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LAGRANGIANS FOR PLASMAS IN DRIFT-FLUID APPROXIMATION

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Abstract

For drift waves and related instabilities conservation laws can play a crucial role. In an ideal theory these conservation laws are guaranteed when a Lagrangian can be found from which the equations for the various quantities result by Hamilton's principle. Such a Lagrangian for plasmas in drift-fluid approximation was obtained by a heuristic method in a recent paper by Pfirsch and Correa-Restrepo. In the present paper the same Lagrangian is derived from the exact multi-fluid Lagrangian via an iterative approximation procedure which resembles the standard method usually applied to the equations of motion. That method, however, does not guarantee all the conservation laws to hold.

1 Introduction

For drift waves and related instabilities conservation laws for momentum or angular momentum and energy can play a crucial role. This is discussed in a recent paper by Pfirsch and Correa-Restrepo [1] within the framework of collisional multi-fluid theories. The central quantity in this discussion is a Lagrangian for the ideal theory which describes the plasma as multi-drift-fluids, i.e. fluids which obey equations of motion parallel to the magnetic field and perform all kinds of drift motions perpendicular to **B**. Pfirsch and Correa-Restrepo obtained this Lagrangian in a heuristic way via a "recipe" from the Littlejohn-Wimmel Lagrangian for single particle motion in drift approximation [2, 3]. Since this was not a derivation, one had to check whether the thus obtained Lagrangian yielded the correct Euler-Lagrange equations. Although heuristic considerations are often used in order to find a Lagrangian, it is also interesting in the present case to obtain the Lagrangian for drift-fluids as an approximation to the known exact Lagrangian of the multi-fluid theory. This is done in the present paper. The method resembles to a certain degree the standard iterative approximation procedure for the equations of motion, but now applied to the Lagrangian. It exhibits, in particular, the often small, but for the conservation laws to hold essential differences of the two methods. Both methods are restricted to motions with small dependence on time and distance, as compared with corresponding gyro-frequencies and radii.

The starting point for the derivation is not directly the usual form of the multi-fluid Lagrangian, but an equivalent one which corresponds to the phase space Lagrangian for particles. This form was already important for the "recipe" in Ref.[1].

In section 2 the "phase space" Lagrangian for multi-fluid theories is presented, and its equivalence with the usual one proved. Section 3 contains the iterative approximation scheme and the derivation of the drift-fluid Lagrangian together with a comparison with the standard method which is usually applied to the equations of motion.

2 Phase space Lagrangian for multi-fluid plasmas

The phase space Lagrange density for a multi-fluid plasma is

$$\mathcal{L} = \sum_\nu \mathcal{L}_\nu + \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2) \ , \tag{1}$$

$$\begin{aligned} \mathcal{L}_\nu &= n_\nu(\mathbf{x},t)\mathbf{v}_\nu(\mathbf{x},t)\cdot\left(m_\nu\mathbf{q}_\nu(\mathbf{x},t) + \frac{e_\nu}{c}\mathbf{A}(\mathbf{x},t)\right) \\ &\quad -n_\nu\frac{m_\nu}{2}\mathbf{q}_\nu^2 - \frac{p_\nu}{\gamma_\nu - 1} - n_\nu e_\nu\Phi(\mathbf{x},t) \ . \end{aligned} \tag{2}$$

Here, ν denotes the particle species. The "canonical momentum" is expressed as $m_\nu\mathbf{q}_\nu(\mathbf{x},t)+\frac{e_\nu}{c}\mathbf{A}(\mathbf{x},t)$ with the help of an independent vector field $\mathbf{q}_\nu(\mathbf{x},t)$. All other notations are standard. In Hamilton's principle the constraints on n_ν (mass conservation), p_ν (entropy conservation), and $\mathbf{v}_\nu$ (kinematic constraint) can be satisfied by expressing the variations of these quantities by arbitrary displacements $\boldsymbol{\zeta}_\nu$ as

$$\begin{aligned} \delta n_\nu = -\nabla\cdot(n_\nu\boldsymbol{\zeta}_\nu) \ , \ \delta p_\nu = -\boldsymbol{\zeta}_\nu\cdot\nabla p_\nu - \gamma_\nu p_\nu\nabla\cdot\boldsymbol{\zeta}_\nu \\ \delta\mathbf{v}_\nu = \dot{\boldsymbol{\zeta}}_\nu + \mathbf{v}_\nu\cdot\nabla\boldsymbol{\zeta}_\nu - \boldsymbol{\zeta}_\nu\cdot\nabla\mathbf{v}_\nu. \end{aligned} \tag{3}$$

