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**Statistical Theory of Subcritically-Excited Strong Turbulence in Inhomogeneous Plasmas (IV)**

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**Abstract**

A statistical theory of nonlinear-nonequilibrium plasma state with strongly developed turbulence and with strong inhomogeneity of the system has been developed. A Fokker-Planck equation for the probability distribution function of the magnitude of turbulence is deduced. In the statistical description, both the contributions of thermal excitation and turbulence are kept. From the Fokker-Planck equation, the transition probability between the thermal fluctuation and turbulent fluctuation is derived. With respect to the turbulent fluctuations, the coherent part to a certain test mode is renormalized as the drag to the test mode, and the rest, the incoherent part, is considered to be a random noise. The renormalized operator includes the effect of nonlinear destabilization as well as the decorrelation by turbulent fluctuations. The equilibrium distribution function describes the thermal fluctuation, self-sustained turbulence and the hysteresis between them as a function of the plasma gradient. The plasma inhomogeneity is the controlling parameter that governs the turbulence. The formula of transition probability recovers the Arrhenius law in the thermodynamical equilibrium limit. In the presence of self-noise, the transition probability deviates form the exponential law and provides a power law. Application is made to the submarginal interchange mode turbulence, being induced by the turbulent current-diffusivity, in inhomogeneous plasmas. The power law dependence of the transition probability is obtained on the distance between the pressure gradient and the critical gradient for linear instability. Thus a new type of critical exponent is explicitly deduced in the phenomena of subcritical excitation of turbulence. The method provides an extension of the nonequilibrium statistical physics to the far-nonequilibrium states.

## §1. Introduction

Strong turbulence in high temperature plasmas is one of the most challenging problems of statistical physics for systems far from thermodynamic equilibrium. In particular, the strong turbulence in inhomogeneous plasmas casts problems such as the very high level of fluctuations which far exceeds the level of thermal fluctuations and causes the violation of the equipartition law of energy, the existence of internal driving source associated with inhomogeneity, and transitions between various different turbulent states.[1] Such a far-nonequilibrium state is sustained by the flow of energy, momentum and particles, being an open system. Near thermodynamical equilibrium, principles that govern fluctuations (i.e., equipartition of energy, Einstein relation, fluctuation-dissipation (FD) theorem, etc.) are established.[2-4] These standard methods are not sufficient for the understanding of these turbulent characteristics, and the extension of statistical theory is necessary.

An important issue of the plasma turbulence is the phenomena related to its subcritical excitation. Subcritical excitation means the transition to a different turbulent state. Under many circumstances, the presence of submarginal instability has been predicted either by theoretical study or direct nonlinear simulation.[5-19] The study of subcritical excitation of strong turbulence is a challenging theoretical problem. The importance has also been known in experiments. Very abrupt symmetry-breaking perturbations, which include both microscopic turbulence and global perturbations, have often been observed (known as 'trigger event') and the temporal change of the growth rate of perturbation cannot be described from the slow variation of the global parameters that govern the linear growth rates. See a review.[20] For the understanding of such trigger events, acceleration of the growth rate by nonlinearlity has been considered to be a key. (See, e.g., ref.21 and references in ref.20.) The statistical occurrence of transition was predicted.[22] The deductive theory is necessary to fully address the statistical property of the transition in turbulence.

Problem of the transition probability has been discussed in many area of physics; e.g., in the deductive analysis of the chemical reaction or the study of nucleation. (See reviews ref. 23 and 24.) The generalization of Arrhenius law[25] has been derived. There, the characteristic dependence on temperature is recovered; i.e., the exponential function and the argument of which is inversely proportional to temperature. Such dependence on temperature is deduced from the thermal noise, and is an example of application of near-equilibrium thermodynamics. To address the issue of transition between turbulent states, the extension of statistical theory for far-nonequilibrium system is necessary .

There has been much work on the turbulence motivated from the problem of neutral fluids. Theoretical methods such as renormalization or direct interaction approximations have been developed, and are surveyed in, e.g., ref.26. One method is a formulation of Fokker-Planck equation for the turbulent fluctuations, and has been applied to the analysis of homogeneous turbulence.[27, 28] In plasma turbulence research, a line of statistical theory for inhomogeneous turbulence has been developed, being based upon the renormalization[29] and random coupling model,[30] so that Langevin equation for turbulent spectrum was derived.[31-33] Recently, the statistical theory for plasma turbulence has been formulated, in which effects of thermal excitation and turbulent self-noise effects are incorporated in addition to the effects of collisional drag and turbulent drag.[34-37] A Langevin equation is deduced to Fokker-Planck equation, and the Fokker-Planck equation of a further-reduced variable is made for the macroscopic quantity. The solution for a stationary state is obtained. The result for the most probable state is expressed in terms of a principle of 'minimum renormalized dissipation rate', which is given by the ratio of the nonlinear decorrelation rate of fluctuation energy (a total dissipation rate) and the random excitation rate which includes both the thermal noise and turbulent self-noise effects. This result is an extension of Prigogine's principle of minimum entropy production[38] to the system of far from equilibrium state. In addition, the condition for the turbulence transition is derived, which is analogous to the Maxwell's construction in the phase transition physics in thermodynamical equilibrium.[39]

In this article, we study the transition probability between turbulent states, based upon a Fokker-Planck equation for the probability distribution function of the magnitude of turbulence. The Fokker-Planck equation for the coarse-grained fluctuation quantity is employed, and the formula of transition probability between different turbulence states is derived. This formula is a generalization of the Arrhenius law to that in the far-nonequilibrium system. It recovers the Arrhenius law in the thermodynamical equilibrium limit. The transition probability between the thermal fluctuation and turbulent fluctuation is obtained, which shows much weaker dependence on the heat bath temperature than the Arrhenius law. It is found that the turbulent-self noise term accelerates the transition. In a strong turbulent limit, the transition probability from turbulent branch to thermal one does not depend on the temperature, and is governed by the gradient parameter. The dependence of transition probability on the distance of gradient from the critical gradient for linear instability is also studied. A power-law dependence on the control parameter is obtained. Thus a new class of critical exponent is deduced for the problem of transition in turbulent states.

Constitution of this article is as follows. In §2, the derivation of statistical equation for plasma turbulence is surveyed. Analysis of the transition probability is developed in §3. In §4, summary and discussion are given.

## §2. Basic Equation and Statistical Approach

### *2.1 Plasma model and basic equation*

We consider a slab plasma which is inhomogeneous in the x-direction and is immersed in an inhomogeneous and sheared magnetic field. The magnetic field is given as $\boldsymbol{B} = B_0(0, sx, 1)$ with $B_0(x) = \left(1 + \Omega' x + \cdots\right)B_0$. In this system, a collective mode, interchange mode, can be subcritically excited due to the turbulent current diffusivity.[1, 16, 40] The turbulent system of this dissipative instability has been discussed in preceding articles.[34-37] In the following, refs. 35, 36 and 37 are called I, II, and III, respectively. The reduced set of equations for the electrostatic potential $\phi$, current $J$ and pressure $p$ is employed to describe the system.[41] Quantities that are averaged over the $(y, z)$-plane are denoted by suffix 0, as $p_0$ and $\phi_0$; We set $\phi = \phi_0 + \tilde{\phi}$, $J = J_0 + \tilde{J}$ and $p = p_0 + \tilde{p}$. The pressure and electrostatic potential could be inhomogeneous (i.e., inhomogeneous in the $\hat{x}$-direction) in the global scale, but the flow velocity shear is not taken into account here, in order to keep the transparency of the argument. (Introduction of $\Delta\phi_0$ does not change the fundamental structure of the theory.) Parameters $\nabla p_0$ and $\Omega'$ represent the inhomogeneity of the system. The scale separation is introduced, in this article, between the dynamics of the micro fluctuations and macroscopic structures: $|p_0^{-1}\,\partial p_0/\partial t| << |\tilde{p}^{-1}\,\partial\tilde{p}/\partial t|$, and $|p_0^{-1}\nabla p_0| << |\tilde{p}^{-1}\nabla\tilde{p}|$. The symbol ~ which denotes the fluctuating field components is suppressed in the following for the simplicity of expression. The product of pressure gradient and magnetic field inhomogeneity,

$$G_0 = \Omega' p_0', \tag{1}$$

denotes the driving parameter (i.e., one of the main control parameters in this system), being fixed in the evolution of fluctuating fields under the assumption of the time-space scale separation.

The derivation of statistical equation, which has been developed in previous articles, is briefly surveyed. The dynamical equations of fluctuation fields are given as

$$\frac{\partial}{\partial t} f + \mathcal{L}^{(0)} f = \mathcal{N}(f) + \tilde{S}_{th}, \tag{2}$$

where $\mathcal{L}^{(0)}$ denotes the linear operator

$$\mathcal{L}^{(0)} = \begin{pmatrix} -\mu_c \nabla_\perp^2 & -\nabla_\perp^{-2}\nabla_{\parallel} & -\nabla_\perp^{-2}\Omega' \frac{\partial}{\partial y} \\ \xi\nabla_{\parallel} & -\mu_{ec}\nabla_\perp^2 & 0 \\ -\frac{dp_0}{dx}\frac{\partial}{\partial y} & 0 & -\chi_c\nabla_\perp^2 \end{pmatrix}, \tag{3}$$

$f$ denotes the fluctuating field,

$$f = \begin{pmatrix} \phi \\ J \\ p \end{pmatrix}. \tag{4}$$

and $\mathcal{N}(f)$ stands for the nonlinear terms

$$\mathcal{N}(f) = -\begin{pmatrix} \nabla_\perp^{-2}[\phi, \nabla_\perp^2 \phi] \\ [\phi, J] \\ [\phi, p] \end{pmatrix}. \tag{5}$$

The bracket $[f,g]$ denotes the Poisson bracket,

$$[f, g] = (\nabla f \times \nabla g)\cdot \boldsymbol{b},$$

$(\boldsymbol{b} = \boldsymbol{B}_0/B_0)$, $\Delta_\perp = \nabla_\perp^2$, $\Omega'$ is the average curvature of the magnetic field, $\Psi$ is the vector potential, and $1/\xi$ denotes the finite electron inertia, $1/\xi = (\delta/a)^2$ , $\delta$ being the collisionless skin depth. Length, time, static potential and pressure are normalized to the global plasma size $a$, the Alfven transit time $\tau_{Ap} = a/v_{Ap}$, $av_{Ap}B_0$ and $B_0^2 R/2a\mu_0$, respectively ($a$ and $R$ are minor and major radii of torus, $v_{Ap} = B_0(2\mu_0 m_i n_i)^{-1/2} aRq^{-1}$, $m_i$ is the ion mass, and $n_i$ is the ion density; see ref.16 for details). (It is also noted that the study of the response to $\mathcal{L}^{(0)}$ corresponds to the conventional application of RDT in neutral fluid.) In studying the subcritical excitation, the electron inertia effect should be kept, but the classical resistivity is neglected for the simplicity of the argument. The interchange mode has a quasi-2 dimensional nature, $|\nabla_{||}^2| << |\nabla_\perp^2|$; nevertheless, the existence of small but finite $\nabla_{||}$ is essential.

