Energy related conservation law for fluids and multi-fluid plasmas with equilibrium flow

M. UNVERZAGT*

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Within the framework of dissipationless multi-fluid theory for plasmas all nonlinear perturbations of a stationary equilibrium that could be created from the equilibrium without breaking local entropy conservation are considered. Describing the fluid perturbations by Eulerian displacement vectors reveals a general symmetry and through Noether's theorem leads to a conservation law. Its relation to the law of energy conservation is discussed since it will be important for the generalization of the energy principle known for static equilibria in magnetohydrodynamics to an energy principle of stationary equilibria in the framework of a multi-fluid theory.

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I. INTRODUCTION

Finding symmetries and the corresponding conservation laws of a physical system is fundamental to the understanding of the system. Conservation laws can often be used to gain information about the stability of particular solutions of the equations of motion, as demonstrates the energy principle[1] known for static equilibria in the framework of dissipationless magnetohydrodynamic theory.

This work considers stationary equilibria in the framework of a dissipationless multi-fluid theory. All nonlinear perturbations that could be created from an equilibrium without breaking local entropy conservation are taken into account. The focus of attention is a new conservation law that plays a role for equilibria with flow and is strongly related to the energy conservation law.

As a first approach the conservation law is derived in section II from the momentum transport equation within the framework of a dissipationsless hydrodynamic theory. As only a certain structure of the momentum transport equation is crucial for this derivation, the conservation law can easily be transfered to more general models.

The second approach gives insight to the meaning of the new conservation law. It starts in section III by introducing a dissipationless multi-fluid theory capable of describing a plasma. The dynamics of the perturbations (which could be created from an equilibrium without breaking local entropy conservation) is defined by a Lagrangian.

This Lagrangian description of the (nonlinear) perturbation dynamics is used in section IV to reveal the symmetry responsible for the conservation law. Thereby, it is of great help to describe the perturbations by Eulerian displacement vectors, especially as they provide a simple picture of the symmetry (see figure 1).

Section V shows the relation of the new conservation law to the law of energy conservation.

Section VI discusses how the Lagrangian could be modified to describe approximations of the dynamics which are commonly used for the description of plasmas.

The new conservation law could be used to formulate an energy principle for stationary equilibria in the framework of a multi-fluid theory.**

II. CONSERVATION LAW

In this section a conservation law is derived within a dissipationless hydrodynamic theory. As will be discussed at the end of the section, the result can easily be transfered to more general fluid models capable of describing a plasma.

Let Ψ define the state of the hydrodynamic system:

$$\Psi\,[\boldsymbol{x},t] = (n, \boldsymbol{v}, p)\,[\boldsymbol{x},t]\,. \tag{1}$$

n is the density, $\boldsymbol{v}$ the velocity and p the pressure of the fluid. $\boldsymbol{x}$ denotes a point in

**The main idea of such an energy principle would be to consider only such perturbations that can be created from the equilibrium without breaking particle conservation, entropy conservation, the circulation theorems (see section III.C.) and the new conservation law. The restriction to these dynamically accessible perturbations would allow us to write the second order energy expression within a perturbation theory as a function of first order perturbations.

space and t denotes a time. Let us assume that the local thermodynamic properties of the system are completely described by the two equations of state

$$p = nT, \qquad s = \frac{1}{\gamma - 1} \ln p n^{-\gamma} \tag{2}$$

with the temperature T (in energy units), the entropy per particle s and a constant quotient of the specific heats γ (ideal gas: $\gamma = 5/3$). The internal energy u per particle and the enthalpy per particle h can be derived using the thermodynamic equations of state (2):

$$du = T\,ds - p\,d\frac{1}{n} = \frac{dT}{\gamma - 1}, \qquad u = \frac{T}{\gamma - 1}, \tag{3}$$

$$h = u + \frac{p}{n} = \frac{\gamma}{\gamma - 1} T. \tag{4}$$

The dynamics of the system shall be defined by three transport equations

$$\frac{\partial}{\partial t} n + \nabla \cdot \boldsymbol{v} n = 0, \tag{5}$$

$$n \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) m \boldsymbol{v} = -\nabla p, \tag{6}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) s = 0 \tag{7}$$

and suitable boundary conditions.

By using

$$-\frac{1}{n} \nabla p = T \nabla s - \nabla h, \tag{8}$$

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \nabla \frac{\boldsymbol{v}^2}{2} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \tag{9}$$

and introducing the momentum per particle $\boldsymbol{p}$ and the Bernoulli function U as abbreviations

$$\boldsymbol{p} = m \boldsymbol{v}, \tag{10}$$

$$U = \frac{m}{2}|\boldsymbol{v}|^2 + h, \tag{11}$$

we can write the momentum transport equation (6) divided by a nonzero density ($n \neq 0$) in the following form:

$$\frac{\partial}{\partial t}\boldsymbol{p} - \boldsymbol{v} \times (\nabla \times \boldsymbol{p}) = T\nabla s - \nabla U. \tag{12}$$

For the following discussion two special states $\boldsymbol{\Psi} = (n, \boldsymbol{v}, \boldsymbol{p})$ and $\widehat{\boldsymbol{\Psi}} = (\hat{n}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{p}})$ will be important: a stationary equilibrium $\boldsymbol{\Psi}$ defined by $\frac{\partial}{\partial t}\boldsymbol{\Psi}[\boldsymbol{x}, t] = 0$ and a state $\widehat{\boldsymbol{\Psi}}$ that could be created from the equilibrium $\boldsymbol{\Psi}$ without breaking the entropy conservation law (7) during the creation process.

According to equations (5), (12) and (7) the equilibrium equations are:

$$\nabla \cdot n\boldsymbol{v} = 0, \tag{13}$$

$$-\boldsymbol{v} \times (\nabla \times \boldsymbol{p}) = T\nabla s - \nabla U, \tag{14}$$

$$\boldsymbol{v} \cdot \nabla s = 0. \tag{15}$$

We can describe the relation between $\widehat{\boldsymbol{\Psi}}$ and $\boldsymbol{\Psi}$ by relating fluid points $\boldsymbol{x}$ of the equilibrium to fluid points $\hat{\boldsymbol{x}}$ of the state $\widehat{\boldsymbol{\Psi}}$ through Eulerian displacement vectors $\boldsymbol{\xi}$:

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\xi}[\boldsymbol{x}, t], \tag{16}$$

$$\hat{\boldsymbol{v}}[\hat{\boldsymbol{x}}, t] = \left(\frac{\partial}{\partial t} + \boldsymbol{v}[\boldsymbol{x}] \cdot \nabla\right)\hat{\boldsymbol{x}} = \boldsymbol{v}[\boldsymbol{x}] + \boldsymbol{v}[\boldsymbol{x}] \cdot \nabla\boldsymbol{\xi}[\boldsymbol{x}, t] + \dot{\boldsymbol{\xi}}[\boldsymbol{x}, t], \tag{17}$$

$$\hat{s}\,[\hat{\boldsymbol{x}}, t] = s\,[\boldsymbol{x}]\,. \tag{18}$$

Notation: $f\,[\boldsymbol{x}, t]$ defines the value the physical quantity f has at point $\boldsymbol{x}$ at time t and $g\,[\hat{\boldsymbol{x}}, t]$ is the value the physical quantity g has at point $\hat{\boldsymbol{x}}$ at time t. This is a useful convention as one might become confused by the fact that $g[\hat{\boldsymbol{x}}, t]$ could be seen as a function of $\boldsymbol{x}, t$ because of equation (16).

