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Statistical Theory of Subcritically-Excited Strong Turbulence in Inhomogeneous Plasmas (I)

Statistical Theory of Subcritically-Excited Strong Turbulence in Inhomogeneous Plasmas (I)

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A statistical description is developed for a self-sustained subcritical turbulence in inhomogeneous plasmas. Interchange mode in the presence of inhomogeneous magnetic field, which reveals a submarginal strong turbulence, is considered. Nonlinear dispersion relation is extended to a Langevin equation for a dressed test mode, in which nonlinear interactions are kept as renormalized drag and random self noise. Based upon the assumption that the random noise has a faster time scale, the solutions are obtained for the fluctuation level, decorrelation rate, auto- and cross-correlation functions and spectrum. They are expressed as nonlinear functions of non-equilibrium parameters like gradient. Extended fluctuation-dissipation theorem (Einstein relation) is described as statistical relations. Then the Langevin equation is reformulated into a Fokker-Planck equation of the probability distribution function. The steady state probability function is solved. Imposing the constraint of the self-noise, the power-law distribution with respect to the fluctuation amplitude is obtained in the tail of the distribution function.

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1. Introduction

The strong turbulence in high temperature plasmas has attracted attentions motivated by the fusion research or by the observation on extra-terrestrial phenomena like solar flare. The study of such turbulence is an important issue of physics: the problems of statistical physics for systems far from thermal equilibrium remain quite open, in contrast to those near thermal equilibrium in which the principles that govern fluctuations (i.e., equipartition of energy, Einstein relation, fluctuation-dissipation (FD) theorem, etc.) are established [1].

The strong plasma turbulence is governed by the $V \nabla V$ Lagrange-nonlinearity, which is universal to various fluid systems. In the past, turbulence spectrum has been discussed by choosing the distribution of the Gibbsian thermal equilibrium, where attention is focused on the conservation property of the $V \nabla V$ term [2,3]. Several efforts have been done for the case of plasma turbulence, taking examples of drift waves [4-7]. For a given 'temperature' of the system, that is introduced as a coefficient in the exponent of Boltzmann-Gibbs distribution, a Fourier spectrum of fluctuations has been derived. Power-laws in the wave-number space have been derived for the energy spectrum. In this framework, however, nonequilibrium property, which causes the deviation from Gibbsian thermal equilibrium, is not incorporated. The determination of the 'temperature' in turbulent plasmas remains open question.

For the study of strongly unstable plasmas, nonlinear theory has been developed and applied to experiments (e.g., [8-11]), based upon the methodology of clump and two-point correlation functions [12]. Another approach, to solve the renormalized nonlinear dispersion relations by the method of dressed-test mode, was proposed to obtain the mean characteristics of turbulence and turbulent-driven transport [13]. This method has allowed to analyze the strong subcritical turbulence. These analyses have given (at least partly successfully) understanding of the anomalous transport and improved confinement in toroidal plasmas. In particular, the roles of pressure gradient and the gradient of radial electric field have been considerably

clarified [8-11, 14, 15] (see [16] for a review). Nevertheless, it is still far from clarifying the statistical physics of strong turbulence in plasmas.

An attempt was made to study the statistical dynamics of homogeneous turbulence in neutral fluid [17]. Starting from Liouville's equation in the presence of external random forcing, and introducing the concepts of turbulence viscosity (Heisenberg type viscosity) as a drag and the turbulent diffusion in a functional space, a Fokker-Planck equation is formulated for the truncated nonlinearities. The power law distribution of k-Fourier space was examined with an assumption of the decomposition of nonlinear interactions. Efforts have been made including studies to resolve a divergence which appears in the energy equation [18,19].

Within the statistical theory of turbulence, one of the most successful methods would be the one based on the DIA (direct interaction approximation) with RCM (random coupling model) [20]. Extension to the two-scale DIA for nonequilibrium situations is also made [21,22].

One way to study the statistical nature is to utilize a Langevin equation. An effort has been made to derive the Langevin equation of weak turbulence [23] by the method of characteristic functional combined with the time asymptotic method of Bogoliubov and Mitropolski [24]. Based on the DIA method, a Langevin equation for turbulence is to be derived also for the strong turbulence. To this end various kinds of closure models to formulate the Langevin equation have been proposed for plasmas and the validity of models has been examined through a comparison study with numerical simulations [25-28]. The usefulness of this approach has been investigated for the case of linearly-unstable drift wave waves.

In this article, a statistical description and analyses are developed for a self-sustained strong turbulence which is caused by the subcritical excitation of interchange mode. Langevin equation for the dressed test mode, in which the drag term and the random noise term due to nonlinear interactions are kept, is formulated. Imposing ansatz (1) of large degrees of freedom in the turbulence and (2) of the random self-noise, the level and decorrelation rate of turbulence and the auto- and cross-correlation

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functions are solved. Thus the extended FD-theorem (Einstein relation) is explicitly described by the nonequilibrium-parameter that characterizes the gradient of the system [29]. The method is applied to the current-diffusive interchange mode (CDIM) turbulence [13]. Then the Langevin equation is reformulated into a Fokker-Planck equation of the probability distribution function. The steady state probability function is solved. Imposing the constraint of the self-noise, the power-law distribution with respect to the fluctuation amplitude is obtained in the tail of the distribution function.

The method proposed here provides one way to explore the physics of far-nonequilibrium systems with strong instabilities.

2. Basic Equation and Statistical Approach

2.1 Plasma Model and Basic Equation

We consider a slab plasma which is inhomogeneous in the x-direction and is immersed in an inhomogeneous and sheared magnetic field. The magnetic field is given as $B = B_0(0, sx, 1)$ with $B_0(x) = (1 + \Omega' x + \cdots)B_0$. In this system, the current-diffusive interchange mode (CDIM) can be subcritically excited [30]. In the dynamics of its mode, the electron viscosity prohibits the free-motion of electrons along the magnetic field line, and makes the system be nonlinearly unstable. This dissipative instability system is of our interest. To describe the system, the reduced set of equations for the electrostatic potential ϕ, current J and pressure p is employed[31]. The electron inertia effect is kept, because this effect is amplified by the nonlinear shielding effect of the turbulence [32,33]. The classical resistivity is neglected for the simplicity of the argument. Three field equations are: equation of motion:

$$\frac{\partial}{\partial t}\Delta_\perp \phi + [\phi, \Delta_\perp \phi] = \nabla_\parallel J + (\Omega' \times \hat{b}) \cdot \nabla p + \mu_{vc} \Delta_\perp^2 \phi, \tag{1}$$

Ohm's law:

$$\frac{\partial}{\partial t}\Psi = -\nabla_\parallel \phi - \xi^{-1}(\partial J/\partial t + [\phi, J] - \mu_{ec}\Delta_\perp J) \tag{2}$$

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and energy balance equation:

$$\frac{\partial}{\partial t} p + [\phi, p] = \chi_c \Delta_\perp p. \tag{3}$$

The bracket $[f,g]$ denotes the Poisson bracket,

$$[f, g] = (\nabla f \times \nabla g) \cdot b,$$

$(b = B_0/B_0)$, $\Delta_\perp = \nabla_\perp^2$, Ω' is the average curvature of the magnetic field, Ψ is the vector potential, and $1/\xi$ denotes the finite electron inertia, $1/\xi = (\delta/a)^2$, δ being the collisionless skin depth. The transport coefficients μ_{vc}, μ_{ec}, χ_c are the ion viscosity, the electron viscosity and the thermal diffusivity, respectively. The suffix c denotes the contributions from thermal fluctuations (collisional diffusion). Length, time, static potential and pressure are normalized to the global plasma size a, the Alfven transit time $\tau_{Ap} = a/v_{Ap}$, $a v_{Ap} B_0$ and $B_0^2 R/2a\mu_0$, respectively (a and R are minor and major radii of torus, $v_{Ap} = B_0(2\mu_0 m_i n_i)^{-1/2} aRq^{-1}$, m_i is the ion mass, and n_i is the ion density; see [30] for details). In this paper, the electrostatic approximation is employed, i.e., the inductive electric field in the Ohm's law and the nonlinear terms of the form $[\Psi, \cdots]$ are neglected. (The influence of the inductive electric field could be important, if one studies the case that the plasma pressure is high. See [34] for the extension to the plasma with high pressure.) The CDIM has a quasi-2 dimensional nature, $|\nabla_\parallel^2| << |\nabla_\perp^2|$; nevertheless, the small but finite $\nabla_\parallel$ is essential.

The dynamics of micro fluctuations are studied in the presence of the global inhomogeneity of the plasma pressure. Quantities that are averaged over the (y, z)-plane are denoted by the suffix 0, as p_0 and ϕ_0.

$$\phi = \phi_0 + \tilde{\phi}, \ J = J_0 + \tilde{J} \text{ and } p = p_0 + \tilde{p} \tag{4}$$

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The pressure and electrostatic potential could be inhomogeneous (i.e., inhomogeneous in the $\hat{x}$-direction) in the global scale. Parameters ∇p_0 and $\nabla_\perp^2 \phi_0$ represent the inhomogeneity of the system.

The scale separation is introduced, in this article, between the dynamics of the micro fluctuations and macroscopic structures:

$$\left| p_0^{-1} \frac{\partial}{\partial t} p_0 \right| << \left| \tilde{p}^{-1} \frac{\partial}{\partial t} \tilde{p} \right|, \tag{5-1}$$

$$\left| p_0^{-1} \nabla p_0 \right| << \left| \tilde{p}^{-1} \nabla \tilde{p} \right|. \tag{5-2}$$

With the help of this assumption of space-time scale separation, the dynamical equations of fluctuation fields are given as

$$\frac{\partial}{\partial t} \Delta_\perp \tilde{\phi} + [\phi_0, \Delta_\perp \tilde{\phi}] + [\tilde{\phi}, \Delta_\perp \tilde{\phi}] = \nabla_\parallel \tilde{J} + (\Omega' \times \hat{b}) \cdot \nabla \tilde{p} + \mu_{vc} \Delta_\perp^2 \tilde{\phi} \tag{6}$$

$$\frac{\partial}{\partial t} \tilde{J} + [\phi_0, \tilde{J}] + [\tilde{\phi}, \tilde{J}] = -\xi \nabla_\parallel \tilde{\phi} + \mu_{ec} \Delta_\perp \tilde{J} \tag{7}$$

$$\frac{\partial}{\partial t} \tilde{p} + [\tilde{\phi}, p_0] + [\phi_0, \tilde{p}] + [\tilde{\phi}, \tilde{p}] = \chi_c \Delta_\perp \tilde{p} \tag{8}$$

In this set of equation, parameters ∇p_0 and $\nabla_\perp^2 \phi_0$ are assumed constant, so that the dynamics is decoupled from those of global structures. Equations for global structure are given

$$\frac{\partial}{\partial t} \Delta_\perp \phi_0 - \mu_{vc} \Delta_\perp^2 \phi_0 = -\overline{[\tilde{\phi}, \Delta_\perp \tilde{\phi}]} \tag{9}$$

$$\frac{\partial}{\partial t} p_0 - \chi_c \Delta_\perp p_0 = -\overline{[\tilde{\phi}, \tilde{p}]} \tag{10}$$

where the symbol $\overline{[X, Y]}$ denotes the average of the quantity $[X, Y]$ over the magnetic surface, i.e., the (y, z)-plane. Global magnetic field is kept constant, and the equation for J_0 is not considered for simplicity.