We consider the thermal excitation of fluctuations, $\tilde{S}_{th}$. In the thermal fluctuations, coherent parts to the microscopic interchange mode are given by the collisional transport coefficients $\mu_c$, $\mu_{ec}$ and $\chi_c$ (the ion viscosity, electron viscosity and thermal diffusivity, respectively). Incoherent parts are considered to be a random noise and expressed as $\tilde{S}_{th}$.[4)]

### *2.2 Langevin equation*

In order to describe the turbulent characteristics, the system which has a large number of degrees of freedom and has many positive Lyapunov exponents is considered. A part of the Lagrangean nonlinearity is considered to cause the turbulent drag to this collective mode and this part is renormalized to the eddy-viscosity type nonlinear transfer rate $\gamma_j$. The other part is regarded as a random noise, which has a faster decorrelation time than $\gamma_j$ according to RCM.[30] As has been discussed in I-III, a projection operator $\mathcal{P}$ is introduced to divide the nonlinear interactions into the drag and others. The nonlinear drag term is written in an apparent linear term as

$$\mathcal{P}\mathcal{N}(f) = \begin{pmatrix} \mu_N \nabla_\perp^2 f_1 \\ \mu_{Ne} \nabla_\perp^2 f_2 \\ \chi_N \nabla_\perp^2 f_3 \end{pmatrix} = -\begin{pmatrix} \gamma_1 f_1 \\ \gamma_2 f_2 \\ \gamma_3 f_3 \end{pmatrix} \tag{6}$$

and the rest part, $\tilde{S} = (1 - \mathcal{P})\mathcal{N}(f)$, is considered to be a random (self) noise. Then a Langevin equation is derived as[32-37]

$$\frac{\partial}{\partial t} f + \mathcal{L} f = \tilde{S} + \tilde{S}_{th} \tag{7}$$

with

$$\mathcal{L}_{ij} = \mathcal{L}_{ij}^{(0)} + \gamma_i \delta_{ij} \tag{8}$$

($\delta_{ij}$ is the Kronecker's delta). Notation here follows the convention in ref.33. In this article, suffix $i, j = 1,2,3$ denotes the i-th or j-th field. In the following, Fourier transformation is used, and $k, p, q$ describes the wave number of Fourier components. Suffix $k, p, q$ is often omitted unless confusion is caused.

The operator to the k-th component, $\mathcal{L}_k$, $\mathcal{L}_k f_k = \mathcal{L}_{0,k} f_k - \mathcal{P}_k \mathcal{N}_k(\boldsymbol{f})$, is the renormalized operator, which includes the renormalized transfer rates of

$$\gamma_{i,k} = -\sum_{\Delta} M_{i,kpq} M^*_{i,qkp} \theta^*_{qkp} \left| \tilde{f}^2_{1,p} \right| . \tag{9}$$

The self-noise, $\tilde{\boldsymbol{S}} = \left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3\right)^T$, has a much shorter correlation time as is discussed in I-III, and is approximated to be given by the Gaussian white noise term $w(t)$. The self-noise term for the k-th component is expressed as

$$\tilde{S}_{i,k} = w(t) \tilde{g}_{i,k} , \tag{10a}$$

$$\tilde{g}_{i,k} \equiv \sum_{\Delta} M_{i,kpq} \sqrt{\theta_{kpq}} \zeta_{1,p} \zeta_{i,q} . \tag{10b}$$

In these expressions, summation $\Delta$ indicates the constraint $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$. The explicit form of the nonlinear interaction matrix is given as, e.g.,

$$M_{1,kpq} = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{b} \frac{\left(p_\perp^2 - q_\perp^2\right)}{k_\perp^2} , \quad \text{or} \quad M_{(2,3),kpq} = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{b}, \tag{11}$$

and the propagator satisfies the relation $(\partial/\partial t + \mathcal{L}(k) + c.p.)\theta_{kpq} = 1$, where c.p. indicates the counter part, i.e., $\mathcal{L}(p) + \mathcal{L}(q)$ .[33] The term $\zeta_{j,p}$ in a random noise represents the j-th field of $q$-component in the nonlinear term $\mathcal{N}$, and their correlation functions satisfy the average relations of the mode, which we call an Ansatz of equivalence in correlation in the following, as

$$\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle \tag{12}$$

with

$$\langle \zeta_{i,p} \zeta_{j,q} \rangle \propto \delta_{pq} \tag{13}$$

where the bracket $\langle \ \rangle$ indicates the statistical average.

The thermal excitation is also assumed to be a Gaussian white noise,

$$\tilde{S}_{th,i} = w(t)\tilde{g}_{th,i} \tag{14}$$

The statistical independence between the incoherent parts of thermal and turbulent fluctuations is also imposed, that is,

$$\langle \tilde{S}_{th,i} \tilde{S}_j \rangle = 0 \tag{15}$$

### *2.3 One branch approximation*

The basic set of equation describes the unstable (or least stable) branch, but also contains two others branches of plasma mode. These additional two branches are strongly damped, and are excited up to much smaller amplitudes. Therefore the excitation of the two stable branches is neglected. Based on this approximation, the Langevin equation is reduced to that for only one branch. The detailed procedure of decomposition is described in the previous articles I-III.

The matrix $\exp[-\mathcal{L}(t-\tau)]$ was explicitly expressed as

$$\exp[-\mathcal{L}(t-\tau)] = \boldsymbol{A}\exp(-\lambda_1(t-\tau)) + \boldsymbol{A}^{(2)}\exp(-\lambda_2(t-\tau)) + \boldsymbol{A}^{(3)}\exp(-\lambda_3(t-\tau)) \tag{16}$$

where the elements of matrix $\boldsymbol{A}$ are given as

$$
\boldsymbol{A} = \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \begin{pmatrix} (\bar{\gamma}_e-\lambda_1)(\bar{\gamma}_p-\lambda_1) & \dfrac{-ik_{\parallel}(\bar{\gamma}_p-\lambda_1)}{k_\perp^2} & \dfrac{-ik_y\Omega'(\bar{\gamma}_e-\lambda_1)}{k_\perp^2} \\ -ik_{\parallel}\xi(\bar{\gamma}_p-\lambda_1) & \dfrac{\xi k_{\parallel}^2}{k_\perp^2}\dfrac{(\lambda_1-\bar{\gamma}_p)}{(\lambda_1-\bar{\gamma}_e)} & \dfrac{-\xi k_{\parallel}k_y\Omega'}{k_\perp^2} \\ ip_0'k_y(\bar{\gamma}_e-\lambda_1) & \dfrac{k_{\parallel}k_y p_0'}{k_\perp^2} & \dfrac{G_0k_y^2}{k_\perp^2}\dfrac{(\bar{\gamma}_e-\lambda_1)}{(\lambda_1-\bar{\gamma}_p)} \end{pmatrix}
\tag{17}
$$

and $-\lambda_m$ ($m = 1,2,3$ and $\lambda_1 < \lambda_2 < \lambda_3$) represents the eigenvalue of the non-normal matrix $\mathcal{L}$, which gives the homogeneous solution of eq.(7) if $\mathcal{L}$ is constant. The eigenvalue is determined by:

$$
\det(\lambda \boldsymbol{I} + \mathcal{L}) = 0 \tag{18}
$$

and $\boldsymbol{I}$ is a unit tensor. The eigenvector with the eigenvalue $-\lambda_1$ corresponds to the least stable branch, the decay time of which is the longest. Others with $(-\lambda_2, -\lambda_3)$ denote highly-stable branches, which decay much faster. (Elements $A_{ij}^{(2,3)}$ are also obtained in a similar way, and are given in I and II, being not repeated here.)

In the one branch approximation, only the pole of $(s+\lambda_1)^{-1}$ is kept. Then, the Langevin equation is deduced to that of one field, e.g., $f_1 = \phi$, as is discussed in I and III,

$$
\frac{\partial}{\partial t}\phi + \lambda_1\phi = \tilde{s} \tag{19}
$$

with the source of

$$
\tilde{s}_k = w(t)\left(g_k + g_{th,\,k}\right) \tag{20}
$$

The magnitude of the noise term, for which both the contributions of the turbulent self noise and thermal noise are retained, is given by use of the matrix $\boldsymbol{A}$ as

$$g_k = \Re e\left(\sum_{j=1}^{3} A_{1j} g_{j,k}\right), \tag{21}$$

and

$$g_{th,k} = \sum_{j=1}^{3} A_{1j} g_{th,j,k} \tag{22}$$

By retaining the real part in eq.(21), the possible problem of complex quantity of $g_{i,k}$ is eliminated, and the diffusion process is assured in the Fokker-Planck equation. The coefficient $g_k$ is statistically independent for each $k$-component, $\langle g_k g_{k'} \rangle = \langle\!\langle g_k^2 \rangle\!\rangle \delta_{k,k'}$, $\langle g_{th,k} g_{th,k'} \rangle = \langle\!\langle g_{th,k}^2 \rangle\!\rangle \delta_{k,k'}$ and $\langle g_k g_{th,k'} \rangle = 0$.