The derivation of the conservation law starts with the momentum transport equation (12) for the state $\widehat{\boldsymbol{\Psi}}$ at a point $\hat{\boldsymbol{x}}$ and a time t:

$$\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{p}} - \hat{\boldsymbol{v}} \times \left(\frac{\partial}{\partial \hat{\boldsymbol{x}}} \times \hat{\boldsymbol{p}}\right) = \hat{T}\frac{\partial}{\partial \hat{\boldsymbol{x}}}\hat{s} - \frac{\partial}{\partial \hat{\boldsymbol{x}}}\hat{U}, \tag{19}$$

where $\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}}$ denotes the partial time derivative at constant $\hat{\boldsymbol{x}}$. The scalar product of equation (19) is then taken from the left with $\nabla\hat{\boldsymbol{x}}$:

$$(\nabla\hat{\boldsymbol{x}}) \cdot \left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}} + \hat{\boldsymbol{v}} \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}}\right)\hat{\boldsymbol{p}} - (\nabla\hat{\boldsymbol{p}}) \cdot \hat{\boldsymbol{v}} = \hat{T}\nabla\hat{s} - \nabla\hat{U}. \tag{20}$$

According to equations (16) and (17), there are the following equivalent representations of the 'total time derivative' of the momentum $\hat{\boldsymbol{p}}$:

$$\left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}} + \hat{\boldsymbol{v}} \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}}\right)\hat{\boldsymbol{p}}[\hat{\boldsymbol{x}}, t] = \left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \nabla\right)\hat{\boldsymbol{p}}[\hat{\boldsymbol{x}}[\boldsymbol{x}, t], t]. \tag{21}$$

Using equations (17) and (21), equation (20) can be transformed to

$$\begin{aligned} &\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} ((\nabla\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) + (\nabla\hat{\boldsymbol{x}}) \cdot (\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{p}}) - (\nabla\hat{\boldsymbol{p}}) \cdot (\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{x}}) \\ = \;& \hat{T}\nabla\hat{s} - \nabla\left(\hat{U} - \dot{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{p}}\right) \end{aligned} \tag{22}$$

and with

$$\begin{aligned} &(\nabla\hat{\boldsymbol{x}}) \cdot (\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{p}}) - (\nabla\hat{\boldsymbol{p}}) \cdot (\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{x}}) \\ &= \boldsymbol{v} \cdot \nabla((\nabla\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) - (\nabla(\nabla\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) \cdot \boldsymbol{v} \\ &= -\boldsymbol{v} \times (\nabla \times ((\nabla\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}})) \end{aligned} \tag{23}$$

to

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} & ((\nabla \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) - \boldsymbol{v} \times (\nabla \times ((\nabla \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}})) \\ = \quad & \hat{T} \nabla \hat{s} - \nabla \left(\hat{U} - \dot{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{p}} \right). \end{aligned} \tag{24}$$

Using the crucial assumption (18), we arrive at the equation

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} & ((\nabla \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) \\ = \quad & \boldsymbol{v} \times (\nabla \times ((\nabla \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}})) + \hat{T} \nabla s - \nabla \left(\hat{U} - \dot{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{p}} \right). \end{aligned} \tag{25}$$

On its right hand side $-\nabla \left(\hat{U} - \dot{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{p}} \right)$ is the only force, contributing in the direction of a field line C of the equilibrium velocity $\boldsymbol{v}$.

Before looking at the general case we will investigate the case where C is a closed field line. Integrating equation (25) along a total closed field line leads to a conservation law:

$$\begin{aligned}\frac{d}{dt}\left(\oint_{\hat{C}} d\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{p}}[\hat{\boldsymbol{x}}, t] \right) &= \frac{d}{dt}\left(\oint_{C} d\boldsymbol{l} \cdot (\nabla \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}[\hat{\boldsymbol{x}}, t] \right) \\ &= - \oint_{C} d\boldsymbol{l} \cdot \nabla \left(\hat{U} - \dot{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{p}} \right) \\ &= 0. \end{aligned} \tag{26}$$

Now we investigate the general case where the field lines C need not be closed. Let $f[\boldsymbol{x}]$ be some equilibrium quantity that is constant along the equilibrium path of a fluid point:

$$\boldsymbol{v} \cdot \nabla f = 0. \tag{27}$$

The scalar product of equation (25) with $f n \boldsymbol{v}$ becomes a local conservation law by making use of equations (13), (15) and (27):

$$\frac{\partial}{\partial t}\left(fn\boldsymbol{v}\cdot(\nabla\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{p}}\right)+\nabla\cdot\left(fn\boldsymbol{v}\left(\hat{U}-\dot{\boldsymbol{\xi}}\cdot\hat{\boldsymbol{p}}\right)\right)=0. \tag{28}$$

Taking the integral of equation (28) over a volume V, on the surface of which the vector $fn\boldsymbol{v}$ has nowhere a normal component, leads to the following conservation law:

$$\frac{d}{dt}\left(\int_V d^3x\, f[\boldsymbol{x}]n[\boldsymbol{x}]\boldsymbol{v}[\boldsymbol{x}]\cdot(\nabla\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{p}}[\hat{\boldsymbol{x}},t]\right)=0. \tag{29}$$

Which assumptions leading to this conservation law are crucial? The first crucial assumption is the restriction (18) to such a perturbation of the equilibrium that can be created without breaking the entropy conservation law. The second crucial assumption is the form of the equation (19). As no use is made of the expression for the momentum $\hat{\boldsymbol{p}}$, the discussion can easily be transfered to a more general fluid theory that is capable of describing a plasma.

III. DISSIPATIONLESS MULTI-FLUID THEORY FOR PLASMAS

A. Dynamics

Let

$$\boldsymbol{\Psi}[\boldsymbol{x},t]=(\boldsymbol{E},\boldsymbol{B},n_\nu,\boldsymbol{v}_\nu,p_\nu)[\boldsymbol{x},t] \tag{30}$$

describe the state of a system. $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ the magnetic field. n_ν is the density, $\boldsymbol{v}_\nu$ the velocity, and p_ν the pressure of a fluid species ν. The regions with

$n_\nu \neq 0$ are denoted as $V_{P\nu}$. All regions $V_{P\nu}$ contribute to the plasma, which possibly has surface currents and surface charges. The regions where $n_\nu = 0$ for all species ν form the vacuum.