In the following analysis, the statistical properties of fluctuations are discussed based on Eqs.(6)-(8). The symbol ~ which denotes fluctuating quantity is suppressed.

2.2 Derivation of Langevin Equation

Langevin equation is derived by use of the renormalized eddy viscosity type model and a random coupling model (RCM). The basic set of equations, Eqs.(6), (7) and (8), has the form

$$\partial f / \partial t + \mathcal{L}^{(0)} f = \mathcal{N}(f), \tag{11}$$

where $\mathcal{L}^{(0)}$ denotes the linear operator

$$\mathcal{L}^{(0)} = \begin{pmatrix} -\mu_c \nabla_\perp^2 & -\nabla_\perp^{-2}\nabla_\parallel & -\nabla_\perp^{-2}\Omega' \dfrac{\partial}{\partial y} \\[2mm] \xi \nabla_\parallel & -\mu_{ec}\nabla_\perp^2 & 0 \\[2mm] -\dfrac{dp_0}{dx}\dfrac{\partial}{\partial y} & 0 & -\chi_c \nabla_\perp^2 \end{pmatrix} \tag{12}$$

$$f = \begin{pmatrix} \phi \\ J \\ p \end{pmatrix}. \tag{13}$$

and $\mathcal{N}(f)$ stands for the nonlinear terms.

We consider a test mode f_k and treat the nonlinear terms as follows: (Fourier transformation is used, and the suffix k implies the k-th Fourier component.) The system which has numerous degrees of freedom and has many positive Lyapunov exponents is considered. A test mode (dressed-test mode) in the presence of turbulence is subject to the random force of other turbulent modes and is shielded by the turbulence drag effect. A part of the Lagrangean nonlinearity on f_k is considered to

cause the drag and is renormalized, by using the direct interaction approximation (DIA), to the eddy-viscosity type nonlinear transfer rate $(\gamma_{j,\,k})$ in the k-space. The other part of Lagrangean nonlinearity is regarded as a random noise, which has a faster decorrelation time than $(\gamma_{j,\,k})$ according to RCM.

By choosing these procedures, a Langevin equation for a dressed test mode is formulated. If one symbolically writes

$$\partial f_{i,\,k}/\partial t + \sum_{j=1}^{3} L^{(0)}_{ij,\,k} f_{j,\,k} = \mathcal{N}_{i,\,k} \equiv \sum_{p,q} M_{i,\,kpq} f_{1,\,p} f_{i,q} \tag{14}$$

where $\mathcal{N}_{i,\,k}$ is the nonlinear interactions that generates $f_{i,\,k}$, then the Langevin equation was derived as [20,26,35]

$$\partial f/\partial t + Lf = \tilde{S} \tag{15}$$

with

$$L_{ij,\,k} = L^{(0)}_{ij,\,k} + \gamma_{i,\,k} \delta_{ij} \tag{16}$$

(δ_{ij} is the Kronecker's delta) and

$$\tilde{S}_k = \begin{pmatrix} \tilde{S}_{1,k} \\ \tilde{S}_{2,k} \\ \tilde{S}_{3,k} \end{pmatrix}. \tag{17}$$

Notation here follows the convention in [35]. In this article, suffix $i,\,j = 1,2,3$ denotes the i-th or j-th field, and $k,\,p,\,q$ describes the wave number. Suffix $k,\,p,\,q$ is often omitted unless confusion is caused. When a projection operator $\mathcal{P}_k$ is introduced to divide the nonlinear interactions into the drag and others, Eqs.(15)-(17) may be written as

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$$\partial f_k/\partial t + \mathcal{L}_0 f_k - \mathcal{P}_k N_k(f) = (1_k - \mathcal{P}_k)N_k(f) \qquad (15')$$

$(1_k$ is a unit operator) with

$$\mathcal{P}_k N_k(f) = - \begin{pmatrix} \gamma_{1,k} f_{1,k} \\ \gamma_{2,k} f_{2,k} \\ \gamma_{3,k} f_{3,k} \end{pmatrix} \qquad (16')$$

and

$$\tilde{S} = (1_k - \mathcal{P}_k)N_k(f) \qquad (17')$$

From the definition of $\mathcal{P}_k$, $\mathcal{P}_k(1_k - \mathcal{P}_k)\mathcal{N}_k = 0$ and $\mathcal{P}_k\mathcal{P}_k = \mathcal{P}_k$. (See also appendix.)

The operator $\mathcal{L}$, $\mathcal{L}_k f_k = \mathcal{L}_{0,k} f_k - \mathcal{P}_k \mathcal{N}_k(f)$, is the renormalized operator, which includes the effective transfer rates expressed as

$$\gamma_{i,k} = - \sum_\Delta M_{i,kpq} M^*_{i,qkp} \theta^*_{qkp} \left| \tilde{f}^2_{1,p} \right| . \qquad (18)$$

The random self-noise is assumed to have a much shorter correlation time, and is approximated to be given by the Gaussian white noise term $w(t)$ as

$$\tilde{S}_{i,k} = w(t) \sum_\Delta M_{i,kpq} \sqrt{\theta_{kpq}} \zeta_{1,p} \zeta_{i,q} . \qquad (19)$$

In these expressions, summation Δ indicates the constraint $k + p + q = 0$. The explicit form of the nonlinear interaction matrix is given as, e.g.,

$$M_{1,kpq} = ((p \times q) \cdot b)(p_\perp^2 - q_\perp^2)k_\perp^{-2} \qquad (20\text{-}1)$$

or

$$M_{(2, 3), kpq} = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{b}, \qquad\qquad (20\text{-}2)$$

and the propagator satisfies the relation

$$(\partial/\partial t + \mathcal{L}(k) + c.p.)\theta_{kpq} = 1. \qquad\qquad (21)$$

where c.p. indicates the counter part, i.e., $\mathcal{L}(p) + \mathcal{L}(q)$ [35]. The term $\zeta_{j, p}$ in a random noise represents the j-th field of q-component in the nonlinear term $\mathcal{N}_k$; therefore their correlation functions satisfy the average relations, which we call an Ansatz of equivalence in correlation in the following, as

$$\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle \qquad\qquad (22)$$

and

$$\langle \zeta_{i, p} \zeta_{j, q} \rangle \propto \delta_{pq} \qquad\qquad (23)$$

where the bracket $\langle \ \rangle$ indicates the statistical average.

It has been pointed out that the "realizability condition" (i.e., the second order moments $\langle f_i f_i \rangle$ should be positive definite) might not be satisfied in this form of Langevin equation [25]. A similar but alternative form of Langevin equation has been proposed to satisfy the realizability for drift waves. As is explained in [28,35], this problem rises from the wave propagation nature of the mode. In the present analysis, the corresponding linear mode (interchange mode) is purely growing mode, and θ_{kpq} is considered to be real. Owing to this basis, the form of Eq.(15) is employed after [35].

In introducing the nonlinear transfer rate, it is assumed that 1) the contribution from the shorter wave-length components to the test mode plays the dominant role and that 2) the time rates (γ_j), $(\gamma_v, \gamma_e, \gamma_p)$, which are nonlinear transfer rates in the k-space,

are expressed in terms of the diffusion coefficients (μ_v, μ_e, χ) as $\gamma_v = \mu_v k_\perp^2$, $\gamma_j = \mu_e k_\perp^2$, and $\gamma_p = \chi k_\perp^2$. Explicit derivation of the renormalized coefficients (μ_v, μ_e, χ) for the case of CDIM turbulence is given in [32]. The formalism in a previous work of the dressed-test mode is deduced from Eq.(15) by neglecting the noise term $\tilde{S}$.

This system has a strong instability source due to the presence of inhomogeneities, and the product of pressure gradient and magnetic field inhomogeneity,

$$G_0 = \Omega' p_0', \qquad (24)$$

denotes the driving parameter. From the assumption of the time-scale separation, this parameter is fixed in time in this article.

This system also describes the submarginal turbulence, where the enhanced dissipation due to the nonlinear transfer rate can be an origin of the dissipative instability [13, 15, 30]. Solving Eq.(15), we shall determine in this article the typical nonlinear decorrelation rate of the turbulence and transfer rate in the fluctuation spectrum, simultaneously.

3. Langevin Equation and Statistical Characteristics

3.1 Solution of Langevin Equation

In order to solve the Langevin equation (15), an ansatz of large number of freedom in random modes, N, is introduced. The renormalized term γ_j in $\mathcal{L}$ is the statistical sum of contributions from N components, so that its relative variation in time becomes $O(N^{-1/2})$ times smaller in comparison with that of f_k. Therefore, in solving the time evolution of fluctuating f_k, $\mathcal{L}$ is approximated to be constant in time in the limit of $N \to \infty$. Namely, another time scale separation ansatz is introduced in addition to Eq.(5). With the help of this ansatz, Eq.(15) is solved by use of the Laplace transformation. The general solution is formally given as

$$f(t) = \sum_m exp\,(-\lambda_m t) f(0) + \int_0^t exp\,[-\mathcal{L}(t-\tau)]\tilde{S}(\tau)\,d\tau \qquad (25)$$

where $-\lambda_m$ ($m = 1,2,3$ and $\lambda_1 < \lambda_2 < \lambda_3$) represents the eigenvalue of the non-normal matrix $\mathcal{L}$, which gives the homogeneous solution of Eq.(15) if $\mathcal{L}$ is constant. The eigenvalue is determined by:

$$Det(\lambda I + \mathcal{L}) = 0 \qquad (26)$$

and I is a unit tensor. The eigenvalue $-\lambda_1$ corresponds to a branch of CDIM which drives the strong turbulence, and others $(-\lambda_2, -\lambda_3)$ denote highly-stable branches. Since Eq.(26) is a third order equation of λ, one can also write as

$$Det(\lambda I + \mathcal{L}) = (\lambda + \lambda_1)(\lambda + \lambda_2)(\lambda + \lambda_3). \qquad (27)$$

This equation (27) provides a relation between λ_1, γ_j and global parameters such as G_0.

