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Non local effects of ICRH on the singularities of the e.m. field

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Abstract

The non–local effect of ICRH on the singularities of the e.m. field of a tokamak (Vlasov) plasma is investigated and compared with the effect of the finite Larmor radius. The parameter region where one of the two dominates is derived. In order to obtain these results, first the location of the singularities (when the thermal velocity is zero) and the form of the field in their neighbourhood are derived.

Introduction

The aim of this paper is to study the effect of ICRH on the singularities of the e.m. field of a tokamak (Vlasov) plasma. The singularities have two causes:

- Particles can be in resonance, somewhere in the plasma, with the external magnetic field, causing the displacement current, and hence the dielectric tensor, to diverge at zero temperature.
- The solutions of the partial differential (Maxwell) equations can be singular (at zero temperature) for some finite value of the coefficients, i.e. also if the displacement current is finite.

The singularities disappear when $v_t \neq 0$ through the effect of the finite Larmor radius, and because a resonance in the plasma, as in ICRH, modifies the dielectric tensor far from the resonance region as well. The study of the second effect requires a new approach to determining the singularities at zero temperature. The electric field is not written as a series with a given first singular term, but is determined in a two–dimensional strip (toroidal periodicity being assumed) by requiring that the components of the electric field parallel and perpendicular to the curve (not yet known) on which the solution is singular satisfy some inequalities (Section 1). The form of the singularities is then obtained by solving the equations (Section 2). The effect of ICRH is considered in Section 3 and then compared with the effect of the finite Larmor radius in Section 4. Finally, the direction of the energy flux in the neighbourhood of the resonance region (without and with the effect of ICRH) is deduced in Section 5.

1. The equations

The problem is simplified by approximating the toroidal plasma by a straight cylinder in the z direction and postulating that all quantities be periodic in z, with period $2\pi R$. The toroidal effects considered are those due to the magnetic field $B_z = B_o\big(1 - (r/R)\cos\theta\big)$ and $B_\theta = B_o r/qR$. The surface where the gyrofrequency Ω_i of the ion species to be heated is equal to the frequency ω of the ICRH waves is denoted by $\theta = \theta_r(r)$. The value of the magnetic field encountered by a particle at a point (r,θ) is approximated by the value at the gyrocentres. Moreover, trapped or quasi–trapped particles are neglected since it is assumed that $r \ll R$ (see Cattanei & Croci, 1977). For the part of dielectric tensor ϵ_{ij} due to the particles with gyrofrequency different from Ω_i, and for the contribution of the ions to be heated to the displacement current j^- (the component that rotates as the electrons) it is assumed that the thermal velocity is zero. The electric displacement is then written in the form (with $(r,\theta,z) \to (1,2,3)$ for simplicity of notation)

$$
\begin{aligned}
D_1 &= \epsilon_{11}E_1 + \epsilon_{12}E_2 - (2\pi i/\omega)j^+ , \\
D_2 &= \epsilon_{21}E_1 + \epsilon_{22}E_2 - (2\pi/\omega)j^+ .
\end{aligned} \tag{1}
$$

The displacement current j^+ (the component that rotates as the ions) is connected with the electric field (via the Vlasov equation) by an integral operator, where the electric field appears (as a consequence of the approximation described before) in the form $E_j\big(\theta + vs\,, r(v,s)\big)$ ($v \equiv v_z/v_t$; $s \equiv v_t t'/qR$ is the normalized time along the characteristics and v_t is the thermal velocity of the ions). It is then assumed that $s \ll 1$, which is equivalent to the assumption that the resistivity, although not local, is independent of the value of the field at a poloidal distance comparable to r; since for $t' = 1/\Omega_i$ one has $s = \rho/qR \ll 1$ ($\rho \equiv v_t/\Omega_i$), the integration interval over s can be chosen so that (although $s \ll 1$) the ions gyrate many times in the time interval considered. Since, moreover, the most important contribution to the integral over v is due to the interval $v \lesssim 1$, it is reasonable to write E_j in the operator that defines j^+ in the form

$$
E_j\big(\theta + vs\,, r(\cdot)\big) \approx E_j(\theta\,, r) + vs\ \partial_\theta E_j(\theta\,, r) + O\big(\rho^2(v,s)\big)\ \partial^2 E_j/\partial r^2 .
$$

