

Cut-offs and mode conversions  
in a Vlasov plasma

*Riccardo Croci*

IPP III/218

June 1997



**MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK**

**85748 GARCHING BEI MÜNCHEN**

**MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK**  
**GARCHING BEI MÜNCHEN**

**Cut-offs and mode conversions  
in a Vlasov plasma**

*Riccardo Croci*

IPP III/218

June 1997

*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem  
Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über  
die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt*

## Abstract

A consistent description of the propagation of an e.m. field perturbation through a succession of cut-offs and mode conversions in a Vlasov plasma is derived. Whereas the quantitative description confirm some qualitative expectations, the results about the transmitted energy are less obvious. In particular, the transmitted energy is larger when the direction of the incident wave is such that the cut-off is encountered first, than for the opposite direction. Moreover, the transmission coefficients for forwards and backwards waves strongly depends on the plasma parameters.

## 1. Introduction

A consistent description of the propagation of an e.m. field perturbation through a succession of cut-offs and mode conversions is of obvious interest in HF heating problems — in particular ICRH, where the presence of such ‘relevant points’ is essential. It is moreover recalled that the presence of cut-offs and/or mode conversions forbids using the Kirchhoff’s law  $E = (1 - e^{-\tau}) I_{BB}$  ( $\tau$  is the optical depth and  $I_{BB}$  is the black-body emissivity), which is sometimes applied outside its validity range (see e.g. Shvets & Swanson, 1993). When a group of ‘relevant points’ is given, it is enough to consider a plasma slab with the  $x$ -direction normal to the cut-offs and mode conversion curves (with abscissa  $x_q$ ) contained in it; the inhomogeneity in the slab is due to density and/or equilibrium magnetic field variations, and is assumed to be weak. This is an appropriate description also for ion cyclotron heating problems (see e.g. Perkins, 1977). The plasma is described by the Maxwell and the linearized Vlasov equations. The displacement currents are derived in Section 2 from the Vlasov equation with the ansatz  $E_j(x) \exp(i \int k_x(x) dx)$  for the electric field (with  $E_j$  slowly varying with  $x$ ). They are a good approximation when  $\rho(x, \mathbf{v}, t) |\partial_x k_x / k_x| \ll 1$ , which is a less stringent condition than the often used  $\rho |k_x| \ll 1$ . Then a nonlinear differential equation for  $k_x(x)$  is obtained from the Maxwell equations. The approximate solutions of this equation are derived with usual methods in Section 3 by considering the derivatives of  $k(x)$  as small quantities. The (well known) fact that these solutions are not valid in the neighbourhood of the ‘relevant points’ shows that it is wrong to deduce an equation for the ‘relevant points’ by letting  $ik_x \rightarrow d/dx$  in the local dispersion relation developed in powers of  $k^2$  (as done, for example, in Swanson 1995), thereby ignoring the presence of the derivatives of  $k(x)$ . The correct solutions of the nonlinear equation in the intervals of interest are derived in Section 4 for a particular group of ‘relevant points’ consisting of a cut-off and two mode conversions. This example has been chosen because it is of interest in ICR heating problems (see e.g. Perkins 1977) and is, moreover, complex enough to illustrate the method. The solutions valid in the various intervals are connected by a ‘matched asymptotic procedure’ (see e.g. Murray, Asymptotic Analysis). Preliminary, necessary results are obtained in Section 5. The connection is done in Section 6 when the incoming perturbation reaches first the cut-off and then the mode conversions (source in  $-\infty$ ), and in Section 7 when the source is in  $+\infty$ . The Poynting vectors for the transmitted waves in the two cases are derived in Section 8. The results are summarized in the Conclusion.

## 2. The equation

For a consistent description of the propagation of an e.m. field perturbation through a succession of cut-offs and mode conversions it is sufficient to solve the problem in a plasma slab with the  $x$ -direction normal to the cut-off and mode conversion surfaces. Variations in  $x$  are due either to the dependence of the equilibrium magnetic field on the tokamak major radius  $R$  (the situation to be found in the ion-ion hybrid resonance heating), or to minor-radius variations of the density (as in the ion-cyclotron mode conversion). Then one has (see also Perkins, 1977)  $k_{\parallel} = k_z + (r/qR)k_x \cos \theta$ , where  $r$  is the tokamak minor-radius,  $\theta$  is the poloidal variable and  $q$  is the tokamak safety factor. Let the electric field be of the form (with  $k$  for  $k_x$  from now on):

$$E_j(x) \exp\left(i \int^x k(x) dx\right).$$

The dependent variables are  $k(x)$  and two of the three  $E_j$ ; in this paper the choice is

$$E_{\parallel} = \text{const}, \quad P_x \equiv E_x/E_{\parallel}, \quad P_y \equiv E_y/E_{\parallel}.$$

In the integrals over time and velocity that appear in the displacement current the upper limit of the integral over  $x$  is  $x + \rho(t, x, \mathbf{v})$ . It will be assumed that

$$\int^x \approx \int^{x+\rho} -\rho k \quad \text{and} \quad E_j(x + \rho) \approx E_j(x).$$

The first of these approximations is correct if  $\rho|\partial_x k/k| \ll 1$ . The correction to the second is of order  $\rho^2$ , and will be neglected. In this way one obtains the local approximation of the dielectric tensor  $\epsilon_{ij}$ , which can be used also for  $\rho|k| \geq 1$  (if  $\rho|\partial_x k/k| \ll 1$ ). In accordance with the assumption the  $P_j$  are given by the local approximation, that is by:

$$P_x = \frac{n^2 - \epsilon_{33}}{n_{\parallel} n}, \quad P_y = \frac{\epsilon_{12}(n^2 - \epsilon_{33})}{n_{\parallel} n(\epsilon_{22} - n_{\parallel}^2 - n^2)}. \quad (1)$$