The matrix $exp\,[-\mathcal{L}(t-\tau)]$ in Eq.(25) is explicitly expressed in terms of λ_1, λ_2 and λ_3 as

$$(exp[-\mathcal{L}(t-\tau)])_{ij} = A_{ij}exp(-\lambda_1(t-\tau)) + A_{ij}^{(2)}exp(-\lambda_2(t-\tau)) + A_{ij}^{(3)}exp(-\lambda_3(t-\tau)) \qquad (28)$$

where the elements of matrix A are given as

$$A = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}
\begin{pmatrix}
(\gamma_e - \lambda_1)(\gamma_p - \lambda_1) & \dfrac{-ik_{||}(\gamma_p - \lambda_1)}{k_\perp^2} & \dfrac{-ik_y\Omega'(\gamma_e - \lambda_1)}{k_\perp^2} \\[2ex]
-ik_{||}\xi(\gamma_p - \lambda_1) & \dfrac{\xi k_{||}^2}{k_\perp^2}\dfrac{(\lambda_1 - \gamma_p)}{(\lambda_1 - \gamma_e)} & \dfrac{-\xi k_{||}k_y\Omega'}{k_\perp^2} \\[2ex]
ip_0'k_y(\gamma_e - \lambda_1) & \dfrac{k_{||}k_y p_0'}{k_\perp^2} & \dfrac{G_0 k_y^2}{k_\perp^2}\dfrac{(\gamma_e - \lambda_1)}{(\lambda_1 - \gamma_p)}
\end{pmatrix}$$

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Elements $A_{ij}^{(2,3)}$ are also obtained in a similar way,

$$
A^{(2)} = \frac{1}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} \times
\begin{pmatrix}
(\gamma_e - \lambda_2)(\gamma_p - \lambda_2) & \dfrac{-ik_\parallel(\gamma_p - \lambda_2)}{k_\perp^2} & \dfrac{-ik_y\Omega'(\gamma_e - \lambda_2)}{k_\perp^2} \\[2mm]
-ik_\parallel\xi(\gamma_p - \lambda_2) & \dfrac{\xi k_\parallel^2}{k_\perp^2}\dfrac{(\lambda_2 - \gamma_p)}{(\lambda_2 - \gamma_e)} & \dfrac{-\xi k_\parallel k_y\Omega'}{k_\perp^2} \\[2mm]
ip_0'k_y(\gamma_e - \lambda_2) & \dfrac{k_\parallel k_y p_0'}{k_\perp^2} & \dfrac{G_0 k_y^2}{k_\perp^2}\dfrac{(\gamma_e - \lambda_2)}{(\lambda_2 - \gamma_p)}
\end{pmatrix}
\tag{30}
$$

and

$$
A^{(3)} = \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \times
\begin{pmatrix}
(\gamma_e - \lambda_3)(\gamma_p - \lambda_3) & \dfrac{-ik_\parallel(\gamma_p - \lambda_3)}{k_\perp^2} & \dfrac{-ik_y\Omega'(\gamma_e - \lambda_3)}{k_\perp^2} \\[2mm]
-ik_\parallel\xi(\gamma_p - \lambda_3) & \dfrac{\xi k_\parallel^2}{k_\perp^2}\dfrac{(\lambda_3 - \gamma_p)}{(\lambda_3 - \gamma_e)} & \dfrac{-\xi k_\parallel k_y\Omega'}{k_\perp^2} \\[2mm]
ip_0'k_y(\gamma_e - \lambda_3) & \dfrac{k_\parallel k_y p_0'}{k_\perp^2} & \dfrac{G_0 k_y^2}{k_\perp^2}\dfrac{(\gamma_e - \lambda_3)}{(\lambda_3 - \gamma_p)}
\end{pmatrix}
\tag{31}
$$

The matrices A, $A^{(2)}$ and $A^{(3)}$ have properties as: $AA = A$, $A^{(2)}A^{(2)} = A^{(2)}$,

$A^{(3)}A^{(3)} = A^{(3)}$, $AA^{(2)} = A^{(2)}A = 0$, $AA^{(3)} = A^{(3)}A = 0$ and

$A^{(2)}A^{(3)} = A^{(3)}A^{(2)} = 0$.

Substitution of $\tilde{S}_{i,k}$ (Eq.(19)) into Eq.(25) gives us the solution of $f_i(t)$.

3.2 Statistical Quantities

3.2.1 Statistical Average

According to the standard procedure of statistical physics, the square average is calculated. We are interested in the long-time-averaged values. For this purpose, the initial condition in Eq.(25) is unimportant and is neglected. We write

$$f_i(t)f_j(t) = \int_0^t d\tau \int_0^t d\tau' \{exp[-\mathcal{L}(t-\tau)]\tilde{S}(\tau)\}_i \{exp[-\mathcal{L}(t-\tau')]\tilde{S}(\tau')\}_j^* \quad (32)$$

where the relation Eq.(19) for $\tilde{S}$ and Eq.(28) should be substituted. Since $\tilde{S}$ is assumed to be Gaussian white noise, the relation

$$\langle \tilde{S}_i(\tau)\tilde{S}_j(\tau')\rangle \propto \delta(\tau-\tau') \quad (33)$$

holds, and we have

$$\langle f_i f_j\rangle = \frac{1}{2\lambda_1}A\sigma A^{T*} + \frac{1}{2\lambda_2}A^{(2)}\sigma A^{(2)T*} + \frac{1}{2\lambda_3}A^{(3)}\sigma A^{(3)T*}$$
$$+ \frac{1}{\lambda_1+\lambda_2}(A\sigma A^{(2)T*} + A^{(2)}\sigma A^{T*}) + \frac{1}{\lambda_1+\lambda_3}(A\sigma A^{(3)T*} + A^{(3)}\sigma A^{T*})$$
$$+ \frac{1}{\lambda_2+\lambda_3}(A^{(2)}\sigma A^{(3)T*} + A^{(3)}\sigma A^{(2)T*}) \quad (34)$$

where elements of tensor σ is expressed as

$$\sigma_{ij} = \langle \tilde{S}_i \tilde{S}_j \rangle \quad (35)$$

The term for the least stable branch is taken as the dominant contribution. In other words, in obtaining the long time averaged value, rapidly-decaying terms (the second and third terms in the right hand side of Eq.(28)) are neglected without loosing the generality. We have the relation

$$\langle f_i f_j\rangle = \frac{1}{2\lambda_1}A\sigma A^{T*}$$
$$= \frac{1}{2\lambda_1}\sum_{i',j'} A_{ii'}\langle S_{i'}S_{j'}\rangle A_{jj'}^*. \quad (36)$$

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Equation (36) relates the fluctuation level, the correlation rate and random noise.

It is noted that this system includes the interference among the field components and among the branches. If one explicitly writes the first component of the branch of λ_1

$$f_1 \propto \tilde{S}_1 + \frac{ik_{\parallel}}{(\gamma_e - \lambda_1)} \tilde{S}_2 + \frac{ik_y \Omega'}{(\gamma_p - \lambda_1)} \tilde{S}_3 \tag{37}$$

one sees that not only the random force in the vorticity equation, $\tilde{S}_1$, but also those in the current and pressure equations, $\tilde{S}_2$ and $\tilde{S}_3$, excite the fluctuation in the velocity. The partition among the branches of λ_1, λ_2 and λ_3 is governed by the ratio $1/\lambda_1$, $1/\lambda_2$, and $1/\lambda_3$ as is shown in Eq.(33). The rate of interference is also calculated for the branches of λ_2 and λ_3 as

$$f_1^{(2)} \propto \tilde{S}_1 + \frac{ik_{\parallel}}{(\gamma_e - \lambda_2)} \tilde{S}_2 + \frac{ik_y \Omega'}{(\gamma_p - \lambda_2)} \tilde{S}_3 , \tag{38-1}$$

$$f_1^{(3)} \propto \tilde{S}_1 + \frac{ik_{\parallel}}{(\gamma_e - \lambda_3)} \tilde{S}_2 + \frac{ik_y \Omega'}{(\gamma_p - \lambda_3)} \tilde{S}_3 , \tag{38-2}$$

respectively. The superscripts (2) and (3) denote the second and third branches, respectively.

3.2.2 Ansatz of Equivalence of Correlation and Extended Fluctuation Dissipation Theorem

Equation (36) describes the relation between the decorrelation rate and the random noise. This relation, Eq.(36), is rewritten as

$$\lambda_1 = \frac{1}{2\langle f_1 f_1 \rangle} \sum_{i', j'} A_{1i'} \langle S_{i'} S_{j'} \rangle A_{1j'}^* \tag{39}$$

and shows that the decorrelation rate is expressed in terms of the random noise. Therefore, this relation is considered as an extension of the fluctuation-dissipation theorem of the second kind.

The relation with FD theorem may be seen by considering the limit of thermal equilibrium. The plasma velocity of the system of consideration is given by the $E \times B$ velocity, and is in proportion to f_1. The auto-correlation $\langle f_{1,k} f_{1,k} \rangle$ is proportional to the kinetic energy. When the system is in a thermal equilibrium and the equi-partition law holds, the average kinetic energy is given by $k_B T$, i.e., $\rho \langle f_{1,k} f_{1,k} \rangle = k_B T$ holds (ρ being the proportionality constant). Substituting this relation into Eq.(39) one sees

$$\lambda_1 = \frac{\rho}{2 k_B T} \sum_{i', j'} A_{1i'} \langle S_{i'} S_{j'} \rangle A_{1j'}^* \qquad \text{(thermal fluctuation)} \qquad (40)$$

Equation (36") describes the relation between the decorrelation rate, noise and temperature. In this article, the equipartition law is not assumed, but the properties in the nonequilibrium turbulence are investigated.

Terms $\sigma_{ij} = \langle \tilde{S}_i \tilde{S}_j \rangle$ could be given in terms of correlation functions $\langle \zeta_i \zeta_j \rangle$. The average $\langle \zeta_{i,p} \zeta_{j,q} \zeta_{i,p'} \zeta_{j',q'} \rangle$ $(p + q = k, \; p' + q' = k)$ is decomposed as

$$\langle \zeta_{i,p} \zeta_{j,q} \zeta_{i,p'} \zeta_{j',q'} \rangle = \langle \zeta_{i,p}^2 \rangle \langle \zeta_{j,q} \zeta_{j',q} \rangle \delta_{pp'} \delta_{qq'} + \langle \zeta_{i,p} \zeta_{j',p} \rangle \langle \zeta_{i,q} \zeta_{j,q} \rangle \delta_{pq'} \delta_{qp'}$$

$$(41)$$

based on the random coupling approximation. This yields relations

$$\langle S_1 S_1 \rangle = 2 \sum_q M_{1,kpq}^2 \theta_{kpq} \langle \zeta_{1,p}^2 \rangle \langle \zeta_{1,q}^2 \rangle, \qquad (42\text{-}1)$$

$$\langle S_1 S_2 \rangle = 2 \sum_q M_{1,kpq} M_{2,kpq} \theta_{kpq} \langle \zeta_{1,p}^2 \rangle \langle \zeta_{1,q} \zeta_{2,q} \rangle, \qquad (42\text{-}2)$$

$$\langle S_1 S_3 \rangle = 2 \sum_q M_{1,kpq} M_{3,kpq} \theta_{kpq} \langle \zeta_{1,p}^2 \rangle \langle \zeta_{1,q} \zeta_{3,q} \rangle, \qquad (42\text{-}3)$$

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$$\langle S_2 S_2 \rangle = \sum_q M_{2,\,kpq} M_{2,\,kpq} \theta_{kpq} \left\{ \langle \zeta_{1,p}^2 \rangle \langle \zeta_{2,q}^2 \rangle + \langle \zeta_{1,p} \zeta_{2,p} \rangle \langle \zeta_{1,q} \zeta_{2,q} \rangle \right\}, \quad (42\text{-}4)$$

$$\langle S_3 S_3 \rangle = \sum_q M_{3,\,kpq}^2 \theta_{kpq} \left\{ \langle \zeta_{1,p}^2 \rangle \langle \zeta_{3,q}^2 \rangle + \langle \zeta_{1,p} \zeta_{3,p} \rangle \langle \zeta_{1,q} \zeta_{3,q} \rangle \right\} \qquad (42\text{-}5)$$

and

$$\langle S_2 S_3 \rangle = \sum_q M_{2,\,kpq} M_{3,\,kpq} \theta_{kpq} \left\{ \langle \zeta_{1,p}^2 \rangle \langle \zeta_{2,q} \zeta_{3,q} \rangle + \langle \zeta_{1,p} \zeta_{3,p} \rangle \langle \zeta_{1,q} \zeta_{2,q} \rangle \right\}. \quad (42\text{-}6)$$

We here employ the Ansatz of equivalence of correlation Eq.(22), i. e., $\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle$. Substituting the expression Eq.(42) into Eq.(36), a closed set of equations for λ_1 and $\langle f_i f_j \rangle$ is obtained. The correlation $\langle \tilde{S}_i \tilde{S}_j \rangle$ in $(1/2\lambda_1) A \sigma A^{T*}$ of Eq.(36) is expressed in terms of $\langle f_i f_j \rangle$. Equations(26), (28) and (36) constitute the closed set of equations that determines the nonlinear decorrelation rate, λ_m, the nonlinear transfer rate γ_j and correlation functions $\langle f_i f_j \rangle$, simultaneously. This process corresponds to the extended Fluctuation dissipation theorem.

