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A NONLINEAR EQUATION FOR DRIFT WAVES

H. Tasso

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A nonlinear equation for drift waves

H. Tasso
Max-Planck-Institut für Plasmaphysik
Euratom Association
85748 Garching bei München, Federal Republic of Germany

Abstract

A nonlinear drift wave equation is derived for a hot plasma confined by a strong magnetic field, the temperature gradient being perpendicular to the magnetic field and unidirectional. This equation which is obtained through a long wave-length approximation, is the two-dimensional Korteweg-deVries equation. It can be transformed approximately in the two-dimensional modified Korteweg-deVries equation via the Miura transformation. A bi-Hamiltonian formulation of the drift wave equation is given, which implies its complete integrability. This derivation of the nonlinear drift wave equation gives a strong support to k-spectrum calculations published previously by the author.

0.1 Introduction

In a series of papers (see [1] and references therein), the author was led to consider two-dimensional extensions of the Korteweg-de Vries (KdV) and modified KdV (mKdV) equations, in order to explain the observed k-spectra of drift waves. It will be shown in this paper that these models are not purely "ad hoc" but can be derived from two-fluid plasma theory using long wavelength scalings and some other appropriate assumptions. The plan of the paper is as follows. Section 2 contains the essentials of the derivation from the two-fluid plasma equations. Section 3 is devoted to the passage from the 2-D KdV equation to the 2-D mKdV model. A bi-Hamiltonian formulation of the 2-D KdV equation is given in section 4. Some concluding remarks can be found in section 5.

0.2 Derivation from the two-fluid model

The equations of motion of a two-fluid plasma (see e.g. [1]) are given by

$$n_i m_i\left(\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i\right) = e n_i(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i, \tag{0.1}$$

$$n_e m_e\left(\frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e\right) = -e n_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e. \tag{0.2}$$

Consider the application of system (1,2) to a low pressure plasma immersed in a strong magnetic $\mathbf{B}$ pointing in the z-direction. Assume quasi-neutral electrostatic perturbations with electrons isothermal along the magnetic field. Neglect in a first approximation the inertia of ions and electrons and solve (1,2) for $\mathbf{v}_i$, $\mathbf{v}_e$ and n_e. This situation augmented with continuity and Maxwell equations is well represented by the following system of equations.

$$n_i \mathbf{v}_{i\perp} = \frac{n_i \nabla\phi \times \mathbf{B}}{B^2} - \frac{\nabla p_i \times \mathbf{B}}{eB^2}, \tag{0.3}$$

$$n_e \mathbf{v}_{e\perp} = \frac{n_e \nabla\phi \times \mathbf{B}}{B^2} + \frac{\nabla p_e \times \mathbf{B}}{eB^2}, \tag{0.4}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_{i\perp}) = 0, \tag{0.5}$$

$$
\begin{aligned}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) &= 0, & (0.6) \\
n_e &= n_0(x) \exp - \frac{e\phi}{kT_e(x)}, & (0.7) \\
n_i &= n_e, & (0.8) \\
\mathbf{B} &= B\mathbf{e}_z, & (0.9)
\end{aligned}
$$

where n_e and n_i are the electron and ion densities, $\mathbf{v}_e$ and $\mathbf{v}_i$ are the electron and ion macroscopic velocities, T_e is the electron temperature, ϕ is the electrostatic potential, x is the coordinate perpendicular to the slab and y is the coordinate perpendicular to both x and z, n_0 is the unperturbed density, e is the charge of the proton, and k is the Boltzmann constant.

At this point it is important to identify the constraints which insure the correctness of the electrostatic approximation [2]. This problem is very involved, in general, but can be circumvented here by taking $p_e = f(n)$, which is an acceptable assumption.

Equations (3) and (4) solve the equations of motions (1,2) for neglected inertia. The parallel motion of the ions is small in view of their large mass. The continuity equations for ions and electrons are expressed by equations (5) and (6), while the electrons behave along z according to a Boltzmann distribution given by equation (7). Quasineutrality, easily restored by the electrons along the field lines, is ensured by equation (8).

Elimination of $\mathbf{v}_i$ and n_i from equation (5) using equations (3) and (8) leads to the following equation for ϕ :

$$
\frac{eB}{kT}\frac{\partial \phi}{\partial t} - (\frac{n'}{n} + \frac{T'}{T}\frac{e\phi}{kT})\frac{\partial \phi}{\partial y} = 0. \qquad (0.10)
$$

The subscripts as well as the explicit indication of x-dependence have been dropped in equation (10). The prime denotes the derivative with respect to x.