From the fluctuation-dissipation (FD) theorem for the thermodynamical equilibrium, the thermal excitation rate is expressed in terms of the temperature as[28)]

$$\sum_{j,j'} A_{1j} A_{1j'}^{*} \tilde{S}_{th,j} \tilde{S}_{th,j'} = 2\mu_{vc} \hat{T}. \tag{23}$$

where the normalized temperature (with an additional dimension of volume) is introduced as

$$\hat{T} = \frac{2\mu_0}{B_p^2} k_{\mathrm{B}} T \,. \tag{24}$$

The thermal contribution to the diffusion term is rewritten as

$$g_{th,k}^2 = 2\mu_{vc} \hat{T} \tag{25}$$

This term is independent of the choice of test mode number $k$, so long as collisional viscosity is independent of the scale size.

Based on the statistical independence of thermal noise and self noise, the magnitude of the total noise is the sum of two statistically-independent noises, and the form

$$\hat{g}_k^2 = g_k^2 + g_{th,\,k}^2 \tag{26a}$$

i.e.,

$$\hat{g}_k^2 = g_k^2 + 2\mu_{vc}\hat{T} \tag{26b}$$

is employed.

### *2.4 Langevin equation of coarse-grained quantity*

We further reduce the Langevin equations for each $k$ -component, eq.(19), and employ the dynamical equations of macro variables. The total fluctuating energy, which is the quantity integrated over some finite-size volume,

$$\mathcal{E} \equiv \frac{1}{2} \sum_k k_\perp^2 \phi_k^2 \tag{27}$$

is taken as examples.

From eq.(19), one has

$$\frac{1}{2}\frac{\partial}{\partial t} \sum_k k_\perp^2 \phi_k^2 + \sum_k \lambda_{1,k} k_\perp^2 \phi_k^2 = \sum_k k_\perp^2 \tilde{s}_k \phi_k \ . \tag{28}$$

By introducing a time constant, which denotes the dissipation rate,

$$\Lambda \equiv \frac{\sum_k \lambda_{1,k} k_\perp^2 \phi_k^2}{\sum_k k_\perp^2 \phi_k^2} , \tag{29}$$

the Langevin equation for the total fluctuating energy is given as

$$\frac{\partial}{\partial t} \mathcal{E} + 2\Lambda \mathcal{E} = \sum_k k_\perp^2 \tilde{s}_k \phi_k . \tag{30}$$

### *2.5 Fokker-Planck equation for macro variable (coarse-grained quantity)*

Following I-III, the Fokker-Planck equation for the probability distribution function of the coarse-grained quantity, $P(\mathcal{E})$, is described. (See also refs. 42 and 43 for the basis of reduction of Fokker-Planck equation.)

In the Langevin equation for the average energy $\mathcal{E}$, the magnitude of the statistical source term is written as $g^2 = \sum_k \tilde{s}_k^2 k_\perp^4 \phi_k^2$ , i.e.,

$$g^2 = \sum_k 2\mu_{vc} \hat{T} k_\perp^4 \phi_k^2 + \sum_k \left( \sum_{j=1}^3 A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \tag{31}$$

where eq.(26) is used. It is also useful to introduce an average of the classical decorrelation rate

$$\gamma_m \equiv \frac{\sum_k \gamma_{vc} \mathcal{E}_k}{\mathcal{E}} \tag{32}$$

($\gamma_{vc} = \mu_{vc} k_\perp^2$ ). When the fluctuation level changes, this coefficient might deviate form the value in the limit of thermodynamical equilibrium. The amplitude of the source term is rewritten as

$$g^2 = 4\hat{T} \gamma_m \mathcal{E} + \sum_k \left( \sum_{j=1}^3 A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \tag{33}$$

The Fokker-Planck equation for the probability distribution function $P(\mathcal{E})$ is obtained as

$$\frac{\partial}{\partial t}P(\mathcal{E}) = \frac{\partial}{\partial \mathcal{E}}\left(2\Lambda\mathcal{E} + \frac{1}{2}\, g\,\frac{\partial}{\partial \mathcal{E}}\, g\right)P(\mathcal{E}) \tag{34}$$

## § 3 Transition Probability

Let us obtain the transition probability between two probable states, the distribution functions of which are governed by the Fokker-Planck equation eq.(34).

### *3.1 Flux of probability and transition probability*

The equilibrium probability distribution function (PDF) has been discussed in the preceding article III for the system which has the hysteresis characteristics between the fluctuation level $\phi$ and the control parameter $G_0$. One state is characterized by the thermal fluctuations and the other by turbulent fluctuations. The schematic relation between the thermal branch and turbulent branch is drawn in Fig.1. For the case where the gradient parameter $G_0$ takes the value in between $G_*$ and $G_c$, i.e.,

$$G_* > G_0 > G_c \tag{35}$$

the system has multiple solutions of thermal branch and turbulent branch. The probability distribution function of the steady state has been solved for a model function of $\Lambda(\phi)$. A typical example is shown in Fig.2. The peak near $\phi \simeq 0$ corresponds to the thermal branch, and the peak with finite amplitude corresponds to the turbulent branch. In the following, we call A state for the thermal one and B state for the turbulent one. The associated effective potential $S(\mathcal{E})$

$$S(\mathcal{E}) = \int^{\mathcal{E}} \frac{4\Lambda\mathcal{E}}{g^2}\, d\mathcal{E}\,, \tag{36}$$

where the relation $\mathcal{E} \propto \phi^2$ holds, is drawn in Fig.3. The effective potential $S(\mathcal{E})$ reduces to the potential of the form $-\mathcal{E}/\hat{T}$ in the thermodynamical limit (i.e., limit of $\mathcal{E} \to 0$ ). The states A and B in Fig.3 correspond to those in Fig.1. The number density of state A in the ensemble, $N_A$ , of the finite energy width $\Delta\mathcal{E}_A$ is expressed by the probability distribution function (PDF), $P(\mathcal{E})$ , as

$$N_A = \int_0^{\Delta\mathcal{E}_A} P(\mathcal{E})\, d\mathcal{E} \tag{37}$$

and is evaluated as

$$N_A = P(\mathcal{E}_A)\Delta\mathcal{E}_A\,. \tag{38}$$

In the same way, the number density of state B in the ensemble, $N_B$ ,

$$N_B = \int_{\mathcal{E}_B - \Delta\mathcal{E}_B/2}^{\mathcal{E}_B + \Delta\mathcal{E}_B/2} P(\mathcal{E})\, d\mathcal{E} \tag{39}$$

where $\Delta\mathcal{E}_B$ is the width of energy of the state B. $N_B$ is evaluated as

$$N_B = P(\mathcal{E}_B)\Delta\mathcal{E}_B \tag{40}$$

*3.1.1 Rate equation*

The transition probability in between A and B states, which is governed by the Fokker-Planck equation eq.(34), is of our interest. The procedure how to obtain the transition probability is briefly discussed using the rate equations for $N_A$ and $N_B$ . They are written as

$$\frac{\partial}{\partial t} N_A = -r_{A \to B} N_A + r_{B \to A} N_B + h_A \tag{41}$$

and

$$\frac{\partial}{\partial t} N_B = r_{A \to B} N_A - r_{B \to A} N_B + h_B \ , \tag{42}$$

where $r_{A \to B}$ and $r_{B \to A}$ are the transition probability from the state A to state B and that from the state B to state A, respectively, and $h_A$ and $h_B$ are the sums of source and sink for the state A and state B, respectively. In the following, we call $r_{A \to B}$ "transition probability" and $r_{B \to A}$ "back-transition probability".

Consider the stationary state with source and sink

$$\partial N_A / \partial t = \partial N_B / \partial t = 0 \ . \tag{43}$$

If the sink of state B is large so that the density of state B vanishes,

$$N_B = 0 \ , \tag{44}$$

then the transition condition satisfies the relation

$$r_{A \to B} = \frac{h_A}{N_A} \ . \tag{45}$$

This relation suggests that the transition probability can be calculated by calculating the necessary source term from Fokker-Planck equation with the condition eq.(44).

*3.1.2 Flux of probability and probability of transition*

The source (sink) rate is connected to the flux of probability $w$ in the $\mathcal{E}$ space, which is governed by the Fokker-Planck equation eq.(34). Fokker-Planck equation, in the presence of some external source $h$, is rewritten as

$$\frac{\partial}{\partial t}P(\mathcal{E}) + \frac{\partial}{\partial \mathcal{E}} w = h \tag{46}$$

with the flux of probability

$$w = -\left(2\Lambda\mathcal{E} + \frac{1}{2} g \frac{\partial}{\partial \mathcal{E}} g\right)P(\mathcal{E}) . \tag{47}$$

If one integrates eq.(46) in the region $0 < \mathcal{E} < \Delta\mathcal{E}_A$, one has

$$\frac{\partial}{\partial t}N_A + w(E_A) = h_A \equiv \int_0^{\Delta\mathcal{E}_A} h \, d\mathcal{E} . \tag{48}$$

In a steady state, one has

$$w(E_A) = h_A , \tag{49}$$

and the flux of probability $w$ is independent of $\mathcal{E}$ and is constant in the region where the source does not exist. Therefore eq.(45) is written as

$$r_{A\to B} = \frac{w}{N_A} \tag{50}$$

with the condition (44). The flux of probability $w$ is determined by the probability by which the barrier at C of effective potential is overcome. (See also Fig.3.)

By use of eq.(36), $w$ can be rewritten in terms of $S(\mathcal{E})$, $P(\mathcal{E})$ and $g$ as

$$w = -\frac{1}{2}\, g \exp\{-S(\mathcal{E})\} \frac{\partial}{\partial \mathcal{E}}\left[ gP(\mathcal{E})\exp\{S(\mathcal{E})\}\right] \tag{51}$$

Since $w$ is constant, we obtain the flux of probability by the integration from the A state to B state as

$$w = \frac{\left[ gP(\mathcal{E})\exp\{S(\mathcal{E})\}\right]\Big|_{B}^{A}}{2\int_{A}^{B} \frac{1}{g} \exp\{S(\mathcal{E})\}\, d\mathcal{E}} \,. \tag{52}$$

Analytic estimate of the denominator of eq.(52) is possible by use of the method of steepest descent in the vicinity of the state C.[42] Expansion

$$\frac{1}{g}\exp\{S(\mathcal{E})\} = \exp\{S(\mathcal{E}) - \ln(g)\} = \exp\left\{ S(\mathcal{E}_C) - \ln\left(g(\mathcal{E}_C)\right) - \frac{\alpha^2}{2}\left(\mathcal{E} - \mathcal{E}_C\right)^2 + \cdots \right\} \tag{53}$$

provides the evaluation

$$2\int_{A}^{B} \frac{1}{g} \exp\{S(\mathcal{E})\}\, d\mathcal{E} = \frac{2\sqrt{2\pi}}{\alpha} \frac{1}{g(\mathcal{E}_C)} \exp\{S(\mathcal{E}_C)\} \,. \tag{54}$$

Recalling the boundary condition at B state eq.(44), $N_B = 0$ , i.e.,

$$P(\mathcal{E}_B) = 0 \,, \tag{55}$$

the flux of probability associated with the $A \rightarrow B$ transition is given from eq.(52) as

$$w_{A\rightarrow B} = \frac{\alpha}{2\sqrt{2\pi}}\, g(\mathcal{E}_C) g(\mathcal{E}_A)\, P(\mathcal{E}_A) \exp\{S(\mathcal{E}_A) - S(\mathcal{E}_C)\} \,. \tag{56}$$

(The suffix $A \to B$ denotes the case of $A \to B$ transition.)

