MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Cylindrical Ideal Magnetohydrodynamic Equilibria with Incompressible Flows

G.N. Throumoulopoulos and H. Tasso

IPP 6/343 September 1996

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Cylindrical ideal magnetohydrodynamic equilibria with incompressible flows

G. N. Throumoulopoulos[†] and H. Tasso[*]

[†] *Section of Theoretical Physics,*
Physics Department, University of Ioannina,
GR 451 10 Ioannina, Greece

[*] *Max-Planck-Institut für Plasmaphysik,*
EURATOM Association, D-85748 Garching, Germany

September 1996

Abstract

It is proved that (a) the solutions of the ideal magnetohydrodynamic equation, which describes the equilibrium states of a cylindrical plasma with purely poloidal flow and arbitrary cross sectional-shape [G. N. Throumoulopoulos and G. Pantis, to appear in Plasma Phys. and Contr. Fusion **38** (1996)], are also valid for incompressible equilibrium flows with the axial velocity component being a free surface quantity and that (b) for the case of isothermal incompressible equilibria the magnetic surfaces necessarily have circular cross-section.

I. Introduction

In a recent paper [1] it is proved that, if the ideal MHD stationary flows of a cylindrical plasma with arbitrary cross-sectional shape are purely poloidal, they must be incompressible. This property considerably simplifies the equilibrium problem, i.e. it turns out that the equilibrium is governed by an elliptic partial differential equation for the poloidal magnetic flux function ψ which is amenable to several classes of analytic solutions. For an arbitrary flow, i.e. when the velocity has non-vanishing axial and poloidal components, the equilibrium becomes much more complicated. With the adoption of a specific equation of state, e. g. isentropic magnetic surfaces [2], the symmetric equilibrium states in a two-dimensional geometry are governed by a partial differential equation for ψ which contains five surface quantities (i.e. quantities solely dependent on ψ), in conjuction with a nonlinear algebraic Bernoulli equation. The derivation of analytic solutions of this set of equations is difficult.

In the present note we study the equilibrium of a cylindrical plasma with incompressible flows and show that the incompressibility condition makes it possible to construct analytic equilibria which constitute a generalization of those obtained in Ref. [1]. This is the subject of Sec. II. The special class of incompressible equilibria with isothermal magnetic surfaces is examined in Sec. III. Section IV summarizes our conclusions.

II. Equilibrium equations and analytic solutions

The ideal MHD equilibrium states of plasma flows are governed by the following set of equations, written in standard notations and convenient units:

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P, \tag{2}$$

$$\nabla \times \mathbf{E} = 0, \tag{3}$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{6}$$

The system under consideration is a cylindrical plasma with flow and arbitrary cross-sectional shape. For this configuration convenient coordinates are ξ, η and z with unit basis vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$ and $\mathbf{e}_z$, where $\mathbf{e}_z$ is parallel to the axis of symmetry and ξ and η are generalized coordinates pertaining to the poloidal cross- section. The equilibrium quantities do not depend on z. The divergence-free fields, viz.

the magnetic field $\mathbf{B}$, the current density $\mathbf{j}$ and the mass flow $\rho\mathbf{v}$, can be expressed in terms of the stream functions $\psi(\xi,\eta)$, $F(\xi,\eta)$, $B_z(\xi,\eta)$ and $v_z(\xi,\eta)$ as

$$\mathbf{B} = B_z\mathbf{e}_z + \mathbf{e}_z \times \nabla\psi, \tag{7}$$

$$\mathbf{j} = \nabla^2\psi\mathbf{e}_z - \mathbf{e}_z \times \nabla B_z \tag{8}$$

and

$$\rho\mathbf{v} = \rho v_z\mathbf{e}_z + \mathbf{e}_z \times \nabla F. \tag{9}$$

Constant ψ surfaces are the magnetic surfaces. Equations (1)-(6) can be reduced by means of certain integrals of the system which are shown to be surface quantities. To identify two of these quantities, the time-independent electric field is expressed by $\mathbf{E} = -\nabla\Phi$ and Ohm's law (6) is projected along $\mathbf{e}_z$ and $\mathbf{B}$, yielding respectively,

$$\mathbf{e}_z \cdot (\mathbf{e}_z \times \nabla F) \times (\mathbf{e}_z \times \nabla\psi) = 0 \tag{10}$$

and

$$\mathbf{B} \cdot \nabla\Phi = 0. \tag{11}$$

Equations (10) and (11) imply that $F = F(\psi)$ and $\Phi = \Phi(\psi)$. Two additional surface quantities are found from the component of Eq. (6) perpendicular to a magnetic surface:

$$\frac{B_z F'}{\rho} - v_z = \Phi', \tag{12}$$

and from the component of the momentum conservation equation (2) along $\mathbf{e}_z$:

$$B_z - F'v_z \equiv X(\psi). \tag{13}$$

(The prime denotes differentiation with respect to ψ.) Solving the set of equations (12) and (13) for B_z and v_z, one obtains

