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Abstract:

This paper discusses some general aspects of Marfe formation and bifurcation of temperature in stellarators. The starting point is the three-dimensional heat conduction equation with a conduction matrix depending on temperature and position. The existence, uniqueness and stability of steady state solutions are discussed. Marfes in stellarators predominantly exist on rational magnetic surfaces if the shear is low. Outside the last magnetic surface bifurcation of temperature can arise on flux bundles between two divertor target plates. The number of solutions depends on the non-linearity of the boundary conditions and the details of the radiation function, especially its dependence on the temperature. Numerical solutions of the one-dimensional heat conduction equation in a slab and a cylindrical fusion plasma, using the shooting method, illustrate the various aspects of bifurcation.

1. Introduction

MARFE formation is a frequently observed phenomenon in the edge region of tokamak plasmas. It is attributed to a thermal instability, where a local decrease of temperature leads to an increase of radiation and consequently to a further cooling of this region. If parallel thermal conduction is insufficient to compensate for these radiation losses the instability leads to a stable and localised radiative region, called MARFE. In tokamaks the MARFE is an axisymmetric phenomenon and poloidally localised to the X-point region of the poloidal magnetic field.

There exists a large collection of papers on the MARFE phenomenon, both experimental and theoretical ones. Wesson and Hender[1] consider the stability of a radial temperature profile against poloidal perturbations, however the issue of multiple solutions of the temperature equation is not addressed. Kesner and Freidberg[2] also start from a two-dimensional equation and discuss the stability of radial temperature. Although in tokamaks MARFE formation often are the precursors of disruptive instabilities the authors argue that in a reactor plasma controlled MARFE formation may help to radiate a significant fraction of the power leaving the plasma. In Stacey's theory[3] the transition from a poloidally symmetric profile to a MARFE and furthermore to a detached plasma has been analysed.

Stellarator configurations are inherently 3-dimensional, which poses some difficulties to the description of temperature profiles. In the boundary region of stellarators the magnetic field is ergodic and seeded with islands. Magnetic field lines may begin and terminate on divertor target plates or on the wall of the vacuum chamber. As in tokamaks this situation can be described by the one-dimensional heat conduction equation with the perpendicular fluxes as source or sink terms. In contrast to tokamaks no use can be made from the up-down symmetry, instead the asymmetry of the flux bundle which is limited by two target plates has to be taken into account.

The edge region is characterised by a radiative layer, which - as has been observed in tokamaks - may lead to a detachment of the plasma. To investigate this effect in stellarators we must start from the 3-dimensional form of the heat conduction equation and impose the appropriate boundary condition. Thermal conductivity and radiation losses are non-linear functions of the temperature, which in general is the reason for multiple solutions and bifurcations. Convective flow can be another cause of bifurcation, which is well known from the Bénard convection, however, this effect can only be described by including the equation of motion. For reason of simplicity plasma convection will be neglected in the following.

In a fusion plasma bifurcation of temperature occurs as a consequence of the temperature dependent heating rate and the temperature dependence of the transport coefficients. Furthermore, the radiation of impurities is tightly connected with bifurcation of the temperature profile, thus leading to a rich spectrum of bifurcaction phenomena.

In the following paper several issues related to the 3-dimensional geometry of stellarator equilibria will be discussed. Furthermore, some numerical examples will be presented.

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[1] J.A. Wesson, T.C. Hender, Nuclear Fusion 33, 1019 (1993)

[2] J. Kesner, J.P. Freidberg, Nuclear Fusion 35, 115 (1995)

[3] W.M. Stacey, Phys. Plasmas 3, 2673 (1996)

2. Basic Equations

In the following we adopt the model of a plasma with equal electron and ion temperatures and consider the heat conduction equation. We neglect the convection terms in the heat conduction equation and start from the equation

$$-\nabla \bullet \chi(x,T) \bullet \nabla T = H(x,T) - Q(x,T) \tag{2.1}$$

where $H(x,T)$ is the heating term and $Q(x,T)$ the radiation loss of the electrons. Retaining the convection term would include another reason for bifurcation as is well known from Bénard convection.

In addition to the heat conduction equation, boundary conditions must be imposed. The standard boundary condition is a fixed temperature on the boundary. However, this is a too simple approximation to the physics determining the plasma wall interaction. In general there exists a non-linear relation between the temperature and the temperature gradients on the wall.

$$F(A \bullet \chi(x,T) \bullet \nabla T, T) = 0 \tag{2.2}$$

A is some given vector on the boundary, it could be the normal vector.

Equation (2.1) is applicable to a fusion plasma where $H(x,T)$ is the alpha-particle heating term and $Q(x,T)$ the radiative loss which can be Bremsstrahlung, cyclotron radiation or impurity radiation. In plasma experiments the heating term is the external heating and the radiation is mainly determined by impurities. A characteristic feature of the non-linear terms is that they exhibit one or several maxima in temperature. This occurs in the radiation function and in the alpha-power heating term. This non-linearity is the main reason for multiple solutions, which means that there are several solutions

$$T_1(x), \ldots, T_n(x) \tag{2.3}$$

to the same boundary. Which one of these solutions is verified in a particular experiment is determined by stability of the solutions and the experimental scenario to reach the stable solutions. A mathematical theory of an equation similar to eq. (2.1) is presented in ref.[4]

3. Existence of solutions

Equation (2.1) is a quasi-linear elliptic differential equation, which can be solved if appropriate boundary conditions are imposed. The boundary conditions depend on the physics of the boundary layer and they are non-linear, in general. Let us, at first, consider a toroidal domain Ω and impose Dirichlet boundary conditions $T = \text{const} = T_a$. In a magnetised plasma the tensor of thermal conduction

$$\chi = \chi_\perp I + (\chi_\parallel - \chi_\perp) b{:}b \tag{3.1}$$

[4] J.B. Keller, s. Antman, Bifurcation Theory and Nonlinear Eigenvalue Problems, W.A. Benjamin, inc. New York, Amsterdam 1969.

is highly anisotropic and also depends on the temperature. $\boldsymbol{b}$ is the unit vector parallel to field lines. Both functions in eq. (3.1) are positive and bounded and the parallel thermal conductivity is much larger than the perpendicular conductivity. This implies that the matrix χ is positive definite. Both, perpendicular and parallel thermal conductivities are functions of the temperature, the classical parallel thermal conductivities grow with the exponent 5/2. Furthermore, we impose the restriction that heating and radiation losses are bounded functions of their arguments. In general the radiation function is

$$Q(\boldsymbol{x},T) = n n_z L(T) \tag{3.2}$$

where n is the electron density and n_z the density of the impurity ion. $L(T)$ is a bounded function of temperature which depends on the ion species. Since $L(T)$ is positive, continuous and bounded it has at least one maximum at a temperature T_{max}. The density of impurities is a function of the spatial coordinate $\boldsymbol{x}$. Its special shape depends on the impurity species and the diffusion process of impurities.

In plasma experiments or in fusion plasmas the heating term is also a bounded function of $\boldsymbol{x}$ and T. Equation (2.1) is valid in any magnetic field, there are no restrictions and existence of magnetic surfaces is not required. In particular, this equation is appropriate to compute the temperature profile in the boundary region of stellarators, where the magnetic field exhibits island structure and ergodicity.

Since the parallel thermal conductivity is by orders of magnitude larger than the perpendicular conductivity, the temperature will be almost constant along magnetic field lines. In regions of nested magnetic surfaces this results in constant temperature on magnetic surfaces. However, on rational magnetic surfaces closed field lines are decoupled from each other and the temperature can vary on rational magnetic surfaces. The variation depends on the shear, which will localize such an inhomogeneity of the temperature to the neighbourhood of the rational surfaces.

In order to facilitate the following analysis we introduce dimensionless variables. Let a be a characteristic length scale of the system, then all spatial coordinates are defined in units of a ($\boldsymbol{x} \to \boldsymbol{x}/a$). Furthermore, let T_0 be a reference temperature and χ_0 a reference value of the thermal conductivity, then the temperature is measured in units of T_0 ($T \to T/T_0$). Such a reference temperature is defined by T_{max}, which is the temperature where the radiation function $L(T)$ has a maximum. Let N be a characteristic plasma density and N_Z a number to characterise the density of impurities, then the dimensionless functions are $n \to n/N$ and $n_z \to n_z/N_Z$. With L_o as characteristic value of the radiation the dimensionless radiation function is $L(T) \to L(T)/L_o$. In dimensionless units we write the heating function as $H \to h = Ha^2/T_o\chi_o$. In dimensionless units the heat conduction equation is

$$-\nabla \bullet \chi(\boldsymbol{x},T) \bullet \nabla T = h(\boldsymbol{x},T) - \lambda\, p(\boldsymbol{x},T) \tag{3.3}$$

where λ is defined by

$$\lambda = \frac{a^2}{\chi_o T_o} N N_Z L_o \tag{3.4}$$

and $p(\boldsymbol{x},T)$ by

$$Q(\boldsymbol{x},T) \to \lambda n(\boldsymbol{x}) n_Z(\boldsymbol{x}) L(T) =: \lambda p(\boldsymbol{x},T) \tag{3.5}$$

The parameter λ is introduced as a variable or control parameter and the aim of the analysis is to study how the solution $T(x,\lambda)$ depends on this control parameter. Either the electron density N or the density N_z of impurities can be interpreted as the control parameter λ. The control parameter grows with the size of the system. Very often the plasma pressure is considered as the independent parameter. If $nT \sim \lambda$, the function $L(T)$ is replaced by $L(T)/T$, where L(T) is of the order unity.

In a later chapter the time dependent heat conduction equation will be considered in the stability analysis of stationary solutions. Also the time is measured in units of t_o, where the time reference is $t_o = a^2/(\chi_o N)$.

The choice of the control parameter is arbitrary and only determined by the special field of interest. In case of a fusion plasma the heating function consists of an external heating term and the alpha-particle heating. In such a case one may consider the external heating as a variable and introduce a control parameter, which is proportional to the external heating power. In the boundary region of a plasma the radiation is of special interest and the control parameter is chosen as described above. Another choice could be a parameter in the non-linear boundary conditions.

In general, the thermal conductivity depends on the temperature and on the spatial coordinate x :$\chi = \chi(x,T)$. The assumptions made on the matrix elements of χ are:

1) All elements are monotonically increasing with temperature
2) The thermal conductivity is positive definite for $T \geq 0$:

$$\chi_{ik}(x,T)y_i y_k \geq a(T) y_k^2 \;\; ; \;\; a(T) > 0 \quad \forall T \tag{3.6}$$

3) All coefficients in χ have continuous derivatives up to second order.

$a(T)$ is a non-zero and monotonically increasing function of T. Because of these properties the operator

$$L[T] = -\nabla \bullet \chi(x,T) \bullet \nabla \tag{3.7}$$

is uniformly elliptic for any given T and for any positive T we can define the Green's function by

$$-\nabla \bullet \chi(x,T) \bullet \nabla G(x,y) = \delta(x-y) \tag{3.8}$$

The right hand side is positive which implies that the Green's function is positive. G depends on the specific choice of T because of $\chi = \chi(x,T)$. Using the Green's function the inverse operator $A =: L^{-1}$ can be written as

$$A[T] = \iiint_\Omega G(x,y) \ldots d^3y \tag{3.9}$$

Since equation (2.1) is non-linear, existence of a solution satisfying the boundary condition is not a trivial matter. Utilising the Green's function of the operator the equation (2.1) can also written in the form.

$$T = \iiint_\Omega G(x,y)\big(h(y,T) - \lambda p(y,T)\big) d^3y \tag{3.10}$$

or in the more abstract form

$$T = A\left[T\right]\left(h(T) - \lambda p(T)\right) \tag{3.11}$$

The inverse operator $A = L^{-1}$ generated by the Green's function is compact. The product of a compact operator and a bounded operator is also compact and therefore the Leray-Schauder fixed-point theorem[5] can be used to prove the existence of a solution. The procedure to construct a solution is the following: The iterative scheme is defined by

$$-\nabla \bullet \chi(x, T_n) \bullet \nabla T_{n+1} = h(x, T_n) - \lambda p(x, T_n) \tag{3.12}$$

This a linear and inhomogeneous equation and can by solved by

$$T_{n+1} = \iiint_{\Omega} G_n(x,y)\left(h(y, T_n) - \lambda p(y, T_n)\right) d^3 y \tag{3.13}$$

The existence theorem is based on the proof that this sequence of functions converges. Since only positive solutions are acceptable this imposes an upper limit on the radiation losses. Thus, there exist a domain $[0,\lambda^*]$ where the solutions of eq. (3.11) are positive. To start this iteration scheme one must choose an initial function $T_0(\mathbf{x})$. This choice determines which one of the possible solutions will be reached at the end of the iteration procedure.