### *3.1.3 Back transition*

The probability flux associated with the $B \to A$ transition is obtained by the same procedure with the boundary condition

$$P(\mathcal{E}_A) = 0 \, . \qquad (57)$$

The result is obtained by replacing A and B in eq.(56) as

$$w_{B \to A} = \frac{\alpha}{2\sqrt{2\pi}} \, g(\mathcal{E}_C) g(\mathcal{E}_B) \, P(\mathcal{E}_B) \exp\left\{ S(\mathcal{E}_B) - S(\mathcal{E}_C) \right\} \qquad (58)$$

## *3.2 Transition probability*

The transition probability is further analyzed by use of the model formula of $\Lambda(\mathcal{E})$ in the effective potential $S(\mathcal{E})$. The model formula of $\Lambda(\mathcal{E})$ gives the explicit expression of the curvature term $\alpha$ in eqs.(56) and (58). The function $\Lambda(\mathcal{E})$ is Taylor expanded in the vicinity of $\mathcal{E}_C$ as

$$\Lambda(v) = \Lambda(v_{*1}) - \Lambda_0'(v - v_{*1}) + \cdots \qquad (59a)$$

($v = \sqrt{\mathcal{E}}$ and $v_{*1} = \sqrt{\mathcal{E}_C}$ ). A model like

$$\Lambda(v) = \Lambda_0 - \Lambda_0' \, v = \Lambda_0' \, (v_{*1} - v) \qquad (59b)$$

has been successfully applied to study the subcritical excitation of turbulent branch in III, which is drawn in Fig.4. Based upon this model formula, the equilibrium distribution function of probability and the effective potential have been obtained in III. They are illustrated in Figs.2 and 3, respectively.

In evaluating $S(\mathcal{E})$ near $\mathcal{E} \simeq \mathcal{E}_C$, as in the case of eq.(53), the change of $\Lambda$ is considered to contribute the most dominant term as

$$S(\mathcal{E}) \simeq S(\mathcal{E}_c) + \frac{8v_{*1}^3}{g^2(\mathcal{E}_c)} \int_{v_{*1}}^{v} \Lambda \, dv \tag{60}$$

The substitution of the Taylor-expansion eq.(59a) into eq.(60) gives the relation

$$S(\mathcal{E}) \simeq S(\mathcal{E}_c) - \frac{4\Lambda_0' v_{*1}^3}{g^2(\mathcal{E}_c)} \left(v_{*1} - v\right)^2 . \tag{61}$$

Rewriting $v$ and $v_{*1}$ in terms of $\mathcal{E}$ and $\mathcal{E}_C$, we obtain the relation

$$S(\mathcal{E}) \simeq S(\mathcal{E}_c) - \frac{\lambda_0}{g^2(\mathcal{E}_c)} \left(\mathcal{E}_c - \mathcal{E}\right)^2 \tag{62}$$

where quantity $\lambda_0$ has the dimension of the time rate and is given as

$$\lambda_0 = \Lambda_0' v_{*1} . \tag{63}$$

In the model like eq.(59b), the relation

$$\lambda_0 = \Lambda_0 \tag{64}$$

holds by definition. The curvature $\alpha$ in eq.(53) is explicitly written as

$$\alpha^2 = \frac{2\lambda_0}{g^2(\mathcal{E}_c)} . \tag{65}$$

Substituting eq.(65) into eq.(56), we obtain the flux of probability $w_{A\to B}$ as

$$w_{A\to B} = \frac{\sqrt{\lambda_0}}{2\sqrt{\pi}}\, g\left(\mathcal{E}_A\right) P\left(\mathcal{E}_A\right) \exp\left\{S\left(\mathcal{E}_A\right) - S\left(\mathcal{E}_C\right)\right\} . \tag{66}$$

The transition rate can be introduced once the flux of probability is obtained. They have the relation eq.(50). Substitution of eqs.(38) and (66) into eq.(50) gives

$$r_{A\to B} = \frac{\sqrt{\lambda_0}}{2\sqrt{\pi}} \frac{1}{\Delta \mathcal{E}_A}\, g\left(\mathcal{E}_A\right) \exp\left\{S\left(\mathcal{E}_A\right) - S\left(\mathcal{E}_C\right)\right\} . \tag{67}$$

The flux of probability associated with the back-transition (from B state to A state) is also given by eq.(58). By the help of eq.(65), the flux $w_{B\to A}$ is given as

$$\left| w_{B\to A} \right| = \frac{\sqrt{\lambda_0}}{2\sqrt{\pi}}\, g\left(\mathcal{E}_B\right) P\left(\mathcal{E}_B\right) \exp\left\{S\left(\mathcal{E}_B\right) - S\left(\mathcal{E}_C\right)\right\} . \tag{68}$$

(The symbol of absolute value indicates that the flux $w_{B\to A}$ is from large value of $\mathcal{E}$ to small values of $\mathcal{E}$.) The transition rate is also given by

$$r_{B\to A} = \frac{\left| w_{B\to A} \right|}{N_B} . \tag{69}$$

Combining eqs.(40), (68), and (69), we finally obtain the rate of back-transition as

$$r_{B\to A} = \frac{\sqrt{\lambda_0}}{2\sqrt{\pi}} \frac{1}{\Delta \mathcal{E}_B}\, g\left(\mathcal{E}_B\right) \exp\left\{S\left(\mathcal{E}_B\right) - S\left(\mathcal{E}_C\right)\right\} . \tag{70}$$

### *3.3 Competition of forward and backward transitions*

The transition probability between A state and B state are to be discussed, examining the competition between the forward transition probability (from A to B) and the backward transition probability (from B to A).

The ratio of these transition probabilities is given by eqs.(67) and (70) as

$$\frac{r_{A\to B}}{r_{B\to A}} = \frac{\Delta\mathcal{E}_B\, g\left(\mathcal{E}_A\right)}{\Delta\mathcal{E}_A\, g\left(\mathcal{E}_B\right)} \exp\left\{S\left(\mathcal{E}_A\right) - S\left(\mathcal{E}_B\right)\right\} . \tag{71}$$

If this ratio exceeds unity, i.e., $r_{A\to B} > r_{B\to A}$ , the transition from A state to B state dominantly occurs. If, on the contrary, the ratio is below unity, $r_{A\to B} < r_{B\to A}$ , the back transition is dominant. The equi-probability condition

$$r_{A\to B} = r_{B\to A} \tag{72}$$

is given by

$$\exp\left\{S\left(\mathcal{E}_B\right) - S\left(\mathcal{E}_A\right)\right\} = \frac{\Delta\mathcal{E}_B\, g\left(\mathcal{E}_A\right)}{\Delta\mathcal{E}_A\, g\left(\mathcal{E}_B\right)} , \tag{73}$$

or

$$S\left(\mathcal{E}_B\right) - S\left(\mathcal{E}_A\right) = \ln\left(\Delta\mathcal{E}_B\, g\left(\mathcal{E}_A\right)\right) - \ln\left(\Delta\mathcal{E}_A\, g\left(\mathcal{E}_B\right)\right) . \tag{74}$$

If the logarithmic contribution is regarded as a small contribution and is neglected, the rough estimate of the boundary is given by

$$S\left(\mathcal{E}_B\right) - S\left(\mathcal{E}_A\right) = 0 . \tag{75}$$

In fact, the effective potential difference between A and B, i.e., $S(\mathcal{E}_B) - S(\mathcal{E}_A)$, dominantly dictates the transition probability. The function $S(\mathcal{E})$ has been introduced as the renormalized dissipation rate as the thermodynamical function of turbulent state. (See the article III.) This fact is also confirmed from the study of the transition probability.

### *3.4 Transition probability for subcritical excitation*

Based on the analysis of §3.2, the time rate for the transition from the thermal fluctuation to the turbulent fluctuation is calculated.

#### *3.4.1 Transition from thermal fluctuations*

In the state A, the distribution function is approximated by the Boltzmann distribution. Therefore, the width of distribution of fluctuation energy, $\Delta\mathcal{E}_A$, is given by the relation

$$\Delta\mathcal{E}_A = \hat{T} \ . \tag{76}$$

The magnitude of the thermal noise term, eq.(33) of §2, was calculated in III as $g(\mathcal{E}_A)^2 = 4\gamma_m T^2$, i.e.,

$$g(\mathcal{E}_A) = 2\sqrt{\gamma_m}T \ . \tag{77}$$

Substituting eqs.(76) and (77) into eq.(67), one obtains

$$r_{A\to B} = \frac{\sqrt{\Lambda_0\gamma_m}}{\sqrt{\pi}} \exp\left\{ -S(\mathcal{E}_C)\right\} \ . \tag{78}$$

In deriving eq.(78), the relation $S(\mathcal{E}_A) = 0$ which is evident from the definition of $S(\mathcal{E})$ was used.

This result is an extension of the theory of transition probability which has been based on the thermal excitations. The Arrhenius law is one of the typical examples of the thermodynamical equilibrium statistics. The formula (78) provides the more general solution as is discussed in the following sections.

*3.4.2 Thermodynamical limit*

Before describing the subcritical excitation to the turbulent state, we study the thermodynamical limit of eq.(78). The classical formula like Arrhenius law is naturally recovered.