$$B_z = \frac{\rho[X(\psi) - F'(\psi)\Phi'(\psi)]}{\rho - (F'(\psi))^2} \tag{14}$$

and

$$v_z = \frac{F'(\psi)X(\psi) - \Phi'(\psi)}{\rho - (F'(\psi))^2}. \tag{15}$$

With the aid of Eqs. (10)-(13), the components of Eq. (2) along $\mathbf{B}$ and perpendicular to a magnetic surface are put in the forms, respectively,

$$\mathbf{B} \cdot \left[\nabla\left(\frac{v_z^2}{2} + v_z\Phi' + \frac{|\nabla F|^2}{2\rho^2}\right) + \frac{\nabla P}{\rho}\right] = 0 \tag{16}$$

and

$$\nabla\cdot\left[\left(1-\frac{(F')^2}{\rho}\right)\nabla\psi\right]+\frac{F''F'|\nabla\psi|^2}{\rho}+\frac{B_z\nabla B_z\cdot\nabla\psi}{|\nabla\psi|^2}$$
$$+\rho\frac{\nabla\psi}{|\nabla\psi|^2}\cdot\left[\nabla\left(\frac{(F')^2|\nabla\psi|^2}{2\rho^2}\right)+\frac{\nabla P}{\rho}\right]=0. \tag{17}$$

It is pointed out here that Eqs. (16) and (17) are valid for any equation of state for the plasma.

In order to reduce the equilibrium equations further, we employ the incompressibiliy condition

$$\nabla\cdot\mathbf{v}=0. \tag{18}$$

Equation (1) then implies that the density is a surface quantity

$$\rho=\rho(\psi), \tag{19}$$

and, consequently, Eqs. (14) and (15) yield

$$B_z=B_z(\psi),\quad v_z=v_z(\psi). \tag{20}$$

By means of Eqs. (19) and (20), Eq. (16) can be integrated to yield an expression for the pressure:

$$P=P_s(\psi)-\frac{F'^2}{2\rho}|\nabla\psi|^2. \tag{21}$$

We note here that, unlike in static equilibria, in the presence of flow magnetic surfaces do not coincide with isobaric surfaces, because Eq. (2) implies that $\mathbf{B}\cdot\nabla P$ in general differs from zero. In this respect, the term $P_s(\psi)$ is the static part of the pressure which does not vanish when F' is set to zero; Eqs. (14), (15) and (17) have a singularity when

$$\frac{(F')^2}{\rho}=1. \tag{22}$$

On the basis of Eq. (9) for $\rho\mathbf{v}$ and the definitions $v_{Ap}^2\equiv\frac{|\nabla\psi|^2}{\rho}$ for the Alfvén velocity associated with the poloidal magnetic field and the Mach number $M^2\equiv\frac{v^2}{v_{Ap}^2}$, Eq. (22) can be written as $M^2=1$.

If it is now assumed that $\frac{(F')^2}{\rho}\neq 1$ and Eq. (21) is inserted into Eq. (17), the latter reduces to the *elliptic* differential equation

$$\left[1-\frac{(F')^2}{\rho}\right]\nabla^2\psi+\frac{F'}{\rho}\left(\frac{F'}{2}\frac{\rho'}{\rho}-F''\right)|\nabla\psi|^2+\left(P_s+\frac{B_z^2}{2}\right)'=0. \tag{23}$$

The absence of any hyperbolic regime in Eq. (23) can be understood by noting that, as is well known from gas dynamics, the flow must be compressible to allow the equilibrium differential equation to depart from ellipticity. Equation (23) *does not contain the axial velocity* v_z and is identical to the equation governing cylindrical equilibria with purely poloidal flow [1]. By using the ansatz $\frac{\rho'}{\rho} = 2\frac{F''}{F'}$, which implies that $\frac{(F')^2}{\rho} \equiv M_c^2 = \text{const.}$, Eq. (23) is reduced to

$$\nabla^2\psi + \frac{1}{1-M_c^2}\left(P_s + \frac{B_z^2}{2}\right)' = 0. \tag{24}$$

This is similar in form to the equation governing static equilibria; the only explicit reminiscence of flow is the presence of M_c. Equation (24) can be linearized for several choices of $P_s + \frac{B_z^2}{2}$ and a variety of analytic solutions of the linearized equation can be derived. In particular, the exact solutions for a circular cylindrical plasma obtained in Ref. [1] are also valid for incompressible equilibrium flows with a free axial velocity $v_z(\psi)$.

The singularity $M_c^2 = 1$ is the limit at which the confinement can be assured by the axial current $\nabla^2\psi$ alone. For $M_c^2 > 1$ the derivative of $\frac{B_z^2}{2}$ must partly compensate for the pressure gradient.