4. Bifurcation of Solutions

The iterative procedure described above does not provide uniqueness of the solutions. Since the heating term and thermal conductivity are non-linear functions of the temperature, multiple solution may exist. A further source of multiple solutions is the non-linearity of the radiation function $p(x, T)$. In order to study this effect we start form equation (2.1) in the form

$$-\nabla \bullet \chi(x, T) \bullet \nabla\, T = h(x, T) - \lambda p(x, T) \tag{4.1}$$

First we consider the more simple case where the conductivity does not depend on the temperature.

$$-\nabla \bullet \chi(x) \bullet \nabla\, T = h(x, T) - \lambda p(x, T) \tag{4.2}$$

Let us assume that two solutions $T_0(x)$ and $T_1(x)$ of this equation exist. For the difference $f = T_1(x) - T_0(x)$ we obtain

$$-\nabla \bullet \chi(x) \bullet \nabla f = g(x, T_0 + f) - g(x, T_0) \tag{4.3}$$

with

$$g = h(x, T) - \lambda p(x, T)$$

If $g(x, T)$ is a decreasing function of T this equation has no solution and the solution of eq. 4.1 is unique. The proof is by contradiction: the operator on the left-hand side is positive definite. If g is monotonically decreasing, the equation

[5] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of the Second Order*, Springer Verlag 1983, p. 286

$$\iiint \nabla f \bullet \chi(x) \bullet \nabla f \, d^3x = \iiint f\{g(x, T_0 + f) - g(x, T_0)\} d^3x \tag{4.4}$$

leads to a contradiction. The right hand side is negative.

This case arises if the heating term does not depend on the temperature and the radiation loss term increases with temperature. However, this is only true if the radiation is Bremsstrahlung radiation. Radiation by impurity atoms decreases at high temperature, the radiation function has one or more maxima in T. In a fusion plasma the heating term grows with temperature until a maximum is reached and the heating decreases again. Also in this case there can be several solutions of the heat conduction equation.

When two solutions coalesce at a specific value of λ, this defines the bifurcation or branch point. Since the difference f is small in the neighbourhood of this branch point the linearisation yields the equation

$$-\nabla \bullet \chi(x) \bullet \nabla \delta f = g_T(x, T_0, \lambda) \delta f \tag{4.5}$$

To indicate the smallness of f we use δf in the following. Instead of eq. 4.5 we consider the linear eigenvalue problem

$$-\nabla \bullet \chi(x) \bullet \nabla \delta f - g_T(x, T_0, \lambda) \delta f = E \delta f \tag{4.6}$$

The eigenvalues E are real since the operator on the left-hand side is self-adjoint. The bifurcation point is defined by $E(\lambda_0) = 0$. This eigenvalue can be computed by minimising the functional

$$F(\delta f) = \iiint \nabla f \bullet \chi(x) \bullet \nabla f \, d^3x \; - \iiint g_T(x, T_0, \lambda)(\delta f)^2 d^3x \tag{4.7}$$

Since the transport matrix is positive definite all eigenvalues of the operator $L = -\nabla \cdot \chi \cdot \nabla$ are positive and there exists a smallest eigenvalue E_0. This allows one to formulate a more stringent condition for uniqueness: If the condition following holds

$$\iiint g_T(x, T_0, \lambda)(\delta f)^2 d^3x \le E_0 \iiint (\delta f)^2 d^3x \tag{4.8}$$

there is no bifurcation point. If a test function can be found which makes the functional (4.7) negative, a bifurcation point exists. Applying this to the case of temperature-independent heating yields the condition

$$-\lambda \iiint p_T(x, T_0, \lambda)(\delta f)^2 d^3x \le E_0 \iiint (\delta f)^2 d^3x \tag{4.9}$$

In conclusion, if the radiation is small enough, the temperature profile is unique.

In the general case the thermal conductivity also depends on temperature. If this dependence is the same for all components of the thermal conductivity

$$\chi(x, T) = \chi_0(x) T^{\alpha} \; ; \; \alpha > 0 \tag{4.10}$$

the temperature dependence can be removed by defining a new dependent variable

$$U = \frac{1}{\alpha + 1} T^{\alpha+1} \;\; ; \;\; T(U) = \left((\alpha + 1)U\right)^{1/\alpha+1} \tag{4.11}$$

which yields the equation

$$-\nabla \bullet \chi_o(x) \bullet \nabla U = h(x, T(U)) - \lambda p(x, T(U)) \tag{4.12}$$

which has the same structure as eq. 4.1.

In general, however, the perpendicular thermal conduction has another temperature dependence as the perpendicular thermal conduction. The transformation above eliminates the temperature dependence of the parallel conductivity but not in the perpendicular conductivity. In this case we get ($\alpha = 5/2$)

$$-\nabla \bullet \chi(x, U) \bullet \nabla U = h(x, T(U)) - \lambda p(x, T(U)) \tag{4.13}$$

with

$$\chi = \chi_\perp(x, U)\boldsymbol{I} + \left(\chi_{\parallel}(x) - \chi_\perp(x, U)\right)\boldsymbol{b{:}b}$$

The derivative with respect to the temperature is

$$\chi_U = \frac{\partial}{\partial U}\left(\chi_\perp(x, U)\boldsymbol{I} - \chi_\perp(x, U)\boldsymbol{b{:}b}\right) \tag{4.14}$$

Linearisation with respect to small δU yields instead of eq.

$$-\nabla \bullet \chi(x, U_0) \bullet \nabla \delta U - \nabla \bullet \delta U \chi_U(x, U_0) \bullet \nabla U_0 = g_U(x, U_0, \lambda)\delta U \tag{4.15}$$

The second term on the left hand does not define a self-adjoint operator, the Hermitian part is

$$L_H = -\frac{1}{2}\nabla \bullet \left(\chi_U \nabla U_0\right) \tag{4.16}$$

And the anti-Hermitian part

$$L_A = -\nabla U_0 \bullet \chi_U \bullet \nabla - \frac{1}{2}\nabla \bullet \left(\chi_U \nabla U_0\right) \tag{4.17}$$

If there exists a finite solution of eq. (4.15) the following integral relation holds

$$0 = \iiint \nabla \delta U \bullet \chi(x, U_0) \bullet \nabla \delta U \, d^3x - \iiint \left(g_U(x, T(U_0, \lambda) - \frac{1}{2}\nabla \bullet \left(\chi_U \nabla U_0\right)\right)(\delta U)^2 d^3x \tag{4.18}$$

If the right hand side is positive for all test functions δU a bifurcation does not exist.

5. Stability of Solutions

In order to investigate the stability of the various solutions we consider the time-dependent heat conduction equation in the form

$$n\frac{\partial T}{\partial t} - \nabla \bullet \chi(x,T) \bullet \nabla T = h(x,T) - \lambda p(x,T) \tag{5.1}$$

The thermal conductivity depends on temperature which, in general, cannot be eliminated by transforming the dependent variable. The parallel conductivity is proportional to $T^{5/2}$. In the following we assume that the same scaling holds for the perpendicular conductivity, the general case will be investigated later. The non-linearity in the conductivity can be eliminated as has been shown in the previous section, and get the equation for U

$$nT_U(U)\frac{\partial U}{\partial t} - \nabla \bullet \chi_o(x) \bullet \nabla U = h(x,T(U)) - \lambda p(x,T(U)) \tag{5.2}$$

χ_O is a temperature-independent transport matrix which still reflects the strong asymmetry between parallel and perpendicular thermal conduction. Let $U_O(x)$ be one solution of the time-independent equation. The stability is determined by the linearised equation

$$nT_U(U_o)\frac{\partial u}{\partial t} - \nabla \bullet \chi_o(x) \bullet \nabla u = -V(x,U_o)u \tag{5.3}$$

with

$$V(x,U_o) = -\frac{\partial}{\partial U}\left[h(x,T(U_o)) - \lambda p(x,T(U_o))\right] \tag{5.4}$$

The perturbation u is defined by $u = U - U_o$. Making the ansatz $u \to u(x)\, exp(-\gamma t)$ yields the Hermitian eigenvalue equation

$$nT_U(U_o)\gamma u = -\nabla \bullet \chi_o(x) \bullet \nabla u + V(x,U_o)u \tag{5.5}$$

This is a type of "Schrödinger equation" with the potential V. The eigenvalues γ are real, negative eigenvalues correspond to unstable solutions. Since this is a Hermitian problem the eigenvalues can also be computed by minimising the functional

$$F[u] = \iiint_\Omega \left\{\nabla u \bullet \chi_o(x) \bullet \nabla u + V(x,U_o)u^2\right\} d^3x \tag{5.6}$$

under the auxiliary condition

$$\iiint_\Omega nT_U(U_o)u^2 d^3x = 1$$

Ω is the plasma domain. Let γ_O be the lowest eigenvalue of

$$nT_U(U_o)\gamma u = -\nabla \bullet \chi_o(x) \bullet \nabla u \tag{5.7}$$

Furthermore we assume that the potential is absolutely bounded

$$\left|V(x, U_o)\right| \leq M(\lambda) \tag{5.8}$$

The upper bound M may still depend on the control parameter λ. In this case the solution is stable if the following condition is satisfied

$$M(\lambda) \leq \gamma_o$$

Since the operator

$$L[T] = -\nabla \bullet \chi_o(x) \bullet \nabla \tag{5.9}$$

is uniformly elliptic the lowest eigenvalue γ_O is non-zero. The solution $U_O(x,\lambda)$ depends on the control parameter λ. Since transition from stability to instability can only occur via $\gamma = 0$, the marginal stable solution defines the bifurcation point.