If the fluctuation is governed by the thermal excitation and the turbulent self-noise is absent, then the noise term is expressed in terms of heat bath temperature

$$g^2 = 4\gamma_{vc} T^2 \tag{79}$$

Then the renormalized dissipation integral $S(\mathcal{E})$ is given as

$$S(\mathcal{E}) = \frac{1}{\hat{T}} \int^{\mathcal{E}} \frac{\Lambda}{\gamma_m} d\mathcal{E} . \tag{80}$$

The coefficient $\exp\{-S(\mathcal{E}_C)\}$ is expressed as

$$\exp\{-S(\mathcal{E}_C)\} \propto \exp\left\{-\frac{\Delta Q}{\hat{T}}\right\} \tag{81}$$

with the effective potential difference

$$\Delta Q = \int_0^{\mathcal{E}_C} \frac{\Lambda}{\gamma_m} d\mathcal{E} . \tag{82}$$

When the fluctuation is composed of only thermal fluctuations, then the decorrelation rate is given as

$$\Lambda = \gamma_m \tag{83}$$

and the effective potential difference is obtained as

$$\Delta Q = \int_0^{\mathcal{E}_C} d\mathcal{E} = \mathcal{E}_C \ . \tag{84}$$

Substituting eqs.(81) and (84) into eq.(78), the transition probability is given as

$$r_{A \to B} = \frac{1}{\sqrt{\pi}} \sqrt{\gamma_m \Lambda_0} \exp\left\{ -\frac{\mathcal{E}_C}{\hat{T}} \right\} \tag{85}$$

For the thermal fluctuations, if the rate $\Lambda_0$ is approximated as

$$\Lambda_0 = \gamma_m \ , \tag{86}$$

then the transition probability is given by a simple formula

$$r_{A \to B} = \frac{1}{\sqrt{\pi}} \gamma_m \exp\left\{ -\frac{\mathcal{E}_C}{\hat{T}} \right\} \ . \tag{87}$$

In these formula, eqs.(85) and (87), the exponential dependence $\exp\left( -\mathcal{E}_C / \hat{T} \right)$ is the strongest, and one recovers the relation

$$\ln\left( r_{A \to B} \right) \propto -\frac{1}{T} \tag{88}$$

which is the Arrhenius law.

*3.4.3 Transition to turbulent fluctuations*

In the presence of turbulent self-noise, both the thermal excitation noise and the self-noise term are important in determining the transition probability. Cooperative effects of the thermal excitation and self-noise excitation are studied.

Based on the scaling property of the random self-noise term, the dependence of the second term of eq.(33) of §2 was derived in III as

$$\sum_k \left( \sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2 \propto \mathcal{E}^{5/2} . \tag{89}$$

Based on this property, the term $g^2$ was written as

$$g^2 = 4\hat{T}\gamma_m \mathcal{E} + \bar{\bar{g}}_0^2 \left( \frac{\mathcal{E}}{\mathcal{E}_{eq}} \right)^{5/2} . \tag{90}$$

In eq.(90), the coefficient $\bar{\bar{g}}_0^2$ depends on the plasma parameter, and $\mathcal{E}_{eq}$ is an average energy at the state B.

By use of this random source term, the probability is calculated. The integral $S(\mathcal{E})$ is given as

$$S(\mathcal{E}) = \int_0^{\mathcal{E}} \frac{4\Lambda \mathcal{E}}{4\hat{T}\gamma_m \mathcal{E} + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2} \mathcal{E}^{5/2}} d\mathcal{E} . \tag{91}$$

It is sometimes convenient to use the form

$$S(\upsilon) = \int_0^{\upsilon} \frac{2\,\Lambda(\upsilon)\,\upsilon}{\hat{T}\gamma_m + \bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2} \upsilon^3} d\upsilon \tag{92}$$

where $\upsilon$ is fluctuation velocity, $\mathcal{E} = \upsilon^2$ .

Renormalized dissipation rate is calculated by specifying the form of the decorrelation rate. After III, we employ the simplest model which describes the subcritical excitation, eq.(59b),

$$\Lambda(v) = \Lambda_0 - \Lambda_0' v \qquad (0 < v < v_c) .$$

(See Fig.4.) By substituting eq.(59b) into eq.(92), the integral $S(\mathcal{E})$ is obtained. The explicit form was derived in III as

$$\exp\{-S(v)\} = \frac{\left(vd^{-1}+1\right)^{3b_1}}{\left(v^3 d^{-3}+1\right)^{a_1}} \exp\left(-2\sqrt{3}b_1\left\{\arctan\left(\frac{2v-d}{\sqrt{3}d}\right)+\frac{\pi}{6}\right\}\right) \tag{93}$$

where the following abbreviations are used:

$$a_1 = \left(1 - \frac{2d}{v_{*1}}\right) b_1 , \tag{94}$$

$$b_1 = \frac{\Lambda_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{-5/2}} \frac{1}{3d} , \tag{95}$$

and

$$d^3 = \hat{T}\gamma_m \bar{\bar{g}}_0^{-2} \mathcal{E}_{eq}^{5/2} . \tag{96}$$

By use of the integral form eq.(93) the transition probability is obtained. Substituting eq.(93) into eq.(78), one finds

$$r_{A\to B} = \frac{\sqrt{\Lambda_0 \gamma_m}}{\sqrt{\pi}} \frac{\left(v_{*1}d^{-1}+1\right)^{3b_1}}{\left(v_{*1}^3 d^{-3}+1\right)^{a_1}} \exp\left(-2\sqrt{3}b_1\left\{\arctan\left(\frac{2v_{*1}-d}{\sqrt{3}d}\right)+\frac{\pi}{6}\right\}\right) \tag{97}$$

where $\mathcal{E}_C = v_{*1}^2$ holds.

In the model of eq.(59), it is natural to choose

$$\Lambda_0 = \gamma_m \quad . \tag{98}$$

By the help of this, the transition probability is given as

$$r_{A \to B} = \frac{\gamma_m}{\sqrt{\pi}} \frac{\left(v_{*1} d^{-1} + 1\right)^{3b_1}}{\left(v_{*1}^3 d^{-3} + 1\right)^{a_1}} \exp\left( -2\sqrt{3} b_1 \left\{ \arctan\left( \frac{2v_{*1} - d}{\sqrt{3} d} \right) + \frac{\pi}{6} \right\} \right) . \tag{99}$$

From this formula, one sees the fact that the transition rate which is normalized to the time rate $\gamma_m$ , $r_{A \to B}/\gamma_m$ , is controlled by the parameter

$$v_{*1}/d \; .$$

*3.4.4 Case of weak fluctuations*

First, the limiting formula in the case of weak fluctuations are discussed. The limit of weak fluctuation is that the level of fluctuations at the saddle point C is low so that the relation

$$v_{*1} << d \tag{100}$$

holds.

In this limiting case, one has a Taylor expansion as

$$\arctan\left( \frac{2v - d}{\sqrt{3} d} \right) \simeq \frac{-\pi}{6} + \frac{\sqrt{3}}{2} \frac{v}{d} + \frac{\sqrt{3}}{4} \left( \frac{v}{d} \right)^2 + \cdots \tag{101}$$

Substituting eq.(101) into eq.(99), one has

$$r_{A\to B}=\frac{\gamma_m}{\sqrt{\pi}}\left(1-3b_1\left(\frac{v_{*1}}{d}\right)^2+\cdots\right) \tag{102}$$

i.e.,

$$r_{A\to B}=\frac{\gamma_m}{\sqrt{\pi}}\left(1-\frac{v_{*1}^2}{\hat{T}}+\cdots\right) \tag{103}$$

This form converges, in the limit of eq.(100), to the Gaussian form eq.(87), $r_{A\to B}\propto\exp\left(-\mathcal{E}_C/\hat{T}\right)$, of the thermodynamical equilibrium limit.

*3.4.5 Case of strong turbulence*

It is most interesting to study the case that the turbulence level is strong at the saddle point C, i.e.,

$$v_{*1}>>d\,. \tag{104}$$

In this limit, one has

$$\frac{\left(v_{*1}d^{-1}+1\right)^{3b_1}}{\left(v_{*1}^3d^{-3}+1\right)^{a_1}}\to 1 \tag{105}$$

and

$$\arctan\left(\frac{2v-d}{\sqrt{3}d}\right)+\frac{\pi}{6}\to\frac{2}{3}\pi\,. \tag{106}$$

Then the dependence is derived from eq.(99) as

$$r_{A \to B} = \frac{\gamma_m}{\sqrt{\pi}} \exp\left(-\frac{4\pi}{\sqrt{3}} b_1\right) \tag{107}$$

in the limit of eq.(104).

This result shows that the transition probability remains to be finite even in a limit of $\mathcal{E}_C/d^2 \to \infty$. Transition probability is explicitly calculated from eq.(99). Figure 5 illustrates the dependence of the function

$$F(X) = (X+1)^{1/2} (X^3+1)^{1/3X - 1/6} \exp\left(-\frac{1}{\sqrt{3}}\left\{\arctan\left(\frac{2}{\sqrt{3}}X - \frac{1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\}\right) \tag{108}$$

by which the transition probability is expressed with variable $\upsilon_{*1}/d \equiv X$ as

$$r_{A \to B} = \frac{\gamma_m}{\sqrt{\pi}} \left\{F(\upsilon_{*1}/d)\right\}^{6b_1} \tag{109}$$

Characteristic dependence of this function is observed. The Taylor expansion is discussed in the previous subsection,

$$F(X) \simeq 1 - \frac{1}{2}X^2 \qquad X << 1 \tag{110}$$

and the asymptotic limit of $F(X) \to \exp\left(-2\pi/3\sqrt{3}\right)$ at $X \to \infty$ holds as is shown by eq.(107). Important feature is seen in a region of large argument, i.e., this function is fitted to a power law as

$$F(X) \simeq X^{-1/3} \qquad 1 < X < 10\,. \tag{111}$$

The transition probability in the region of this large argument, which is obtained by substituting eq.(111) into eq.(109), is explicitly obtained as

$$r \sim \frac{\gamma_m}{\sqrt{\pi}}\left(\frac{v_{*1}}{d}\right)^{-2b_1} = \frac{\gamma_m}{\sqrt{\pi}}\left(\frac{\mathcal{E}_C}{d^2}\right)^{-b_1} . \tag{112}$$

The transition probability is given in a form of power law in the region $1 < \mathcal{E}_C/d^2 < 100$ .