The evolution of the state $\boldsymbol{\Psi}$ shall be defined by the following transport equations for each fluid species ν inside the volume $V_{P\nu}$

$$\frac{\partial}{\partial t} n_\nu + \nabla \cdot \boldsymbol{v}_\nu n_\nu = 0, \tag{31}$$

$$m_\nu n_\nu \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \boldsymbol{v}_\nu = -\nabla p_\nu + e_\nu n_\nu \boldsymbol{E} + e_\nu n_\nu \boldsymbol{v}_\nu \times \boldsymbol{B}, \tag{32}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) s_\nu = 0 \tag{33}$$

with the entropy per particle $s_\nu = \frac{1}{\gamma_\nu - 1} \ln \left(p_\nu n_\nu^{-\gamma_\nu} \right)$ and the Maxwell equations:

$$\nabla \cdot \boldsymbol{B} = 0, \tag{34}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial}{\partial t} \boldsymbol{B} = 0, \tag{35}$$

$$\nabla \cdot \varepsilon_0 \boldsymbol{E} = \sum_\nu e_\nu n_\nu, \tag{36}$$

$$\nabla \times \frac{1}{\mu_0} \boldsymbol{B} = \sum_\nu e_\nu n_\nu \boldsymbol{v}_\nu + \frac{\partial}{\partial t} \varepsilon_0 \boldsymbol{E}. \tag{37}$$

The electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ shall be described by means of single-valued electromagnetic potentials Φ and $\boldsymbol{A}$ (This property will play a role in equations (64) and (84)):

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla \Phi - \frac{\partial}{\partial t} \boldsymbol{A}. \tag{38}$$

Similar to the transformation from equation (6) to equation (12), we transform equation (32):

$$\frac{\partial}{\partial t} \boldsymbol{p}_\nu - \boldsymbol{v}_\nu \times (\nabla \times \boldsymbol{p}_\nu) = T_\nu \nabla s_\nu - \nabla U_\nu, \tag{39}$$

$$\boldsymbol{p}_\nu = m_\nu \boldsymbol{v}_\nu + e_\nu \boldsymbol{A}, \tag{40}$$

$$T_\nu = \frac{p_\nu}{n_\nu}, \tag{41}$$

$$U_\nu = \frac{1}{2} m_\nu |\boldsymbol{v}_\nu|^2 + \frac{\gamma_\nu}{\gamma_\nu - 1} T_\nu + e_\nu \Phi \tag{42}$$

where $\boldsymbol{p}_\nu$ is the canonical momentum per particle, T_ν the temperature, and U_ν the Bernoulli function.

B. Total energy

The total energy of the system is

$$E[\Psi] = \sum_\nu \int_{V_{P_\nu}} d^3x_\nu \left(n_\nu \frac{m_\nu}{2} |\boldsymbol{v}_\nu|^2 + \frac{p_\nu}{\gamma_\nu - 1} \right) + \int_{V_{\text{fix}}} d^3x \left(\frac{\varepsilon_0}{2} |\boldsymbol{E}[\boldsymbol{x}, t]|^2 + \frac{|\boldsymbol{B}[\boldsymbol{x}, t]|^2}{2\mu_0} \right). \tag{43}$$

V_{fix} denotes a fixed volume that encloses the plasma (for all times) and possibly vacuum regions.

The total energy E is a conserved quantity if the boundary conditions do not allow energy to flow across the surface ∂V_{fix} of the volume V_{fix}. This is the case, if in addition to the assumption that no plasma leaves or enters volume V_{fix}, the Poynting vector $\frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$ has no normal component on the surface ∂V_{fix}, which holds for an infinitely conducting wall.

It is remarkable that the total energy E is still a conserved quantity in the framework of Braginskii's equations[2] with all dissipation processes taken into account (provided that no heat current is flowing across ∂V_{fix}).

C. Circulation theorems

Taking the curl of equation (39), leads to the following equation:

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}_\nu - \nabla \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega}_\nu) = (\nabla T_\nu) \times (\nabla s_\nu) \tag{44}$$

with the gauge invariant quantity

$$\boldsymbol{\Omega}_\nu = \nabla \times \boldsymbol{p}_\nu = m_\nu \nabla \times \boldsymbol{v}_\nu + e_\nu \boldsymbol{B}. \tag{45}$$

From equation (44) one can derive generalizations[3,4] of the well known hydrodynamic circulation theorems of Helmholtz[5] and Ertel.[6] The generalization to hydrodynamics lies in the fact that the kinetical momentum $m_\nu \boldsymbol{v}_\nu$ is replaced by the canonical momentum $\boldsymbol{p}_\nu$. Taking the conservation laws (31) and (33) into account, one can easily derive the generalized Ertel theorem from equation (44):

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right) \frac{\boldsymbol{\Omega}_\nu \cdot \nabla s_\nu}{n_\nu} = 0. \tag{46}$$

If $(\nabla T_\nu) \times (\nabla s_\nu) = 0$ for a species ν (for example: T_ν or s_ν being a constant or T_ν being a function of s_ν), the more powerful generalized Helmholtz theorem results from equation (44):

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}_\nu - \nabla \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega}_\nu) = 0. \tag{47}$$

Since $\nabla \cdot \boldsymbol{\Omega}_\nu = 0$, the equation (47) states that the flux of $\boldsymbol{\Omega}_\nu$ through any surface moving with the fluid of species ν is conserved.[7] It is remarkable that equation (47) is reduced to the flux theorem of magnetohydrodynamics if one neglects the inertia of fluid species ν by setting $m_\nu = 0$.

D. Equilibrium and its perturbations

From now on let $\Psi = (\boldsymbol{E}, \boldsymbol{B}, n_\nu, \boldsymbol{v}_\nu, p_\nu)$ be an equilibrium state, defined by $\frac{\partial}{\partial t}\Psi\,[\boldsymbol{x}, t] = 0$. Let $\widehat{\Psi} = \left(\hat{\boldsymbol{E}}, \hat{\boldsymbol{B}}, \hat{n}_\nu, \hat{\boldsymbol{v}}_\nu, \hat{p}_\nu\right)$ define a state that could be created from Ψ without breaking local entropy conservation. This property can be described[8] by means of Eulerian displacement vectors $\boldsymbol{\xi}_\nu$:

$$\hat{\boldsymbol{x}}_\nu = \boldsymbol{x} + \boldsymbol{\xi}_\nu\,[\boldsymbol{x}, t]\,, \quad J_\nu\,[\boldsymbol{x}, t] = \det\left(\frac{\partial \hat{\boldsymbol{x}}_\nu}{\partial \boldsymbol{x}}\right), \tag{48}$$

$$\hat{\boldsymbol{v}}_\nu\,[\hat{\boldsymbol{x}}_\nu, t] = \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right) \hat{\boldsymbol{x}}_\nu\,[\boldsymbol{x}, t]\,, \tag{49}$$

$$\hat{s}_\nu[\hat{\boldsymbol{x}}, t] = s_\nu[\boldsymbol{x}] \quad \text{or} \quad \hat{p}_\nu \hat{n}_\nu^{-\gamma_\nu}[\hat{\boldsymbol{x}}, t] = p_\nu n_\nu^{-\gamma_\nu}[\boldsymbol{x}]. \tag{50}$$

The perturbation of $(\Phi, \boldsymbol{A}, n_\nu)$ at a point $\boldsymbol{x}$ and time t is described by $(\phi, \boldsymbol{a}, \tilde{n}_\nu)$:

$$\left(\hat{\Phi}, \hat{\boldsymbol{A}}, \hat{n}_\nu\right)[\boldsymbol{x}, t] = (\Phi, \boldsymbol{A}, n_\nu)\,[\boldsymbol{x}] + (\phi, \boldsymbol{a}, \tilde{n}_\nu)\,[\boldsymbol{x}, t]\,. \tag{51}$$

The size of the perturbation is not subject to restrictions.