We would like to note the partition to the highly-damped branches of λ_2 and λ_3. In principle these branches are also excited to the finite levels. This analysis is also done by forming the closed set of equations, i.e., Eq.(22) $\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle$ [Ansatz of equivalence of correlation], Eq.(42) [decomposition] and Eq.(34). Expression becomes much more tedious, but the logic of the analysis is unaltered. The following analysis is performed by neglecting the excitation of the highly-damped branches in order to keep a transparency of the argument.

3.2.3 Auto-Correlation Functions

For the analytic insight of the problem, we employ an approximation that the cross-correlations $\langle f_i f_j \rangle$ $(i \neq j)$ are smaller than the auto-correlations $\langle f_i f_i \rangle$. By this ordering the equation Eq.(36) is decomposed as

$$
\begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} = \frac{1}{2\lambda_1} \begin{pmatrix} |A_{11}^2| & A_{12}A_{21}^* & A_{13}A_{31}^* \\ A_{21}A_{12}^* & |A_{22}^2| & A_{23}A_{32}^* \\ A_{31}A_{13}^* & A_{32}A_{23}^* & |A_{33}^2| \end{pmatrix} \sum_q \begin{pmatrix} 2M_{1,kpq}M_{1,kpq}\theta_{kpq}I_1(p)I_1(q) \\ M_{2,kpq}M_{2,kpq}\theta_{kpq}I_1(p)I_2(q) \\ M_{3,kpq}M_{3,kpq}\theta_{kpq}I_1(p)I_3(q) \end{pmatrix}
$$

$$(43)$$

and

$$
\begin{pmatrix} J_{12}(k) \\ J_{13}(k) \\ J_{23}(k) \end{pmatrix} = \frac{1}{2\lambda_1} \begin{pmatrix} A_{11}A_{12}^* & A_{12}A_{22}^* & A_{13}A_{32}^* \\ A_{11}A_{13}^* & A_{12}A_{23}^* & A_{13}A_{33}^* \\ A_{21}A_{13}^* & A_{22}A_{23}^* & A_{23}A_{33}^* \end{pmatrix} \sum_q \begin{pmatrix} 2M_{1,kpq}M_{1,kpq}\theta_{kpq}I_1(p)I_1(q) \\ M_{2,kpq}M_{2,kpq}\theta_{kpq}I_1(p)I_2(q) \\ M_{3,kpq}M_{3,kpq}\theta_{kpq}I_1(p)I_3(q) \end{pmatrix}
$$

$$
+ \frac{1}{2\lambda_1} \begin{pmatrix} A_{11}A_{22}^* & A_{11}A_{32}^* & A_{12}A_{32}^* \\ A_{11}A_{23}^* & A_{11}A_{33}^* & A_{12}A_{33}^* \\ A_{21}A_{23}^* & A_{21}A_{33}^* & A_{22}A_{33}^* \end{pmatrix} \sum_q \begin{pmatrix} 2M_{1,kpq}M_{2,kpq}\theta_{kpq}I_1(p)J_{12}(q) \\ 2M_{1,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{13}(q) \\ M_{2,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{23}(q) \end{pmatrix}
$$

$$(44)$$

where simplified notations

$$ I_j(k) = \langle f_{j,k} f_{j,k} \rangle \tag{45-1} $$

$$ J_{ij}(k) = \langle f_{i,k} f_{j,k} \rangle \qquad (i \neq j) \tag{45-2} $$

are used. In Eq.(44), quadratic terms of cross-correlations $J_{ij}J_{i'j'}$ are neglected. The equations for auto-correlation functions are closed within themselves, and the cross-correlation functions are given by the auto-correlation and cross-correlation functions. The latter would be an extended FD theorem of the first kind, that is the transport quantity (cross-correlation) is expressed in terms of the auto-correlation functions.

The relation Eq.(36) is rewritten in terms of the auto-correlation functions, in a symbolic form, as

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$$\begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} = \frac{1}{\lambda_1} \sum_P I_1(k) \, \mathcal{R} \begin{pmatrix} I_1(q) \\ I_2(q) \\ I_3(q) \end{pmatrix} \tag{46}$$

where the matrix $\mathcal{R}$ is expressed in terms of the matrix A and other coupling coefficients as

$$\mathcal{R}_{ij} = \left(1 + \delta_{j1}\right) M_{j, \, kpq}^2 \theta_{kpq} A_{ij} A_{ji}^* \tag{47}$$

Besides the trivial solution, i.e., $\langle f_i f_i \rangle = 0$, the consistent solution is obtained from Eq.(46). For an analytic estimate, let us assume that the spectrum average is a smooth function, and the ratio between two moments is given by a coefficient C_0 as

$$\frac{\left(1 + \delta_{j1}\right) \sum_q M_{j, \, kpq} M_{j, \, kpq} \theta_{kpq} I_1(p) I_j(q)}{\sum_q M_{1, \, kpq} M_{1, \, kpq}^* \theta_{qkp}^* I_1(p) I_j(k)} \simeq C_0 . \tag{48}$$

The denominator of Eq.(48) is rewritten using the relation Eq.(18) for γ_ν, and we have

$$\left(1 + \delta_{1j}\right) \sum_q M_{j, \, kpq} M_{j, \, kpq} \theta_{kpq} I_1(p) I_j(q) = C_0 \gamma_\nu I_j(k). \tag{49}$$

With this analytic estimate, Eq.(46) is simplified as

$$\begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} = \frac{1}{\lambda_1} \, \hat{\mathcal{R}} \begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} \tag{50}$$

with an approximated matrix $\hat{\mathcal{R}}$, i.e.,

$$\hat{\mathcal{R}}_{mn} \simeq C_0 \gamma_\nu A_{mn} A_{nm}^* . \tag{51}$$

In an explicit form, one has

$$
\begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} = \frac{C_0 \gamma_\nu}{2\lambda_1} \begin{pmatrix} |A_{11}^2| & A_{12}A_{21}^* & A_{13}A_{31}^* \\ A_{21}A_{12}^* & |A_{22}^2| & A_{23}A_{32}^* \\ A_{31}A_{13}^* & A_{32}A_{23}^* & |A_{33}^2| \end{pmatrix} \begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix}
\tag{52}
$$

Equation (50) (or (52)) is a simplified form of the extended fluctuation-dissipation theorem. The nontrivial solution exists, if λ_1 satisfies the following secular equation

$$
det\left[\lambda_1 I - \hat{\mathcal{R}}\right] = 0 .
\tag{53}
$$

Equation (53) together with Eq.(26) determines the decorrelation rate λ_1 and transfer rates γ_j and closes the analysis. For instance, once the eigenvalue λ_1 is expressed in terms of γ_j by use of Eq.(53), the substitution of the relation $\lambda_1[\gamma_j]$ into Eq.(26) provides the solution of γ_j together with λ_1. If γ_j is obtained, the average fluctuation amplitude $\langle f_i(t)f_i(t) \rangle$ is obtained by solving the integral equation (the first component of Eq.(18))

$$
\gamma_\nu = -\sum_\Delta M_{1,\,kpq} M_{1,\,qkp}^* \theta_{qkp}^* \left| \tilde{f}_{1,p}^2 \right|
\tag{54}
$$

as has been performed in [36].

The correlation function $\langle f_i(t)f_i(t + \tau) \rangle$ is given, using the decorrelation rate λ_1, as

$$
\langle f_i(t)f_i(t + \tau) \rangle = \langle f_i f_i \rangle exp\left(-\lambda_1 |\tau|\right) .
\tag{55}
$$

Namely, the power spectrum $I(\omega)$ is given by the Lorentzian distribution

$$
I(\omega) \propto \lambda_1 \left(\omega^2 + \lambda_1^2\right)^{-1} .
\tag{56}
$$

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The test-particle diffusion coefficient is also directly calculated.

3.2.4 Cross-Correlation and Fluxes

Cross-correlation function is estimated from Eq.(44). From the same spirit of introducing the spectrum average C_0 for the auto-correlation function, let us introduce a parameter C_{cross} for the averages of cross-correlations as,

$$\sum_q 2M_{1,kpq}M_{2,kpq}\theta_{kpq}I_1(p)J_{12}(q) \simeq C_{cross}\gamma_v J_{12}(k) , \qquad (57\text{-}1)$$

$$\sum_q 2M_{1,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{13}(q) \simeq C_{cross}\gamma_v J_{13}(k) , \qquad (57\text{-}2)$$

$$\sum_q M_{2,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{23}(q) \simeq C_{cross}\gamma_v J_{23}(k) . \qquad (57\text{-}3)$$

Namely, the approximate relation holds for the second term in the right hand side of Eq.(44) as

$$\frac{1}{2\lambda_1}\begin{pmatrix} A_{11}A_{22}^* & A_{11}A_{32}^* & A_{12}A_{32}^* \\ A_{11}A_{23}^* & A_{11}A_{33}^* & A_{12}A_{33}^* \\ A_{21}A_{23}^* & A_{21}A_{33}^* & A_{22}A_{33}^* \end{pmatrix}\sum_q\begin{pmatrix} 2M_{1,kpq}M_{2,kpq}\theta_{kpq}I_1(p)J_{12}(q) \\ 2M_{1,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{13}(q) \\ M_{2,kpq}M_{3,kpq}\theta_{kpq}I_1(p)J_{23}(q) \end{pmatrix}$$

$$\simeq \frac{C_{cross}\gamma_v}{2\lambda_1}\,M_c\begin{pmatrix} J_{12}(k) \\ J_{13}(k) \\ J_{23}(k) \end{pmatrix} \qquad (58\text{-}1)$$

with

$$M_c \equiv \begin{pmatrix} A_{11}A_{22}^* & A_{11}A_{32}^* & A_{12}A_{32}^* \\ A_{11}A_{23}^* & A_{11}A_{33}^* & A_{12}A_{33}^* \\ A_{21}A_{23}^* & A_{21}A_{33}^* & A_{22}A_{33}^* \end{pmatrix} . \qquad (58\text{-}2)$$

