IPP-Report

Max-Planck-Institut
für Plasmaphysik

G.O. Spies

Comment on "The ideal magnetohydrodynamic continuous spectrum in a cylindrical screw pinch: a question of completeness"

IPP 6/348
May 1998

G.O. Spies

Comment on "The ideal magnetohydrodynamic continuous spectrum in a cylindrical screw pinch: a question of completeness"

IPP 6/348
May 1998

In a recent investigation, Thyagaraja et al[1] raked up an old dispute regarding the frequency spectrum of one-dimensional magnetohydrostatic equilibria (circular cylinders or plane slabs): They claim that, for a fixed two-dimensional wave vector in the pressure surfaces, the continuous spectrum contains not only the Alfvén continuum and cusp continuum, but also the two magnetosonic continua postulated a quarter of a century ago by Grad,[2] but then refuted by Appert et al[3] and Goedbloed.[4] They also claim that the cusp continuum disappears in the homogeneous limit. It is now shown that there are no magnetosonic continua (the singularities that could give rise to these being removable), and that no continuum disappears in the homogeneous limit (the Alfvén continuum becoming an eigenvalue of infinite degeneracy, and the cusp continuum an accumulation point of eigenvalues).

We start from the equations of linearized ideal magnetohydrodynamics, Fourier-decomposed in time by assuming a time dependence $\exp i\omega t$ (the frequency ω then acting as an eigenvalue parameter), and written in units such that the vacuum permeability equals unity:

$$i\omega\rho\,\mathbf{u} + \nabla p + \mathbf{B} \times \operatorname{curl}\mathbf{b} + \mathbf{b} \times \operatorname{curl}\mathbf{B} = 0, \tag{1}$$

$$i\omega p + \mathbf{u} \cdot \nabla P + \gamma P \operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$i\omega\mathbf{b} - \operatorname{curl}(\mathbf{u} \times \mathbf{B}) = 0. \tag{3}$$

Here, $\gamma = 5/3$ is the ratio of specific heats, ρ is the equilibrium mass density, P and $\mathbf{B}$ are the equilibrium pressure and magnetic field vector, p and $\mathbf{b}$ are the corresponding perturbations, and $\mathbf{u}$ is the perturbing velocity vector. The domain is a torus, so that all physical quantities are periodic in two coordinates. The radial boundary conditions (i. e. those at the surface) need not be specified because the continuous spectrum, unlike discrete eigenfrequencies, does not depend on these.

Attention is restricted to plane slab equilibria because these, while sharing with a circular cylinder the essential features of the continuous spectrum, are simpler. Equilibrium quantities thus depend only on one 'radial' Cartesian coordinate x, and perturbing quantities depend on the two ignorable coordinates y and z only through a factor $\exp i(\mathbf{k}\cdot\mathbf{x})$, where $\mathbf{x}$ is the position vector, and the two components k_y and k_z of the wave vector $\mathbf{k}$ take discrete values (viz. multiples of some inverse lengths). Following Appert et al,[3] we express the perturbing pressure p in terms of the perturbing total pressure $p_+ = p + \mathbf{B}\cdot\mathbf{b}$ and then eliminate all dependent variables except $u = u_x$ and $v = i\omega p_+$. Equations (1)-(3) then reduce to the two first-order ordinary differential equations

$$B_+^2\, A\, C\, u' + \Delta\, v = 0, \qquad v' - A\, u = 0, \tag{4}$$

where the primes denote the derivatives with respect to x, $B_+^2 = B^2 + \gamma P$ (with $B^2 = |\mathbf{B}|^2$),

$$A = \omega^2\rho - F^2, \qquad C = \omega^2\rho - \beta F^2 \tag{5}$$

(with $F = \mathbf{k}\cdot\mathbf{B}$ and $\beta = \gamma P/B_+^2$), and

$$\Delta = \omega^4\rho^2 - B_+^2 k^2\omega^2\rho + \gamma P k^2 F^2 \tag{6}$$

(with $k^2 = |\mathbf{k}|^2$). Once the system (4) is solved for $u(x)$ and $v(x)$, the two remaining components of $\mathbf{u}$ are computed algebraically from

$$\omega^2\rho\,(\mathbf{k}\cdot\mathbf{u}) = i(k^2 v + F^2 u'), \qquad \omega^2\rho\, A\,(\mathbf{B}\cdot\mathbf{u}) = iF(Dv + B^2 A u') \tag{7}$$

(with $D = \omega^2\rho - k^2B^2$).