6. General Stability Analysis

We return to the time-dependent equation with the thermal conduction in its general form. Perpendicular and parallel thermal conduction have different dependencies on temperature. Linerarising the equation leads to

$$n\frac{\partial \delta T}{\partial t} - \nabla \bullet \left\{\chi \bullet \nabla \delta T + \delta T \chi_T \bullet \nabla T_0\right\} = h_T(x, T_0)\delta T - \lambda p_T(x, T_0)\delta T \tag{6.1}$$

χ_T is the derivative of the transport matrix with respect to the temperature. The exponential ansatz yields

$$-\gamma\, n\delta T - \nabla \bullet \left\{\chi \bullet \nabla \delta T + \delta T \chi_T \bullet \nabla T_0\right\} = f_T(x, T_0)\delta T \tag{6.2}$$

with

$$f_T(x, T_0) = h_T(x, T_0) - \lambda p_T(x, T_0)$$

The operator L

$$L\delta T = -\nabla \bullet \left\{\chi \bullet \nabla \delta T + \delta T \chi_T \bullet \nabla T_0\right\} \tag{6.3}$$

is not Hermitian. The Hermitian part is

$$L_H = -\nabla \bullet \chi \bullet \nabla - \frac{1}{2}\nabla \bullet (\chi_T \nabla T_0) \tag{6.4}$$

and the anti-Hermitian part

$$L_A = -\nabla T_0 \bullet \chi_T \bullet \nabla - \frac{1}{2}\nabla \bullet (\chi_T \nabla T_0) \tag{6.5}$$

The eigenvalue equation,

$$-\gamma\, n\delta T + L_H \delta T + L_A \delta T = f_T(x, T_0)\delta T \tag{6.6}$$

in general, has complex eigenvalues $\gamma = \gamma_r + i\omega$. The growthrate is given by

$$\gamma_r = \left(\delta T, L_H \delta T\right) - \left(\delta T, f_T(x, T_o)\delta T\right) \tag{6.7}$$

and the frequency by

$$\omega = \left(\delta T, L_A \delta T\right) \tag{6.8}$$

Here we have used the abbreviation

$$(f,g) = \iiint_\Omega f^* g d^3x$$

The equilibrium is thermally stable if

$$\left(\delta T, L_H \delta T\right) - \left(\delta T, f_T(x, T_o)\delta T\right) \geq 0 \quad \forall \delta T \tag{6.9}$$

A sufficient condition for stability is if the effective potential

$$V(x, T_o) = -\frac{1}{2}\nabla\bullet(\chi_T \nabla T_o) - h_T(x, T_o) + \lambda p_T(x, T_o) \tag{6.10}$$

is positive for all $\mathbf{x}$. The equation for the eigenvalue λ is

$$-\nabla\bullet\left\{\chi\bullet\nabla\delta T + \delta T \chi_T \bullet \nabla T_o\right\} = h_T(x, T_o)\delta T - \lambda p_T(x, T_o)\delta T \tag{6.11}$$

It should be noted that the general case has oscillatory solutions in contrast to the special case, where perpendicular and parallel thermal conduction depend on temperature with the same power. In this special case the solutions are either purely growing or purely damped.

Let us return the conduction equation 5.1 and let us consider a parallel heat conduction in the form

$$\chi_{\parallel} = \chi_o T^{5/2} \tag{6.12}$$

Introducing the new variable U yields the equation

$$nT_U(U)\frac{\partial U}{\partial t} - \nabla\bullet\chi_\perp(x, U)\bullet\nabla_\perp U - \nabla\bullet\, \mathbf{b{:}b}\chi_o(x)\bullet\nabla U = G(x, U, \lambda) \tag{6.13}$$

In the parallel term of the heat conduction the dependent variable has been eliminated. The heating term and the radiation term on the right hand side are summarised in the G-function. The eigenvalue equation of the complex growth rate is

$$L_o \delta U + L_I \delta U = nT_U(U_o)\gamma\, \delta U \tag{6.14}$$

The operators in this equation are

$$L_o\delta U = -\nabla\bullet\chi_\perp(x,U_o)\bullet\nabla\delta U - \nabla\bullet b{:}b\chi_o(x)\bullet\nabla\delta U = G_U(x,U_o,\lambda)\delta U \tag{6.15}$$

and

$$L_1\delta U = \nabla\bullet\frac{\partial\chi_\perp}{\partial U}\delta U\bullet\nabla U_o \tag{6.16}$$

This operator is not self-adjoint and it can be written as the sum of a Hermitian and an anti-Hermitian operator $L_1 = L_H + L_A$.

$$L_H\delta U = -\delta U\frac{1}{2}\nabla\bullet\left(\frac{\partial\chi_\perp}{\partial U}\nabla U_o\right) \tag{6.17}$$

and

$$L_A\delta U = -\nabla\bullet\frac{\partial\chi_\perp}{\partial U}\delta U\nabla U_o + \delta U\frac{1}{2}\nabla\bullet\left(\frac{\partial\chi_\perp}{\partial U}\nabla U_o\right) \tag{6.18}$$

The real part of the growth rate is given by

$$\left(\delta U,(L_o + L_H)\delta U\right) = \gamma_r\left(\delta U, nT_U(U_o)\delta U\right) \tag{6.19}$$

And the frequency is determined by the anti-Hermitian part of the operator

$$\left(\delta U, L_A\delta U\right) = \omega\left(\delta U, nT_U(U_o)\delta U\right) \tag{6.20}$$

Since the parallel conductivity is by orders of magnitude larger than the perpendicular conductivity, the frequency ω is much smaller than the growth rate. In practice the anti-Hermitian term of the stability operator can be neglected.

7. Non-linear Boundary Conditions

The boundary conditions in the previous section are T = const. In the stability analysis this leads to $\delta T = 0$. In general, there exists a non-linear relation between the temperature and the temperature gradient on the boundary. These conditions are determined by the physics at the target plates. In the following we analyse this case in the one-dimensional approximation. We consider the nonlinear thermal conduction equation in one dimension

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\chi(x,T)\frac{\partial T}{\partial x} + H(x,T,\lambda) \tag{7.1}$$

in a domain [a,b] and impose the boundary condition

$$\frac{\partial T}{\partial x} = A(x,T)\,|_{x=a} \quad ; \quad \frac{\partial T}{\partial x} = B(x,T)\,|_{x=a} \tag{7.2}$$

where A and B are some non-linear functions of the arguments. As an example we consider the heat conduction parallel to field lines and take into account that the conductivity is proportional to $T^{5/2}$. As described in the previous section this non-linearity can be eliminated by introducing a new variable $U \sim T^{7/2}$. Linearising around a steady state solution yields

$$nT_U(U_o)\frac{\partial \delta U}{\partial t} = \frac{\partial}{\partial x}\chi_o\frac{\partial \delta U}{\partial x} + H_U(x, U_o, \lambda)\delta U \tag{7.3}$$

By linearising the boundary conditions (7.2) we obtain the boundary conditions for the perturbation δU.

$$\frac{\partial \delta U}{\partial x} = \alpha\, \delta U|_{x=a} \quad ; \quad \frac{\partial \delta U}{\partial x} = \beta\, \delta U|_{x=b} \tag{7.4}$$

where α and β are constants. The boundary conditions are of mixed type. Let us define the Hilbert space $L_2[a,b]$ of functions $u(x)\varepsilon C^2[a,b]$ which satisfy the boundary conditions (7.4). The operator

$$L = \frac{\partial}{\partial x}\chi_o\frac{\partial}{\partial x} + H_U(x, U_o) \tag{7.5}$$

is self-adjoint in $L_2[a,b]$. By partial integration and using the boundary conditions (7.4) we can show that the relation holds $(g,Lf) = (f,Lg)$ for all f,g ε $C^2[a,b]$. The brackets (,) denote the integration over the domain [a, b]. The exponential ansatz

$$\delta U = u(x)\, exp\,(-\gamma t) \tag{7.6}$$

leads to the eigenvalue equation

$$-\gamma\, nT_U(U_o)u(x) = Lu(x) \tag{7.7}$$

The eigenvalues are real, the solution is either purely growing or exponentially decaying. Given the solution $u(x)$ the eigenvalue is given by

$$\gamma = -\frac{(u,Lu)}{(u,T_U(U_o)u))} = \frac{\chi_o\alpha u^2(a) - \chi_o\beta u^2(b) + \int_a^b \left\{\chi_o\left(\frac{\partial u}{\partial x}\right)^2 - H_U(x,U_o)u^2\right\}dx}{\int_a^b T_U(U_o)u^2 dx} \tag{7.8}$$

Since L is self-adjoint the eigenvalue is the minimum of the functional $-(u,Lu)$ with the normalisation $(u,nT_U(U_o)) = 1$.
The function H in these equations includes the heating terms and the radiation losses. Defining a parameter in H as a control parameter λ, the bifurcation point can be found from $\gamma(\lambda) = 0$.

8. Cylindrical Plasma

A straight cylindrical plasma is often used as a model for a tokamak plasma. Let us consider a cylindrical plasma in a straight magnetic field oriented along the z-axis. Because of the invariance in z-direction the equation (2.1) is two-dimensional. Furthermore, we assume that the perpendicular thermal conductivity has a temperature dependence, which can be approximated by a power law. In this case the temperature in χ can be eliminated by introducing a new dependent variable $T \to U$, as it has been described in the previous section. However, in the following we write T instead of U. Furthermore, we neglect the spatial dependence of the thermal conduction and write χ = const. Let be $T_O(r,\lambda)$ one solution of

$$-\chi \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T_o}{\partial r} = h(r) - \lambda p(r, T_o) \tag{8.1}$$

This solution is independent of the azimutal angle θ. In principle there can be more than one solution, it depends on the control parameter λ where these multiple solutions occur. If λ is small enough the radiation is negligible and the solution of eq. 8.1 is unique. Above a certain value of radiation three solutions exist for the same λ. The general behaviour is sketched in Fig. 1, numerical examples will be presented in the following chapter.

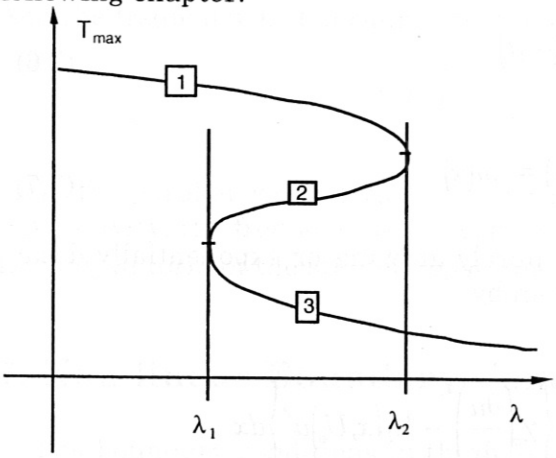

Fig. 1: Maximum temperature vs control parameter λ. In the domain $\lambda_1 < \lambda < \lambda_2$ three solutions exist. In this figure 3 branches can be identified: The high-temperature branch 1 from $\lambda = 0$ to $\lambda = \lambda_2$, the low-temperature branch, starting at $\lambda = \lambda_1$ and the intermediate branch between λ_1 and λ_2.