This yields the following expression for j^+ (see Croci, 1995):

$$
j^+ = -(qR\omega_{pi}^2/16\pi^{3/2}v_t)\big(GE^+ + i(\partial_{n_z}G)\,\partial_\theta E^+\big) + O\big(\rho^2(v,s)\big)\ \partial^2 E^+/\partial r^2 , \tag{2}
$$

where G is a function of (r,θ,n_z) as defined in the paper cited; different approximations of it can also be found there. It is convenient to introduce the quantities

$$
\epsilon \equiv i\,(qR\omega_{pi}^2/8\pi^{1/2}\omega v_t)G\,, \qquad \epsilon' \equiv (qR\omega_{pi}^2/8\pi^{1/2}\omega v_t)\partial G/\partial n_z\,.
$$

Equation (2) can thus be written in the form

$$j^+ = \epsilon E^+ + \epsilon' \partial_\theta E^+ + O(\epsilon\rho^2)\partial^2 E^+/\partial r^2 .$$

The second term is correct if $|\epsilon' \partial_\theta E^+| \ll |\epsilon E^+|$, a condition that is verified by the solutions to be derived, as will be seen in Section 4.

The approximations of G are given inside the zone delimited by $(\theta-\theta_r)^2 = (\alpha|\sin\theta_r|)^{-1}$ (with $\alpha \equiv qr/\rho$) – the resonance zone – and outside it. Outside the resonance zone one has

$$G \approx -i(\pi^{1/2}/n_z)\, Z\big((\cos\theta - \cos\theta_r)\alpha/n_z\big) \rightarrow -i(\pi^{1/2}/n_z)\, Z\big(\alpha\Theta/n_z\big) ,$$

where Z is the Plasma Dispersion Function. If the condition $n_z \lesssim \alpha\Theta$ is verified, outside the resonance zone one has

$$G \approx i\,\frac{\pi^{1/2}}{\alpha\Theta}\left(1 + \frac{n_z^2}{2\alpha^2\Theta^2}\right) .$$

Thus ϵ' is imaginary, and is zero for $n_z = 0$ (besides the obvious $v_t = 0$); ϵ is real. The solution will first be derived for $n_z \lesssim \alpha\Theta$; a discussion of this condition follows in Section 4.

In the resonance zone one has

$$G \approx \frac{\pi^{1/2}\Gamma(1/4)}{4(\alpha|\sin\theta_r|)^{1/2}} + \frac{\pi^{1/2} n_z}{\alpha|\sin\theta_r|} .$$

Thus here ϵ is imaginary and ϵ' is real.

An essential approximation used in Sections 2 and 3 is that the term proportional to ρ^2 in j^+ is neglected in relation to the term proportional to ϵ'. The explicit form of the solution then allows one to determine (in Section 4) the interval of the parameters where the approximation is verified, in particular the interval of n_z.

Since the coupling of E_3 with $E_{1,2}$ is neglected although $k_z \neq 0$, the Maxwell equations are

$$B_1 = -n_z E_2 , \quad B_2 = n_z E_1 , \quad B_3 = i(c/r\omega)\big(\partial_r(rE_2) - \partial_\theta E_1\big) ,$$

$$D_1 - n_z^2 E_1 = -i(c/r\omega)\partial_\theta B_3 , \quad D_2 - n_z^2 E_2 = i(c/\omega)\partial_r B_3 . \tag{3}$$