### *3.5 Back transition probability*

Transition probability for the back transition, i.e., the transition from the turbulent fluctuation to the thermal fluctuation, is also calculated. Evaluations of the width of distribution at the state B and of the integral $S(\mathcal{E})$ are performed, and back transition probability is obtained.

#### *3.5.1 Width of distribution at state B*

In the preceding article III, the width of distribution at the state B, $\Delta\mathcal{E}_B$ , was obtained. The result is quoted as

$$\Delta\mathcal{E}_B = 2v_{*2}\Delta v \tag{113}$$

and

$$\Delta v_{*2} = \frac{g(\mathcal{E}_B)}{\sqrt{\mathcal{E}_B\gamma_B}} . \tag{114}$$

In the denominator of eq.(114), the relations

$$\mathcal{E}_{eq} = \mathcal{E}_B \quad \text{and} \quad \gamma_{eq} = \gamma_B \tag{115}$$

are substituted. From eqs.(113) and (114), one has

$$\Delta \mathcal{E}_B = \frac{2g(\mathcal{E}_B)}{\sqrt{\gamma_B}} \quad , \tag{116}$$

where the relation $v_{*2} = \sqrt{\mathcal{E}_B}$ is used. Equation (116) provides a relation

$$\frac{1}{\Delta \mathcal{E}_B} g(\mathcal{E}_B) = \frac{\sqrt{\gamma_B}}{2} \, . \tag{117}$$

Substitution of eq.(117) into eq.(70) provides the formula for the back transition probability as

$$r_{B \to A} = \frac{\sqrt{\Lambda_0 \gamma_B}}{4\sqrt{\pi}} \exp\left\{ S(\mathcal{E}_B) - S(\mathcal{E}_C) \right\} \, . \tag{118}$$

*3.5.2 Probability for the back-transition*

In order to obtain the analytic form of the integral $S(\mathcal{E})$, the model of decorrelation rate of III is employed as (see Fig.4)

$$\Lambda(v) = \Lambda_0 - \Lambda_0' v \qquad (0 < v < v_c) \tag{119a}$$

$$\Lambda(v) = -\overline{\Lambda}_0 + \Lambda_1' v \qquad (v_c < v). \tag{119b}$$

In addition to eq.(93), the term $\exp\left\{ - S(\mathcal{E}_B) \right\}$ is calculated as

$$\exp\left\{ - S(\mathcal{E}_B) \right\} = \frac{\left( v_{*2}^3 d^{-3} + 1 \right)^{a_2}}{\left( v_{*2} d^{-1} + 1 \right)^{3b_2}} \frac{\left( v_c d^{-1} + 1 \right)^{3b_1 + 3b_2}}{\left( v_c^3 d^{-3} + 1 \right)^{a_1 + a_2}} \times$$

$$\exp\left\{2\sqrt{3}b_2\left(\arctan\left(\frac{2v_{*2}-d}{\sqrt{3}d}\right)-\arctan\left(\frac{2v_c-d}{\sqrt{3}d}\right)\right)-2\sqrt{3}b_1\left(\arctan\left(\frac{2v_c-d}{\sqrt{3}d}\right)+\frac{\pi}{6}\right)\right\}$$

(120)

where

$$a_2 = b_2\left(1-\frac{2d}{v_{*2}}\right) \tag{121}$$

and

$$b_2 = \frac{\overline{\Lambda}_0}{\bar{\bar{g}}_0^2 \mathcal{E}_{eq}^{5/2}}\frac{1}{3d} . \tag{122}$$

An exponential factor $\exp\left\{S(\mathcal{E}_B)-S(\mathcal{E}_c)\right\}$ is given as

$$\exp\left\{S(\mathcal{E}_B)-S(\mathcal{E}_c)\right\} = \frac{\left(v_{*1}d^{-1}+1\right)^{3b_1}}{\left(v_{*1}^3d^{-3}+1\right)^{a_1}}\frac{\left(v_{*2}d^{-1}+1\right)^{3b_2}}{\left(v_{*2}^3d^{-3}+1\right)^{a_2}}\frac{\left(v_c^3d^{-3}+1\right)^{a_1+a_2}}{\left(v_cd^{-1}+1\right)^{3b_1+3b_2}}$$

$$\times\exp\left\{-2\sqrt{3}b_1\left(\arctan\left(\frac{2v_{*1}-d}{\sqrt{3}d}\right)-\arctan\left(\frac{2v_c-d}{\sqrt{3}d}\right)\right)\right\}$$

$$\times\exp\left\{-2\sqrt{3}b_2\left(\arctan\left(\frac{2v_{*2}-d}{\sqrt{3}d}\right)-\arctan\left(\frac{2v_c-d}{\sqrt{3}d}\right)\right)\right\} . \tag{123}$$

*3.4.3 Case of strong turbulence*

As is the case of transition from thermal fluctuations, we study the limit of strong turbulence,

$$v_{*2} >> d \qquad v_{*1} >> d . \tag{124}$$

In this limit, one has the limiting form

$$\exp\left\{S(\mathcal{E}_B)-S(\mathcal{E}_c)\right\}=\frac{v_{*1}^{3b_1-3a_1}}{v_c^{3b_1-3a_1}}\frac{v_{*2}^{3b_2-3a_2}}{v_c^{3b_2-3a_2}}\ . \tag{125}$$

Substituting this result eq.(125) into eq.(118), the probability for the back transition is also obtained as

$$r_{B\to A}=\frac{\sqrt{\Lambda_0\gamma_B}}{4\sqrt{\pi}}\frac{v_{*1}^{3b_1-3a_1}}{v_c^{3b_1-3a_1}}\frac{v_{*2}^{3b_2-3a_2}}{v_c^{3b_2-3a_2}}\ , \tag{126}$$

or

$$r_{B\to A}=\frac{\sqrt{\Lambda_0\gamma_B}}{4\sqrt{\pi}}\left(\frac{v_{*1}}{v_c}\right)^{\eta_1}\left(\frac{v_{*2}}{v_c}\right)^{\eta_2}\ . \tag{127}$$

with

$$\eta_1=3(b_1-a_1)=\frac{2\Lambda_0'}{\bar{\bar{g}}_0^2\mathcal{E}_{eq}^{5/2}} \tag{128a}$$

$$\eta_2=3(b_2-a_2)=\frac{2\Lambda_1'}{\bar{\bar{g}}_0^2\mathcal{E}_{eq}^{5/2}}\ . \tag{128b}$$

This result of back-transition probability is also expressed in a form of the power law. It is also noticed that the unit of time rate is given by

$$\sqrt{\Lambda_0\gamma_B}\ ,$$

being accelerated in comparison with $\gamma_m$. It should also be noticed that the power indices $\eta_1$ and $\eta_2$ in the expression $r_{B\to A}$ do not include the temperature $T$. This is because the random noise is dominated by the self-noise of turbulence in the state B: The deviation of the mean value of fluctuations is governed by the turbulent self-noise and is not influenced by the thermal excitation.

### *3.6 Example of subcritical interchange mode turbulence*

In this section, the result is applied to the case of subcritical interchange mode turbulence. By taking an example of plasma turbulence, the transition probability is explicitly expressed in terms of global plasma parameters. The importance of the gradient, i.e., the nonequilibrium parameter, is also illuminated.

#### *3.6.1 Turbulent state*

Statistical quantities in the turbulent state (state B) have been calculated in the preceding articles II and III. By specifying the volume of integral for $\mathcal{E}$, the self-noise term was evaluated. According to III, we choose the region of $L^2$ (across the main magnetic field) for the coarse-graining of fluctuation energy. By use of this scale, the amplitude of the self noise was given as

$$\bar{\bar{g}}_0^2 \simeq \frac{8C_0}{3} \gamma_0 \mathcal{E}_{eq}^2 \left( k_0 \frac{L}{a} \right)^{-2} \tag{129}$$

and

$$\mathcal{E}_B = \frac{1}{4} G_0^2 \left( \frac{\delta}{sa} \right)^2 \left( \frac{L}{a} \right)^2 \tag{130}$$

where $k_0$ is a typical wave number of turbulent fluctuations

$$k_0 \simeq G_0^{-1/2} \frac{sa}{\delta} \tag{131}$$

and $\gamma_0 = \gamma_B$ is the characteristic eddy damping rate at the state B

$$\gamma_0 \simeq G_0^{1/2} \, . \tag{132}$$

(Detailed argument of the typical wave number and decorrelation rate, the $k$ - and $\omega$ - spectra of equilibrium turbulent sate is given in refs.44-47 and is not repeated here.) The magnitude of the noise source term is expressed as

$$\frac{\bar{\bar{g}}_0^2}{\mathcal{E}_{eq}^{5/2}} \simeq \frac{16C_0}{3} k_0^{-1}\left(\frac{L}{a}\right)^{-3} . \tag{133}$$

In expressions (129) and (133), the coefficient $C_0$ is of the order unity, and is defined by the ratio[36,37]

$$C_0 \simeq \frac{-\int_{>k} \frac{\left(\left(\boldsymbol{p}\times\boldsymbol{q}\right)\cdot\boldsymbol{b}\right)^2}{k_\perp^4}\left(\frac{p_\perp^2-q_\perp^2}{k_\perp^2}\right)^2\left(\frac{p_\perp}{k_\perp}\right)^{-\beta}\left(\frac{p_\perp q_\perp}{k_\perp^2}\right)^{-\alpha}\frac{\mathrm{d}\boldsymbol{p}_\perp}{k_\perp^2}}{\int_{>k} \frac{\left(\left(\boldsymbol{p}\times\boldsymbol{q}\right)\cdot\boldsymbol{b}\right)}{k_\perp^2}\left(\frac{p_\perp^2-q_\perp^2}{k_\perp^2}\right)\frac{\left(\left(\boldsymbol{k}\times\boldsymbol{p}\right)\cdot\boldsymbol{b}\right)}{k_\perp^2}\left(\frac{k_\perp^2-p_\perp^2}{q_\perp^2}\right)\left(\frac{p_\perp}{k_\perp}\right)^{-\alpha-\beta}\frac{\mathrm{d}\boldsymbol{p}_\perp}{k_\perp^2}} . \tag{134}$$

In performing the integral of (134), $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the relation $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$ .