To investigate the stability of the one-dimensional solution we employ the method outlined in chapter 4. If the control parameter λ is sufficiently small the eigenvalue E of the linearised operator is positive and the upper branch in Fig. 1 is stable. The stability persists until the branch point λ_2 is reached. In the branch point the eigenvalue E is zero. Let be $T_O + \delta T(r,\theta)$ a second solution which depends on the poloidal angle θ. The non-linear equation for δT is

$$-\chi\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \delta T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \delta T}{\partial \theta^2}\right) + \lambda p(r, T_o + \delta T)\delta T = h(r)$$

The linearised equation is

$$-\chi\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \delta T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \delta T}{\partial \theta^2}\right) + \lambda p_T(r, T_o(r,\lambda))\delta T = E\delta T \tag{8.2}$$

which can be reduced to an one-dimensional equation by the ansatz

$$\delta T = u(r)\cos(m\theta) \tag{8.3}$$

$$-\chi\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \chi\frac{m^2}{r^2}u + \lambda p_T(r,T_o(r,\lambda))u = Eu \tag{8.4}$$

The spectrum of this eigenvalue problem has two "quantum number", a radial quantum number and an azimutal number m. The eigenvalues can be computed by minimising the functional

$$E = Min\int_0^a\left\{\chi\frac{1}{r}\left(\frac{\partial u}{\partial r}\right)^2 + \chi\frac{m^2}{r^2}u^2 + \lambda p_T(r,T_o(r,\lambda))u^2\right\}rdr \tag{8.5}$$

The eigenvalues E depend on the integer m and we find that

$$E(m,\lambda) > E(n,\lambda) \quad if\ m > n \tag{8.6}$$

In particular, all eigenvalues with m different from zero are larger than E $(m = 0)$. The branch points (or bifurcation points) of poloidally symmetric solutions are defined by $E(0,\lambda) = 0$. From the result (8.6) we may draw the conclusion:
If symmetric (or one-dimensional) solutions are unique, bifurcation into poloidally asymmetric solutions does not occur.
The two functions $E(0,\lambda)$ and $E(1,\lambda)$ are sketched in the following figure

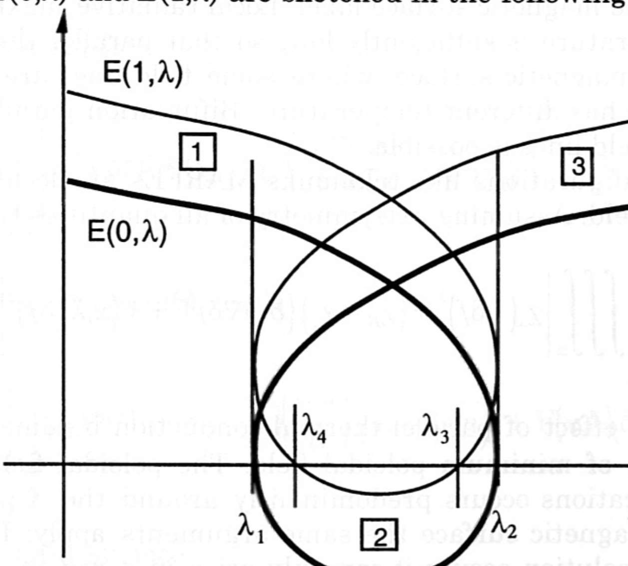

Fig. 2: General shape of the eigenvalues $E(0,\lambda)$ (thick solid curve) and $E(1,\lambda)$. Between the branch points λ_3 and λ_4 the non-axisymmetric solution exists.

This result suggests the following scenario for MARFE formation if the impurity level is increased. At low values of λ the solution is poloidally symmetric and the temperature is high. Above the first bifurcation point λ_1 a poloidally symmetric solution with low temperatures is possible. This corresponds to a radiative layer. In the domain between

λ_3 and λ_4 an asymmetric MARFE is possible and above λ_4 the plasma jumps back into a radiative mantle. This scenario has been described in the paper by Stacey[6].

The eigenvalues E are computed by minimisation of a quadratic functional. The following conclusion holds:

If a test function can be found which makes the functional negative, negative eigen-values exist and the system has at least one bifurcation point.

9. Toroidal plasma

In the following we consider a general toroidal plasma with magnetic surfaces in a subdomain of Ω. We adopt the approximation as described in chapter 4. The eigenvalue E is computed by minimising the functional

$$F(\delta f,\lambda) = \iiint_\Omega \left[\chi_\perp(\nabla\delta f)^2 + (\chi_{II} - \chi_\perp)(\boldsymbol{b}\bullet\nabla\delta f)^2 + V(\boldsymbol{x},\lambda)(\delta f)^2\right] d^3\boldsymbol{x} \tag{9.1}$$

where V is given by eq. 5.6.

Some conclusions can be drawn from this equation. Since the parallel conductivity is much larger than the perpendicular conductivity, the eigen-function will be nearly constant on magnetic surfaces. If there are multiple solutions, the temperature in all cases is a function of the magnetic surface label. Local radiative maxima (MARFEs) only can occur if the temperature is sufficiently low, so that parallel thermal conduction is small. Outside the last magnetic surface, where some field lines are limited by material targets, every field line has different temperature. Bifurcation parallel to field lines and also perpendicular to field lines is possible.

In axisymmetric configurations like tokamaks MARFEs are localised to the X-point region of the poloidal field. Assuming axisymmetry of all quantities the functional F is

$$F(\delta f,\lambda) = \iiint_\Omega \left[\chi_\perp(\nabla\delta f)^2 + (\chi_{II} - \chi_\perp)(\boldsymbol{b}_p\bullet\nabla\delta f)^2 + V(\boldsymbol{x},\lambda)(\delta f)^2\right] d^3\boldsymbol{x} \tag{9.2}$$

This shows that the effect of parallel thermal conduction becomes very weak if δf is localised to the region of minimum poloidal field. The poloidal field is zero at the X-points. Thus the bifurcations occurs predominently around the X-point of the poloidal field. Inside the last magnetic surface the same arguments apply: If a bifurcation to a poloidally asymmetric solution occurs it can only arise in a region of low temperature and small poloidal magnetic field.

X-point MARFEs are axisymmetric in tokamaks. Any toroidal asymmetry of δf leads to larger values of $E(\lambda) = \text{Min } F$. Using the ansatz

$$\delta f = u(r,z)\cos(m\varphi)$$

we get

6 see ref. 3 (W.M. Stacey, Phys. Plasmas 3, 2673 (1996))

$$F(\delta f,\lambda) = \iiint_{\Omega}\left[\chi_{\perp}(\nabla\delta f)^2 + (\chi_{II} - \chi_{\perp})\left((b_p\bullet\nabla\delta f)^2 + (b_t\bullet\nabla\delta f)^2\right) + V(x,\lambda)(\delta f)^2\right] d^3x \qquad (9.3)$$

This leads to the same conclusion as in the cylindrical plasma

$$E(m,\lambda) > E(n,\lambda) \quad if\ m > n \qquad (9.4)$$

Here m,n are the toroidal and poloidal "quantum numbers". Since the toroidal magnetic field in tokamaks is larger than the poloidal field toroidal asymmetry will lead to a strong effect of the parallel thermal conduction and toroidally localised MARFEs will be inhibited. In conclusion, if bifurcated solutions exist, the axisymmetric solutions are the "first" ones to occur. This a similar conclusion as has been drawn by Stacey.

In stellarators the situation is different since the region outside the last magnetic surface exhibits islands and stochastic regions. Inside the last magnetic surface any solution of equation (1) is three-dimensional, the only approximation which can be made is constant temperature on magnetic surfaces. Let be $T_O(\psi,\lambda)$ a solution of (1). To describe a bifurcation we start from eq. (9.1) and utilise the Hamada coordinate system (s,θ,φ) to describe the magnetic field. This yields

$$\boldsymbol{B}\cdot\nabla = \psi'(s)\left(e_t\cdot\nabla + \iota(s)e_p\cdot\nabla\right) = \psi'(s)\left(\frac{\partial}{\partial\varphi} + \iota(s)\frac{\partial}{\partial\theta}\right) \qquad (9.5)$$

The functional (9.3) becomes

$$F(\delta f,\lambda) = \iiint_{\Omega}\left[\chi_{\perp}(\nabla\delta f)^2 + (\chi_{II} - \chi_{\perp})\left(\frac{\psi'}{B}\right)^2\left(\frac{\partial\delta f}{\partial\varphi} + \iota(s)\frac{\partial\delta f}{\partial\theta}\right)^2 + V(x,\lambda)(\delta f)^2\right] d^3x \qquad (9.6)$$

a test function $\delta f \sim u(s)\cos(m\varphi - n\theta)$ yields

$$F(\delta f,\lambda) = \iiint_{\Omega}\left[\chi_{\perp}(\nabla\delta f)^2 + (\chi_{II} - \chi_{\perp})\left(\frac{\psi' u}{B}\right)^2 (m - \iota(s)n)^2 + V(x,\lambda)(\delta f)^2\right] d^3x \qquad (9.7)$$

If there exists a resonant surface

$$M - \iota(s_o)N = 0$$

we choose a test function with $m=M$ and $n=N$ and expand around the resonant surface. This yields

$$F(\delta f,\lambda) = \iiint_{\Omega}\left[\chi_{\perp}(\nabla\delta f)^2 + (\chi_{II} - \chi_{\perp})\left(\frac{\psi' u}{B}\right)^2 (\iota'(s_o)n)^2 (s - s_o)^2 + V(x,\lambda)(\delta f)^2\right] d^3x \qquad (9.8)$$

Thus, the effect of parallel heat conduction can be minimised if the perturbation is localised to the neighbourhood of resonant surfaces. Low shear and low poloidal mode number alleviate the occurence of bifurcated solutions. If resonant MARFEs occur in stellarators they exhibit an $n = N$ asymmetry where N is defined by the condition $\iota = M/N$. In the case of $m = 0$ and $n = 1$ which corresponds to MARFEs in tokamaks we obtain the functional

$$F(\delta f,\lambda) = \iiint_\Omega \left[\chi_\perp (\nabla \delta f)^2 + (\chi_\parallel - \chi_\perp)\left(\frac{\psi' u}{B}\right)^2 (\iota)^2 + V(x,\lambda)(\delta f)^2 \right] d^3x \tag{9.9}$$

This kind of non-resonant MARFEs does not need to be localised to resonant surfaces. Eq. (9.9) is valid in tokamaks and stellarators, small rotational transform in the region of low temperature alleviates the occurence of non-resonant $n = 1$ MARFEs. Small rotational transform is just another formulation of small poloidal field.

Radiation cooling in magnetic islands is another case of bifurcation. We assume that there is no heating inside the island and consider the impurity content as the control parameter. The temperature on the boundary of the island is fixed. Radiation lowers the temperature inside the island below the value at the boundary. In magnetic islands we may assume that δf is constant on magnetic surfaces and thus the effect of parallel heat conduction is minimised. This would approximate the functional by

$$F(\delta f,\lambda) = \iiint_\Omega \left[\chi_\perp (\nabla \delta f)^2 + V(x,\lambda)(\delta f)^2 \right] d^3x \tag{9.10}$$

Since the δf is constant on magnetic surfaces the problem reduces to a one-dimensional eigenvalue problem.

10. Numerical Calculations

10.1. Plasma slab

As an example we consider a plasma slab with a fixed temperature on one side and a fixed temperature gradient on the other side. There is no heating in the slab. This may be considered as a model for the boundary of a plasma with a fixed power input on the inner side and a constant temperature on the outer side. The equation (χ = const) is

$$\frac{\partial^2 T}{\partial r^2} = \lambda p(r,T) \tag{10.1.1}$$

The boundary conditions in the domain [0,1] are $T(1) = 0$ and $T'(0) = a$. Explicitely the radiation function is

$$\lambda p(r,T) = \frac{d^2 n n_z L(T)}{\chi} \tag{10.1.2}$$

where d is the width of the slab. We normalize the temperature to the maximum temperature of the radiation function and introduce $L(T_{max}) = L_o$ as the reference radiation. The electron density is normalized to its maximum value in the slab and the impurity density is normalized in a similar procedure. This yields

$$p(r,T)=\frac{n(r)n_z(r)}{NN_z}\frac{L(T)}{L_0} \tag{10.1.3}$$

and the dimensionless control parameter

$$\lambda=\left(\frac{d^2NN_zL_0}{\chi T_{max}}\right) \tag{10.1.4}$$

Applying these normalisation makes the balance equation (10.1.1) dimensionless. The radiation function p is smaller than 1 and has its maximum at $T = 1$. In order to solve the equation analytically we simplify this equation by

$$\frac{\partial^2 T}{\partial r^2}=\lambda p(T) \tag{10.1.5}$$

i.e. the radiation function depends only on the temperature. An equation of this type has been analysed by Capes et al. as a model for parallel thermal conduction in the scrape-off-layer of tokamaks[7]. Let us consider equation (10.1.1) with the more general radiation function $p(r,T)$ and let $T_O(\lambda,x)$ be a solution of this equation. The linearised eigenvalue equation is

$$-\frac{\partial^2\delta T}{\partial r^2}+\lambda p_T(r,T_o)\delta T=E\delta T \tag{10.1.6}$$

and the boundary conditions are

$$\frac{d\delta T}{dr}=0\,,\; r=0 \quad ; \quad \delta T=0\,,\; r=1 \tag{10.1.7}$$

Furthermore, the normalising condition is

$$\int_0^1(\delta T)^2dr=1 \tag{10.1.8}$$

The eigenvalue is given by

$$E=Min\int_0^1\left\{\left(\frac{\partial\delta T}{\partial r}\right)^2+\lambda p_T(r,T_o)(\delta T)^2\right\}dr \tag{10.1.9}$$

where the test functions δT obey the conditions (10.1.7) and (10.1.8). If a test function exists which makes the functional negative, then there also exists a negative eigenvalue E. As long as the temperature T_O is smaller than the temperature T_{max} p_T is positive and a bifurcation is not possible. This implies that the input power into the radiative slab which is proportional to the derivative $|a|$ must be small enough.