E. Lagrangian

The dynamics of the dissipationless multi-fluid theory can be defined through a Lagrangian.[9] Such a Lagrangian theory makes it straightforward to connect a conservation law to a symmetry with the help of Noether's theorem. Therefore, we introduce the Lagrangian $L\,[t] = \int d^3x \mathcal{L}\,[\boldsymbol{x}, t]$ with the Lagrangian density

$$\mathcal{L}[\boldsymbol{x}, t] \quad = $$

$$\sum_\nu J_\nu[\boldsymbol{x}, t]\left(\hat{n}_\nu[\hat{\boldsymbol{x}}_\nu, t]\frac{m_\nu}{2}\,|\hat{\boldsymbol{v}}_\nu[\hat{\boldsymbol{x}}_\nu, t]|^2 - \frac{\hat{p}_\nu[\hat{\boldsymbol{x}}_\nu, t]}{\gamma_\nu - 1}\right.$$

$$-e_\nu \hat{n}_\nu[\hat{\boldsymbol{x}}_\nu, t] \left(\hat{\Phi}[\hat{\boldsymbol{x}}_\nu, t] - \hat{\boldsymbol{v}}_\nu[\hat{\boldsymbol{x}}_\nu, t] \cdot \hat{\boldsymbol{A}}[\hat{\boldsymbol{x}}_\nu, t] \right)$$

$$+\lambda_\nu [\boldsymbol{x}, t] \left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}_\nu} \hat{n}_\nu [\hat{\boldsymbol{x}}_\nu, t] + \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot (\hat{\boldsymbol{v}}_\nu \hat{n}_\nu [\hat{\boldsymbol{x}}_\nu, t]) \right) \Bigg)$$

$$+\frac{\varepsilon_0}{2} \left| \frac{\partial}{\partial t} \hat{\boldsymbol{A}}[\boldsymbol{x}, t] + \nabla \hat{\Phi}[\boldsymbol{x}, t] \right|^2 - \frac{\left| \nabla \times \hat{\boldsymbol{A}}[\boldsymbol{x}, t] \right|^2}{2\mu_0}. \tag{52}$$

Particle conservation is incorporated into the Lagrangian by means of the Lagrange multipliers λ_ν. In combination with the equations (48)–(51) this Lagrangian defines completely the equations of motion for the perturbation as proven in appendix A.

IV. SYMMETRY AND CONSERVATION LAW

The plasma model described in section III provides the set of equations $\{$(39), (48)–(50), $\nabla \cdot n_\nu \boldsymbol{v}_\nu = 0$, $\boldsymbol{v}_\nu \cdot \nabla s_\nu = 0\}$, corresponding to the set of equations $\{$(12), (16)–(18), (13), (15)$\}$ we used in section II. In addition we introduce an equilibrium quantity $f_\nu[\boldsymbol{x}]$ that can be chosen to be any function with the property $\boldsymbol{v}_\nu \cdot \nabla f_\nu = 0$ corresponding to the property (27). Realizations of f_ν are:

$$f_\nu = 1, \; f_\nu = s_\nu \text{ or } f_\nu = \frac{1}{n_\nu} \boldsymbol{\Omega}_\nu \cdot \nabla s_\nu. \tag{53}$$

As shown in section II a conservation law (29) holds. Yet we do not know what the related symmetry is. The answer is provided in the following section.

A. Displacement of a single equilibrium path

Let $\boldsymbol{X}_\nu[\tau]$ denote the path of a fluid point of species ν following the equilibrium motion, defined by some initial condition and

$$\frac{d}{d\tau}\boldsymbol{X}_\nu[\tau] = \boldsymbol{v}_\nu[\boldsymbol{X}_\nu[\tau]]. \tag{54}$$

The displacement of the path $\boldsymbol{X}_\nu[\tau]$ can be described by Eulerian displacement vectors $\boldsymbol{\xi}_\nu[\boldsymbol{x}, t]$. Let $\hat{\boldsymbol{X}}_\nu[\tau, t]$ define the displaced path:

$$\hat{\boldsymbol{X}}_\nu[\tau, t] = \boldsymbol{X}_\nu[\tau] + \boldsymbol{\xi}_\nu[\boldsymbol{X}_\nu[\tau], t]. \tag{55}$$

The same displacement of the path could be described by different Eulerian displacement vectors $\boldsymbol{\xi}_\nu^{\tau'}[\boldsymbol{x}, t]$, defined by the following equation:

$$\boldsymbol{\xi}_\nu^{\tau'}[\boldsymbol{X}_\nu[\tau], t] = \hat{\boldsymbol{X}}_\nu[\tau + \tau', t] - \boldsymbol{X}_\nu[\tau]. \tag{56}$$

An infinitesimal transformation from $\boldsymbol{\xi}_\nu$ to $\boldsymbol{\xi}_\nu^{\delta\tau}$ is illustrated by figure 1.

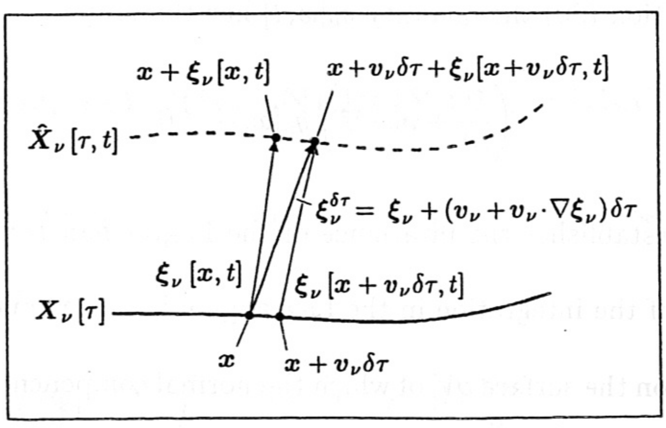

figure 1: The figure shows the equilibrium path $\boldsymbol{X}_\nu[\tau]$ of a fluid point of species ν and the displaced path $\hat{\boldsymbol{X}}_\nu[\tau, t]$. The displacement of the path can equally well be described by $\boldsymbol{\xi}_\nu[\boldsymbol{x}, t]$ and $\boldsymbol{\xi}_\nu^{\delta\tau}[\boldsymbol{x}, t]$.