With the notation  $P'_j \equiv (d/dx)P_j$  one has  $P'_j = \partial_x P_j + 2nn'\partial_{n^2} P_j$ ; thus the first term will be neglected because the direct dependence on  $x$  is slow. The Maxwell equations (with  $\epsilon_{13} = \epsilon_{23} = 0$  for simplicity):

$$\begin{aligned} (\epsilon_{11} - n_{\parallel}^2)E_x(x) + \epsilon_{12}E_y(x) - i(c/\omega)n_{\parallel}\partial_x E_{\parallel}(x) &= 0, \\ \epsilon_{21}E_x(x) + (\epsilon_{22} - n_{\parallel}^2 + (c^2/\omega^2)\partial_{xx})E_y(x) &= 0, \\ -i(c/\omega)n_{\parallel}\partial_x E_x(x) + (\epsilon_{33} + (c^2/\omega^2)\partial_{xx})E_{\parallel}(x) &= 0, \end{aligned} \quad (2)$$

thus become:

$$\begin{aligned} (\epsilon_{11} - n_{\parallel}^2)P_x + \epsilon_{12}P_y(x) + n_{\parallel}n &= 0, \\ \epsilon_{21}P_x + (\epsilon_{22} - n_{\parallel}^2 - n^2 + i(c/\omega)n' + 2in(c/\omega)(P'_y/P_y) + (c^2/\omega^2)(P''_y/P_y))P_y &= 0, \\ (n_{\parallel}n - in_{\parallel}(c/\omega)(P'_x/P_x))P_x + \epsilon_{33} - n^2 + i(c/\omega)n' &= 0. \end{aligned} \quad (3)$$

Since  $P'_j \approx 2nn'\partial_{n^2} P_j$ , and the  $P_j$  are given, the solvability condition of the system (3) is the following nonlinear differential equation for  $n$ :

$$(\epsilon_{11} - n_{\parallel}^2)(\epsilon_{22} - n_{\parallel}^2 - n^2 + i(c/\omega)n' + 2in(c/\omega)(P'_y/P_y) + (c^2/\omega^2)(P''_y/P_y)).$$

$$\begin{aligned} & \cdot (\epsilon_{33} - n^2 + i(c/\omega)n') + \epsilon_{12}^2 (\epsilon_{33} - n^2 + i(c/\omega)n') - n_{\parallel} n (n_{\parallel} n - i n_{\parallel} (c/\omega) (P'_x/P_x)) \cdot \\ & \cdot (\epsilon_{22} - n_{\parallel}^2 - n^2 + i(c/\omega)n' + 2in(c/\omega)(P'_y/P_y) + (c^2/\omega^2)(P''_y/P_y)) = 0. \end{aligned} \quad (4)$$

Equation (4) will also be written as  $D = i\Delta$ , where  $\Delta$  contains the derivatives of  $n$ , so that

$$D \equiv (\epsilon_{11} - n_{\parallel}^2)(\epsilon_{22} - n_{\parallel}^2 - n^2)(\epsilon_{33} - n^2) + \epsilon_{12}^2 (\epsilon_{33} - n^2) - n_{\parallel}^2 n^2 (\epsilon_{22} - n_{\parallel}^2 - n^2) = 0 \quad (5)$$

is the local dispersion relation. Usually  $n'$  is treated as a small correction in equation (4); then an approximate solution is obtained by adding to the solution  $k_o^2(x)$  of  $D = 0$  the correction  $\delta$  given by:

$$\delta/ = i\beta, \quad \text{where } \beta = \Delta(k_o^2)/D'(k_o^2), \quad (6)$$

and the prime of  $D$  denotes partial derivation with respect to  $k^2$ . Since  $k \approx k_o + \delta/2k_o$ , this solution is acceptable only far enough from cut-offs ( $k_o = 0$ ) and of mode conversion points ( $D' = 0$ ). In the neighbourhood of such points  $n'$  cannot be considered as a mere correction, so that it is wrong to use there differential equations derived from the local dispersion relation by the replacement  $ik \rightarrow d/dx$ , as is often done (see for example Swanson, 1995). In this paper equation (4) is solved correctly by retaining only the terms of  $\Delta$  linear in  $n'$ . This approximation is justified because the derived solution satisfies  $|k'/k^2| \ll 1$ , as will be shown in Section 4. The approximate form of  $\Delta$  is thus (with a change of notation for  $\Delta$ ):

$$\begin{aligned} \Delta n' = & ((\epsilon_{11} - n_{\parallel}^2)(\epsilon_{22} + \epsilon_{33} - n_{\parallel}^2 - 2n^2) + \epsilon_{12}^2 - n_{\parallel}^2 n^2) + \\ & + 4n^2 (\epsilon_{33} - n^2 - n_{\parallel}^2 n^2) (\partial_{n^2} P_y/P_y) + 2n_{\parallel}^2 n^3 (\epsilon_{22} - n_{\parallel}^2 - n^2) (\partial_{n^2} P_x/P_x). \end{aligned} \quad (7)$$

### 3. The approximate solution

Some more details about the approximate solution will be useful in the following sections. We begin with the cut-off, which is assumed to be at the point  $x = x_c$ . Let us introduce an interval  $S_c$  (of yet undefined width) that contains the point  $x_c$ , and an interval  $s_c$  that does not contain  $x_c$  but is partly superposed to  $S_c$ . The approximation deduced in this section is valid in  $s_c$ ; in  $s_c \cap S_c$  it is equal to the asymptotic approximation of the true solution valid in  $S_c$ , as will be shown in the next section. In  $s_c \cap S_c$  one can obviously write

$$D(k_o^2, x) \approx k_o^2 D'(0, x_c) + (x - x_c) D_x, \quad \text{and} \quad D'(k_o^2, x) \approx D'(0, x_c). \quad (8)$$

Since it can be seen that  $\Delta(n_{\parallel} = 0, n = 0) = D'$ , to simplify the exposition it will be assumed  $D' \approx \Delta$ , so that the effect on the electric field of the correction of  $k_o$  due to  $\partial_x n$  is the factor  $|k_o|^{-\beta_c/2}$ , with  $\beta_c \approx 1$ . The approximate solution is valid in the interval  $|z|^{3/2} \gg 1$ , where  $z \equiv (D_x/\Delta\beta_c)^{1/3}(x - x_c)$ . It can be extended formally to the whole  $s_c$  if the  $x$  dependence of  $D$  is weak.