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7 H. Capes, Ph. Ghendrih. A. Samain, Phys. Fluids B4, (1992) 1287

In summary: If the input power into the slab is small ($|a| < 1$) the temperature is smaller than 1 and the solution is unique. Raising the input power so that $|a| > 1$ may lead to multiple solutions, there are at least two stable solutions, a "high temperature" solution and a "low temperature" solution (Fig. 3). The intermediate solution is unstable.

10.2. Numerical Example

We consider the following simple example where the radiation function is modelled as

$$p(r,T)) = (1-r)\, T \, exp\left(-(T - T_{max})^2 / w\right) \tag{10.1.1}$$

The slab ranges from r = 0 to $r = 1$ at the boundary. T_{max} and w are constants. The boundary conditions are $T = 0$ at $r = 1$ and $T'(r = 0) = -a$ at the inner boundary. The derivative of T at the inner side is proportional to the power input into the slab. There is no heating inside the slab. The equation for the temperature is

$$\chi \frac{\partial^2 T}{\partial r^2} = \lambda p(r,T) \tag{10.2.2}$$

The solution for $\lambda = 0$ is $T_0 = a(1\text{-}r)$. The solution is a straight line in radius r. Increasing the parameter λ reduces the temperature, and above a certain value of λ three solutions occur. An example is shown in the next figure.

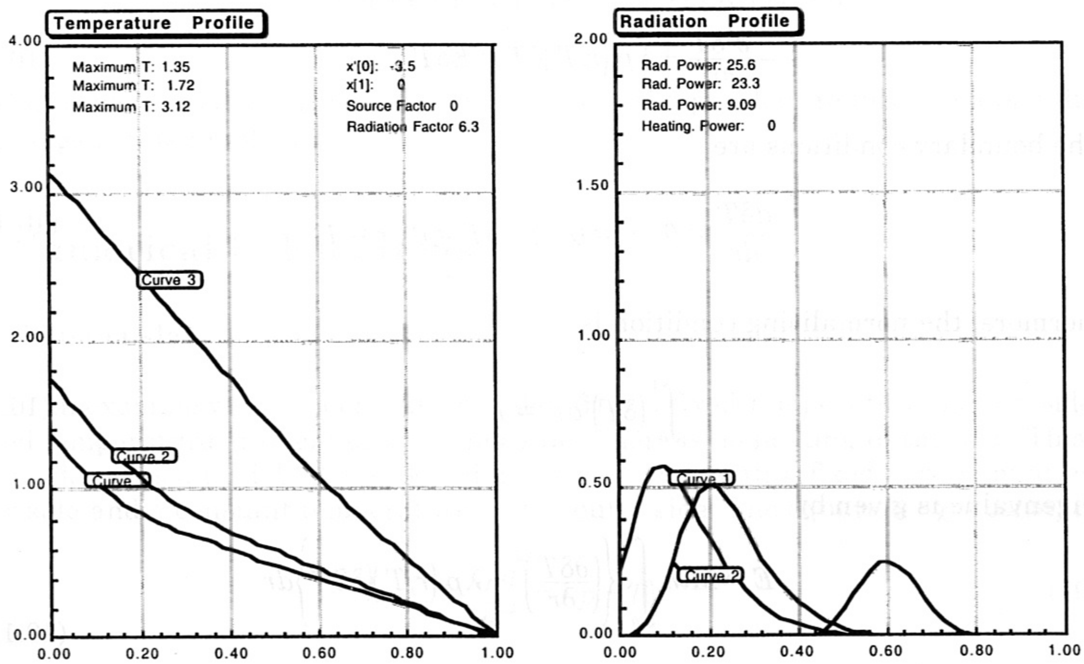

Fig. 3: Temperature profile in a slab with radiation. $T_{max} = 1$, $w = 0.1$, $\chi = 0.5$. control parameter $\lambda = 6.3$. $T'(0) = -3.5$. The right figure shows the radiation profile.

The existence of multiple solutions depends on the input power. If the input power ($|T'(0)|$) is too low only one solution exists. Above a certain value of the input power three solutions are possible. In this case two bifurcation points on the λ-axis exist. In the regime between these points three solutions exist as shown in Fig. 3. The low temperature solution has the highest radiation losses.

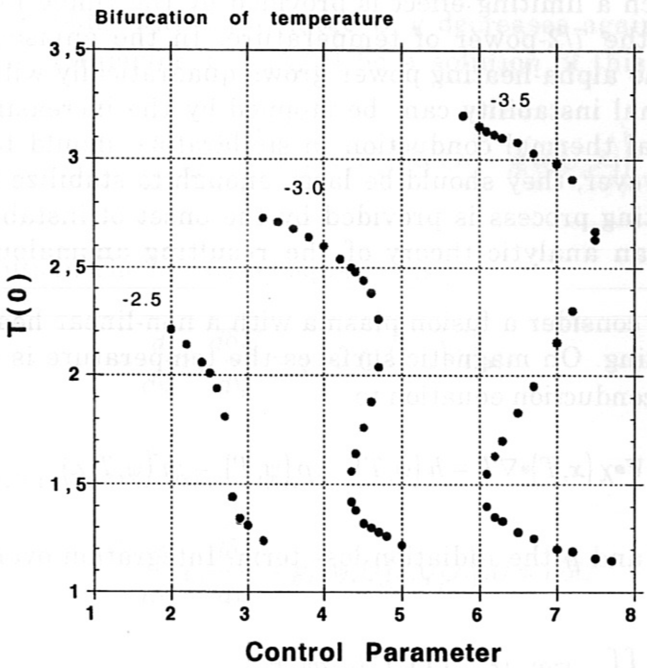

Fig. 4: Maximum temperature T(0) as function of the control parameter λ. The label at the curves is proportional to the input power (-T' ~ power).

This figure shows that at low heating power multiple solutions do not exist and bifurcations do not occur. If the heating power is large enough the system can jump from a low temperature solution to a high temperature solution. A small reduction of the impurity level may cause a transition from the low temperature solution to a high temperature solution. Typically the temperature increases by a factor of 2 to 3.

10.3 Fusion Plasma in Stellarators

In a fusion plasma several non-linear mechanisms may cause bifurcations of the temperature profile. The thermal conductivity depends on temperature, the alpha-heating power grows with temperature and the radiation losses also depends on temperature. The alpha-particle heating gives rise to the thermal instability of the fusion plasma which can only be limited by a strong increase of the thermal conductivity with temperature. Such a limiting effect is provided by the ripple losses in stellarators which increase with the 7/2-power of temperature. In the envisaged reactor regime around $T = 20$ keV the alpha-heating power grows quadratically with the temperature, in this case the thermal instability can be stopped by the increasing ripple losses. In conclusion, neoclassical thermal conduction in stellarators should be small enough to allow for ignition, however, they should be large enough to stabilize the plasma around 20 keV. Another limiting process is provided by the onset of instabilities at the MHD beta limit, however, an analytic theory of the resulting anomalous transport is not available.

In the following we consider a fusion plasma with a non-linear heating term provided by alpha-particle heating. On magnetic surfaces the temperature is constant $T = T(\psi)$. This reduces the heat conduction equation to

$$-\nabla\bullet\chi\left(\boldsymbol{x},T\right)\bullet\nabla T = h\left(\psi,T\right)-\lambda p\left(\psi,T\right) =: h^{*}\left(\psi,T,\lambda\right) \qquad (10.3.1)$$

h is the heating term and p the radiation loss term. Integration over magnetic surfaces yields

$$-\iint \chi_{\perp}\nabla T\bullet d\boldsymbol{f} = \iiint h^{*}\left(\psi,T\right) d^{3}\boldsymbol{x}$$

$$-\iint \chi_{\perp}\nabla\psi\bullet d\boldsymbol{f}\frac{dT}{d\psi} = \iiint h^{*}\left(\psi,T\right)\frac{df}{\left|\nabla\psi\right|}\,d\psi \qquad (10.3.2)$$

Differentiating with respect to ψ we get

$$-\frac{d}{d\psi}\bar{\chi}_{\perp}\frac{dT}{d\psi} = h^{*}\left(\psi,T\right)\iint\frac{df}{\left|\nabla\psi\right|} \qquad (10.3.3)$$

with

$$\bar{\chi}_{\perp} = \iint \chi_{\perp}\nabla\psi\bullet d\boldsymbol{f} \qquad (10.3.4)$$

Let us assume that the perpendicular thermal conductivity is proportional to some power of the temperature

$$\bar{\chi}_{\perp} = \chi_{o}T^{\alpha} \qquad (10.6.5)$$

where χ_0 is a function of ψ . Introducing a new dependent variable $U \sim T^{\alpha+1}$ we obtain

$$-\frac{d}{d\psi}\chi_{o}\frac{dU}{d\psi} = h^{*}\left(\psi,T(U)\right)\iint\frac{df}{\left|\nabla\psi\right|} \qquad (10.3.7)$$

We define

$$g(\psi,U,\lambda)=:h^*(\psi,T(U),\lambda)\iint\frac{df}{|\nabla\psi|} \tag{10.3.8}$$

and consider the equation

$$-\frac{d}{d\psi}\chi_o\frac{dU}{d\psi}=g(\psi,U,\lambda) \tag{10.3.9}$$

for small U g is increasing with U, at large U g decreases again. The parameter λ is considered as control parameter. Let $U_O(\psi)$ be a solution of this equation. Bifurcation occurs if

$$-\frac{d}{d\psi}\chi_o\frac{d\delta U}{d\psi}=\left(g(\psi,U_o+\delta U,\lambda)-g(\psi,U_o,\lambda)\right) \tag{10.3.10}$$

has a non-trivial solution. The linearised equation is

$$-\frac{d}{d\psi}\chi_o\frac{d\delta U}{d\psi}=g_U(\psi,U_o(\lambda,\psi),\lambda)\delta U \tag{10.3.11}$$

Instead of this equation we consider again

$$-\frac{d}{d\psi}\chi_o\frac{d\delta U}{d\psi}-g_U(\psi,U_o(\lambda,\psi),\lambda)\delta U=E\delta U \tag{10.3.12}$$

The eigenvalue E is given by

$$E(\lambda)=Min\int_0^{\psi_O}\left\{\chi_o\left(\frac{d\delta U}{d\psi}\right)^2-g_U(\psi,U_o(\lambda,\psi),\lambda)(\delta U)^2\right\}d\psi \tag{10.3.13}$$

If the control parameter is sufficiently low the eigenvalue is positive and a bifurcation does not exist. Increasing the parameter leads to decrease of the $E(\lambda)$ and there exists a first bifurcation at $\lambda=\lambda_1$. At very large values of the control parameter the solution U_O approaches the maximum of $g(U)$ and g_U decreases to zero. For large values of λ we again expect positive E and a second bifurcation point $\lambda=\lambda_2$.