Equation (10) is essentially the inviscid Burgers equation in which x appears as a parameter. It is the simplest model of a nonlinear drift wave equation [3]. The nonlinearity is due to the temperature gradient of the electrons, which is present in any confined hot plasma, and is called "scalar nonlinearity" in the literature. In regions of flat temperature profiles, equation (10) becomes linear. In this case, however, higher-order terms due to ion inertia produce a so-called "vector nonlinearity", which is, in essence, two-dimensional and first appeared in Ref. [4].

The solutions of the inviscid Burgers equation are known to develop infinitely steep gradients at finite times, which can be prevented by adding some of the neglected physical terms such as ion inertia or gyroviscosity, thus limiting attention to nondissipative effects.

A first attempt [5] to take such terms into account was to consider the case of cold ions and concentrate on the first inertial term in equation (1). On the assumption of solutions with weak x-dependence, the correction due to ion inertia is obtained by iteration of equation (1), inserting in the inertial term the approximate solution given by equation (3) for zero ion pressure. This leads to

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} = \frac{\nabla\phi \times \mathbf{B}}{B^2} + \mathbf{e}_y \frac{m_i}{eB^2}\frac{\partial^2 \phi}{\partial y \partial t}. \quad (0.11)$$

Let us insert $\mathbf{v}$ from equation (11) in equation (5), using equations (7) and (8) to obtain

$$\frac{eB}{kT}\frac{\partial\phi}{\partial t} - (\frac{n'}{n} + \frac{T'}{T}\frac{e\phi}{kT})\frac{\partial\phi}{\partial y} - \frac{m_i}{eB}(\frac{\partial^3\phi}{\partial t\partial y^2} + \frac{e}{kT}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial t}) = 0. \quad (0.12)$$

Equation (12) becomes identical to equation (6) of Ref. [5] if ϕ is replaced by $-\phi$, and if solutions of the form $\phi(y - ut)$ are sought. The discussion of such solutions led to the existence of drift solitons [5] and other nonlinear waves.

In the long wave-length approximation, the dispersive term of equation (12) can be simplified by replacing the time derivative through $\frac{kTn'}{eBn}\frac{\partial}{\partial y}$. Furthermore, the nonlinear part of the dispersive term can be neglected for small values of ϕ. In the frame travelling at the drift velocity $v_d = \frac{kTn'}{eBn}$, equation (12) becomes the KdV equation. Finally, if we remove the assumption of weak x dependence for ϕ and neglect again the nonlinear contributions in the dispersive term, equations (11,12) become

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} = \frac{\nabla\phi \times \mathbf{B}}{B^2} + \frac{m_i}{eB^2}\nabla\frac{\partial\phi}{\partial t}, \quad (0.13)$$

$$\frac{\partial\phi}{\partial t} - \frac{T'}{BT}\phi\frac{\partial\phi}{\partial y} - \frac{m_i(kT)^2 n'}{(eB)^3 n}\frac{\partial}{\partial y}\nabla^2\phi = 0. \quad (0.14)$$

The coefficients of equation (14) have usually an x-dependence which is much weaker as the x-dependence of ϕ. This is why those coefficients will be considered constant throughout the paper. For appropriate boundary conditions e.g. vanishing of ϕ at large values of x,y, or periodic in y and vanishing for large x, equation (14) can be written in the form

$$\phi_t = \frac{\partial}{\partial y}\frac{\delta H}{\delta \phi}, \tag{0.15}$$

where the Hamiltonian H is

$$H = \int [A\phi^3 + B(\nabla\phi)^2]dxdy, \tag{0.16}$$

and A and B are appropriate constants. Equation (14) is the 2-D KdV equation for drift waves. It represents the lowest approximation to those waves in confined hot plasmas. Its modified form has been used in [1] to obtain 2-D k-spectra which agree with observations on large tokamaks. In the next section, the passage to the modified KdV equation will be investigated. Note that a 3-D KdV equation has been obtained by a similar procedure in the context of ion acoustic waves propagating along the magnetic field [6]. See also Ref. [7].

0.3 Modified 2-D KdV equation

As explained in [1], the Hamiltonian (16) of equation (14) is not bounded. To overcome this difficulty in the 1-D case, Miura transformation [8] has been used to obtain the 1-D mKdV equation which has a simple bounded Hamiltonian, the 1-D Ginzburg-Landau [9] Hamiltonian. It will be shown in this section that a 2-D equation of the kind of (14) can be transformed by Miura transformation in almost the 2-D mKdV equation up to quadratic dispersive terms which can be neglected on the same physical grounds as in section 2. Let us use the "standard" form of equation (14)

$$u_t - 6uu_y + u_{yyy} + u_{yxx}. \tag{0.17}$$

Miura transformation $u = v^2 + v_y$ leads to

$$(2v + \frac{\partial}{\partial y})[v_t - 6v^2 v_y + v_{yyy} + v_{yxx}] + 2(v_y v_x)_x. \tag{0.18}$$