In the case of a mode conversion at  $x = x_p$  with  $k^2 = k_p^2$ , intervals  $S_p$  and  $s_p$  are introduced similarly to before. In  $s_p \cap S_p$  one can write

$$\begin{aligned} D(k_o^2, x) &\approx (x - x_p)D_{/x} + (k_o^2 - k_p^2)^2 D''(k_p^2, x_p)/2, \\ D'(k_o^2, x) &\approx (k_o^2 - k_p^2)D''(k_p^2, x_p). \end{aligned} \quad (9)$$

Hence the correction of the integral of  $k_o$  over  $x$  due to  $\partial_x k$  is:

$$-\int \frac{(k_o^2)' \Delta}{4k_o^2 D'} dx \rightarrow -\frac{\Delta}{4D''} \int \frac{(k_o^2)'}{k_o^2(k_o^2 - k_p^2)} dx \rightarrow (\beta_p/2) \ln \frac{k_o^2}{k_o^2 - k_p^2}, \quad (10)$$

where  $\beta_p \equiv \Delta/2k_p^2 D''$ . It is important to note that  $\beta_p$  is not necessarily positive, so that — in the region where  $k_o^2 > 0$  — the modulus of the electric field can increase or decrease by increasing  $|x - x_p|$ , whereas in the case of a cut-off it can only decrease by increasing  $|x - x_c|$ . The solution now derived is valid in the interval  $|z|^{3/2} \gg 1$  (with  $z \equiv (D_{/x}/\Delta\beta_p)^{1/3}(x - x_p)$ ) of  $s_p \cap S_p$ , but not in the whole  $s_p$ , because equation (9) implies  $D'(k_p^2, x_p) = 0$ .

#### 4. The correct solution

The solution will be derived for a succession of cut-offs and mode conversion points that is physically interesting (it is found in the ion-ion hybrid heating, see for example Perkins), and is complex enough to illustrate the method. The first 'relevant point' is a cut-off at  $x = x_c$ , with  $k_o^2 > 0$  for  $x < x_c$ . Then two mode conversions follow, in  $x = x_m$  and  $x = x_n$ , with  $k_o^2(x_m) = -k_m^2$  and  $k_o^2(x_n) = k_n^2$ ;  $k_o^2$  is positive for  $x > x_n$ . Thus the local dispersion relation has two real  $k^2$  solutions in  $x < x_m$  and in  $x > x_n$ . The intervals introduced in the previous section are now  $s_1, S_2, s_3, S_4, s_5, S_6, s_7$ ; the intervals  $S_j$  contain the corresponding relevant point.

With the notation  $k_o^2 = k_p^2 + 2k_p k_1$  for the solutions in the neighbourhood of the mode conversion points, the following properties are immediate consequences of the local dispersion relation:

- In order that  $k_o^2 > 0$  in  $x < x_c$  it must be  $D_{/x} > 0$ .
- In order that  $k_o^2$  be real in  $s_3 \cap S_4$  where  $k_p^2 = -k_m^2$  it must be  $k_1^2 < 0$ , and thus (since  $x < x_m$ )  $D''_m D_{/x} > 0$ . The sign of  $\beta_m$  depends on the sign of  $\Delta(k_m^2)$ .
- In  $S_4 \cap s_5$  one has  $x > x_m$ , and thus  $k_1^2 > 0$ .
- In  $s_5 \cap S_6$  it must be  $k_1^2 < 0$  and thus  $D''_n D_{/x} < 0$ . Since it is assumed that  $\Delta$  has no zeros between  $x_m$  and  $x_n$ ,  $\beta_m$  and  $\beta_n$  have the same sign.

In the neighbourhood of the cut-off in  $x_c$  equation (4) becomes

$$k^2 D' + (x - x_c) D_{/x} = i\Delta \partial_x k, \quad (11)$$

a Riccati equation. It is reduced to a linear differential equation by the ansatz

$$\beta_c F'/F = ik, \quad \text{that is} \quad E_z \propto F^{\beta_c}.$$

As  $\beta_c = \Delta/D' \approx 1$  (see preceding section), one obtains for  $F$  the Airy equation

$$F'' - (x - x_c)(D_{/x}/D')F = 0. \quad (12)$$

The argument of the Airy functions is one of the three values obtained by multiplying the (real) quantity  $z \equiv (x - x_c)(-D_{/x}/D')^{1/3}$  with the third roots of unity. The final result being the same,  $z$  is chosen as argument.

The equation valid in the neighbourhood of the mode conversion points is

$$(k^2 - k_p^2)^2 D''/2 + (x - x_p)D_{/x} = i\Delta \frac{\partial k}{\partial x}. \quad (13)$$

It is convenient to introduce the function  $K \equiv k^2 - k_p^2$  (so that  $k \approx k_p + K/2k_p \equiv k_p + k_1$ ); with it equation (13) becomes

$$K^2 D''/2 + (x - x_p)D_{/x} - (i\Delta/2k_p) \frac{\partial K}{\partial x} = 0, \quad (14)$$

or

$$k_1^2 + (x - x_p)D_{/x}/2k_p^2 D'' - (i\Delta/2k_p^2 D'') \frac{\partial k_1}{\partial x} = 0. \quad (15)$$

This is again a Riccati equation; the corresponding linear Airy equation is obtained with the ansatz  $ik_1 = \beta_p F'/F$ , where  $\beta_p = \Delta/2k_p^2 D''$ . The Airy functions have the argument  $z \equiv (x - x_c)(-D_{/x}/\Delta\beta_p)^{1/3}$ . The electric field is given by

$$E_z \propto e^{ik_p x} F^{\beta_p}.$$

Only few properties of the Airy functions are needed in the following. The Airy functions are entire functions of their argument. Their asymptotic approximations for real argument are (with  $d \equiv 1/2\pi^{1/2}|z|^{1/4}$  and  $y \equiv (2/3)|z|^{3/2}$ ):

$$\text{Ai}(z) \approx (d/2)e^{-y}, \quad \text{Bi}(z) \approx de^y \quad \text{for } z > 0;$$

$$\text{Ai}(z) \approx d \sin(y + \pi/4), \quad \text{Bi}(z) \approx d \cos(y + \pi/4) \quad \text{for } z < 0.$$