B. Symmetry

Since the entropy is constant along the displaced path $\hat{\boldsymbol{X}}_\nu[\tau, t]$, the transformation discussed in the previous section might be a symmetry transformation of the physical system. We therefore define the following infinitesimal transformation (ν denotes one fluid species, $\mu \neq \nu$ denotes all other species):

$$\delta\boldsymbol{\xi}_\nu = \boldsymbol{\xi}_\nu^{\delta\tau} - \boldsymbol{\xi}_\nu = (\boldsymbol{v}_\nu \cdot \nabla\hat{\boldsymbol{x}}_\nu)\,\delta\tau, \quad \delta\dot{\boldsymbol{\xi}}_\nu = \frac{\partial}{\partial t}\delta\boldsymbol{\xi}_\nu, \tag{57}$$

$$\delta\left(\phi, \boldsymbol{a}, \tilde{n}_\nu, \tilde{n}_\mu, \boldsymbol{\xi}_\mu, \dot{\boldsymbol{\xi}}_\mu\right) = 0. \tag{58}$$

The quantity $\delta\tau$ is generally a function of $\boldsymbol{x}$ and t but proportional to a quantity $\delta\tilde{\tau}$ independent of $\boldsymbol{x}$ and t:

$$\delta\tau\,[\boldsymbol{x}, t] = g_\nu\,[\boldsymbol{x}, t]\,\delta\tilde{\tau} \tag{59}$$

with

$$g_\nu\,[\boldsymbol{x}, t] = f_\nu\,[\boldsymbol{x}]\,\frac{n_\nu\,[\boldsymbol{x}]}{\hat{n}_\nu\,[\hat{\boldsymbol{x}}_\nu, t]\,J_\nu\,[\boldsymbol{x}, t]}. \tag{60}$$

Through its definition (60) the quantity $g_\nu[\boldsymbol{x}, t]$ has the property

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right) g_\nu[\boldsymbol{x}, t] = 0. \tag{61}$$

Appendix B establishes the invariance of the Lagrangian L under the transformation (57)–(60) if the integration in the Lagrangian is carried out over a volume V belonging to $V_{P\nu}$, on the surface ∂V of which the normal component of the equilibrium current $f_\nu n_\nu \boldsymbol{v}_\nu$ is zero everywhere.

This invariance can be regarded as the definition of a symmetry.

C. Conservation law

Through Noether's theorem the symmetry leads to a conservation law:

$$0 = \delta L = \frac{d}{dt}\sum_{\kappa}\int_V d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\xi}}_\kappa}\cdot \delta\boldsymbol{\xi}_\kappa\right) = \delta\tilde{\tau}\frac{d}{dt} I_{\nu,V,f_\nu} \tag{62}$$

with

$$I_{\nu,V,f_\nu} = \int_V d^3x\, f_\nu[\boldsymbol{x}] n_\nu[\boldsymbol{x}]\, \boldsymbol{v}_\nu[\boldsymbol{x}]\cdot(\nabla\hat{\boldsymbol{x}}_\nu)\cdot\hat{\boldsymbol{p}}_\nu[\boldsymbol{x}+\boldsymbol{\xi}_\nu, t]. \tag{63}$$

The conserved quantity (63) is invariant under a gauge transformation with $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$, in which χ is a single-valued function:

$$\begin{aligned}
I'_{\nu,V,f_\nu} - I_{\nu,V,f_\nu} &= \int_V d^3x\, e_\nu f_\nu n_\nu \boldsymbol{v}_\nu\cdot(\nabla\hat{\boldsymbol{x}}_\nu)\cdot\frac{\partial}{\partial\hat{\boldsymbol{x}}_\nu}\chi[\hat{\boldsymbol{x}}_\nu, t]\\
&= \int_V d^3x\, e_\nu f_\nu n_\nu \boldsymbol{v}_\nu\cdot\nabla\chi[\boldsymbol{x}+\boldsymbol{\xi}_\nu, t]\\
&= \int_V d^3x\, \nabla\cdot(e_\nu f_\nu n_\nu \boldsymbol{v}_\nu \chi[\boldsymbol{x}+\boldsymbol{\xi}_\nu, t])\\
&= \int_{\partial V} d\boldsymbol{f}\cdot e_\nu f_\nu n_\nu \boldsymbol{v}_\nu \chi[\boldsymbol{x}+\boldsymbol{\xi}_\nu, t]\\
&= 0.
\end{aligned} \tag{64}$$

V. RELATION TO THE ENERGY CONSERVATION LAW

The total energy $E[\widehat{\Psi}]$, defined by equation (43), can be expressed in the following form, as show in appendix C:

$$\begin{aligned}
E[\widehat{\Psi}] &= \sum_\nu \int_{V_{P_\nu}} d^3x\, \hat{\boldsymbol{v}}_\nu[\boldsymbol{x}+\boldsymbol{\xi}_\nu, t]\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\xi}}_\nu}\\
&+ \int_{V_{\text{fix}}} d^3x \left(\dot{\boldsymbol{a}}\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{a}}} - \mathcal{L} - \varepsilon_0\nabla\cdot\hat{\Phi}\hat{\boldsymbol{E}}\right).
\end{aligned} \tag{65}$$

The reader should focus his attention on the first term in the energy expression (65). (He should not be distracted by the well known[10] term $\nabla \cdot \hat{\Phi}\hat{\boldsymbol{E}}$.)

In view of the energy principle we will restrict our investigation to such perturbations that could be created from the equilibrium without breaking local entropy and particle conservation.[11] The latter property can be expressed by equation

$$n_\nu[\boldsymbol{x}] = \hat{n}_\nu[\hat{\boldsymbol{x}}, t]\, J_\nu[\boldsymbol{x}, t], \tag{66}$$

such that the perturbation can completely be described by the quantities $\phi, \boldsymbol{a}, \boldsymbol{\xi}_\nu$.

Making use of the fact that the Lagrangian is only time dependent through the perturbation, one would then rather be guided to the following energy expression than to expression (65):

$$\begin{aligned} E_1 = \sum_\nu \int_{V_{P\nu}} d^3x\, \dot{\boldsymbol{\xi}}_\nu \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\xi}}_\nu} \\ + \int_{V_{\text{fix}}} d^3x \left(\dot{\boldsymbol{a}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} - \mathcal{L} - \varepsilon_0 \nabla \cdot \hat{\Phi}\hat{\boldsymbol{E}} \right). \end{aligned} \tag{67}$$

The energy difference between E and E_1 is given by the conserved quantities $I_{\nu,V_{P\nu},f_\nu=1}$, defined by equation (63):

$$E = E_1 + E_2, \qquad E_2 = \sum_\nu I_{\nu,V_{P\nu},f_\nu=1}. \tag{68}$$

It is remarkable that the conserved quantity $I_{\nu,V_{P\nu},f_\nu=1}$ can itself be a sum of conserved quantities $I_{\nu,V,f_\nu=1}$, depending on the properties of the equilibrium.

VI. APPROXIMATIONS

There are many possibilities to simplify the plasma model treated in sections II–V without breaking the symmetry discussed in section IV.

If one leaves out the electric energy density $\frac{\varepsilon_0}{2}\left|\hat{\boldsymbol{E}}\right|^2$ in the Lagrangian density (52), one arrives at a quasineutral approximation (instead of equation (36): $\sum_\nu e_\nu n_\nu = 0$) without a displacement current (instead of equation (37): $\nabla \times \boldsymbol{B} = \mu_0 \sum_\nu e_\nu n_\nu \boldsymbol{v}_\nu$). By further restricting the vector potential to be time-independent, one arrives at an electrostatic approximation.