The last term of equation (18) is a dispersive quadratic term and can be neglected. If we want v and the square bracket to vanish at infinity, we have to equate the square bracket to zero, to obtain the 2-D mKdV equation. This equation can be written in the form

$$v_t = \frac{\partial}{\partial y}\frac{\delta H}{\delta v}, \tag{0.19}$$

where

$$H = \frac{1}{2}\int [v^4 + v_y^2 + v_x^2] dx dy. \tag{0.20}$$

Hamiltonian (20) was precisely the starting point for the calculation of the 2-D space correlation function and the 2-D spectra of Ref. [1]. This Hamiltonian is not only an "ad hoc" reasonable assumption, but is now derived from the two-fluid theory of plasmas.

Let us note that Hamiltonian (20) is "massless" since the quadratic term in v^2 is missing. However, by adding and substracting this quadratic term in (20), one can define an "effective mass" which helps to justify the assumed exponential behaviour of the correlation function of Ref. [1]. The larger the "effective mass" is, the nearer to $r \equiv \sqrt{x^2+y^2} = 0$ will the exponential behaviour be true. The mild divergence at $r = 0$ will presumably not be suppressed in this way, but a residual viscosity and a dispersion with fifth order derivatives will do it.

This section gives a strong support to the tools used in the calculations of the spectrum $S(k)$ found in Ref. [10] and explained in [1]. We reproduce here the result obtained there:

$$S(k) = \frac{\mu}{(\sqrt{\mu^2 + k_x^2 + k_y^2})^3}. \tag{0.21}$$

0.4 Bi-Hamiltonian formulation of 2-D KdV equation

The nonlinear drift wave equation derived in this paper is the 2-D KdV equation (14). It has the Hamiltonian formulation (15,16). Like in the 1-D case the second Hamitonianian is easy to extend to the 2-D equation. In the

1-D case, the second formulation is given by

$$u_t = [u, H_E] = -(\frac{\partial^3}{\partial y^3} - 4u\frac{\partial}{\partial y} - 2u_y)\frac{\delta H_E}{\delta u}, \tag{0.22}$$

where

$$H_E = \frac{1}{2}\int u^2 dy. \tag{0.23}$$

Since the second dimension x in equation (14) appears only in the linear dispersive operator, it suffices to add that third derivative to the previous "cosymplectic" operator of the 1-D case. Indeed, it is known (see [11]) that a linear antisymmetric operator added to the "cosymplectic" operator of a given generalized Poisson bracket will not change the property of the new bracket to fulfil Jacobi identity. This means that the second Hamiltonian formulation is given by

$$u_t = [u, H_E] = -(\frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial y \partial x^2} - 4u\frac{\partial}{\partial y} - 2u_y)\frac{\delta H_E}{\delta u}, \tag{0.24}$$

with $H_E = \frac{1}{2}\int u^2 dxdy$.

The bi-Hamiltonian property has important implications as proved in [12]. In particular, it means that equation (14) has an infinity of constants of motion and is probably completely integrable. Moreover, a whole hierarchy of higher order equations can be constructed by recurrence [12].

The bi-Hamiltonian property of equation (14) does not imply the same property for the 2-D mKdV equation of section 3. Indeed, the Miura transformation fails to transform in an exact way the 2-D KdV into the 2-D mKdV equation. In order to obtain the 2-D mKdV equation, we had to neglect the quadratic dispersive terms on physical grounds. A modification of the Miura transformation does not seem to be successful either.

0.5 Concluding remarks

The derivation of tractable nonlinear wave equations out of complex systems of physical equations is important for the modelling and interpretation of phenomenological observations. Such derivations specialize on particular scalings which determine the kind of nonlinear equation to be applied in specific cases.

For drift waves in a hot confined plasma the nonlinearity due to a strong equilibrium temperature gradient [3] is dominant. The dispersion, however, is essentially linear and two-dimensional around the magnetic field. Scalar nonlinearity, long wave-length scaling and linear third derivative dispersion are the main ingredients of the derivation of equation (14) of this paper. It is not surprising anymore that this equation is the 2-D KdV equation or its modified form as guessed before by the author [10]. Note that in another context [6] the equation for nonlinear ion acoustic waves along the magnetic field was derived in a similar way, which led to the 3-D KdV equation.

The bi-Hamiltonian property proved in section 4 for the 2-D KdV equation can easily be extended to the 3-D KdV equation of Ref. [6] since the derivative with respect to the third variable does not enter the nonlinear part of the equation.

The derivation given in this paper for the nonlinear drift wave equation gives much more confidence to the calculations of the k-spectrum published in [10, 1], and reproduced here in equation (21). Note that the agreement of (21) with ovservations on large tokamaks is quite satisfactory.

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