With  $\phi = \exp(i\pi/3)$  and  $\ell = 3^{1/3}\Gamma(2/3)/\Gamma(1/3) \approx 1.37$ , in  $z = 0$  one has (the prime denotes derivation with respect to  $z$ ):

$$\text{Bi}'/\text{Bi} \equiv \ell, \quad \text{Ai}'/\text{Ai} = -\ell, \quad (\text{Bi} - i\text{Ai})'/(\text{Bi} - i\text{Ai}) \equiv \ell\phi.$$

The following identities can be helpful in evaluating the results:

$$1 - \phi = -\phi^2, \quad 1 - \phi^* = \phi, \quad 1 + \phi = 2i\phi_1(1 - \phi) = (1 + \phi^*)\phi. \quad (16)$$

## 5. Connection of the WKB solutions

Suppose that in  $S_4 \cap s_5$  the solution is  $k_o = ik_m + k_5(x)$  ( $k_m$  and  $k_5$  positive, for definiteness; the notation  $k_5$  instead of  $k_1$  for clarity). It should be connected in  $s_5 \cap S_6$  with one of the solutions valid there, which are of the kind  $k_o = k_n + ik_6(x)$ ; the connection criterium is that  $k_{oR}$  and  $k_{oI}$  do not change their sign in  $s_5$ , so that  $k_n$  and  $k_6$  have both to be positive. Thus an approximation of  $D = 0$  valid in  $s_5$  is required. To this purpose let us consider a point  $x'_m > x_m$  that belongs to  $S_4 \cap s_5$  and define  $y_5 \equiv k_o^2(x'_m)$ . When  $D$  is expanded about  $(x'_m, y_5)$  (where  $D' \approx (x'_m - x_m)D''_m$ ) one obtains the dispersion relation

$$(y - y_5)(x'_m - x_m)D''_m + (x - x'_m)D_{/x} = 0. \quad (17)$$

Is it possible to chose a point  $x'_n < x_n$  of  $s_5 \cap S_6$  — where the solution is  $y_6 \equiv k_o^2(x'_n)$  — such that equation (17) be valid in all  $(x'_m, x'_n)$ ? The answer is positive if one has

$$(y_6 - y_5)(x'_m - x_m)D''_m + (x'_n - x'_m)D_{/x} = 0. \quad (18)$$

Since  $y_6 - y_5 \approx k_m^2 + k_n^2 + 2i(k_mk_5 - k_nk_6)$ , equation (18) is equivalent to the two conditions:

$$k_nk_6 = k_mk_5, \quad (k_m^2 + k_n^2)(x'_m - x_m)D''_m + (x'_n - x'_m)D_{/x} = 0. \quad (19)$$

With the notations  $a_m \equiv x'_m - x_m$  e  $k_5^2 = b_ma_m$  (and similarly for the index  $n$ ) equation (19) can be written

$$k_n^2b_na_n = k_m^2b_ma_m, \quad (k_m^2 + k_n^2)a_mD''_m/D_{/x} + a_n - a_m = x_m - x_n. \quad (20)$$

The solutions of the system (20) determine  $(x'_m, x'_n)$  and thus the required dispersion relation (17).

The connection of the solutions valid in  $S_2 \cap s_3$  and in  $s_3 \cap S_4$  is easier: it is enough to use equation (11) without  $\partial_x k$  (if  $x_c$  and  $x_m$  are not too wide apart).

If in the interval  $S_4 \cap s_5$  the electric field is proportional to

$$\exp\left(i \int_{x_m}^x k_5 dx\right)$$

(the lower limit of the integral is approximated with  $x_m$  instead of being  $x'_m$ ), in  $s_5 \cap S_6$  it is proportional to

$$\exp\left[i \int_{x_m}^{x_n} k_5 dx + i \int_{x_n}^x k_5 dx\right] \rightarrow K_m \exp\left(i \int_{x_n}^x k_5 dx\right).$$

The last exponential is the asymptotic form of the solutions valid in  $S_6$ , as shown in Section 4.

In the interval  $s_3$  a similar procedure leads to the introduction of the corresponding quantity  $K_c$ .



## 6. Source in $-\infty$

We now have all the elements necessary to write  $E_z$  in the various intervals. Throughout the next sections following definitions will be used:

$$(\text{Bi} - i\text{Ai})^\beta \equiv F, \quad \text{Ai}^\beta \equiv G, \quad \text{Bi}^\beta \equiv H.$$

When the source is in  $-\infty$ , in  $x < x_c$  a perturbation propagates in the positive direction and a reflected perturbation propagates in the negative direction, with  $k^2$  that correspond to the upper branch of the local dispersion relation. The asymptotic approximations of the Airy functions show that the incident perturbation is represented by the function  $F$  and the reflected by  $F^*$ . Thus for  $x < x_c$  the electric field is given by

$$E_z e^{i\omega t} = I F^*(z) + R_1 F(z) + R_2 e^{\int_{x_c} k_2 dx}. \quad (21)$$

The third term (with  $k_2$  real positive) describes the reflected field whose  $k$  belongs to the lower branch of  $D = 0$ ; for this branch the WKB approximation is correct. In a symbolic and compact form the first two terms are represented by the triplets

$$\text{in } s_1 \cap S_2^< : (I/F^*, 0, F^*(z)), (R_1/F, 0, F(z)).$$

If the argument is not specified, it is  $z = 0$ . The amplitude normalization simplifies the form of the continuity conditions. The third term does not change in the transition to  $x > x_c$ , because it is well described by the WKB approximation.