### *3.6.2 Threshold condition for nonlinear instability*

Threshold amplitude of the nonlinear instability gives the energy at saddle point $\mathcal{E}_C$ . The study of the subcritical excitation for the interchange turbulence has shown that the threshold value of the fluctuation amplitude is given as[1, 18)]

$$\left.\sqrt{\bar{\bar{\phi}}^2}\right|_C \simeq \mu_{ec}\sqrt{1-\frac{G_0}{G_c}} \tag{135}$$

in the vicinity of stability criterion

$$G_0 \simeq G_c \,. \tag{136}$$

On the other hand, the level of potential fluctuation at the state B was also given as

$$\sqrt{\tilde{\phi}^2}\Big|_B = \chi_N \simeq \frac{G_0^{3/2}}{s^2}\left(\frac{\delta}{a}\right)^2 , \tag{137}$$

and the ratio of fluctuation energy is estimated as

$$\frac{\mathcal{E}_C}{\mathcal{E}_B} = \left(\sqrt{\tilde{\phi}^2}\Big|_c\right)^2 \left(\sqrt{\tilde{\phi}^2}\Big|_B\right)^{-2} \tag{138}$$

for the common wave number. Combining eqs. (135), (137) and (138), one has the expression for $\mathcal{E}_C$ as

$$\frac{\mathcal{E}_C}{\mathcal{E}_B} \simeq \mu_{ec}^2\, s^4 G_0^{-3}\left(\frac{\delta}{a}\right)^{-4}\left(1-\frac{G_0}{G_c}\right). \tag{139}$$

Substituting eqs.(130) and (131) into eq.(139), one has $\mathcal{E}_C$ for typical wave number $k_0$ as

$$\mathcal{E}_C \simeq \frac{\mu_{ec}^2}{4}\, k_0^2\left(\frac{L}{a}\right)^2\left(1-\frac{G_0}{G_c}\right). \tag{140}$$

*3.6.3 Transition probability*

In the case that state C is in the strong turbulence state, the transition probability is explicitly calculated.

Combination of eqs.(96) and (133) provides the form

$$d^3 = \hat{T}\gamma_m \frac{3}{16C_0} k_0 \left(\frac{L}{a}\right)^3 . \tag{141}$$

By use of eqs.(133) and (141), the index factor $b_1$ is also evaluated as

$$b_1 \simeq \frac{1}{3}\left(\frac{3}{16C_0} k_0 \gamma_m\right)^{2/3} \hat{T}^{-1/3}\left(\frac{L}{a}\right)^2 \tag{142}$$

Note that parameters $d$ and $b_1$ are expressed by the combination of thermal fluctuations (e.g., $\hat{T}$ ) and turbulent fluctuations $\left(k_0, C_0\right)$. Combining eqs.(140) and (141), one has the ratio $\mathcal{E}_C/d^2$ in terms of the global parameters as

$$\frac{\mathcal{E}_C}{d^2} \simeq \frac{\mu_{ec}^2}{4} k_0^{4/3}\left(\hat{T}\gamma_m \frac{3}{16C_0}\right)^{-2/3}\left(1 - \frac{G_0}{G_c}\right) . \tag{143}$$

Then the transition probability eq.(112) is given, after substitution of eqs.(143) as

$$r_{A \to B} \sim \frac{\gamma_m}{\sqrt{\pi}}\left(\frac{\mu_{ec}}{2}\right)^{-2b_1} k_0^{-4b_1/3}\left(\hat{T}\gamma_m \frac{3}{16C_0}\right)^{2b_1/3}\left(1 - \frac{G_0}{G_c}\right)^{-b_1} . \tag{144}$$

From the result of eq.(144), one sees that the transition probability is obtained as a function of the critical parameter as

$$\left(1 - \frac{G_0}{G_c}\right) . \tag{145}$$

Important feature is that the probability is expressed in terms of the power law

$$r_{A \to B} \propto \left(1 - \frac{G_0}{G_c}\right)^{-b_1} . \tag{146}$$

The transition probability decreases as the parameter $G_0$ becomes smaller than $G_c$. However, the decay of the probability is much slower, and considerable probability remains for the onset of transition from the thermal fluctuation to the turbulent fluctuations.

## §4 Summary and Discussion

In this article, the statistical theory for the strongly turbulent system is further developed. The transition between two probable states of turbulence is examined based upon the Fokker-Planck equation for the probability distribution function. In the system where the submarginal turbulence (subcritical excitation of turbulence) is possible, the probability distribution function of steady state has more than one peak and the transition from one probable state to another takes place. In order to study the transition probability, the Fokker-Planck equation for the coarse-grained fluctuation quantity, i.e., the averaged amplitude of turbulent energy, is employed. The formula of transition probability between two probable states is derived. This formula is reduced to the Arrhenius law in the thermodynamical equilibrium limit. The transition probability between the thermally-excited branch (thermal fluctuations) and the turbulent branch is calculated for the generalized model formula of subcritical excitation. The transition probability from the thermal fluctuation to the turbulent fluctuation can be much larger than that obtained from Arrhenius law. This is because the transition probability depends weakly on the heat bath temperature and depends on the nonequilibrium parameter (the gradient parameter, $G_0$).

The back transition from the turbulent state to the thermal fluctuation is also examined. Combining these transition probabilities, so-called probability density matrix (statistical operator) can be calculated.

The model formula for subcritical excitation is applied to the actual plasma turbulence of interchange mode. The results are explicitly shown by the global plasma parameters. In this system, the gradient parameter plays an essential role and the transition probability is expressed in terms of the gradient parameter. The distance from the critical point is expressed by the difference between the critical gradient, $1 - G_0/G_c$ , and the dependence of the transition probability on this distance is studied. A power law dependence is deduced. A critical exponent for the transition to turbulence is obtained.

The results thoroughly come from the fact that the turbulent self-noise as well as the turbulent drag have the dependencies on the turbulence amplitude. These amplitude dependencies are the common feature of fully developed turbulence in both plasma and neutral fluids, regardless of the turbulent mode considered. The similarity can be found in gaseous fluids turbulence. Therefore the analogy near the critical point in these cases can be analyzed along the line of this theory.

The critical dependence on the gradient parameter of the system, in turn, is interpreted as the critical size effect under some circumstances. An application of this statistical theory to the physics of molecular dynamics as well as the problems of nucleation is desired, permitting further analysis.

The transition among strong turbulence in plasmas has been reported theoretically and experimentally. In this paper, focus is made on the transition between the thermal fluctuations and turbulent fluctuations. The turbulence-turbulence transition, i.e., the electrostatic turbulence to electromagnetic turbulence, is another issue to be addressed. The work is left for future.

In the calculation of the transition probability, the typical scale size of the turbulent fluctuation is assumed to be microscopic and being decoupled form the global scale size of gradient. In various experimental observations, scale size of fluctuations becomes large. Inclusion of such effects requires further analysis of turbulent interactions between the microscopic fluctuations and macroscopic structures.

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**Dedication**

This article is dedicated to the memory of Prof. R. Kubo.

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**Figure Captions**

**Fig.1** Statistical average of fluctuation level as a function of the pressure gradient. The case of subcritical excitation is shown. In the case of low gradient, only thermal fluctuations are realized. In the higher gradient region, hysteresis is shown, and multiple states of fluctuations (denoted by A and B) could appear in the steady state.

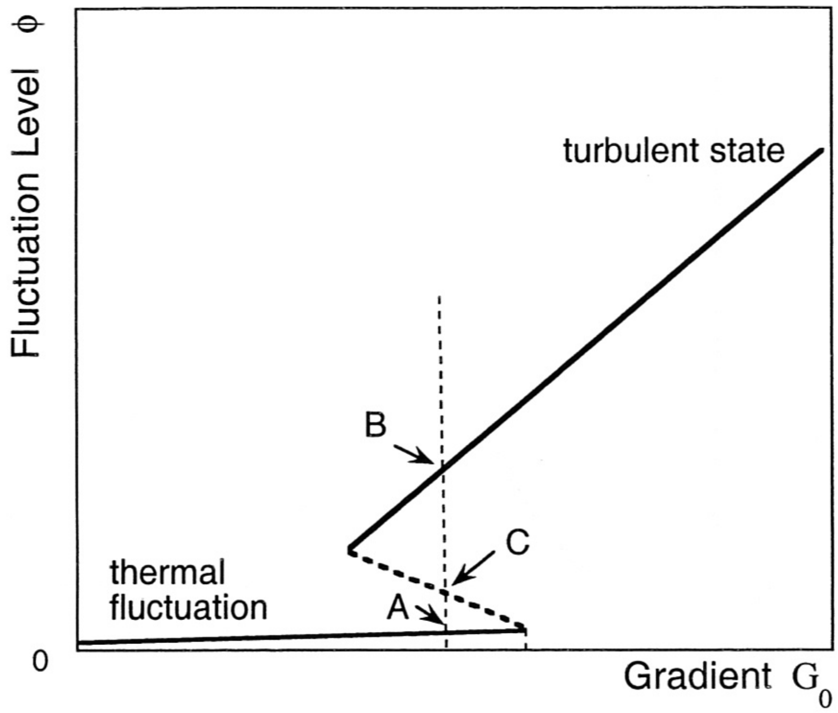
**Fig.2** Schematic drawings of the equilibrium probability distribution function for submarginal turbulence. Probability distribution function $P_{eq}$ has two peaks, i.e., one for thermal fluctuations and the other for self-sustained turbulence.

**Fig.3** Renormalized dissipation rate $S(\upsilon)$ as a function of the fluctuation amplitude. The case in Fig.1 is schematically shown. Owing to the presence of zeros of $\Lambda$, $S(\upsilon)$ takes extremum at $\upsilon = \upsilon_{*1}$ and $\upsilon = \upsilon_{*2}$. $S(\upsilon)$ has local minima at $\upsilon = 0$ and $\upsilon = \upsilon_{*2}$.