These arguments are also valid if the control parameter λ is not a factor but a non-linear parameter in g : $g=g(\psi,U,\lambda)$. This only modifies the function $E(\lambda)$.

Numerical computations

The heat conduction equation is one-dimensional, instead of the flux ψ the averaged radius of the magnetic surfaces can be introduced. The heat conduction equation (10.3.1) then reads

$$-\frac{1}{r}\frac{\partial}{\partial r}r\chi n\frac{\partial T}{\partial r}=h(T,r))-\lambda p(T,r)) \tag{10.3.14}$$

where $h(T,r)$ is the heating term and $p(T,r)$ the radiation loss term. In a fusion plasma the heating is provided by the alpha-particle heating

.

$$h(T,r)) = n^2(r)H(T) \ ; \tag{10.3.15}$$

and the radiation losses by bremsstrahlung and line radiation of impurities. The function $H(T)$ is the product of the reaction rate $<\sigma v>_{DT}$ and the energy released by the fusion process. The analytic approximation to the reaction rate given by Bosch and Hale[8]. In the relevant region of fusion plasmas (10 - 20 keV) this function increases roughly with T^2, which is the reason for the thermal instability of the fusion plasma. In addition to the internal heating term also external heating will be included. The bremsstrahlung is

$$p_{brems} \sim n^2\sqrt{T} \tag{10.3.15}$$

$n(r)$ is the density of the plasma. The model of the density is

$$n(r) = \frac{n(0)}{1 + (r/r_0)^\alpha} \ ; \ r_0 = 0.7, \alpha = 4 \tag{10.3.16}$$

To start the ignition, an external heating is superimposed which is modelled by

$$h_{ex}(r) = A \ exp\left(-r^2/w\right) \tag{10.3.17}$$

A and w are constants which describe the magnitude and the width of the heating profile. The thermal conduction coefficient χ is the sum of an anomalous term and a neoclassical term.

$$\chi = \frac{C}{1 - ar^2} + \left(\frac{T}{14}\right)^{3.5}; \ a = 0.7 \tag{10.3.18}$$

Here a simple model of the neoclassical conductivitay is employed, it scales with $T^{3.5}$ and is normalised to $\chi = 1$ at $T = 14$ keV. The anomalous transport has no temperature dependence. The model used for the line radiation is

$$p\,(r,T) = n_e(r)n_z(r)L(T) \tag{10.3.19}$$

with

$$n_e(r)n_z(r) = \frac{1}{\sqrt{a}} exp\left(-\left(r - r_a\right)^2/a\right) \tag{10.3.20}$$

and

$$L(T) = T \, exp\left(-\left(T - T_{max,1}\right)^2/w_1\right) + c \, T \, exp\left(-\left(T - T_{max,2}\right)^2/w_2\right) \tag{10.3.21}$$

The parameter r_a describes the radial localisation of the radiation profile. The radiation function $L(T)$ has two maxima in this approximation. For reason of simplicity we employ the boundary conditions $T = 0$ at the boundary.

As an example we consider the Helias reactor with major radius 22 m and an average plasma radius of 1.75 m. The density profile is fixed. The following figures show the temperature profiles in a fusion plasma with edge localised radiation losses. The bremsstrahlung is subtracted from the alpha-heating power.

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[8] H.S. Bosch, G.M.Hale, *Nuclear Fusion* **32**, No. 4, (1982), 611

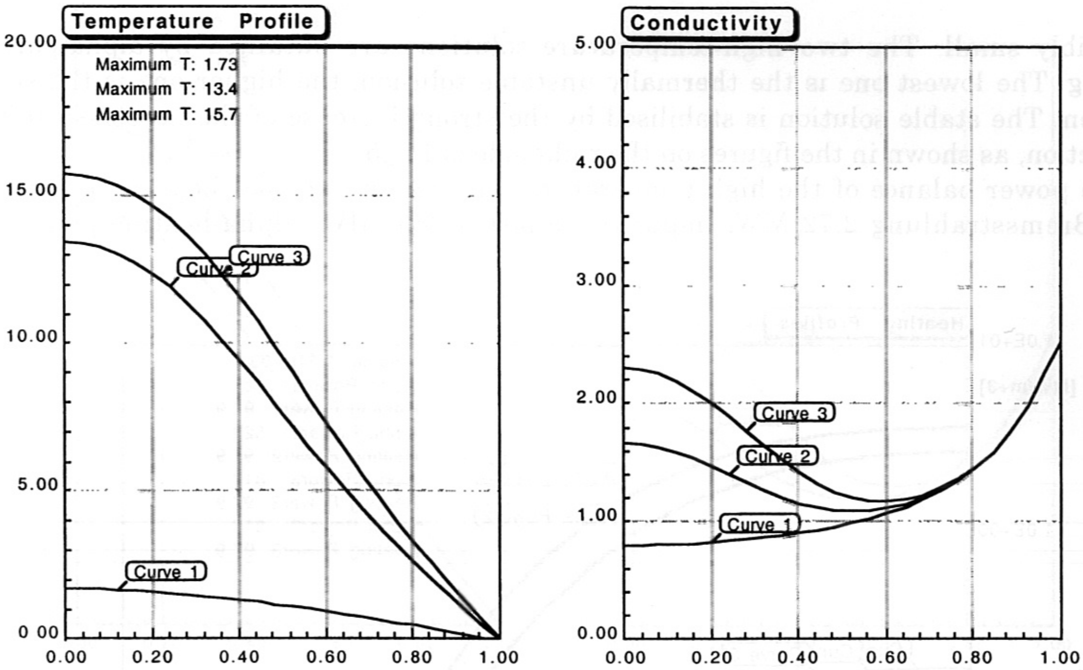

Fig. 5: Temperature profile and thermal conductivity. Density n(0) = 3.0×10^{20} m^{-3} . Heating function: A: 0.2, Width w: 0.4, Radiation function 1: T_{max1}: 0.8, Width1: 0.4, Radiation function 2: Factor: 0.3, T_{max2}: 2, Width2: 0.1. Lambda: 10, Conductivity: Kappa: 0.8, Boundary conditions: T = 0, $n_e(r)nz(r)$: a: 0.02, r_a: 0.8.

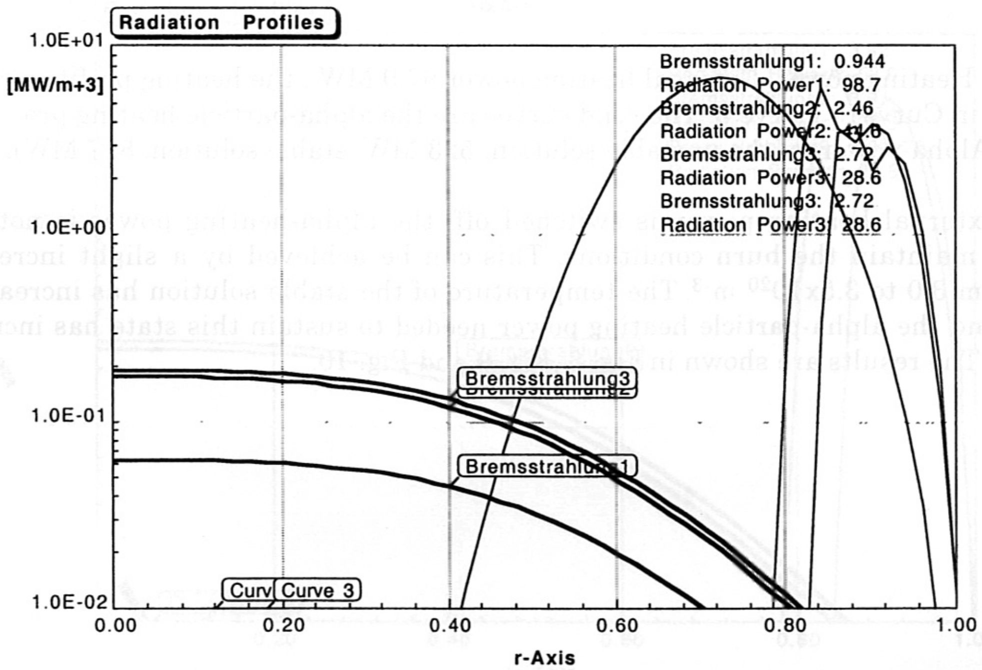

Fig. 6: Radiation losses. Solid curves: Bremsstrahlung: 2.46 MW, 2.72 MW, 2.72 MW. Impurity radiation: 98.7 MW, 44.6 MW, 28.6 MW

Fig. 5 shows three temperature profiles (left figures), which are solutions to the heat conduction equation satisfying the same boundary conditions. The lowest temperature is a solution which is sustained by the external heating only, alpha-particle heating is negligibly small. The two high-temperature solutions are sutained by alpha-particle heating. The lowest one is the thermally unstable solution, the higher one is the stable solution. The stable solution is stabilised by the strong increase of the neoclassical heat conduction, as shown in the figures on the right side of Fig.5.

The power balance of the high temperature solution is: External heating power 97.9 MW, Bremsstrahlung 2.72 MW, impurity radiation 28.6 MW, alpha-heating power 817 MW.

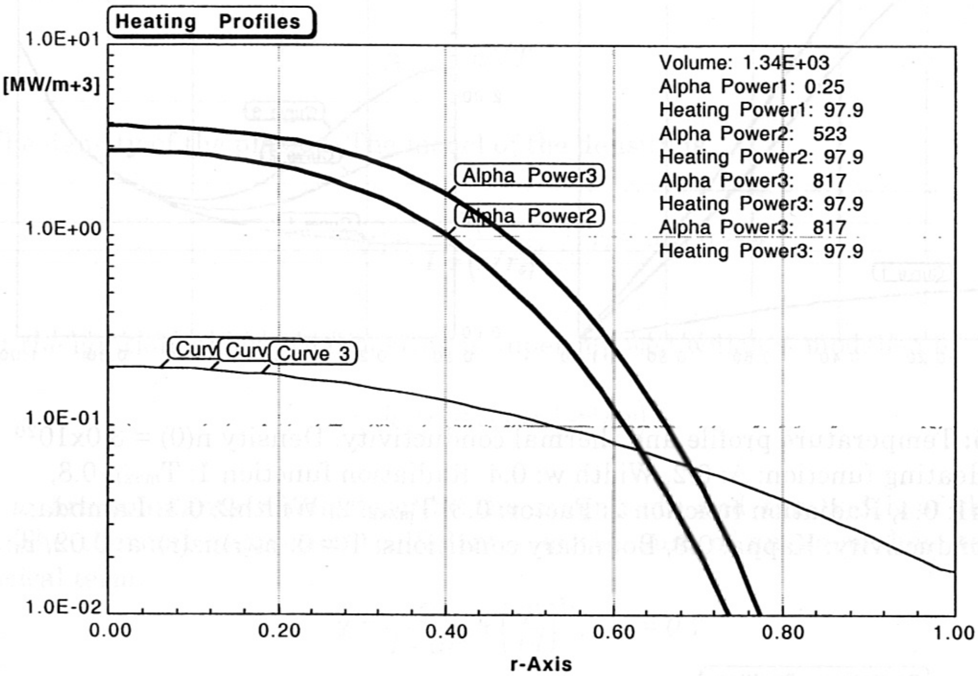

Fig. 7: Heating power. External heating power 97.9 MW , the heating profile is shown in Curve1 - Curve 3. The solid curves are the alpha-particle heating profiles. (Alpha-power: of the unstable solution: 523 MW, stable solution: 817 MW).

If the external heating power is switched off, the alpha-heating power is not large enough to maintain the burn conditions. This can be achieved by a slight increase of density from 3.0 to 3.5×10^{20} m^{-3}. The temperature of the stable solution has increased to 17.5 keV and the alpha-particle heating power needed to sustain this state has increased to 1.4 GW. The results are shown in Fig. 8 ,Fig. 9 and Fig. 10.