The representation (21) cannot be used for  $x > x_c$ ; indeed, the asymptotic approximations show that the first two terms increase exponentially in  $z \gg 1$ , which is physically unacceptable. Thus the field in  $x > x_c$  is written as a linear superposition of the functions  $G$  (exponentially decreasing) and  $H$  (exponentially increasing), the coefficients of the superposition being chosen so that the field and its derivative are continuous in  $x_c$ . Accordingly, the part of the field not proportional to  $R_2$  is described by

$$\text{in } S_2^> \cap s_3 : (c_1/G, 0, G(z)), (c_2/H, 0, H(z)),$$

where the constants  $c_{1,2}$  are yet arbitrary. The continuity conditions are:

$$I + R_1 = c_1 + c_2, \quad I\phi^* + R_1\phi = -c_1 + c_2. \quad (22)$$

It is convenient to solve equations (22) in the form

$$(1 + \phi)R_1 = -(1 + \phi^*)I + 2c_2, \quad (1 + \phi)c_1 = (\phi - \phi^*)I + (1 - \phi)c_2. \quad (23)$$

As we have seen in Section 3, the upper branch of the solutions of  $D = 0$  are  $k_o^2 = -(\beta_m H'/H)^2$  in  $S_2^> \cap s_3$  and  $k_o^2 = -k_m^2 + 2ik_mk_1$ , with  $k_1^2 < 0$ , in  $s_3 \cap S_4^<$ . The connection of these solutions is obtained by assuming that  $k_{oI}$  does not change sign in  $s_3$ . Since the part of the field proportional to  $c_1$  has a negative derivative with respect to  $x$  in  $S_2^> \cap s_3$ , according to our criterium it has to be proportional to  $\exp(-k_m(x - x_m))$  in  $s_2 \cap S_4^<$ . Moreover, the part due to the Airy functions has to give a positive contribution to the  $x$ -derivative because the solution of  $D = 0$  belongs to the upper branch. When  $\beta_m < 0$  the corresponding triplet in  $s_3 \cap S_4$  is thus  $(c_1/HK_c, ik_m, H)$ ;  $K_c$  is the factor introduced in Section 5:

$$K_c = \exp \left[ i \int_{x_c}^{x_m} k_o dx \right],$$

with  $k_{oI} < 0$ . The amplitude of the other two terms of the electric field contain the factor  $\exp(k_m(x - x_m))$ . The triplet with  $c_2$  belongs to the upper branch; thus the part due to the Airy functions must have  $\partial_x < 0$  and therefore the function to be chosen in  $s_3 \cap S_4^<$  is  $G$ . The last triplet belongs to the lower branch, so that the Airy function contribution must have  $\partial_x > 0$ ; thus the function to be chosen is  $H$ . In conclusion, the triplets are

$$\text{in } s_3 \cap S_4^< : (c_1/HK_c, ik_m, H(z)), (c_2K_c/G, -ik_m, G(z)), (R_2K_2/H, -ik_m, H(z)).$$

The triplets for the case  $\beta_m > 0$  are obtained from those for  $\beta_m < 0$  by interchanging the functions  $G$  and  $H$ .

In  $S_4^> \cap s_5$  the Airy functions correspond to propagating perturbations and the triplets are:

$$\text{in } S_4^> \cap s_5 : (d_1/F_4, ik_m, F_4(z)), (d_2/F_4^*, -ik_m, F_4^*(z)).$$

where the  $d_p$  are yet arbitrary and the function  $F_4$  can be  $F$  or  $F^*$ . The continuity in  $x_m$  of the field and of its derivative require (for  $\beta_m$  positive or negative):

$$\begin{aligned} d_1 + d_2 &= c_1/K_c + c_2K_c + R_2K_2, \\ d_1 - d_2 &= c_1/K_c - c_2K_c - R_2K_2. \end{aligned} \quad (24)$$

At a mode conversion one should also require the continuity of the second derivative, which is given by

$$\partial_{xx}f = (-k_p^2f + 2ik_p\partial_x f + \partial_{xx}f) e^{ik_px}.$$

The term proportional to  $k_p^2$  has the same form as the condition for the continuity of the field (the first of the (24)), and therefore disappears. Moreover  $|\partial_{xx}f| \ll |k_p\partial_x f|$ , so that the continuity of the second derivative in  $x_m$  requires

$$d_1\phi_4 - d_2\phi_4^* = \delta_m(-c_1/K_c - c_2K_c + R_2K_2), \quad (25)$$

where  $\delta_m = \beta_m/|\beta_m|$ . It is convenient to solve equations (24) and (25) in the form:

$$\begin{aligned} d_1 &= c_1/K_c, & (1 + \delta_m\phi_4^*)d_2 &= 2c_2K_c + (1 + \delta_m\phi_4)c_1/K_c, \\ (1 + \delta_m\phi_4^*)K_2K_cR_2 &= (1 + \delta_m\phi_4)c_1 + (1 - \delta_m\phi_4^*)K_c^2c_2. \end{aligned} \quad (26)$$

The connection of the solutions valid in  $S_4^> \cap s_5$  with those valid in  $s_5 \cap S_6^<$  is done again by imposing that  $k_{oI}$  and  $k_{oR}$  do not change their sign. One recognises that (when  $\beta_m < 0$ ) the function  $F$  in  $S_4^> \cap s_5$  corresponds to  $k_n$  in  $S_6$ , and  $ik_m$  in  $S_4^> \cap s_5$  corresponds to the function  $G$  in  $s_5 \cap S_6^<$ . On the other hand, the sign of  $k_{oR}$  in  $S_4^> \cap s_5$  is not yet determined, so that the triplets are written in the general form:

$$\text{in } s_5 \cap S_6^< : (d_1/GK_m, \delta_5k_n, G(z)), (d_2K_m/H, -\delta_5k_n, H(z)),$$

where  $\delta_5 = \pm 1$ . The factor  $K_m$  has been introduced in Section 5:

$$K_m = \exp \left[ i \int_{x_m}^{x_n} k_o dx \right],$$

with  $k_{oI} < 0$ . Again, when  $\beta_m$  is positive one has to interchange the functions  $G$  and  $H$ .