**Fig.4** Model of the decorrelation rate $\Lambda$ as a function of fluctuation velocity $\upsilon$. In the low amplitude region, $\Lambda$ is a decreasing function of $\upsilon$, representing the subcritical excitation owing to nonlinear instability. In the large amplitude limit, $\Lambda$ is an increasing function of $\upsilon$, and asymptotic dependence $\Lambda \propto \upsilon$ holds. $\Lambda$ vanishes at two values of $\upsilon$, $\upsilon_{*1}$ and $\upsilon_{*2}$.

**Fig.5** Function $F(X)$. It is a decreasing function of $X$, but remains finite at $X \to \infty$ (a). An expanded view in the region of small argument is shown in (b). Logarithmic scale indicates a fitting to the power law (c).

Fig. 1



Fluctuation Level $\phi$
turbulent state
B
C
thermal
fluctuation
A
0
Gradient $G_0$

Fig.2

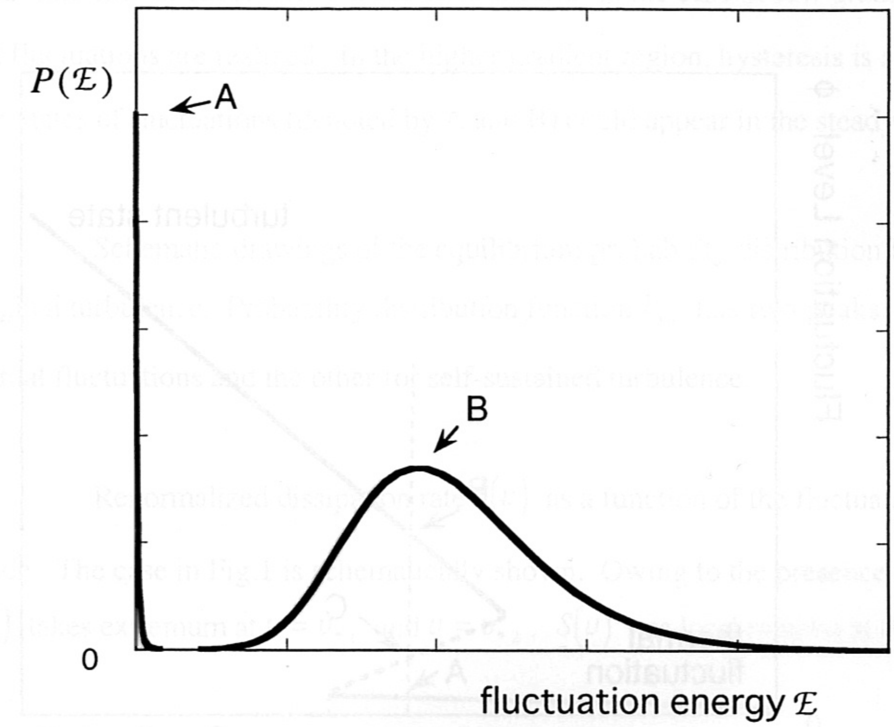


**Fig.3**

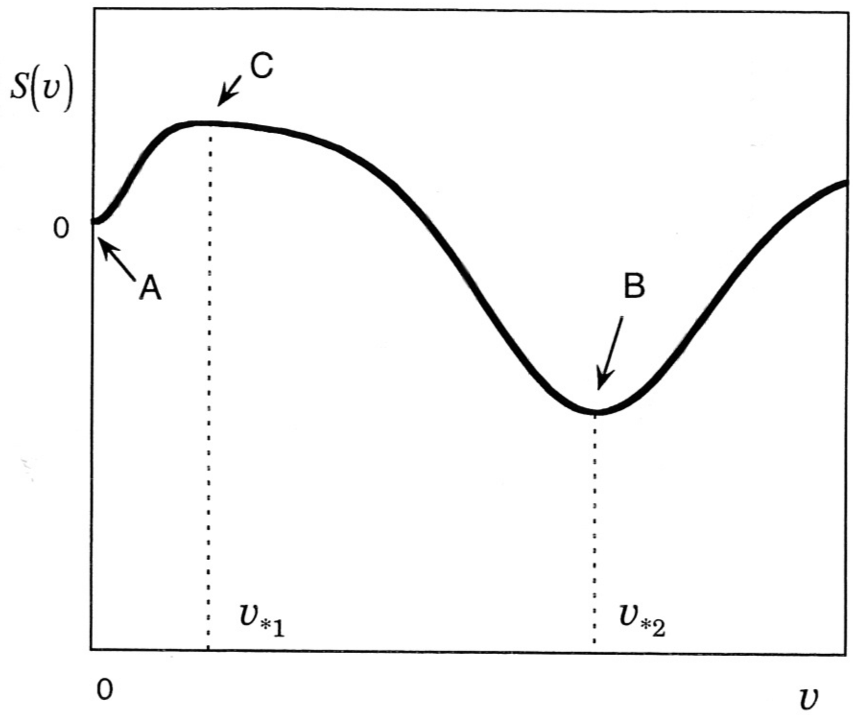


**Fig.4**

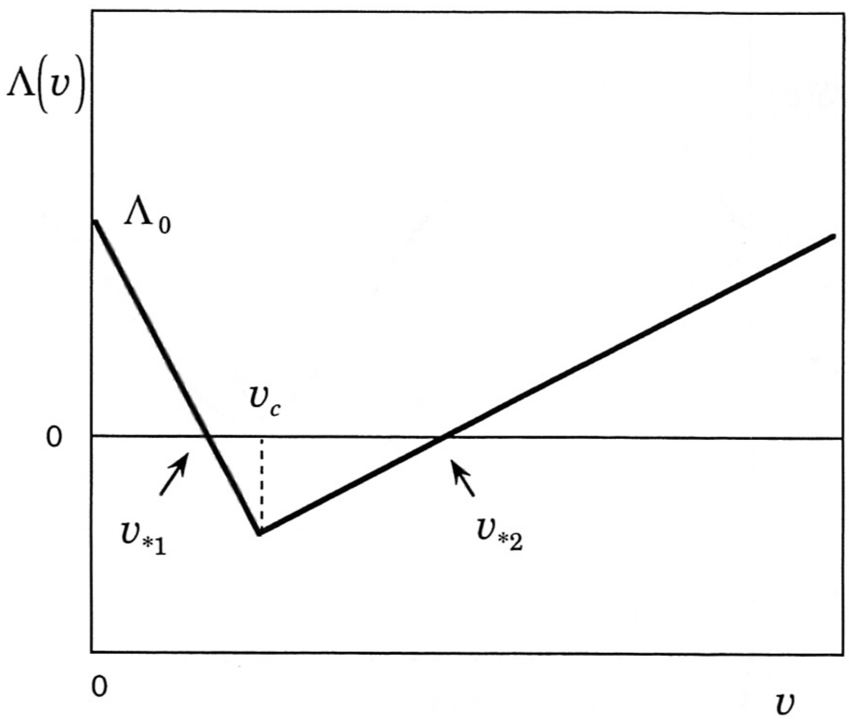


**Fig.5 (a)**

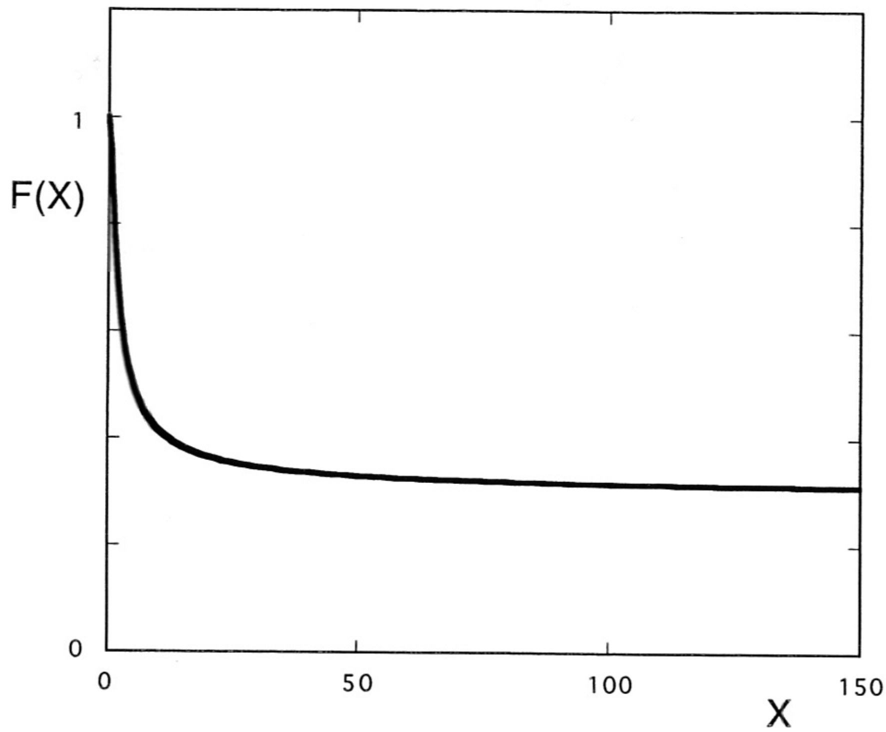


Fig.5 (b)

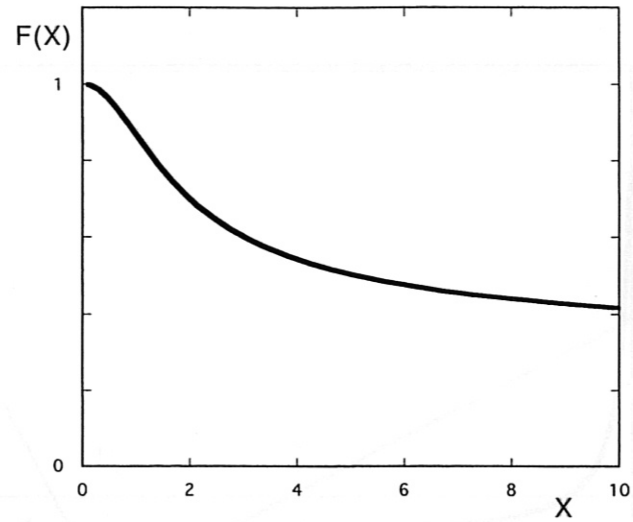



Fig.5(c)