One could neglect the inertia of the plasma's electrons by setting their mass m_e to zero: $m_e = 0$. If for a general state Ψ the relation $(\nabla s_e) \times (\nabla T_e) = 0$ is true, then the zero mass approximation will freeze the magnetic flux into the electron fluid according to equation (47).

Handling a drift approximation is more complicated. Pfirsch and Correa-Restrepo[12,13] introduce a most promising approach to describe the drift-approximation by a fluid Lagrangian similar to a particle Lagrangian in phase space. The equation of motion for the 'canonical momentum per particle' corresponding to this Lagrangian still has the form of equation (39) which is crucial for the conservation law (62) and the generalized circulation theorems (46) and (47).

VII. ACKNOWLEDGEMENTS

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VIII. APPENDICES

A. Hamilton's principle

Following Hamilton's principle we are doing a virtual variation of $(\phi, \boldsymbol{a}, \tilde{n}_\nu, \boldsymbol{\xi}_\nu, \lambda_\nu)$ in the time and space integral over the Lagrangian density (52). The virtual variation of $(\phi, \boldsymbol{a})$ is straightforward and leads to the inhomogeneous Maxwell equations. This appendix will treat the more involved variation of $(\tilde{n}_\nu, \boldsymbol{\xi}_\nu, \lambda_\nu)$.

The virtual variation of $\lambda_\nu [\boldsymbol{x}, t]$ leads to the law of particle conservation in the representation

$$0 = \left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}_\nu} + \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot \hat{\boldsymbol{v}}_\nu \right) \hat{n}_\nu [\hat{\boldsymbol{x}}_\nu, t] \,. \tag{69}$$

After doing partial integrations in the time integral over the Lagrangian and making use of

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \ldots = \left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}_\nu} + \hat{\boldsymbol{v}}_\nu \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \right) \ldots \tag{70}$$

we get an alternative representation:

$$0 = \left(\frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{v}_\nu \right) \hat{n}_\nu [\hat{\boldsymbol{x}}_\nu, t] \, J_\nu [\boldsymbol{x}, t] \,. \tag{71}$$

The virtual variation of $\tilde{n}_\nu [\hat{\boldsymbol{x}}_\nu, t]$ leads to:

$$
\begin{aligned}
0 = & \frac{m_\nu}{2} |\hat{\boldsymbol{v}}_\nu [\hat{\boldsymbol{x}}_\nu, t]|^2 - \frac{p_\nu [\boldsymbol{x}] n_\nu^{-\gamma_\nu} [\boldsymbol{x}]}{\gamma_\nu - 1} \gamma_\nu \hat{n}_\nu^{\gamma_\nu - 1} [\hat{\boldsymbol{x}}_\nu, t] \\
& - e_\nu \hat{\Phi} [\hat{\boldsymbol{x}}_\nu, t] + e_\nu \hat{\boldsymbol{v}}_\nu [\hat{\boldsymbol{x}}_\nu, t] \cdot \hat{\boldsymbol{A}} [\hat{\boldsymbol{x}}_\nu, t] \\
& - \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \lambda_\nu [\boldsymbol{x}, t]. \qquad (72)
\end{aligned}
$$

To prepare for the virtual variation of $\boldsymbol{\xi}_\nu$ we define

$$
J_\nu [\boldsymbol{x}, t] = \frac{1}{6} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} \frac{\partial \hat{x}_{\nu l}}{\partial x_i} \frac{\partial \hat{x}_{\nu m}}{\partial x_j} \frac{\partial \hat{x}_{\nu n}}{\partial x_k}, \qquad (73)
$$

$$
\mathsf{T}_{\nu l i} [\boldsymbol{x}, t] = \frac{1}{2} \sum_{jkmn} \varepsilon_{ijk} \varepsilon_{lmn} \frac{\partial \hat{x}_{\nu m}}{\partial x_j} \frac{\partial \hat{x}_{\nu n}}{\partial x_k} \qquad (74)
$$

and calculate the virtual variation $\delta^{\text{vir}} J_\nu$:

$$
\begin{aligned}
\delta^{\text{vir}} J_\nu &= \sum_{li} \mathsf{T}_{\nu l i} \frac{\partial}{\partial x_i} \delta^{\text{vir}} \hat{x}_{\nu l} \qquad (75) \\
&= \sum_{li} \mathsf{T}_{\nu l i} \frac{\partial}{\partial x_i} \xi_{\nu l}^{\text{vir}} \\
&= \sum_{lij} \mathsf{T}_{\nu l i} \frac{\partial \hat{x}_{\nu j}}{\partial x_i} \frac{\partial}{\partial \hat{x}_{\nu j}} \xi_{\nu l}^{\text{vir}} \\
&= \sum_{lj} J_\nu \delta_{lj} \frac{\partial}{\partial \hat{x}_{\nu j}} \xi_{\nu l}^{\text{vir}} \\
&= J_\nu \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot \boldsymbol{\xi}_\nu^{\text{vir}}.
\end{aligned}
$$

In the following we will do the virtual variation of $\boldsymbol{\xi}_\nu$ inside the time and space integral over the Lagrangian density (52) without localizing $\boldsymbol{\xi}_\nu^{\text{vir}}$:

$$
\begin{aligned}
& \int dt \int d^3x \Bigg(J_\nu \left(\frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot \boldsymbol{\xi}_\nu^{\text{vir}} \hat{n}_\nu \right) \left(\frac{m_\nu}{2} |\hat{\boldsymbol{v}}_\nu|^2 - \frac{p_\nu n_\nu^{-\gamma_\nu}}{\gamma_\nu - 1} \hat{n}_\nu^{\gamma_\nu - 1} \right. \\
& \left. - e_\nu \hat{\Phi} + e_\nu \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{A}} - \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \lambda_\nu \right) \\
& + \hat{n}_\nu J_\nu \Bigg(m_\nu \hat{\boldsymbol{v}}_\nu \cdot \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \boldsymbol{\xi}_\nu^{\text{vir}} - p_\nu n_\nu^{-\gamma_\nu} \boldsymbol{\xi}_\nu^{\text{vir}} \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \frac{\hat{n}_\nu^{\gamma_\nu - 1}}{\gamma_\nu - 1}
\end{aligned}
$$

$$-e_\nu \boldsymbol{\xi}_\nu^{\text{vir}} \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{\Phi} + e_\nu \hat{\boldsymbol{A}} \cdot \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \boldsymbol{\xi}_\nu^{\text{vir}}$$

$$+ e_\nu \hat{\boldsymbol{v}}_\nu \cdot \left(\boldsymbol{\xi}_\nu^{\text{vir}} \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{\boldsymbol{A}} \right) \Bigg) \Bigg). \tag{76}$$