The quantities $\mathbf{q}_\nu$, $\mathbf{A}$ and Φ are to be varied independently. For isothermal species, i.e. $\gamma_\nu = 1$, the thermal energy must be replaced by the free energy:

$$\frac{p_\nu}{\gamma_\nu - 1} \to p_\nu \ln p_\nu \quad \text{with} \quad \delta p_\nu = -\nabla\cdot(p_\nu\boldsymbol{\zeta}_\nu) \ . \tag{4}$$

Variation of $\mathbf{q}_\nu$ in the action integral yields

$$\mathbf{q}_\nu(\mathbf{x},t) = \mathbf{v}_\nu(\mathbf{x},t) \ . \tag{5}$$

When this is inserted in $\mathcal{L}_\nu$, Eq. (2), the usual form of the multi-fluid Lagrange density results, which proves the equivalence of the two forms of Lagrangians.

It is important to note that Eq. (5) is only the result of Hamilton's principle, and that $\mathbf{q}_\nu$ and $\mathbf{v}_\nu$ are primarily independent quantities.

When the Lagrange density (2) is used in Hamilton's principle, the Euler-Lagrange equations corresponding to the variations done in terms of the virtual displacements $\boldsymbol{\zeta}_\nu$ are

$$m_\nu n_\nu \left(\frac{\partial}{\partial t}\mathbf{q}_\nu + \nabla \frac{1}{2}\mathbf{q}_\nu^2 - \mathbf{v}_\nu \times (\nabla \times \mathbf{q}_\nu) \right) = -\nabla p_\nu + e_\nu n_\nu \left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\nu \times \mathbf{B} \right) . \tag{6}$$

With Eq. (5), the known equations of motion result. Variations of Φ and $\mathbf{A}$ yield the inhomogeneous Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi \sum_\nu e_\nu n_\nu \ , \ \nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_\nu e_\nu n_\nu \mathbf{v}_\nu + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} \ . \tag{7}$$

3 Iterative approximation scheme

This scheme will be based on the following representation of $\mathbf{q}_\nu(\mathbf{x}, t)$:

$$\mathbf{q}_\nu = \mathbf{v}_E + q_{\nu\|}\mathbf{b} + \hat{\mathbf{q}}_{\nu\perp} \ ,$$

$$\mathbf{v}_E = c\frac{\mathbf{E}\times\mathbf{B}}{B^2} \ , \ \mathbf{b} = \frac{\mathbf{B}}{B} \ . \tag{8}$$

The variations with respect to $\mathbf{q}_\nu$ are to be done now by varying $q_{\nu\|}$ and $\hat{\mathbf{q}}_{\nu\perp}$, and one obtains again Eq. (5). The representation (8) guarantees that the new quantity $\hat{\mathbf{q}}_{\nu\perp}$ is of the order of drift velocities independent of the frame of reference and scales therefore essentially as gyroradius over macroscopic length.

The approximation scheme makes use of the fact that insertion of the exact solutions for $\hat{\mathbf{q}}_{\nu\perp}$ as functionals of the other variables in the Lagrangian yields a Lagrangian in the variables $\mathbf{v}_\nu$, $q_{\nu\|}$, n_ν, p_ν, Φ, $\mathbf{A}$ which is equivalent to the original one. Equation (5) is then obtained only for the components parallel to $\mathbf{B}$. The components of Eq. (6) perpendicular to $\mathbf{B}$ yield $\mathbf{v}_{\nu\perp}$ as functionals of the other variables, and the parallel components of this equation are the equations for $q_{\nu\|}$. If one uses an approximation for $\hat{\mathbf{q}}_{\nu\perp}$, one obtains correspondingly approximate equations for $\mathbf{v}_{\nu\perp}$ and $q_{\nu\|}$. The iterative approximation scheme is then the following procedure:

Let $\hat{\mathbf{q}}_{\nu\perp}^{(s)}$ be the approximation to $\hat{\mathbf{q}}_{\nu\perp}$ resulting in the s-th iteration step. When used in Eq. (6) one obtains new solutions $\mathbf{v}_{\nu\perp}^{(s+1)}$ for $\mathbf{v}_{\nu\perp}$ as functionals of the other variables and $q_{\nu\|}^{(s+1)}$ for $q_{\nu\|}$. The still existing parallel component

of equation (5) is modified in general because of the possible occurrence of $q_{\nu\|}$ in $\hat{\mathbf{q}}^{(s)}_{\nu\perp}$. It yields

$$v^{(s+1)}_{\nu\|} = q^{(s+1)}_{\nu\|} + \delta v^{(s+1)}_{\nu\|} \ , \tag{9}$$

where $\delta v^{(s+1)}_{\nu\|}$ results from this modification. It will be zero for the approximation considered below. The new $\hat{\mathbf{q}}^{(s+1)}_{\nu\perp}$ is defined by

$$\mathbf{v}^{(s+1)}_{\nu\perp} = \mathbf{q}^{(s+1)}_{\nu\perp} = \mathbf{v}_E + \hat{\mathbf{q}}^{(s+1)}_{\nu\perp} \ . \tag{10}$$

As mentioned above $\hat{\mathbf{q}}_{\nu\perp}$ scales essentially as gyroradius over macroscopic length. An expansion, essentially in this smallness parameter, requires therefore

$$\hat{\mathbf{q}}^{(0)}_{\nu\perp} = 0 \ . \tag{11}$$

This makes the approximate iteration scheme almost the same procedure as the usual one for the equations of motion. It would be exactly the same if $\mathbf{v}_\nu$ on the left hand side of Eq. (6) were approximated by $\mathbf{q}_\nu$.

Insertion of Eq. (11) in the Lagrange density (2) yields the first-order Lagrange density

$$\mathcal{L}_\nu \approx \mathcal{L}^{(1)}_\nu = n_\nu(\mathbf{x},t)\mathbf{v}_\nu(\mathbf{x},t)\cdot\frac{e_\nu}{c}\hat{\mathbf{A}}(\mathbf{x},t) - n_\nu e_\nu\hat{\Phi}(\mathbf{x},t) - \frac{p_\nu}{\gamma_\nu - 1} \ ,$$

$$e_\nu\hat{\mathbf{A}} = e_\nu\mathbf{A} + m_\nu c(\mathbf{v}_E + q_{\nu\|}\mathbf{b}) \ , \ e_\nu\hat{\Phi} = e_\nu\Phi + \frac{1}{2}m_\nu(q^2_{\nu\|} + \mathbf{v}^2_E) \ . \tag{12}$$

This is exactly the Lagrangian found in Ref.[1] by a heuristic method. It has, in particular, the property that $\delta v^{(1)}_{\nu\|}$ in Eq.(9) vanishes.

An important feature of this drift-fluid Lagrangian is the following: The representations (8) introduce new dependences on the potentials $\mathbf{A}$ and Φ. The variations of the potentials related to these new dependences have, however, no effect, since these variations are multiplied by the Euler-Lagrange equations or derivatives of them corresponding to the $\mathbf{q}_\nu$-variations, which means multiplication by zero. The same is true when the exact solutions for $\hat{\mathbf{q}}_{\nu\perp}$ as functionals of the other variables are inserted. With approximate $\hat{\mathbf{q}}_{\nu\perp}$'s, however, the additional variations of $\mathbf{A}$ and Φ yield nonvanishing contributions to the inhomogeneous Maxwell equations which do not occurr in the standard approximation scheme for the equations of motion. They represent some, usually small, polarization charge and current densities and an additional magnetization current density. These are necessary for the conservation laws to hold also for the approximate theory and correct to a certain degree the incomplete description in terms of drift motions.

References

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