The situation in the intervals  $S_6$  and  $s_7$  is more complex. As is well known, the upper branch of the solutions of  $D = 0$  describes backwards waves — a concept that has a meaning only for a ‘wave packet’, and not for only one value of  $\omega$ , as here. However, in order to make the choice as general as possible the triplets are written in a form that describes both possibilities (wave and energy propagation parallel or anti-parallel) at a time; when  $\beta_n$  is negative one can write:

$$\text{in } S_6^> \cap s_7 : (t_1/F_6, \delta_6 k_n, F_6(z)), (t_2/F^*, k_n, F^*(z)),$$

where the  $t_p$  are yet arbitrary;  $\delta_6 = \pm 1$  and  $F_6$  is equal to  $F$  when  $\delta_6 = 1$  and to  $F^*$  when  $\delta_6 = -1$ . The first triplet corresponds to the upper branch of  $D = 0$ . If  $\delta_6 = 1$  it is assumed that the energy propagates in the same direction as the waves. If  $\delta_6 = -1$  the assumption is that the energy propagates in the opposite direction as the waves. The second triplet corresponds to the lower branch of  $D = 0$ .

When  $\beta_n$  is positive one has to chose  $F_6 = F^*$  for  $\delta_6 = 1$  and  $F$  instead of  $F^*$  in the second triplet.

The continuity conditions in  $x_n$  of the field and its first two derivatives can be written in a compact form by introducing the parameter  $\delta_n = \beta_n/|\beta_n|$ ; then they are:

$$\begin{aligned} t_1 + t_2 &= d_1/K_m + d_2 K_m, & \delta_6 t_1 + t_2 &= \delta_5 (d_1/K_m - d_2 K_m), \\ \delta_6 t_1 \phi_6 + t_2 \phi^{\delta_n} &= \delta_n \delta_5 (d_1/K_m + d_2 K_m). \end{aligned} \quad (27)$$

With the definitions

$$A \equiv \delta_n \delta_6 \phi_6 (1 - \delta_5) + \delta_n \phi^{\delta_n} (\delta_5 - \delta_6) - \delta_5 (1 - \delta_6),$$

$$B \equiv \delta_n \delta_6 \phi_6 (1 + \delta_5) - \delta_n \phi^{\delta_n} (\delta_5 + \delta_6) - \delta_5 (1 - \delta_6),$$

equations (27) are solved in the form:

$$\begin{aligned} Ad_1/K_m + Bd_2 K_m &= 0, & (1 - \delta_6) B t_1 &= ((1 - \delta_5) B - (1 + \delta_5) A) d_1/K_m, \\ (1 - \delta_6) B t_2 &= ((\delta_5 - \delta_6) B + (\delta_5 + \delta_6) A) d_1/K_m. \end{aligned} \quad (28)$$

The first two equations (26) and the first of the (28) yield:

$$(1 + \delta_m \phi_4^*) A c_1 + B (2c_2 K_c^2 + (1 + \delta_m \phi_4) c_1) K_m^2 = 0, \quad (29)$$

or:

$$2c_2 K_c^2 = - [(1 + \delta_m \phi_4) + (1 + \delta_m \phi_4^*) A/B K_m^2] c_1. \quad (30)$$

Hence  $|c_2|^2 \ll |c_1|^2$ , and with the second of equations (23) one obtains:

$$c_1/I \approx (\phi - \phi^*)/(1 + \phi), \quad 2c_2K_c^2/I \approx (\phi^* - \phi)(1 + \delta_m\phi_4)/(1 + \phi).$$

With this result the reflection coefficient becomes:

$$\frac{1 + \phi}{1 + \phi^*} R_1/I = -1 + \frac{(1 + \delta_m\phi_4)(\phi^* - \phi)}{(1 + \phi^*)(1 + \phi)K_c^2}. \quad (31)$$

The solution will now be univocally determined by the condition that the reflected energy be less than the incident, i.e. that  $|R_1/I| < 1$ . In fact, with  $R_1$  given by (31) this condition implies  $\delta_m\phi_I\phi_{4I} > 0$ , that is  $\phi_4 = \phi^*$  and  $F_4 = F^*$ . The continuation of the solutions in the interval  $s_5$  then imposes the choice  $\delta_5 = -1$ . The transmission coefficients given by equations (28) are now:

$$(1 - \delta_6)K_cK_mt_1 = 2c_1, \quad (1 - \delta_6)BK_cK_mt_2 = ((-1 - \delta_6)B + (-1 + \delta_6)A)c_1. \quad (32)$$

Since  $B \propto (1 - \delta_6)$  one has to choose  $\delta_6 = -1$ , which means that waves and energy have opposite propagation directions. The choice  $\delta_6 = -1$  implies  $\phi_6 = \phi^{\delta_n}$ , so that equations (32) give, with the help of (16):

$$K_cK_mt_1 = c_1 \rightarrow (\phi - \phi^*)/(1 + \phi), \quad K_cK_mt_2 = -Ac_1/B \rightarrow c_1 \frac{\delta_n - \phi^{\delta_n}}{\delta_n + \phi^{\delta_n}}. \quad (33)$$

Hence  $|t_2/t_1|^2 = |(1 - \phi)/(1 + \phi)|^2 \rightarrow 1/3$  when  $\beta_n$  is positive, and  $|t_2/t_1|^2 = |(1 + \phi^*)/(1 - \phi^*)|^2 \rightarrow 3$  when  $\beta_n$  is negative.

In conclusion, most of the energy is reflected at the cut-off; the transmitted energy is transported preferentially by the waves of the lower branch of  $D = 0$  when  $\beta_n$  is negative, and by the waves of the upper branch when  $\beta_n$  is positive.

## 7. Source in $+\infty$

In this section we consider the case of waves incident from  $+\infty$  on the mode conversion point at  $x = x_n$  along the lower branch of  $D = 0$ . If the problem of the relative propagation directions of energy and waves of the upper branch solutions of  $D = 0$  is left for the moment unsettled, the triplets are

$$\text{in } S_6^> \cap s_7: \quad (I/F, -k_n, F(z)), \quad (R_1/F_6, \delta_6 k_n, F_6(z)), \quad (R_2/F^*, k_n, F^*(z)),$$

where  $\delta_6 = \pm 1$  and, consequently, when  $\beta_n$  is negative  $F_6 = F$  ( $\delta_6 = 1$ ; same propagation direction for energy and waves) or  $F^*$  ( $\delta_6 = -1$ ; opposite propagation directions). The first triplet represents the incident field. The second (third) represents the reflected field described by the upper (lower) branch of  $D = 0$ .

When  $\beta_n$  is positive one should interchange the functions  $F$  and  $F^*$ .