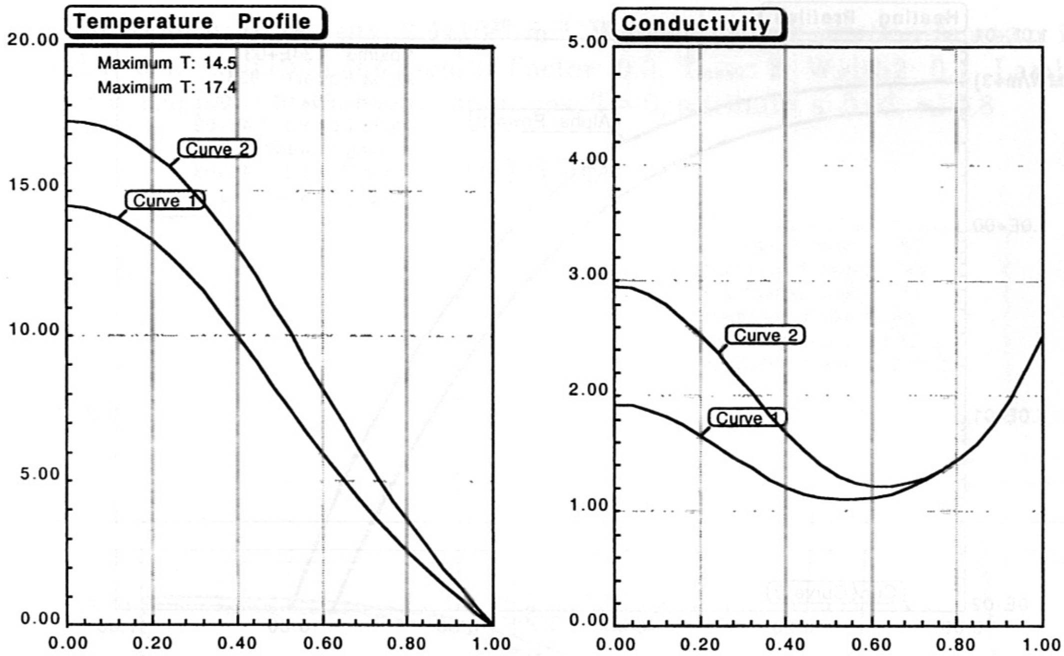

Fig. 8: Temperature profile and thermal conductivity. External heating power zero, density $n(0) = 3.5x10^{20}$ m^{-3}. Heating function: A = 0, Radiation function 1: T_{max1}: 0.8, Width1: 0.4, Radiation function 2: Factor: 0.3, T_{max2}: 2, Width2: 0.1, Lambda: 10, Conductivity: Kappa: 0.8, Boundary conditions: T = 0, $n_e(r)n_z(r)$: a: 0.02, r_a: 0.8.

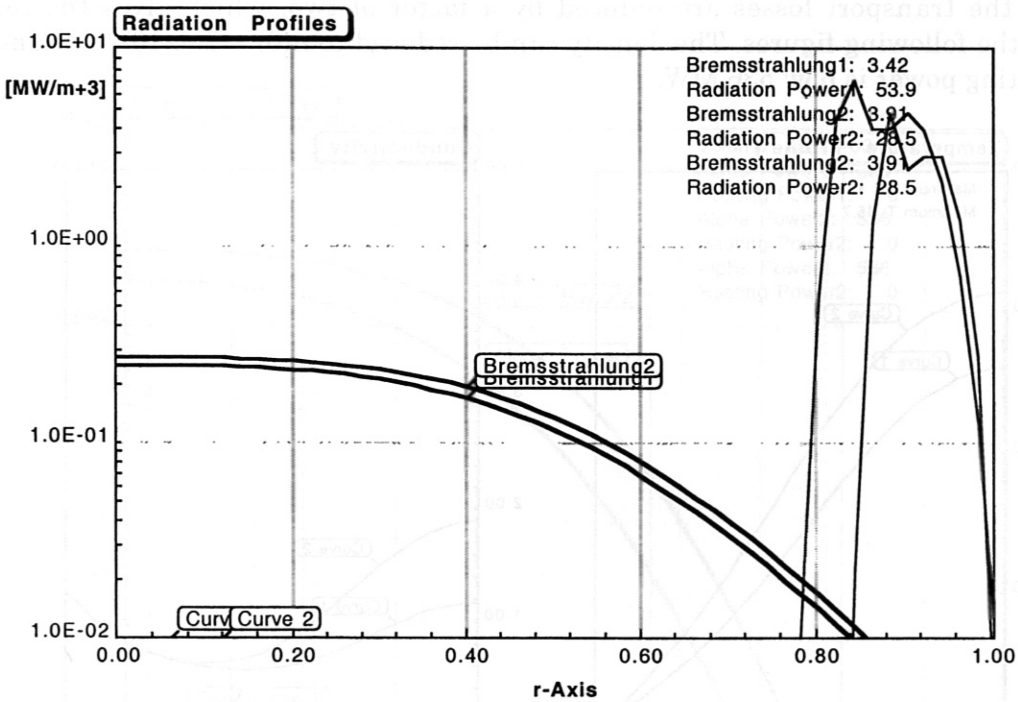

Fig. 9: Radiation profiles

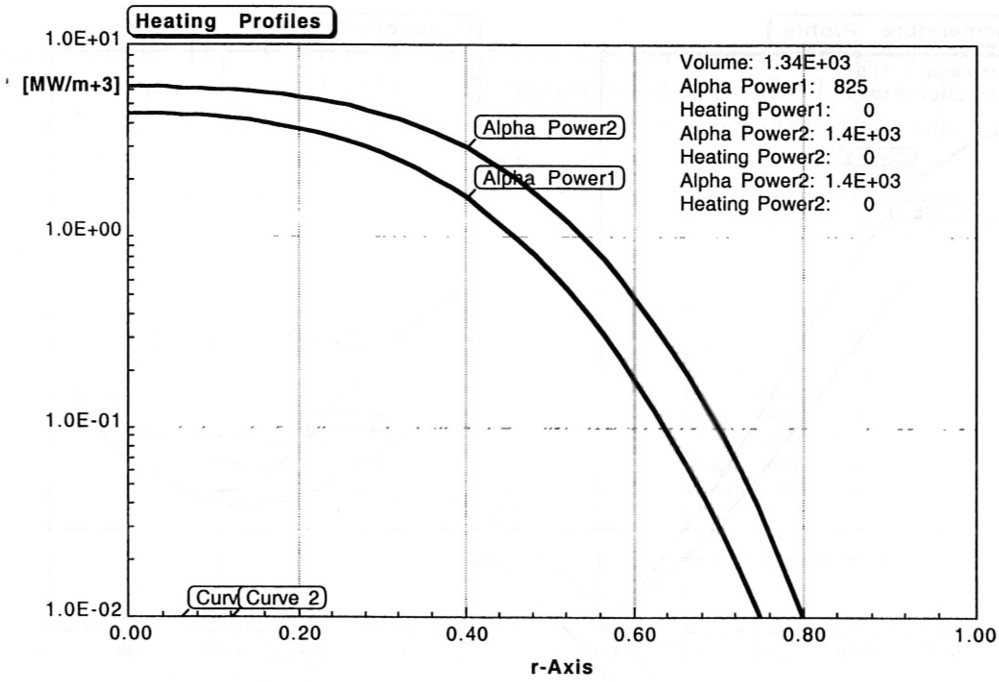

Fig. 10: Heating profiles of alpha particles (external heating zero). Alpha-power: 1.4GW, 825 MW

In this example the alpha-particle heating power is large, the total fusion power would be about 7 GW. Reducing the plasma density would lead to smaller fusion power, however, the transport losses are too large to allow for self-sustaining burn. To demonstrate this case the transport losses are reduced by a factor of two, which gives the results shown in the following figures. The density can be reduced to $n(0) = 2.4 \times 10^{20}\ m^{-3}$ and the alpha-heating power is now 536 MW.

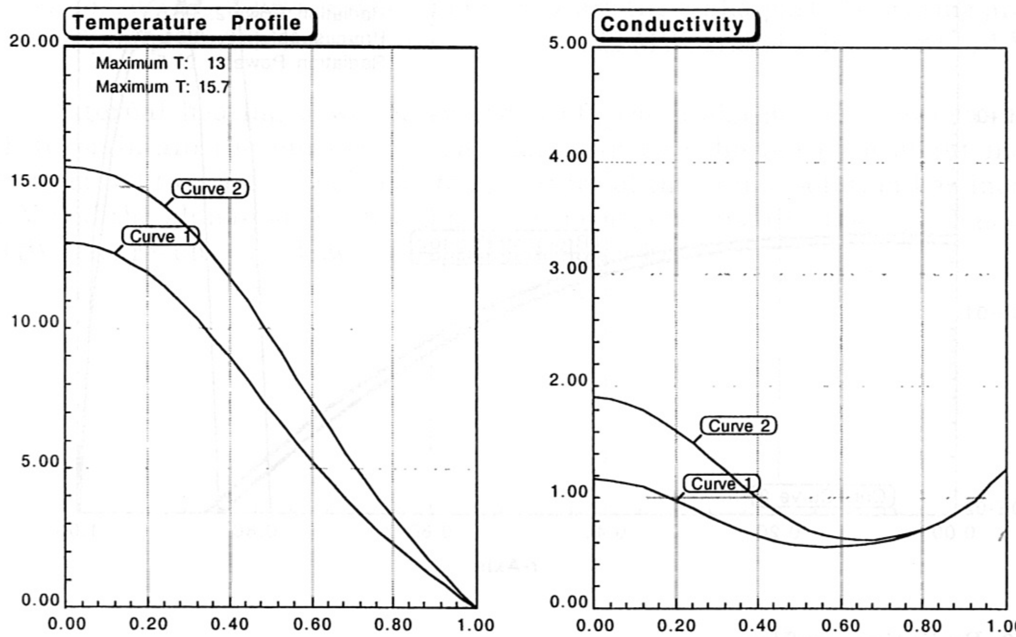

Fig. 11: Temperature profiles and thermal conductivity (Reduced anomalous thermal conduction and density)

Heating function: A: 0, Density: 2.4x10^{20} m^{-3}, Width w: 0.4, Radiation function 1: T_{max1}: 0.8, Width1: 0.4, Radiation function 2: Factor: 0.3, T_{max2}: 2, Width2: 0.1, Lambda: 10. Conductivity: Kappa: 0.4, Boundary conditions: T = 0, $n_e(r)n_z(r)$: a: 0.02, r_a: 0.8