In the next step we do partial integrations, make use of equation (71), localize $\boldsymbol{\xi}_\nu^{\text{vir}}$ at $[\boldsymbol{x}, t]$ and divide by (non zero) $\hat{n}_\nu J_\nu$:

$$0 = -\frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \left(\frac{m_\nu}{2} |\hat{v}_\nu|^2 - \frac{p_\nu n_\nu^{-\gamma_\nu}}{\gamma_\nu - 1} \hat{n}_\nu^{\gamma_\nu - 1} \right.$$

$$\left. - e_\nu \hat{\Phi} + e_\nu \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{A}} - \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \lambda_\nu \right)$$

$$- m_\nu \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \hat{\boldsymbol{v}}_\nu - p_\nu n_\nu^{-\gamma_\nu} \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \frac{\hat{n}_\nu^{\gamma_\nu - 1}}{\gamma_\nu - 1}$$

$$- e_\nu \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{\Phi} - e_\nu \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla \right) \hat{\boldsymbol{A}} + e_\nu \left(\frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{\boldsymbol{A}} \right) \cdot \hat{\boldsymbol{v}}_\nu. \tag{77}$$

Using equations (70) and (72), we can transform equation (77) to

$$0 = \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} p_\nu n_\nu^{-\gamma_\nu} \left(\hat{n}_\nu^{\gamma_\nu - 1} \right)$$

$$+ m_\nu \left(\left. \frac{\partial}{\partial t} \right|_{\hat{\boldsymbol{x}}_\nu} + \hat{\boldsymbol{v}}_\nu \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \right) \hat{\boldsymbol{v}}_\nu + p_\nu n_\nu^{-\gamma_\nu} \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \frac{\hat{n}_\nu^{\gamma_\nu - 1}}{\gamma_\nu - 1} + e_\nu \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{\Phi}$$

$$+ e_\nu \left. \frac{\partial}{\partial t} \right|_{\hat{\boldsymbol{x}}_\nu} \hat{\boldsymbol{A}} - e_\nu \hat{\boldsymbol{v}}_\nu \times \hat{\boldsymbol{B}}. \tag{78}$$

Using the condition (50), we can write equation (78) as the momentum transport equation:

$$0 = \hat{n}_\nu m_\nu \left(\left. \frac{\partial}{\partial t} \right|_{\hat{\boldsymbol{x}}_\nu} + \hat{\boldsymbol{v}}_\nu \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \right) \hat{\boldsymbol{v}}_\nu + \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \hat{p}_\nu$$

$$- e_\nu \hat{n}_\nu \hat{\boldsymbol{E}} - e_\nu \hat{n}_\nu \hat{\boldsymbol{v}}_\nu \times \hat{\boldsymbol{B}}. \tag{79}$$

As the Lagrangian density (52) leads to the correct equations of motion it has been chosen properly.

B. Invariance of the Lagrangian

This appendix shows the invariance of the Lagrangian $L = \int_V d^3x\mathcal{L}$ under the transformation (57)-(61), with the Lagrangian density $\mathcal{L}$ defined by equation (52). We first calculate the transformation of $\left(\hat{\Phi}, \hat{\boldsymbol{A}}, \hat{n}_\nu, \hat{\boldsymbol{v}}_\nu\right)$:

$$\delta\left(\hat{\Phi}, \hat{\boldsymbol{A}}, \hat{n}_\nu\right)[\hat{\boldsymbol{x}}_\nu, t] = \delta\boldsymbol{\xi}_\nu \cdot \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu}\left(\hat{\Phi}, \hat{\boldsymbol{A}}, \hat{n}_\nu\right)[\hat{\boldsymbol{x}}_\nu, t]$$
$$= \delta\tilde{\tau} g_\nu \boldsymbol{v}_\nu \cdot \nabla\left(\hat{\Phi}, \hat{\boldsymbol{A}}, \hat{n}_\nu\right)[\boldsymbol{x} + \boldsymbol{\xi}_\nu, t], \tag{80}$$

$$\begin{aligned}
\delta\hat{\boldsymbol{v}}_\nu[\hat{\boldsymbol{x}}_\nu, t] &= \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right)\delta\boldsymbol{\xi}_\nu \\
&= \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right)(\boldsymbol{v}_\nu \cdot \nabla\hat{\boldsymbol{x}}_\nu)\, g_\nu\delta\tilde{\tau} \\
&= g_\nu\delta\tilde{\tau}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right)(\boldsymbol{v}_\nu \cdot \nabla\hat{\boldsymbol{x}}_\nu) \\
&= \delta\tilde{\tau} g_\nu \boldsymbol{v}_\nu \cdot \nabla\hat{\boldsymbol{v}}_\nu[\boldsymbol{x} + \boldsymbol{\xi}_\nu, t].
\end{aligned} \tag{81}$$

Next we calculate the transformation of J_ν:

$$\begin{aligned}
\delta J_\nu &= \sum_{i,l} \mathsf{T}_{\nu li}\left(\frac{\partial}{\partial x_i}\delta\xi_{\nu l}\right) \\
&= \delta\tilde{\tau}\sum_{i,j,l} \mathsf{T}_{\nu li}\left(\frac{\partial}{\partial x_i} g_\nu v_{\nu j}\frac{\partial \hat{x}_{\nu l}}{\partial x_j}\right) \\
&= \delta\tilde{\tau}\left(g_\nu J_\nu \nabla \cdot \boldsymbol{v}_\nu + g_\nu \boldsymbol{v}_\nu \cdot \nabla J_\nu + (\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{x}}_\nu) \cdot \mathsf{T}_\nu \cdot \nabla g_\nu\right) \\
&= \delta\tilde{\tau}\nabla \cdot (\boldsymbol{v}_\nu g_\nu J_\nu).
\end{aligned} \tag{82}$$

Furthermore, we use the equation

$$J_\nu\left(\left.\frac{\partial}{\partial t}\right|_{\hat{\boldsymbol{x}}_\nu}\hat{n}_\nu + \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot \hat{\boldsymbol{v}}_\nu \hat{n}_\nu\right) = \left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \boldsymbol{v}\right) J_\nu \hat{n}_\nu, \tag{83}$$

which follows from the equations (48), (49), (73) (for example by using the transformation formula for a divergence in space-time). Using the fact that the equilibrium current is source free ($\nabla \cdot n_\nu \boldsymbol{v}_\nu = 0$) and using the equations (80)–(82) we can establish the invariance of the Lagrangian $L[\widehat{\Psi}]$ under the transformation (57)–(60) if the integration in the Lagrangian is carried out over a volume V belonging to V_{P_ν} on which the normal component of the equilibrium current $f_\nu n_\nu \boldsymbol{v}_\nu$ is zero everywhere on the surface ∂V:

$$
\begin{aligned}
\delta L[\widehat{\Psi}] =& \delta \sum_\nu \int_V d^3\hat{x}_\nu \hat{n}_\nu \hat{l}_\nu [\hat{\boldsymbol{x}}_\nu, t] \\
&+ \delta \sum_\nu \int d^3\hat{x}_\nu \lambda_\nu \left(\frac{\partial}{\partial t}\bigg|_{\hat{\boldsymbol{x}}_\nu} + \frac{\partial}{\partial \hat{\boldsymbol{x}}_\nu} \cdot \hat{\boldsymbol{v}}_\nu \right) \hat{n}_\nu [\hat{\boldsymbol{x}}_\nu, t] \\
=& \delta \sum_\nu \int_V d^3x J_\nu \hat{n}_\nu \hat{l}_\nu [\hat{\boldsymbol{x}}_\nu, t] \\
&+ \delta \sum_\nu \int d^3x \lambda_\nu \left(\frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{v}_\nu \right) J_\nu \hat{n}_\nu \\
=& \delta\tilde{\tau} \sum_\nu \int_V d^3x \nabla \cdot \left(g_\nu n_\nu \boldsymbol{v}_\nu \hat{l}_\nu [\boldsymbol{x} + \boldsymbol{\xi}_\nu, t] \right) \\
&+ \sum_\nu \int d^3x \lambda_\nu \left(\frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{v}_\nu \right) \delta \left(J_\nu \hat{n}_\nu \right) \\
=& \delta\tilde{\tau} \sum_\nu \int_{\partial V} d\boldsymbol{f} \cdot g_\nu n_\nu \boldsymbol{v}_\nu \hat{l}_\nu [\boldsymbol{x} + \boldsymbol{\xi}_\nu, t] \\
&+ \delta\tilde{\tau} \sum_\nu \int d^3x \lambda_\nu \left(\frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{v}_\nu \right) \nabla \cdot \left(\boldsymbol{v}_\nu g_\nu J_\nu \hat{n}_\nu \right) \\
=& \delta\tilde{\tau} \sum_\nu \int d^3x \lambda_\nu \left(\frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{v}_\nu \right) \nabla \cdot \left(n_\nu \boldsymbol{v}_\nu f_\nu \right) \\
=& 0
\end{aligned}
\tag{84}
$$

with the abbreviation

$$
\hat{l}_\nu = \frac{m_\nu}{2} |\hat{\boldsymbol{v}}_\nu|^2 - \frac{p_\nu[\boldsymbol{x}] n_\nu[\boldsymbol{x}]}{\gamma_\nu - 1} \hat{n}_\nu^{\gamma_\nu - 1} - e_\nu \hat{\Phi} + e_\nu \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{A}}.
$$

C. Energy contributions

The total energy (43) of a state $\widehat{\Psi}$ is

$$E[\widehat{\Psi}] = \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu \left(\hat{n}_\nu \frac{m_\nu}{2} |\hat{v}_\nu|^2 + \frac{\hat{p}_\nu}{\gamma_\nu - 1}\right)$$
$$= + \int_{V_{\text{fix}}} d^3x \left(\frac{\varepsilon_0}{2} \left|\hat{E}\right|^2 + \frac{\left|\hat{B}\right|^2}{2\mu_0}\right). \tag{85}$$

By using the electromagnetic potentials introduced by equation (38) the energy expression (85) can be written in the following form:

$$\begin{aligned} E[\widehat{\Psi}] \ = \ & \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu \hat{n}_\nu \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{p}}_\nu \\ & - \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu \left(\hat{n}_\nu \frac{m_\nu}{2} |\hat{v}_\nu|^2 - \frac{\hat{p}_\nu}{\gamma_\nu - 1}\right) \\ & + \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu e_\nu \hat{n}_\nu \left(\hat{\Phi} - \hat{v}_\nu \cdot \hat{\boldsymbol{A}}[\boldsymbol{x}, t]\right) \\ & - \sum_\nu \int_{V_{\text{fix}}} d^3x \, e_\nu \hat{n}_\nu[\boldsymbol{x}, t] \hat{\Phi}[\boldsymbol{x}, t] \\ & + \int_{V_{\text{fix}}} d^3x \left(\frac{\varepsilon_0}{2} \left|\frac{\partial}{\partial t} \hat{\boldsymbol{A}}[\boldsymbol{x}, t] + \nabla \hat{\Phi}\right|^2 + \frac{\left|\nabla \times \hat{\boldsymbol{A}}[\boldsymbol{x}, t]\right|^2}{2\mu_0}\right) \end{aligned} \tag{86}$$

By using the inhomogenous Maxwell equation (36) the energy expression (86) can be transformed to

$$
\begin{aligned}
E[\widehat{\Psi}] &= \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu \hat{n}_\nu \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{p}}_\nu \\
&+ \int_{V_{\text{fix}}} d^3x\, \varepsilon_0 \left(\frac{\partial}{\partial t} \hat{\boldsymbol{A}}[\boldsymbol{x},t] + \nabla \hat{\Phi} \right) \cdot \dot{\boldsymbol{a}} \\
&- \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu \left(\hat{n}_\nu \frac{m_\nu}{2} \left|\hat{\boldsymbol{v}}_\nu\right|^2 - \frac{\hat{p}_\nu}{\gamma_\nu - 1} \right) \\
&+ \sum_\nu \int_{\hat{V}_{P_\nu}} d^3\hat{x}_\nu e_\nu \hat{n}_\nu \left(\hat{\Phi} - \hat{\boldsymbol{v}}_\nu \cdot \hat{\boldsymbol{A}}[\boldsymbol{x},t] \right) \\
&+ \int_{V_{\text{fix}}} d^3x \left(-\frac{\varepsilon_0}{2} \left| \frac{\partial}{\partial t} \hat{\boldsymbol{A}}[\boldsymbol{x},t] + \nabla \hat{\Phi} \right|^2 + \frac{\left| \nabla \times \hat{\boldsymbol{A}}[\boldsymbol{x},t] \right|^2}{2\mu_0} \right) \\
&+ \int_{V_{\text{fix}}} d^3x \left(-\nabla \cdot \left(\varepsilon_0 \hat{\Phi} \hat{\boldsymbol{E}} \right) \right). \qquad (87)
\end{aligned}
$$

By using the expression (52) for the Lagrangian density $\mathcal{L}$ and

$$
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\xi}}_\nu} = J_\nu \hat{n}_\nu \hat{\boldsymbol{p}}_\nu, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} = \varepsilon_0 \hat{\boldsymbol{E}}
$$

the energy expression (87) can be given the more compact form (65).

By using equation (49) in the form $\hat{\boldsymbol{v}}_\nu[\hat{\boldsymbol{x}}_\nu, t] = \dot{\boldsymbol{\xi}}_\nu[\boldsymbol{x},t] + \boldsymbol{v}_\nu \cdot \nabla \hat{\boldsymbol{x}}_\nu[\boldsymbol{x},t]$ the energy expression (65) can be split into expression (67) and

$$
E_2[\widehat{\Psi}] = \sum_\nu \int_{V_{P_\nu}} d^3x \, (\boldsymbol{v}_\nu \cdot \nabla \hat{\boldsymbol{x}}_\nu) \cdot J_\nu[\boldsymbol{x},t] \hat{n}_\nu[\hat{\boldsymbol{x}}_\nu,t] \hat{\boldsymbol{p}}_\nu[\hat{\boldsymbol{x}}_\nu,t].
$$

Making use of equation (66) we get $E_2 = I_{\nu,V_{P_\nu},f_\nu=1}$, where $I_{\nu,V_{P_\nu},f_\nu=1}$ is defined by equation (63).

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