In the next interval the triplets for  $\beta_n$  negative are:

$$\text{in } s_5 \cap S_6^<: \quad (c_1/H, -k_n, H(z)), \quad (c_2/G, k_n, G(z)),$$

where the  $c_p$  are yet undetermined. The triplets for  $\beta_n$  positive are again obtained by the interchange of the functions  $G$  and  $H$ .

The continuity conditions in  $x_n$  of the field and its first two derivatives can be written as in the preceding section in a compact form by introducing the parameter  $\delta_n = \beta_n/|\beta_n|$ ; then they are:

$$I + R_1 + R_2 = c_1 + c_2, \quad I - \delta_6 R_1 - R_2 = c_1 - c_2,$$

$$-I\phi^{-\delta_n} + R_1\delta_6\phi_6 + R_2\phi^{\delta_n} = \delta_n(c_1 + c_2). \quad (34)$$

It is convenient to solve these equations in the form:

$$\text{for } \delta_6 = 1 : c_1 = I, R_1 + R_2 = c_2, \delta_n(\phi_6 - \phi^{\delta_n})R_1 = -(1 + \delta_n\phi^{-\delta_n})I - (1 - \delta_n\phi^{\delta_n})c_2, \quad (35)$$

$$\text{for } \delta_6 = -1 : R_2 = c_2, c_1 = I + R_1, (1 + \delta_n\phi_6)R_1 = -(1 + \delta_n\phi^{-\delta_n})I - (1 - \delta_n\phi^{\delta_n})c_2. \quad (36)$$

The argument applied in the previous section for the solutions  $k_o$  in  $s_5$  can now be repeated. When in  $S_4^> \cap s_5$   $\beta_m$  is negative,  $k_n$  in the interval  $S_6$  corresponds to the function  $F$  in  $S_4^> \cap s_5$ , and the function  $G$  in  $s_5 \cap S_6^<$  corresponds to  $ik_m$  in  $S_4^> \cap s_5$ . Thus the triplets for  $\beta_n < 0$  are:

$$\text{in } S_4^> \cap s_5 : (c_1/F^*K_m, -ik_m, F^*(z)), (c_2K_m/F, ik_m, F(z)),$$

where  $K_m$  is the same as in the previous section. In order to obtain the triplets in the case  $\beta_n > 0$  it is enough to interchange  $F$  and  $F^*$ .

In the interval  $s_3 \cap S_4^<$  the possible triplets are:

$$\text{in } s_3 \cap S_4^< : (d_1/G, -ik_m, G(z)), (d_2/H, ik_m, H(z)), \\ (d_3/H, -ik_m, H(z)), (d_4/G, ik_m, G(z)).$$

If  $\beta_m < 0$  the triplets with  $d_{1,2}$  describe perturbations whose exponential behaviour due to  $\pm k_m$  is attenuated by the functions  $G$  and  $H$ ; thus they correspond to the upper branch of  $D = 0$ . The triplet with  $d_4$  corresponds to the lower branch; it has  $\partial_x < 0$ , a property that does not change along the propagation (which is well described by the WKB approximation), and that makes it unacceptable for a reflected wave. Hence it must be  $d_4 = 0$ . The case  $\beta_m > 0$  is again obtained by the interchange of  $G$  and  $H$ . The continuity conditions in  $x_m$  are in any case:

$$d_1 + d_2 + d_3 = c_1/K_m + c_2K_m, \quad d_1 - d_2 + d_3 = c_1/K_m - c_2K_m, \\ \delta_n(d_1 + d_2 - d_3) = c_1\phi^{\delta_n}/K_m - c_2K_m\phi^{-\delta_n}. \quad (37)$$

It is convenient to solve these equations in the form:

$$2d_1 = (1 + \delta_n\phi^{\delta_n})c_1/K_m - (1 + \delta_n\phi^{-\delta_n})K_m c_2, \quad d_2 = K_m c_2, \\ 2d_3 = (1 - \delta_n\phi^{\delta_n})c_1/K_m + (1 + \delta_n\phi^{-\delta_n})K_m c_2. \quad (38)$$

In the next interval  $\beta_c$  is only positive; thus the triplets are

$$\text{in } S_2^> \cap s_3 : (d_1/HK_c, 0, H(z)), (d_2K_c/G, 0, G(z)).$$

The triplet with  $d_3$  characterizes a perturbation corresponding to the lower branch of  $D = 0$ ; it is well described with the WKB method and therefore does not appear. In the last interval the only possible propagation is towards  $-\infty$ , and thus one has

$$\text{in } s_1 \cap S_2^< : (t_1/F, 0, F(z)).$$

The continuity conditions in  $x_c$  are:

$$t_1 = d_1/K_c + d_2K_c, \quad t_1\phi = d_1/K_c - d_2K_c. \quad (39)$$

They yield

$$t_1 = d_1/K_c + d_2K_c, \quad (1 - \phi)d_1 = (1 + \phi)K_c^2d_2. \quad (40)$$

The 8 equations (34), (37) and (40) solve the problem. First the relation between  $c_1$  and  $c_2$  is derived from equations (38) and (40):

$$(1 + \delta_n\phi^{\delta_n})c_1 - (2(1 + \phi)K_c^2K_m^2/(1 - \phi) + (1 + \delta_n\phi^{-\delta_n})K_m^2)c_2 = 0. \quad (41)$$

Therefore  $c_2 \approx (1 + \phi)(1 + \delta_n\phi^{\delta_n})c_1/2(1 + \phi)K_c^2K_m^2$ ; thus  $|c_2|^2 \ll |c_1|^2$ . The choice  $\delta_6 = 1$  is not acceptable, because it would give  $c_1 = I$ ; thus most of the energy should be transmitted, in contradiction with the fact that the transmission coefficient  $t_1 = 2K_cK_m c_2/(1 - \phi)$  is exponentially small.