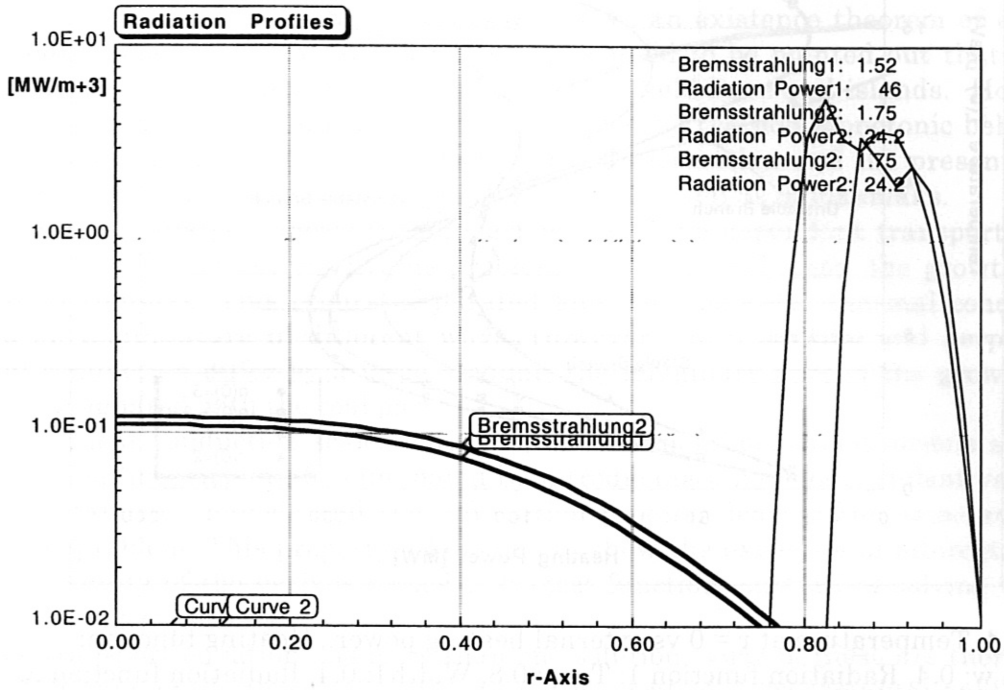

Fig. 12: Bremsstrahlung and line radiation

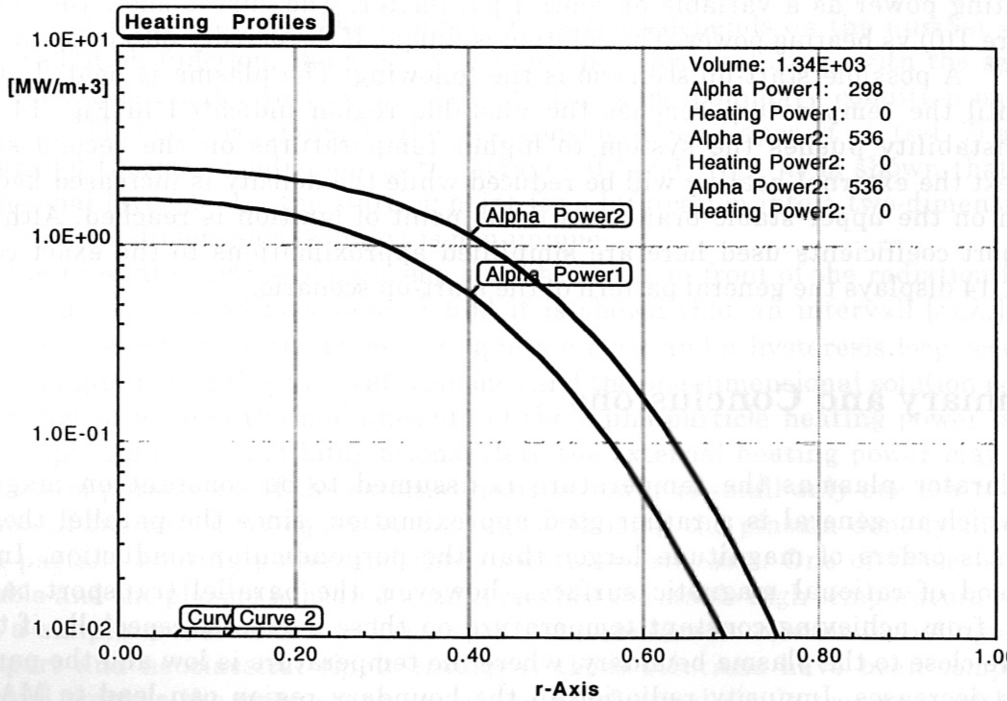

Fig. 13: Heating profiles (Alpha-Power: 298 MW, 536 MW)

The alpha-heating power in this case is 536 MW.

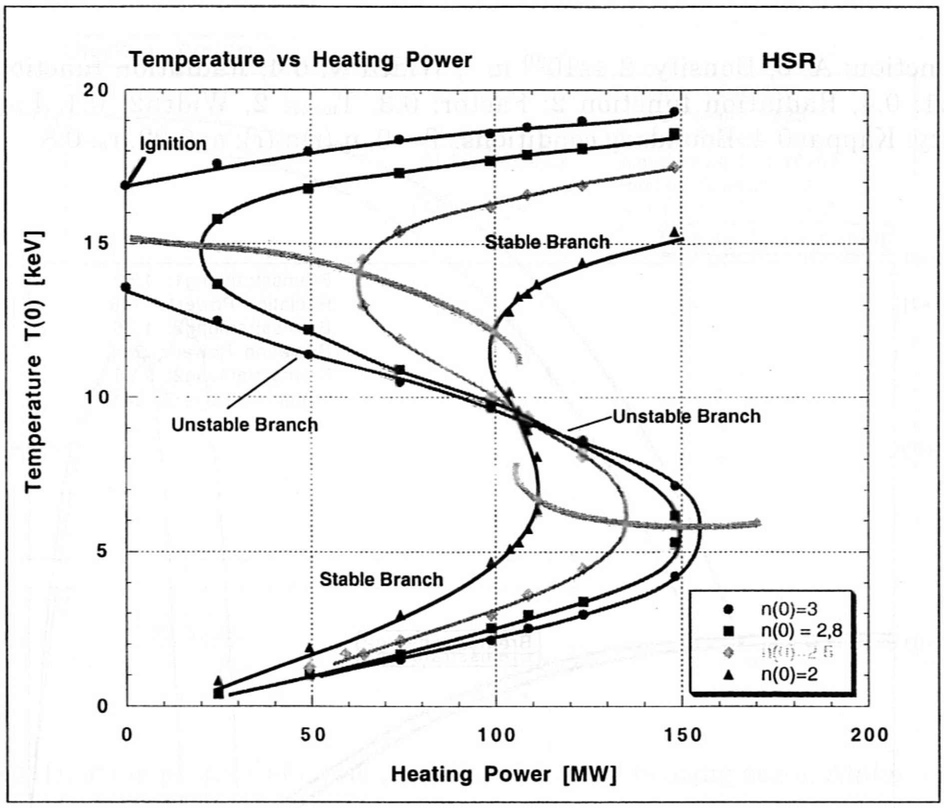

Fig. 14: Temperature at r = 0 vs external heating power., Heating function: Width w: 0.4, Radiation function 1: T_{max1}: 0.8, Width1: 0.4, Radiation function 2: Factor: 0.3, T_{max2}: 2, Width2: 0.1, Lambda: 10. Conductivity: Kappa = 0.6, Boundary conditions: T = 0, $n_e(r)n_z(r)$: a: 0.02, r_a: 0.8. Density n(0) = 2 - $3 x 10^{20}$ m^{-3} .

Fig 14 summarises the result of temperature profiles for various densities and the external heating power as a variable or control parameter. The curves show the central temperature T(0) vs heating power. The solution is unique if the density n(0) is below 2 $x 10^{20}$ m^{-3} . A possible start-up scenario is the following: The plasma is heated at low density until the temperature reaches the unstable region indicated in Fig. 14. The thermal instability pushes the system to higher temperatures on the second stable branch. Next the external heating will be reduced while the density is increased keeping the system on the upper stable branch until the point of ignition is reached. Although the transport coefficients used here are simplified approximations to the exact coefficients, Fig. 14 displays the general pattern of the start-up scenario.

11. Summary and Conclusions

In stellarator plasmas the temperature is assumed to be constant on magnetic surfaces which in general is a rather good approximation, since the parallel thermal conduction is orders of magnitude larger than the perpendicular conduction. In the neigbourhood of rational magnetic surfaces, however, the parallel transport can be prevented from achieving constant temperature on these surfaces, especially if these surfaces are close to the plasma boundary, where the temperature is low and the parallel conduction decreases. Impurity radiation in the boundary region can lead to MARFE formation and the radial extension of this perturbed region also depends on the magnetic shear. Outside the last closed magnetic surface, a large fraction of magnetic field lines is bounded by contact with the wall and target plates.

To analyse the effect of stellarator geometry on thermal transport one must start from the three-dimensional conduction equation, which in general is non-linear due to non-linear heating terms, non-linear radiation loss term and a thermal conductivity, which also depends on the temperature. A further non-linearity is introduced by non-linear boundary conditions.

The existence of steady state solutions is discussed on the basis of the fixed point theorem of Leray and Schauder. This way to prove an existence theorem of non-linear elliptic equations is not new, in this context it should to be pointed out that solutions always exist, even in regions of stochastic magnetic field lines and islands. However, in general, solutions are not unique, which is provided by the non-monotonic behaviour of the radiation function. This is the reason for marfe formation and the present analysis shows that this phenomenon can exist in stellarators as well as in tokamaks.

The stability analysis shows that linearising the time-dependent transport equation leads to a non-Hermitian eigenvalue problem for the growth rate, the growth rate, in general, is complex. This occurs, if parallel and perpendicular thermal conductivities depend on temperature in different ways. However, since parallel and perpendicular thermal conduction differ by a large amount, the imaginary part of the growth rate is negligible compared with the real part.

Parallel heat conduction scales with $T^{5/2}$, and in one-dimensional models along field lines this non-linearity can be eliminated by introducing a new independent variable. In this case stability analysis and the bifurcation problem lead to the same self-adjoint eigenvalue problem. This property allows one to show the existence of bifurcation points and the stability of the various branches by test functions instead of solving the eigenvalue equations explicitely.

If the transport equation has more than one solution, some of these are thermally unstable. Experimentally these solutions can not be verified. Also computational problems occur, if one tries to compute these unstable solutions iteratively or by means of a time-dependent evolution. In one-dimensional geometry, however, the transport equation can be solved be shooting methods, which allow one to compute also these unstable solutions. An example is given in a radiative plasma slab with radiation losses by impurities and a power input from one side. The number of solutions depends on the number of maxima in the radiation function, the present example has three solutions with the same power input on the left hand side of the slab and the same boundary condition on the right hand side. This case is similar to the case considered by Capes et al. (ref. 7) as a model for parallel thermal conduction in the scrape-off layer. Here it is shown that this one-dimensional layer is also the starting point for a bifurcation into a two-dimensional solution, if the one-dimensional solution is non-unique.

In the present example of a plasma slab the factor in front of the radiation function is considered as a control parameter λ and it is shown that an intervall $[\lambda_1,\lambda_2]$ exists, in which three solutions of the transport equation exist and a hysteresis loop occurs. Below a critical input power this intervall vanishes and the one-dimensional solution is unique.

In a fusion plasma the non-linearity of the alpha-particle heating power is a reason for multiple solutions and bifurcations. Here the external heating power may be chosen as a control parameter. If the external heating power is small only one low temperature solution of the transport equation may exist. Raising the plasma density increases the alpha-particle heating power and three solutions can exist. One of these is thermally unstable and the plasma is heated until it reaches the stable high temperature solution.

In a simple model of thermal transport in a stellarator reactor including anomalous transport and neoclassical ripple transport these solutions have been computed. The essential point is the strong increase of the neoclassical ripple transport with temperature, which overcomes the increase of alpha-heating with temperature. In contrast to the radiative layer in the plasma slab, where the decrease of the radiation function with temperature is the cause of the stable high temperature solution, in the fusion plasma the interplay between alpha-heating and non-linear transport stabilises the high temperature solution.

The present analysis did not include convective thermal transport and assumed fixed profiles of density and impurity ions. Changing the temperature profiles certainly will have an effect on density and impurity transport and lead to a slow variation of the control parameter. This effect can be the reason for loss of stability and a sudden transition of the system to another stable branch. This phenomenon demonstrates the limits of the present investigations, however, the basic effects of MARFE formation or the bifurcation of a fusion plasma can be clarified with the heat transport equation and all other parameters fixed.