The first term in the right hand side of Eq.(44) is also approximated, as is done for Eq.(46), as

$$
\frac{1}{2\lambda_1}
\begin{pmatrix}
A_{11}A_{12}^* & A_{12}A_{22}^* & A_{13}A_{32}^* \\
A_{11}A_{13}^* & A_{12}A_{23}^* & A_{13}A_{33}^* \\
A_{21}A_{13}^* & A_{22}A_{23}^* & A_{23}A_{33}^*
\end{pmatrix}
\sum_q
\begin{pmatrix}
2M_{1,kpq}M_{1,kpq}\theta_{kpq}I_1(p)I_1(q) \\
M_{2,kpq}M_{2,kpq}\theta_{kpq}I_1(p)I_2(q) \\
M_{3,kpq}M_{3,kpq}\theta_{kpq}I_1(p)I_3(q)
\end{pmatrix}
$$

$$
\simeq \frac{C_0\gamma_v}{2\lambda_1} M_a
\begin{pmatrix}
I_1(k) \\
I_2(k) \\
I_3(k)
\end{pmatrix}
\tag{59-1}
$$

with

$$
M_a \equiv
\begin{pmatrix}
A_{11}A_{12}^* & A_{12}A_{22}^* & A_{13}A_{32}^* \\
A_{11}A_{13}^* & A_{12}A_{23}^* & A_{13}A_{33}^* \\
A_{21}A_{13}^* & A_{22}A_{23}^* & A_{23}A_{33}^*
\end{pmatrix}.
\tag{59-2}
$$

Substituting Eqs.(58) and (59) into Eq.(44), one obtains the simplified (reduced) relation

$$
\begin{pmatrix}
J_{12}(k) \\
J_{13}(k) \\
J_{23}(k)
\end{pmatrix}
= \frac{C_0\gamma_v}{2\lambda_1}
\left[I - \frac{C_{cross}\gamma_v}{2\lambda_1} M_c \right]^{-1} M_a
\begin{pmatrix}
I_1(k) \\
I_2(k) \\
I_3(k)
\end{pmatrix}.
\tag{60}
$$

With the help of this analytic decomposition between the auto- and cross-correlation functions, the cross-correlation function is explicitly expressed. Cross-correlation function is directly related to the global fluxes in the direction of the gradient. The result corresponds to FD-theorem of the first kind in turbulent plasmas.

The cross-correlation function J_{13} is related to the global heat flux in the $\hat{x}$-direction (across the magnetic surface)

$$
q_{0,x} = \sum_k \langle -ik_y\phi p \rangle_k.
\tag{61}
$$

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The (1,3) component of Eq.(36) is transformed into the second component of Eq.(60), and gives a relation for J_{13} as

$$Im \, J_{13} \simeq - Im(A_{11}A_{31}^*)\gamma_v C_0 \lambda_1^{-1} I_1 \qquad (62)$$

(In deriving this simple form, the matrix $[I - (C_{cross}\gamma_v/2\lambda_1) M_c]^{-1}$ is approximated as a unit tensor based on an assumption of $C_{cross} < C_0$, and a dominant term is retained.) This result is used to calculate the heat flux, providing an estimate

$$q_{0,x} = \sum_k \langle -ik_y \phi p \rangle_k \simeq \sum_k C_0 \lambda_1^{-1} k_\perp^{-2} I_1(k)(-p_0'). \qquad (63)$$

If one introduces a turbulence-driven thermal diffusivity, χ_{turb}, in a form of

$$q_{0,x} = -\chi_{turb} \, p_0', \qquad (64)$$

χ_{turb} is evaluated by the form

$$\chi_{turb} = \sum_k \frac{C_0}{\lambda_1 k_\perp^2} I_1(k). \qquad (65)$$

3.2.5 Spectrum

Spectral function is solved, once the nonlinear transfer rate γ_i is given by solving Eqs.(26) and (53). The average fluctuation amplitude $I_i = \langle f_i(t)f_i(t) \rangle$ is obtained by solving the integral equation Eq.(54) as has been performed in [36].

In the case that γ_v shows only a weak dependence on k, and the spectrum of the kinetic energy of fluctuations,

$$E_1(k) = k_\perp \langle k_\perp^2 f_{1,k} f_{1,k} \rangle, \qquad (66)$$

is found to be deduced to

$$E_l(k_\perp) \propto k_\perp^{-3} \qquad\qquad (67)$$

4. Application to CDIM (Current Diffusive Ballooning Mode)

4.1 Nonlinear Dispersion and Least Stable Branch

Explicit forms of the renormalized operator $\mathcal{L}$ has been derived and the relation Eq.(26) has been solved in an analytic limit for CDIM branch [37].

Let us examine the effect of self-noise, combining Eq.(26) and Eq.(53). The simplest case below is taken,

$$\gamma_v \simeq \gamma_e \simeq \gamma_p. \qquad\qquad (68)$$

This approximation implies that the viscosity of perpendicular ion momentum, electron viscosity and thermal diffusivity are equal in the strong turbulent limit. A detailed study of Eq.(26) has shown that this approximation is valid. An additional approximation, i.e., the small λ_1 limit, is used in this article for the analytic insight of the problem. Then Eq. (26) yields the relations

$$\lambda_2 \simeq \gamma_v \text{ and } \lambda_3 \simeq 2\gamma_v, \qquad\qquad (69)$$

if $\lambda_1 \simeq 0$ is substituted. These simplified forms, Eqs.(68) and (69) are used in evaluating $\hat{\mathcal{R}}_{mn}$, which gives an estimate of Eq.(53) as

$$\lambda_1 \simeq \frac{C_0}{2}\gamma_v . \qquad\qquad (70)$$

On the other hand, the solution of Eq.(26) near $\lambda_1 \simeq 0$ has been obtained in the geometry of inhomogeneous magnetic field associated with the magnetic shear [37].

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Detailed analysis is given in [37]. The deviation of λ_1 is obtained in terms of the eddy viscosity damping rate as

$$\lambda_1 \simeq \frac{1}{2}(\gamma_v - \gamma_*) \tag{71}$$

where rate γ_* was already calculated from the relation $Det \, \mathcal{L} = 0$ for the least stable mode in the absence of noise effect. γ_* is explicitly given in terms of the gradient parameter as

$$\gamma_* \simeq G_0^{1/2} \tag{72}$$

This rate γ_* is related with the effective viscosity on the microscale fluctuation, $\mu_{v\,*}$, and the representative mode number, k_*, as

$$\gamma_* = \mu_{v*} k_*^2 . \tag{73}$$

The effective viscosity and the representative mode number have been estimated as

$$\mu_{v*} = G_0^{3/2} s^{-2} \xi^{-1} , \tag{74}$$

and

$$k_* = \xi^{1/2} s G_0^{-1/2} . \tag{75}$$

4.2 Self-consistent Solution of Turbulence

Equations (26) and (53) are reduced to the approximate equations, Eqs.(71) and (70), respectively. Equations (70) and (71) form a closed set of equations for determining the decorrelation rate λ_1 and the eddy damping rate γ_v in inhomogeneous

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plasma. Figure 1 illustrates schematic relation of Eqs.(70) and (71). The self-consistent solutions of the decorrelation rate and the eddy-damping rate (nonlinear transfer rate) are obtained in terms of the global parameter G_0 as

$$\lambda_I = \frac{C_0}{2(1-C_0)} G_0^{1/2} \tag{76}$$

and

$$\gamma_v = \frac{1}{(1-C_0)} G_0^{1/2} . \tag{77}$$

In this equation, the coefficient C_0 should satisfy the constraint

$$0 < C_0 < 1 \tag{78}$$

in order to satisfy the condition that the decorrelation time should be positive and finite.

The solution of Eq.(53) gives the decorrelation rate λ_I in terms of the random noise. Therefore, it could be interpreted as, in general, the FD-theorem of the second kind for turbulent plasma. The result Eq.(76) for the case of CDIM turbulence gives the decorrelation rate in terms of the global parameters, not of the temperature. The FD-theorem of the second kind in the turbulent plasma is now explicitly given as a formula that relates the decorrelation rate and the 'global' parameter G_0 that specifies the plasma inhomogeneity.

Spectral function has been solved, once the nonlinear transfer rates are given. As was the case of [36], Eq.(77) shows only a weak dependence on k. Following the same procedure in [36], the spectrum of the kinetic energy of fluctuations, $E_I(k) = k_\perp \langle k_\perp^2 f_{1,k} f_{1,k} \rangle$, is deduced as

$$E_I(k_\perp) = \frac{2}{(1-C_0)} G_0 k_\perp^{-3} \tag{79}$$

for $k_\perp > k_*$. (Detail of derivation for given γ_v is described in [36].) This is a partition law in the high-mode number regime ($k_\perp > k_*$) of the turbulent systems, and is not the equi-partition law.

It may be worth noting the validity of the approximation of Eq.(68). If λ_1 is small but finite, Eq.(68) is modified as $\lambda_2 \simeq \gamma_v$ and $\lambda_3 \simeq 2\gamma_v - \lambda_1$. If one substitutes Eq.(76) into these relations, one has

$$\lambda_2 \simeq \frac{1}{(1 - C_0)} G_0^{1/2} , \tag{80-1}$$

$$\lambda_3 \simeq \frac{(4 - C_0)}{2(1 - C_0)} G_0^{1/2} . \tag{80-2}$$

The order relation among three eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3$, is not violated so long as Eq.(78) is satisfied. If C_0 becomes larger, the ratio approaches to $1 : 2 : 3$, and a considerable amount of fluctuation is excited in the branches with λ_2 and λ_3. Under such a circumstance, quantitative estimate in this article, which is obtained by keeping only the first branch, could be modified. Nevertheless, the qualitative conclusion is not altered.

5. Fokker-Planck Equation and Probability Distribution Function

5.1 Formulation of the Fokker-Planck Equation

Probability distribution function is discussed starting from the Langevin Equation. In order to illustrate the white noise in the random source term of the Langevin equation, $\tilde{S}$ is rewritten as

$$\tilde{S}_{i, k} = w(t) g_{i, k} \tag{81}$$

and

$$g_{i,k} \equiv \sum_{\Delta} M_{i,kpq} \sqrt{\theta_{kpq}} \zeta_{1,p} \zeta_{i,q} \tag{82}$$

With this notation, the Langevin equation is written as

$$\partial f / \partial t + \mathcal{L}f = w(t)g \tag{83}$$

From this dynamical equation, the Liouville equation for the probability distribution function $P(f_{i,k}; t)$ is derived. Noting the assumption of the white noise of $w(t)$, $\langle w(t)w(t') \rangle = \delta(t - t')$, the Liouville equation is deduced to a form of the Fokker-Planck equation. If the independence of the noise source holds, i.e., $\langle g_{i,k}g_{j,k'} \rangle = \langle\langle g_{i,k}^2 \rangle\rangle \delta_{i,j}\delta_{k,k'}$ (like the case of neutral fluids [17, 18]), one may have an equation [1, 17, 38]

$$\frac{\partial}{\partial t}P = \sum_{i,k} \frac{\partial}{\partial f_{i,k}} \left(\mathcal{L}f + \frac{1}{2}g_{i,k} \frac{\partial}{\partial f_{i,k}} g_{i,k} \right)P \tag{84}$$

Based on the assumption of statistics of $\{\zeta_{i,k}\}$, the independence of the noise terms between different k-components holds, i.e., $\langle g_{i,k}g_{j,k'} \rangle \propto \delta_{k,k'}$. However, the source terms for different fields, could be statistically dependent with each other: Namely, $\langle g_{i,k}g_{j,k} \rangle$ does not necessarily vanish even if $i \neq j$. The decomposition of the noise source term is necessary before the Fokker-Planck equation is derived. In addition, $g_{i,k}$ could be complex, which deserves consideration.

