in a torus. Numerical solutions of the mapping equations showed the expected effect of Arnold diffusion. This diffusion along the stochastic layer leads to a fast radial transport of those particles, which satisfy the resonance condition. Since Coulomb collisions change the energy of particles, there is a chance that particles diffuse onto the Arnold web and are lost from the confinement region. These computations confirm the existence of Arnold diffusion for passing particles in a torus. How much the total plasma loss is enhanced by this mechanism must be investigated in more detail. The result depends on the amplitude and the Fourier spectrum of the fluctuations. To assess the effect on plasma losses it is necessary to include Coulomb collisions.

Acknowledgements: We should like to acknowledge fruitful discussions with Dr. Ph. Morrison from the University of Texas, Austin