III. Equilibria with isothermal magnetic surfaces

For fusion plasmas the thermal conduction along $\mathbf{B}$ is fast in relation to the heat transport perpendicular to a magnetic surface and therefore equilibria with isothermal magnetic surfaces are of particular interest. The plasma is also assumed to obey the ideal gas law $P = R\rho T$. For this kind of equilibria, Eq. (21) implies that $|\nabla\psi|$ is a surface quantity and consequently from Eq. (17) it is found that $\nabla^2\psi$ is a surface quantity as well. Thus, the incompressible $T = T(\psi)$ equilibria satisfy the set of equations

$$|\nabla\psi|^2 = (g(\psi))^2 \tag{25}$$

and

$$\nabla^2\psi = f(\psi). \tag{26}$$

Equations (25) and (26) imply that, on a magnetic surface, the modulus of the vector $\nabla\psi$, which is perpendicular to this (arbitrary) magnetic surface, and $\nabla^2\psi$, which is related to the variation of $|\nabla\psi|$, are constants. It could therefore be speculated that magnetic surfaces are restricted to being circular. This conjecture can be proved as follows.

The coordinates ξ, η and z are specified to be the Cartesian coordinates x, y and z. With the introduction of the quantities $p = \partial\psi/\partial x$, $q = \partial\psi/\partial y$, $r = \partial^2\psi/\partial x^2$ and $t = \partial^2\psi/\partial y^2$, Eqs. (25) and (26) are written in the forms

$$p^2 + q^2 = g^2 \tag{27}$$

and

$$r + t = f. \tag{28}$$

The set of equations (27) and (28) can be integrated by applying a procedure suggested by Palumbo [3]. Accordingly, considering the functions p and q, which are functions of x and y as functions of x and $\psi(x, y)$, one has

$$r = \left.\frac{\partial p}{\partial x}\right|_y = \frac{\partial p}{\partial x} + p\left.\frac{\partial p}{\partial \psi}\right|_y \tag{29}$$

and

$$t = q\frac{\partial q}{\partial \psi}. \tag{30}$$

(It is noted here that a surface function $\zeta = \zeta(x, y) \equiv \zeta(\psi)$ can be employed instead of ψ.) With the aid of Eqs. (27), (29) and (30), Eq. (28) reduces to $\left.\frac{\partial p}{\partial x}\right|_\psi = f - gg'$ and consequently

$$p = x\,(f - gg') + h(\psi). \tag{31}$$

On a magnetic surface it holds that $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \equiv 0$, and hence

$$\left(\left.\frac{dy}{dx}\right|_\psi\right)^2 = \frac{p^2}{q^2} = \frac{[x\,(f - gg') + h]^2}{g^2 - [x\,(f - gg') + h]^2}. \tag{32}$$

The new quantities $a(\psi) \equiv f - gg'$, $X \equiv ax + h$ and $Y \equiv ay$ are introduced to put Eq. (32) in the form

$$\left(\frac{dY}{dX}\right)^2 = \frac{X^2}{g^2 - X^2}. \tag{33}$$

Equation (33) describes a circle on the (x, y) plane with radius $|g|$ centred at $(-h/a, 0)$.

IV. Conclusions

It has been proved that the ideal MHD equilibrium states of a cylindrical plasma with incompressible flows and arbitrary cross-sectional shape satisfy an elliptic partial differential equation [Eq. (23)] which is identical to the equation governing cylindrical equilibria with purely poloidal flow; the axial flow velocity is a free surface quantity. This equation permits the construction of several classes of analytic solutions. In particular, the exact equlibrium solutions for a circular cylindrical plasma and purely poloidal flow [1] are also valid for the present case. In addition, it has been proved that the magnetic surfaces of isothermal incompressible equilibria must have circular cross-section.

It is interesting to investigate symmetric incompressible equlibria in geometries more realistically representing the magnetic confinement systems, e.g. axisymmetric and straight, helically symmetric configurations. In this respect it may be noted here that, as proved in Ref. [4], the special class of axially symmetric, incompressible $\beta_p = 1$ MHD equilibria with purely poloidal velocity does not exist; the only possible stationary equilibria of this kind are of cylindrical shape.

Acknowledgments

This work was performed during a visit by one of the authors (G.N.T.) to Max-Planck Institute für Plasmaphysik, Garching. The hospitality provided at the said institute is appreciated. G.N.T. acknowledges support by EURATOM (Mobility Contract No 131-83-7 FUSC). One of the authors (H.T.) would like to thank Prof. D. Pfirsch for a useful discussion.

References

[1] G. N. Throumoulopoulos and G. Pantis, Report No. IOA: 03-96, University of Ioannina, Section of Theoretical Physics, Ioannina, Greece; to appear in Plasma Phys. Control. Fusion **38** (1996).

[2] A. I. Morozov and L. S. Solov'ev, Reviews of Plasma Physics **8**, 1 (1980), ed. M. A. Leontovich (New York: Consultants Bureau).

[3] D. Palumbo, Nuovo Cimento B **53**, 507 (1968).

[4] H. Tasso, Phys. Fluids **13**, 1874 (1970).