With respect to the interpretation of the random noise, which gives rise to the diffusion term in Fokker-Planck equation, we follow the model by Stratonovich [39]. Ito's interpretation of the random noise [40] leads to the term $\frac{\partial^2}{\partial f_{i,k}^2} g_{i,k}^2 P$, instead of $\frac{\partial}{\partial f_{i,k}} g_{i,k} \frac{\partial}{\partial f_{i,k}} g_{i,k} P$, in this equation (84) [38, 40]. The Stratonovich definition is employed here after [41].

The dynamical equation describes the evolution of three fields. As a consequence, three branches of the mode (for given k) are included in this system.

Nevertheless, as is shown in the previous section, one branch (current diffusive interchange mode branch) could be strongly excited and other two branches remain highly damped. The nonlinear excitation and statistical nature of this branch are discussed in preceding chapters. Only the pole of $(s + \lambda_1)^{-1}$ is kept. The same approximation is made. Then, the Langevin equation is deduced to that of one field, e.g., $f_1 = \phi$.

$$\frac{\partial}{\partial t} \phi + \lambda_1 \phi = \tilde{s} \tag{85}$$

with the source of

$$\tilde{s}_k = w(t) g_k \tag{86}$$

In this equation, the magnitude of the noise source is given as in the first row of Eq.(29), i.e.,

$$g_k = \Re\left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right), \tag{87}$$

or, in an explicit form, as

$$g_k = \Re\left\{ \frac{(\gamma_e - \lambda_1)(\gamma_p - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)k_1^2} g_{1,k} \right\} + \Re\left\{ \frac{ik_\parallel(\gamma_p - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)k_1^2} g_{2,k} \right\}$$
$$+ \Re\left\{ \frac{ik_y \Omega'(\gamma_e - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)k_1^2} g_{3,k} \right\}. \tag{88}$$

By retaining the real part in Eq.(87), the possible problem of complex quantity of $g_{i,k}$ is eliminated, and the diffusion process is assured in the Fokker-Planck equation. In this one-branch approximation, the coefficient g_k is statistically independent for different k-component, $\langle g_k g_{k'} \rangle = \langle\!|g_k^2|\!\rangle \delta_{k,k'}$. Then the Liouville equation is reduced to a Fokker-Planck Equation as

$$\frac{\partial}{\partial t}P = \sum_k \frac{\partial}{\partial \phi_k}\left(\lambda_{1,k}\phi_k + \frac{1}{2}g_k \frac{\partial}{\partial \phi_k} g_k\right)P . \tag{89}$$

5.2 Steady State Probability Distribution Function

The distribution function provides the statistical average. The average amplitude of fluctuation component ϕ_k is derived from the Fokker-Planck equation, and the result is compared to the solution of the Langevin equation.

5.2.1 Probability Function

Steady state probability function P_{eq} satisfies the Fokker-Planck equation in a stationary limit,

$$\sum_k \frac{\partial}{\partial \phi_k}\left(\lambda_{1,k}\phi_k + \frac{1}{2}g_k \frac{\partial}{\partial \phi_k} g_k\right)P_{eq} = 0 \tag{90}$$

The solution of this equation is obtained by the help of the ansatz of large number of freedom.

A possible form of the equilibrium distribution function satisfies the detailed balance,

$$\left(2\lambda_{1,k}\phi_k + g_k \frac{\partial}{\partial \phi_k} g_k\right)P_{eq} = 0 \qquad \text{(for all } k\text{)} \tag{91}$$

The constant of integral (right hand side) is chosen to be zero, based on the boundary condition that the relation $\phi_k P_{eq} = \partial P_{eq}/\partial \phi_k = 0$ holds as $\phi_k \to \infty$. (The condition that P_{eq} vanishes much faster than ϕ_k^{-1} as $\phi_k \to \infty$ is a necessary condition for the requirement that an integral of P_{eq} could be normalized to unity.)

The detailed-balance equation suggests the form of equilibrium distribution function as

$$P_{eq}(\{\phi_k\}) = \bar{P} \prod_k \frac{1}{g_k} exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k \, g_k^{-2} \, d\phi_k\right\} \tag{92}$$

where $\bar{P}$ is an normalization constant. This equation is an exact solution if g_k is independent of $\phi_{k'}$ if $k' \neq k$. Although $\phi_{k'}$ influences the value of g_k, the influence from one particular $\phi_{k'}$ on g_k is weak. This is because many numbers of fluctuating components statistically contribute to g_k through γ_v, γ_j and γ_p, so that the most sensitive dependence of ϕ_k on P_{eq} appears through the term of $exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k \, g_k^{-2} \, d\phi_k\right\}$. Based on this fact, an approximate solution of the equilibrium distribution function is given as Eq.(92).

5.2.2 Characteristics and Power Law

Several characteristic features are drawn from the equilibrium distribution function of probability, Eq.(92).

The Average

An equilibrium distribution function provides the statistical average. The statistical average of the amplitude ϕ_k, $(\phi_k)_{av} = \sqrt{\langle\phi_k^2\rangle}$ is given as an example. An weighted-integral

$$\langle\phi_k^2\rangle = \int d\phi \, P_{eq}\phi_k^2 \tag{93}$$

yields the average $\langle\phi_k^2\rangle$, where ϕ represents a set of $\{\phi_k\}$ for all the k-th component and the notation $d\phi = \prod_k d\phi_k$ is used. Multiplying ϕ_k to the relation Eq.(91) and integrating the second term by part, one has

$$\int d\phi \, P_{eq}\phi_k^2 = \int d\phi \, g_k P_{eq} \frac{\partial}{\partial\phi_k}\left(\frac{g_k\phi_k}{2\lambda_{1,k}}\right). \tag{94}$$

The ansatz of the large degree of freedom is used to approximate g_k and $\lambda_{1,k}$ to be weak functions of ϕ_k, and the derivative in the right hand side of Eq.(94) is evaluated by

$$\frac{\partial}{\partial \phi_k}\left(\frac{g_k \phi_k}{\lambda_{1,k}}\right) \simeq \frac{g_k}{\lambda_{1,k}} \tag{95}$$

Substituting this evaluation into the right hand side of Eq.(94), one obtains the relation between two moments as

$$\int d\phi \; \phi_k^2 P_{eq} = \int d\phi \; \frac{g_k^2}{2\lambda_{1,k}} P_{eq} \tag{96}$$

In other words, the relation

$$\langle \phi_k^2 \rangle = \left\langle \frac{g_k^2}{2\lambda_{1,k}} \right\rangle \tag{97}$$

is obtained. This relation is considered as the extended FD Theorem of the second kind, and is essentially identical to Eq.(36). The Ansatz of equivalence of correlation, $\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle$ of Eq.(22), should be imposed to estimate $\langle g_k^2 \rangle$ in Eq.(92). This is left for future study.

The Peak

The peak of the equilibrium probability distribution is given by the condition

$$\frac{\partial}{\partial \phi_k} P_{eq} = 0 \; . \tag{98}$$

Noting the detailed balance relation, Eq.(91), i.e.,

$$\frac{\partial}{\partial \phi_k} P_{eq} = -P_{eq} \, g_k^{-2} \cdot \left(2\lambda_{1,k}\phi_k + g_k \frac{\partial}{\partial \phi_k} g_k\right), \tag{99}$$

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the peak of the distribution function is realized for $\phi_k = \phi_{k,p}$ which satisfies the

condition

$$\lambda_{1,k} + \frac{\partial}{\partial \phi_k^2} g_k^2 = 0 \qquad\qquad (\text{at } \phi_k = \phi_{k,p}). \qquad\qquad (100)$$

(The suffix p stands for 'peak', not the wave number here.)

In the vicinity of $\phi_k = \phi_{k,p}$, or Eq.(100), the distribution function is

approximated by use of Taylor expansion with respect to $(\phi_k - \phi_{k,p})$. The distribution

function is approximated as the Gaussian distribution in the vicinity of $\{\phi_{k,p}\}$ as

$$P_{eq}(\{\phi_k\}) \simeq P_0 \exp\left\{-\frac{1}{2}\sum_{k,k'} c_{kk'}(\phi_k - \phi_{k,p})(\phi_{k'} - \phi_{k',p})\right\} \qquad\qquad (101)$$

where $P_0 = \bar{P}g_{k,p}$, $g_{k,p} = g_k(at\ \phi_k = \phi_{k,p})$ and $c_{kk'}$ is a coefficient given by
$c_{kk'} = \left[\partial^2(2g_k^{-2}\lambda_{1,k}\phi_k + g_k^{-1}\partial g_k/\partial \phi_k)/\partial \phi_k \partial \phi_{k'}\right]_{\phi_{k,p}}$.

In the previous analysis, the condition of the nonlinear marginal stability

$$\lambda_1 = 0 \qquad\qquad (102)$$

was used in an estimate of the level of the self-sustained turbulence [15, 32]. If the

random source term is independent of the fluctuation field (e.g., the case of an external

random force), i.e., the relation $\frac{\partial}{\partial \phi_k^2} g_k^2 = 0$ holds. Then the condition, $\lambda_1 = 0$,

provides the exact description for the peak of the equilibrium probability distribution

function.

The Width

The denominator g_k^{-2} in the integrand in the term $\exp\left\{-\int^{\phi_k} 2\lambda_{1,k}\phi_k\ g_k^{-2}\ d\phi_k\right\}$

dictates the width of the distribution function. Near the peak position, the width is

determined by $c_{kk'}$ (Eq.(101)). From the relation Eq.(97), the width of the distribution

is given comparable to the average, i.e., $\left\langle \left(\phi_k - \sqrt{\langle \phi_k^2 \rangle} \right)^2 \right\rangle \sim \langle \phi_k^2 \rangle$. Namely, the variance is comparable to the average value itself.

The Tail Component

The fact that the random noise level, g_k, depends on the turbulence level leads to a possibility of power-law in the tail of probability distribution. This situation is different from the case of thermal fluctuation, where the random noise level is ultimately determined by the temperature.