With equations (1) and (2) the Maxwell equations become (with the notation $\lambda_o \equiv c/r\omega$)

$$(\epsilon_{11} - n_z^2 + \epsilon)E_1 + \epsilon' \frac{\partial}{\partial\theta}E_1 + (\epsilon_{12} + i\epsilon)E_2 + i\epsilon' \frac{\partial}{\partial\theta}E_2 = -i\lambda_o \frac{\partial B_3}{\partial\theta} ,$$

$$(\epsilon_{21} - i\epsilon)E_1 - i\epsilon' \frac{\partial}{\partial\theta}E_1 + (\epsilon_{22} - n_z^2 + \epsilon)E_2 + \epsilon' \frac{\partial}{\partial\theta}E_2 = ir\lambda_o \frac{\partial B_3}{\partial r} . \tag{4}$$

The curve along which the electric field is singular when one has $v_t = 0$ is denoted by $r = r_o(\theta)$, where r_o is real. It is assumed that in its neighbourhood the field depends on r as $r - r_o(\theta)$, so that the Maxwell equations (3) have the following form (the index $/\theta$ denotes the derivative with respect to the θ dependence other than that arising from $r - r_o(\theta)$; the prime denotes either the derivative with respect to r, or the derivative with respect to θ when this is the only variable, as in r_o) :

$$D_1 - n_z^2 E_1 = i\lambda_o(r_o' B_3' - B_{3/\theta}) , \tag{5a}$$

$$D_2 - n_z^2 E_2 = ir\lambda_o B_3' . \tag{5b}$$

The magnetic field is given by

$$-iB_3/\lambda_o = r_o' E_1' + r_o E_2' + E_2 - E_{1/\theta} .$$

It is convenient to replace equation (5a) by a linear combination of equations (5a) and (5b), so that the following equivalent system is obtained:

$$r(D_1 - n_z^2 E_1) - r_o'(D_2 + n_z^2 E_2) = -ir\lambda_o B_{3/\theta} ,$$

$$D_2 - n_z^2 E_2 = ir\lambda_o B_3' . \tag{6}$$

Equations (4) thus become (with $A \equiv (\epsilon_{11} - n_z^2 + \epsilon) r\lambda_o^2$ and $B \equiv (\epsilon_{12} + i\epsilon) r\lambda_o^2$)

$$(rA + r_o' B)E_1 + (-r_o' A + rB)E_2 + (r + ir_o')\epsilon' \partial_\theta E^+ = -(i/\lambda_o) B_{3/\theta} ,$$

$$-BE_1 + AE_2 - i\epsilon' \partial_\theta E^+ = (i/\lambda_o) B_3' . \tag{7}$$

It is useful to introduce the projections of the electric field on the tangent and on the perpendicular to the curve $r = r_o(\theta)$, that is (with $d_o^2 \equiv r_o^2 + r_o'^2$) :

$$d_o F_1 \equiv r_o' E_1 + r_o E_2 , \quad d_o F_2 \equiv r_o E_1 - r_o' E_2 ,$$

with the inverses

$$d_o E_1 = r_o' F_1 + r_o F_2 , \quad d_o E_2 = r_o F_1 - r_o' F_2 .$$

The magnetic field is thus given by

$$-iB_3 d_o/\lambda_o = d_o^2 F_1' + r_o F_1 - r_o' F_2 - d_o\big((r_o' F_1 + r_o F_2)/d_o\big)_\theta . \tag{8}$$

System (7) becomes

$$d_o^2\big(BF_1 + AF_2 + \epsilon' r_o'(F_1' - iF_2')\big) = \Big[d_o^2 F_1' + r_o F_1 - r_o' F_2 - d_o\big((r_o' F_1 + r_o F_2)/d_o\big)_\theta\Big]_{/\theta} ,$$

$$(Ar - Br_o')F_1 - (Ar_o' + Br)F_2 + i\epsilon' r_o'(r_o' + ir_o)(F_1' - iF_2') =$$
$$= \Big[d_o^2 F_1' + r_o F_1 - r_o' F_2 - d_o\big((r_o' F_1 + r_o F_2)/d_o\big)_\theta\Big]' . \tag{9}$$