With  $\delta_6 = -1$  one has  $F_6 = F^{\delta_n}$  and

$$c_2 = (I + R_1)/2(1 + \phi)K_c^2K_m^2.$$

Thus equation (36) for the reflection coefficient  $R_1$  becomes:

$$(1 + \delta_n\phi^{\delta_n} + \epsilon)R_1 = -(1 + \delta_n\phi^{-\delta_n} + \epsilon)I, \quad (42)$$

where  $\epsilon \equiv (1 - \phi)(1 - \delta_n\phi^{\delta_n})(1 + \delta_n\phi^{\delta_n})/2(1 + \phi)K_c^2K_m^2$ .

In conclusion, since  $|1 + \delta_n\phi^{\delta_n}|/|1 + \delta_n\phi^{-\delta_n}| = 1$  and  $R_2 = c_2$ , most of the energy is reflected into the backwards waves. The transmission coefficient that follows from equations (40), (41) and (42) is:

$$K_cK_mt_1 = \delta_n(\phi^{\delta_n} - \phi^{-\delta_n})/(1 + \phi). \quad (43)$$

## 8. Poynting vector

It is not difficult to see that  $\partial_x \langle S_x \rangle = 0$  (the mean is taken over the time) and that

$$(4\pi/c)\langle S_x \rangle = (c/\omega)\langle E_{yR} \partial_x E_{yI} \rangle - \langle E_{\parallel R}(n_z E_{xR} - (c/\omega)\partial_x)E_{\parallel I} \rangle. \quad (44)$$

In order to evaluate this expression one needs the polarizations given by equations (1), where  $P_x$  and  $P_y$  are (almost) imaginary. The Poynting vector is first evaluated in  $x = x_n$  when the energy source is in  $-\infty$ , for the case  $\beta_n < 0$ . Then the amplitude of the upper branch transmitted wave is smaller than that of the lower branch, for which one has in  $x_n$ :

$$E_{\parallel} = t_2 e^{-i(\omega t - k_z z - k_n x)}, \quad \partial_x E_{\parallel} \approx i k_n E_y.$$

Then one easily obtains:

$$\langle E_{yR} \partial_x E_{yI} \rangle \rightarrow -k_n \langle E_{yR}^2 \rangle, \quad \langle E_{\parallel R} E_{xR} \rangle = 0.$$

Since  $\langle E_{yR} E_{yI} \rangle = 0$  and  $\langle E_{\parallel R}^2 \rangle = \langle E_{\parallel I}^2 \rangle = |t_2|^2/2$ , one finally has:

$$(4\pi/c) \langle S_x \rangle = n(1 + P_{xI}^2) |t_2|^2/2. \quad (46)$$

When the source is in  $-\infty$ , in  $x = x_c$  one has only one transmitted wave, with

$$E_{\parallel} = t_1 e^{-i(\omega t + k_z z)}, \quad \partial_x E_{\parallel} \approx \ell \phi(-D_{/x})^{1/3} E_y.$$

The polarizations  $P_{x,y}$  are formally the same as before, with  $n = -i(c/\omega)\ell\phi(-D_{/x})^{1/3}$ ; moreover,  $k_n$  is replaced by  $(\omega/c)n_I$ ; thus one obtains:

$$(4\pi/c) \langle S_x \rangle = -n_I(1 + P_{xI}^2) |t_1|^2/2. \quad (47)$$

The amplitudes  $|t_2|$  and  $|t_1|$  in (46) and (47) have the same order of magnitude, so that the ratio of the two Poynting vectors is approximately given by

$$\frac{\langle S_x \rangle_n}{\langle S_x \rangle_c} \approx \frac{k_n}{\ell \phi_I(-D_{/x})^{1/3}}, \quad (48)$$

a quantity that is clearly much larger than unity.

## Conclusion

In the preceding sections a method to deal with a succession of cut-offs and mode conversions — as can be found in ICRH situations — has been developed. The region of interest can be treated as a slab where equilibrium magnetic field and/or density depend on  $x$ . The possible dependence on the poloidal variable  $\theta$  of the equilibrium magnetic field can be considered as a local parameter. The ansatz  $E_j(x) \exp i \int k dx$  for the components of the electric field (where  $E_j$  vary slowly with  $x$ ) yields a system of differential equations valid for  $\rho |\partial_x k/k| \ll 1$ , and then a nonlinear equation for  $k(x)$ . This equation has been solved in the separate intervals containing the 'relevant points'. For each given position of the source (in  $-\infty$  or in  $+\infty$ ) the solutions valid in the various intervals have been connected by a 'matched asymptotic procedure' (see e.g. Murray) and by continuity conditions at the 'relevant points', thus giving the reflection and transmission coefficients. Some obvious qualitative expectations are confirmed by the quantitative results: when the first 'relevant point' reached by the waves is the cut-off most of the energy is reflected. When the waves reach first a mode conversion along the lower branch of the local dispersion relation (forwards waves) most of the energy is reflected into the upper branch (backwards waves). Other results are less obvious:

- o The energy transmitted when the waves encounter first a mode conversion is much more than that transmitted when the waves encounter first the cut-off. Indeed, their ratio is proportional to  $k_n/D_{/x}^{1/3}$  (the other factor being of the order of unity, see

(48)), and is thus the ratio of the characteristic length of the local dispersion relation to the wavelength at the mode conversion.

- o When the waves are reflected at the cut-off, the ratio of the transmission coefficients for forwards waves,  $|t_2|^2$ , and for backwards waves,  $|t_1|^2$ , can be 3 or 1/3, according to the sign of  $\beta_n$ , that is according to the relative sign of  $D''(k_n^2)$  and of  $\Delta$ , the coefficient of  $n'$  (see equation (7)).

An important formal feature of the problem is that it is solvable only if it is assumed that the waves described by the upper branch solution of the local dispersion relation transport energy in the direction opposite to their phase velocity. An obvious condition, if one remembers that the upper branch waves are backwards — a rather intriguing one, when one notes that this concept presupposes a 'wave packet', and not only one value of  $\omega$ , as is the case here.

## REFERENCES

- Murray, J.D., 1974 *Asymptotic Analysis*, chap. 6, Clarendon  
Perkins, F.W., 1977 *Nuclear Fusion* **17**, 1197  
Shvets, V.F. & Swanson, D.G., 1993 *J. of Plasma Physics* **50**, 163  
Swanson, D.G., 1995 *Rev. of Modern Physics*, 837