Let us consider the scaling properties of variables ϕ_k, $\lambda_{1,k}$ and g_k. Introducing a scaling parameter ℓ and changing the variable as

$$\{\phi_k\} = \{\ell\, \phi_{k,\,0}\}\,, \tag{110}$$

where $\{\phi_{k,\,0}\}$ denotes an initial position, and $\{\phi_k\}$ is varied only in the magnitude, by a factor ℓ, and relative ratio between components $\phi_k/\phi_{k'}$ are fixed. We examine the scaling property. In a large ℓ limit, i.e., the limit of strong nonlinearity, linear terms become smaller than nonlinear terms in $\mathcal{L}_{ij}$, and the matrix element $\mathcal{L}_{ij}$ scales linearly with the nonlinear term γ_i as is shown in Eq.(17). Under this circumstance, the relation Eq.(21) yields the dependence $\mathcal{L}_{ij,\,k}\theta_{kpq} \sim O(1)$, i.e., $\gamma_i\theta_{kpq} \sim O(1) \propto \ell^0$. Equation (18) is interpreted as $\gamma_i \propto \theta_{kpq}\ell^2$. From these relations, one obtains scaling relations

$$\gamma \propto \ell^1 \tag{104-1}$$

$$\theta \propto \ell^{-1} \tag{104-2}$$

and

$$g^2 \simeq O(\phi^4\theta) \propto \ell^3\,. \tag{104-3}$$

In addition to it, Eq. (27) gives the relation $\lambda_i/\gamma_i \propto \ell^0$. Combining it with Eq.(104-1), one obtains

$$\lambda \propto \ell^1, \tag{104-4}$$

From these scaling relations, one has a dependence in an asymptotic limit as

$$\lambda_{1,k}\phi_k \, g_{\bar{k}}^{-2} \, d\phi_k \propto \ell^{-1} \, d\ell \, ,$$

and writes as

$$\sum_k \lambda_{1,k}\phi_k \, g_{\bar{k}}^{-2} \, d\phi_k \sim m_c \ell^{-1} \, d\ell \tag{105}$$

where m_c is introduced as a proportionality constant. The integral $\sum_k \int^{\phi_k} 2\lambda_{1,k}\phi_k \, g_{\bar{k}}^{-2} \, d\phi_k$ has a logarithmic component and is expressed as $\sum_k \int^{\phi_k} 2\lambda_{1,k}\phi_k \, g_{\bar{k}}^{-2} \, d\phi_k \sim m_c \, \ell n(\ell)$. Therefore, a power law dependence in the equilibrium distribution function is found in a large amplitude limit as

$$\bar{P}\prod_k g_{\bar{k}}^{-1} exp\left\{ -\int^{\phi_k} 2\lambda_{1,k}\phi_k \, g_{\bar{k}}^{-2} \, d\phi_k \right\} \propto \ell^{-3/2 - m_c} = \left(\frac{\phi_k}{\phi_{k,\,0}}\right)^{-\left(m_c + \frac{3}{2}\right)} \tag{106}$$

in which the contribution of $\left(\phi_k/\phi_{k,\,0}\right)^{-3/2}$ comes from the denominator $g_{\bar{k}}^{-1}$ in $P_{eq}(\{\phi_k\})$.

Form these considerations, the equilibrium distribution function P_{eq} is approximated as a Gaussian distribution function near the peak, and has a finite tail which is given by a power law distribution. A schematic drawing of the distribution function is illustrated in the Fig. 2. This power law dependence is caused from the nature that the diffusion coefficient of the probabilistic function is determined by the turbulence level.

5.2.3 Accessibility

Accessibility to the equilibrium distribution function, P_{eq}, is shown by constructing a Lyapunov function.

$$\mathcal{H}(t) = \int d\phi \, P(\phi; t) \, \ell n\left(\frac{P(\phi; t)}{P_{eq}(\phi)}\right) \tag{107}$$

where ϕ represents a set of $\{\phi_k\}$ for all the k-th components. Taking the time derivative,

$$\frac{d}{dt}\mathcal{H}(t) = \int d\phi \left[\left\{\frac{d}{dt}P(\phi; t)\right\} \ell n\left(\frac{P(\phi; t)}{P_{eq}(\phi)}\right)\right]$$

$$= \int d\phi \left[\left\{\sum_k \frac{\partial}{\partial \phi_k}\left(\lambda_{1,k}\phi_k + \frac{1}{2}g_k \frac{\partial}{\partial \phi_k} g_k\right)P\right\} \ell n\left(\frac{P(\phi; t)}{P_{eq}(\phi)}\right)\right], \tag{108}$$

and by performing a partial integration, one has

$$\frac{d}{dt}\mathcal{H}(t) = -\int d\phi \left[\sum_k \left(\lambda_{1,k}\phi_k P + \frac{1}{2}g_k \frac{\partial}{\partial \phi_k} g_k P\right)\left\{\frac{\partial}{\partial \phi_k} \ell n\left(\frac{P(\phi; t)}{P_{eq}(\phi)}\right)\right\}\right] \tag{109}$$

The terms in the right hand side of Eq.(109) are calculated, noting identities of Eq.(91) and the relation $g_k \frac{\partial}{\partial \phi_k}(g_k P) = g_k^2 P \frac{\partial}{\partial \phi_k} \ell n(g_k P)$. We have

$$\frac{d}{dt}\mathcal{H}(t) = -\int d\phi \left[\sum_k \frac{g_k^2 P}{2}\left\{\frac{\partial}{\partial \phi_k} \ell n\left(\frac{P(\phi; t)}{P_{eq}(\phi)}\right)\right\}^2\right] \tag{110}$$

The integrand is positive or zero, and an inequality

$$\frac{d}{dt}\mathcal{H}(t) \leq 0 \tag{111}$$

holds. The condition $\frac{d}{dt}\mathcal{H}(t) = 0$ is satisfied if the probability distribution function is equal to the equilibrium distribution function $P(\phi; t) = P_{eq}(\phi)$.

This result shows that the quantity $\mathcal{H}(t)$ plays the role of a Lyapunov function for the evolution of probability distribution function. The fact that the construction of the Lyapunov function is possible means that one of steady state distributions is guaranteed in a time-asymptotic behaviour.

It is not self-evident whether the equilibrium solution is unique or not: Instead, it is highly plausible that there exist multiple solutions that satisfy the steady state condition. This is because that the turbulence-turbulence transition between two nonlinearly-marginal branches has been predicted in the theory of self-sustained turbulence [42].

6. Summary and Discussion

In summary, a new method is proposed to study the nonlinear-nonequilibrium physics for the system with strong (nonlinear-) instability and turbulence. This method consists of (i) derivation of nonlinear Langevin equation by use of renormalization and RCM for the nonlinearity, (ii) homogeneous solution, which we call nonlinear dispersion relation, and the general solution with random self-noise, (iii) the relation between the correlation functions, random noise and decorrelation rate Eq.(36) based upon the statistical average, and (iv) derivation of consistent solution. The quantities such as turbulence level, decorrelation rate, auto- and cross-correlations are explicitly given as functions of the parameter that characterize the nonequilibrium property. The step (iii) is an extension of the FD-theorem to far nonequilibrium systems. In this theoretical framework, the decorrelation rate of turbulence λ_I and eddy-viscosity damping rate γ_v are different. The analytic forms are derived, at the sacrifice of accuracy in numerical factor. Quantitative prediction requires numerical calculation of Eq.(36).

An alternative formulation is presented by deriving an Fokker-Planck equation for the probability distribution function. By use of the one-branch approximation, the noise term is decomposed, and the Liouville equation is approximated to a form of Fokker-Planck equation. The 'drag' and 'diffusion' coefficients in the Fokker-Planck

equation are modelled from the renormalization of the turbulent effects. On the basis of this equation, equilibrium distribution function of turbulence level is derived. The distribution function is approximated as the Gaussian distribution in the vicinity of its peak. The width of the distribution function is in the same order as the mean value itself. The equilibrium distribution function is found to be associated with a small but finite tail component, and has a power law distribution in its large amplitude limit. This power law dependence is caused from the fact that the random noise is the self-noise: namely, the enhancement of the fluctuation level simultaneously increases the noise pumping, establishing a self-sustained tail distribution. It is also noted that the Lyapunov function is constructed for the strongly turbulent plasma. The time derivative of this functional is shown to be negative definite, which indicates that an approach to a certain equilibrium distribution is expected.

Let us discuss the issue of irreversibility in our formulation. In forming the Langevin equation, the nonlinear terms are divided into two parts, say the coherent part and incoherent part. The original nonlinear terms express essentially the time-reversible processes. In our formulation, an ansatz is introduced for the incoherent part to be Gaussian white noise, Eq.(19). This ansatz introduces the irreversibility in the model of nonlinear terms. The system we are considering is irreversible, independence of the time-reversal property of the nonlinear terms: Namely, (1) the molecular dissipation is taken into account and (2) the global system is open and the flows exist for the fixed gradients. Therefore, an apparent irreversibility in a model of nonlinear term is related to the facts that the rate of dissipation is increased by the enhanced flux in space and enhanced dissipation owing to the strong cascade to the much more microscopic scale.

The nonlinear transfer rate γ_* has been obtained in the previous work in the absence of random noise [6, 14]; in this simple treatment, the turbulent decorrelation rate could not be obtained and the statistical description was impossible. The present analysis extends the previous framework for the self-sustained turbulence including the random noise effect consistently. It confirms that the previous simple model has provided a qualitatively appropriate estimate for the nonlinear transfer rates.

Solutions with scaling relations like Eq.(104-1) and (104-2) in the strong turbulence limit belong to a class with the diffusion which is linearly proportional to the Kubo number K (K being a ratio of fluctuating $E \times B$ velocity to [wavelength divided by correlation time]) [43], and are based on the Corrsin approximation [44]. It has been pointed out in the two-dimensional turbulence that the Corrsin approximation is not a good approximation in a large K limit, and that a linear dependence on K may not hold but an weaker power dependence appears [43,45]. This deviation from linear dependence is owing to the trapping of the orbit. In the present study, the fluctuations have a character of quasi-two dimensional one. However, the presence of a magnetic shear prohibits the trapping of fluid elements on a constant-potential surface. From this reason, the linear dependence like Eq.(104-1) is employed here.

Theoretical result of the spectrum Eq.(79), $E_l(k_\perp) \propto k_\perp^{-3}$ seems to be in agreement with the result of the direct numerical simulation of the problem [30], although the resolution of the simulation might be limited. A detailed two-dimensional numerical simulation of a different model equation of interchange mode turbulence, in which the electron nonlinearity in the Ohm's law is not kept, has reported a spectrum of the velocity field as $E_k^V \propto k^{-2.3}$ [46]. The nonlinearlity in the Ohm's law, that causes nonlinear instability, could play crucial role in determining the spectrum. Precise comparison of the theory with direct numerical simulation of the current diffusive interchange mode turbulence is left for future study. The comparison with numerical simulation would also provide a test for the validity of the modelling of the incoherent part of the nonlinear terms.

It should also be noticed that the presence of the tail in the distribution function might give a restriction to the decomposition of the correlations like $\langle \zeta_{i,p} \zeta_{i,q} \zeta_{i,p'} \zeta_{i,q'} \rangle = \langle \zeta_{i,p} \zeta_{i,p} \rangle \langle \zeta_{i,q} \zeta_{i,q} \rangle + \cdots$. We assume a statistical independence between different p and p' components, therefore the contribution of the small tail influences little in the case of $p \neq q$. For the case of $p = p' = q = q'$, the correlation function $\langle \zeta_{i,p} \zeta_{i,q} \zeta_{i,p'} \zeta_{i,q'} \rangle$ could be strongly influenced by a small but finite tail.