A continuous spectrum requires some singularity in the equations (this being necessary, but not sufficient). Since there are obviously no singularities other than the zeroes of A (Alfvén continuum) and C (cusp continuum), there are no other continua. However, if the quantity v is eliminated, then a denominator Δ appears in both the resulting Hain-Lüst equation[5] for the radial velocity u (which led Grad[2] to believe that the two zeroes of Δ give rise to two magnetosonic continua), and the algebraic equations (7) for the other velocity components (which led Thyagaraja et al[1] to the same conclusion). The use of the perturbing total pressure as a dependent variable shows that the zeroes of Δ are removable singularities.

In the homogeneous case, equilibrium quantities are independent of x, and the frequency intervals $A = 0$ and $C = 0$ are single points. It is shown that these points still belong to the continuous spectrum: If $A = 0$, then $v = 0$, and arbitrary functions $u(x)$ are solutions. Hence the Alfvén frequency is an eigenfrequency of infinite degeneracy; it also belongs to the continuous spectrum because the function $u(x)$ need not be square-integrable. If $A \neq 0$, then the dispersion relation is

$$\omega^4\rho^2 - B_+^2 K^2\omega^2\rho + \gamma P K^2 F^2 = 0, \tag{8}$$

where $K^2 = k^2 + \lambda^2$, and λ is an eigenvalue of the equation $u'' + \lambda^2 u = 0$ ($\mathbf{K} = \mathbf{k} + \lambda\nabla x$ is the three-dimensional wave vector). The roots of Eq. (8) are the frequencies of the magnetosonic waves. They depend on the radial boundary conditions because the eigenvalues λ do. The usual periodic boundary conditions $u(x + L) = u(x)$ yield $\lambda = 2\pi n/L$ (with an arbitrary integer n), while the general one-point boundary conditions $u(0) = \beta_1 u'(0)$, $u(L) = -\beta_2 u'(L)$, which are more appropriate for simulating a genuine torus ($\beta = 0$, for instance, corresponds to a perfectly conducting rigid wall, and $\beta^{-1} = 0$ corresponds to an adjacent vacuum), yield

$$(\beta_1 + \beta_2)\lambda \cot \lambda L = 1 - \beta_1\beta_2\lambda^2. \tag{9}$$

The eigenvalues λ, and hence also the magnetosonic frequencies, are thus real, and accumulate at $\pm\infty$, for any boundary conditions of interest. In the limit $K^2 \to \infty$, with fixed F, one root of Eq. (8) diverges and the other approaches the zero of C. The point $C = 0$, though not an eigenfrequency, is thus an accumulation point of eigenfrequencies and hence belongs to the continuous spectrum. It does not depend on the radial boundary conditions even though the eigenfrequencies do.

Acknowledgments

Thanks are due to A. Salat for helpful comments. This work was performed under the terms of the agreement on association between Max-Planck-Institut für Plasmaphysik and Euratom.

References

[1]A. Thyagaraja, C. N. Lashmore-Davies, and R. A. Cairns, UKAEA FUS 385, February 1998, ISBN: 0853111928 (also presented at the Sherwood Theory Conference 1998)

[2]H. Grad, Proc. Natl. Acad. Sci. USA, 70, 3277 (1973)

[3]K. Appert, R. Gruber, and J. Vaclavik, Phys. Fluids 17, 1471 (1974)

[4]J. P. Goedbloed, Phys. Fluids 18, 1258 (1975)

[5]K. Hain and R. Lüst, Z. Naturforsch. 13a, 936 (1958)