According to an ansatz of large degree of freedom, the modification of a particular pair

correlation $\langle \zeta_{i,p}\zeta_{i,p}\zeta_{i,p}\zeta_{i,p}\rangle$ (only one combination among a series) is considered not to cause a considerable modification of the total average of $\langle g_k^2 \rangle$. We in this article do not go into the details of this problem, and leave it to future study.

In this article, statistical property in a steady state is discussed. Based on the Fokker-Planck equation, a study on the dynamical property could be studied. By multiplying ϕ_k^2 to Eq.(89) and integrating it, one has

$$\frac{\partial}{\partial t}\int d\phi \; \phi_k^2 P = \int d\phi \; \phi_k^2 \sum_k \frac{\partial}{\partial \phi_k}\left(\lambda_{1,k}\phi_k + \frac{1}{2}g_k \frac{\partial}{\partial \phi_k} g_k\right)P. \qquad (112)$$

After partial integration of the right hand side, the equation reduces to

$$\frac{\partial}{\partial t}\int d\phi \; I_k P = -2\int d\phi \; \lambda_{1,k} I_k P + \int d\phi \; P \frac{\partial}{\partial I_k}(I_k g_k^2) \qquad (113)$$

$(I_k = \langle \phi_k^2 \rangle)$. This equation can be interpreted as the wave-kinetic equation for the turbulent plasma as

$$\frac{\partial}{\partial t}\langle I_k \rangle = -2\langle \lambda_{1,k} I_k \rangle + \left\langle \frac{\partial}{\partial I_k}(I_k g_k^2) \right\rangle. \qquad (114)$$

where g_k^2 and $\lambda_{1,k}$ are renormalized coefficients. If one assumes that g_k^2 and λ_k is a slowly varying function of I_k, $\partial(I_k g_k^2)/\partial I_k$ is approximated by g_k^2 and the equation is simplified as

$$\frac{\partial}{\partial t}\langle I_k \rangle = -2\langle \lambda_{1,k} \rangle \langle I_k \rangle + \langle g_k^2 \rangle \qquad (115)$$

The steady state solution of this equation agrees with Eq.(36). Analysis of the dynamical evolution is left for the future study.

The present analysis is done by choosing CDIM as one typical example, and the method itself is applied to much wider circumstances, e.g., the problems of various instabilities, other external forces (like flow shear) or turbulence-turbulence transition.

Acknowledgements

Dedication

This article is dedicated to the memory of Prof. R. Kubo.

Appendix: On Separation of Effective Damping Term and Random Term

In formulating a Langevin equation, Eq.(14), the nonlinear term is separated into the effective damping term $\gamma_i f_i$ and the random noise term $\tilde{S}_i$. The process of separation and the relation with the method of dressed test mode are discussed by introducing a model projection operator.

One k-Fourier component of Eq.(11) is chosen as

$$\partial f_k / \partial t + \mathcal{L}^{(0)} f_k = \mathcal{N}_k = \sum_{k'} \mathcal{M}_{kk'} f_{k'} f_{k-k'} \tag{A1}$$

$$\mathcal{L}^{(0)} = \begin{pmatrix} \mu_c k_\perp^2 & i k_\parallel k_\perp^{-2} & i k_y k_\perp^{-2} \Omega' \\ i\tilde{\xi} k_\parallel & \mu_{ec} k_\perp^2 & 0 \\ -i k_y p_0' & 0 & \chi_c k_\perp^2 \end{pmatrix} \tag{A2}$$

where the suffix k is suppressed if not necessary. This component of fluctuation is called the test mode, and the nonlinear term $\mathcal{N}_k$ is divided into two components. The projection operator $\mathcal{P}$ is introduced to filter, from $\mathcal{N}$, the component which is statistically dependent on f_k. The projected term $\mathcal{P}\mathcal{N}_k$ is correlated with f_k. In the eddy damped quasi-normal representation, the proportionality is written as $\mathcal{P}\mathcal{N}_{i,k} = \gamma_{i,k} f_{i,k}$. The rest, $(1 - \mathcal{P})\mathcal{N}$, is statistically independent of f_k, and is called an incoherent, or, random source.

A model projection is considered as follows. The fundamental assumption in the analysis is that the system has large number of positive Lyapunov exponents, and the excited fluctuations are approximately statistically independent of each other. A turbulent state is considered to be specified by a set of all components $\{f_{i,k'}\}$. Choose one component $f_{i,k}$ out of $\{f_{i,k'}\}$. Small but finite correlation between $f_{i,k}$ and $\{f_{i,k'}\}$ exists through the nonlinear interactions between $f_{i,k}$ and background fluctuations. The projection operator $\mathcal{P}$ is introduced to extract this statistically-dependent nonlinear

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contribution. Nevertheless, the set $\{f_{i,\,k'}\}$ is considered to be very close to some set $\{\zeta_{i,\,k'}\}$ which are statistically-independent of $f_{i,\,k}$.

One set of turbulent state, $\{f_{i,\,k'}\}$, is chosen and the process, that the test mode is taken away from this set, is considered. By the reduction of the test mode by the amount of $-\delta f_k$, the modification in the background fluctuations $\{-\delta f_{k''}\}$ appear. The equation that $\delta f_{k''}$ satisfies is

$$\left(\partial/\partial t + \mathcal{L}^{(0)} - \mathcal{N}'\right)\delta f_{k''} = \sum_{k'\,=\,k''\,-\,k} \mathcal{M}_{kk'} f_{k'} \delta f_k \tag{A3}$$

The term $\mathcal{N}'$ is the nonlinear term on $f_{k''}$, which does not include the interaction between f_k (i.e., $\mathcal{N}' = \mathcal{N} - \mathcal{M}_{k,\,k''\,-\,k} f_{k''\,-\,k} f_k$). One solves Eq.(A3) as

$$\delta f_{k''} = \left(\partial/\partial t + \mathcal{L}^{(0)} - \mathcal{N}'\right)^{-1} \mathcal{M}_{k,\,k''\,-\,k} f_{k''\,-\,k} \delta f_k \tag{A4}$$

This is the induced variation of the background fluctuations associated with the change of test mode. This influenced variation, "polarization", is not statistically independent of f_k. With this change, the nonlinear term on the test mode is coherently modified by the amount of

$$\delta \mathcal{N}_k = \sum_{k'} \mathcal{M}_{k,\,-\,k'} f_{-k'} \left(\partial/\partial t + \mathcal{L}^{(0)} - \mathcal{N}'\right)^{-1}_{k+k'} \mathcal{M}_{k,\,k'} f_{k'} \delta f_k \tag{A5}$$

From this relation, we formally pose a model projection operator $\mathcal{P}\mathcal{N}_k$ as

$$\mathcal{P}\mathcal{N}_k = \left\{ \sum_{k'} \mathcal{M}_{k,\,-\,k'} f_{-k'} \left(\partial/\partial t + \mathcal{L}^{(0)} - \mathcal{N}'\right)^{-1}_{k+k'} \mathcal{M}_{k,\,k'} f_{k'} \right\} f_k \tag{A6}$$

In the previous theory, the method of dressed test mode, the statistically independent part $(1 - \mathcal{P})\mathcal{N}_k$ was not kept, but the role of the term $\mathcal{P}\mathcal{N}_k$ was considered important. In evaluating Eq.(A4), further assumption is made to close Eq.(A6). First, it is considered that there are a large number of independent fluctuation components.

This leads the nature that $\mathcal{N}'$ is approximately equal to $\mathcal{N}$. And then $\mathcal{N}'$ in $\left(\partial/\partial t + \mathcal{L}^{(0)} - \mathcal{N}'\right)^{-1}_{k+k'}$ is assumed to be replaced by $\mathcal{PN}_{k+k'}$. This assumption needs a further consideration which is left for our future study. With these procedures, Eq.(A6) is given in a form of recurrent formula with respect to $\mathcal{PN}$ as

$$\mathcal{PN}_k = \left\{ \sum_{k'} \mathcal{M}_{k, -k'} f_{-k'} \left(\partial/\partial t + \mathcal{L}^{(0)}_{k+k'} - \mathcal{PN}_{k+k'}\right)^{-1} \mathcal{M}_{k, k'} f_{k'} \right\} f_k . \qquad \text{(A7)}$$

This is approximated by the effective diffusion operator,

$$\mathcal{PN}_{i, k} = -\mu_{i, k} k_\perp^2 f_{i, k} \qquad \text{(A8)}$$

and the renormalization formula of $\mu_{j, k}$ was given in [32]. In Eq.(14) in the text the notation

$$\mathcal{PN}_{i, k} = \gamma_{i, k} f_{i, k} \qquad \text{(A9)}$$

is used.

The projection of the rest, $(1 - \mathcal{P})\mathcal{N}_k$, is statistically independent of f_k. In the zero-th order approximation, all the fluctuation components are considered to be almost statistically independent. Therefore, an instantaneous amplitude of $\mathcal{N}_k$ is close to that of $(1 - \mathcal{P})\mathcal{N}_k$. The interaction time between two components to generate f_k is short. In the first step, the short interaction time is modelled by the delta-function. These considerations are the basis in modelling the statistically independent part $(1 - \mathcal{P})\mathcal{N}_k$ by the form of Eq.(19) in the text.

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Figure Captions

Fig.1 Relations between the decorrelation rate λ_1 and the viscous damping rate γ_v are given by Eq.(53) (solid line) and Eq.(26) (dashed line). $(\gamma_{sc}, \lambda_{sc})$ denotes the self consistent solution, and γ_* is the estimate which neglected the noise effect [32].

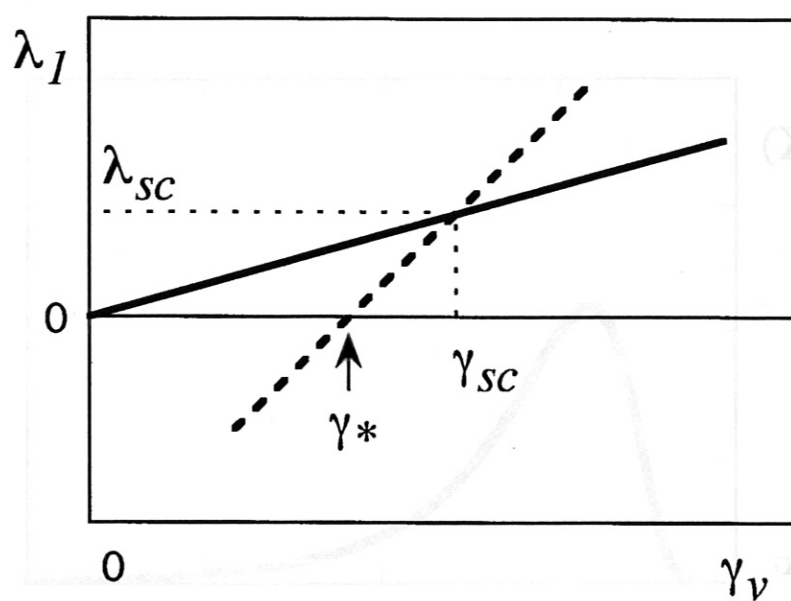**Fig.2** Schematic drawing of the probability distribution function. (Abscissa X denotes the fluctuation level.)

Fig.1

Fig.2