It is obvious that a solution of system (9) can be singular only if $\epsilon' = 0$, i.e. if v_t and/or n_z are equal to zero (the singular point being $X = 0$). However, when ϵ' is different from zero but is sufficiently small, in the plane (X, ϵ') there will be a region that does not contain the singular point $X = 0$, where the solution has the same form (in some asymptotic sense) as the solution for $\epsilon' = 0$. The characterization of the solution we are seeking — which is crucial point to the problem — should distinguish between the two cases and also reproduce properties of the solution without θ dependence (and, for that reason, also without ϵ'). It is then required that the solution satisfy the following inequalities:

$$\text{for } \epsilon' = 0: \quad F_j/r_o F_j' \to 0, \quad F_1/F_2 \to 0;$$

$$\text{for } \epsilon' \neq 0: \quad |F_j/r_o F_j'| \ll 1, \quad |F_1/F_2| \ll 1. \tag{10}$$

The dependence on θ that does not derive from $r - r_o$ is considered as — comparatively — weak; accordingly, the inequalities to be satisfied are less stringent:

$$|F_j| \leq |F_{j/\theta}| \ll |F_j'| \quad \text{for every } v_t. \tag{11}$$

A first consequence of these inequalities is that the condition for the validity of equation (2) becomes

$$|\epsilon' r_o' F_2'| \ll |\epsilon F_2|. \tag{12}$$

Moreover, B_3 (given by equation (8)) is approximated by

$$-iB_3 d_o/\lambda_o \approx d_o^2 F_1' - r_o' F_2 - d_o (r_o F_2/d_o)_{/\theta}.$$

A more far–reaching consequence follows from the condition $|B_{3/\theta}| \ll |d_o B_3'|$ applied to system (9), since then it must hold that

$$d_o |BF_1 + AF_2| \ll |(Ar_o - Br_o')F_1 - (Ar_o' + Br_o)F_2|.$$

This inequality, together with $|F_1/F_2| \to 0$, yields $A = 0$ as the necessary condition for the existence of singularities. The first equation of system (9) is now derived with respect to X and then subtracted from the second, derived with respect to θ. This new equation is used instead of the first equation of system (9). Thus, in the neighbourhood of $A = 0$, where $A \approx XA'$ (and with $|F_1| \ll |F_2|$), one obtains the system of equations

$$d_o^2\big((BF_1)' + (AF_2)' + \epsilon' r_o'(F_1'' - iF_2'')\big) = \big(BrF_2 - i\epsilon' r_o'(r_o' + ir_o)(F_1' - iF_2')\big)_{/\theta},$$

$$BrF_2 + i\epsilon' r_o'(r_o' + ir_o)(F_1' - iF_2') = d_o^2 F_1'' - r_o' F_2' - d_o\big(r_o F_2'/d_o\big)_\theta. \tag{13}$$

A further simplification yields

$$d_o^2\big((BF_1)' + (AF_2)' - i\epsilon' r_o' F_2''\big) = r(BF_2)_{/\theta},$$

$$BrF_2 + \epsilon' r_o'(r_o' + ir_o)F_2' = d_o^2 F_1'' - r_o' F_2' - d_o\big(r_o F_2'/d_o\big)_\theta \, . \tag{14}$$

In order to discuss this system, it is convenient to write the second of equations (14) in the form

$$rBF_2 + \epsilon' r_o'(r_o' + ir_o)F_2' =$$

$$= d_o^2\big((1/B)(BF_1)'' + 2(1/B)'(BF_1)' + (1/B)''(BF_1)\big) - r_o' F_2' - d_o(r_o F_2'/d_o)_{/\theta} \, .$$

By means of the first of equations (14) one finally obtains

$$-i\epsilon'(r_o'/A')F_2'''' + \big(X + 2i\epsilon' r_o'(B'/BA')\big)F_2'' + a_1 F_2' -$$

$$-(B/d_o^2 A')\big(XF_{2/\theta}\big)' + a_2 F_{2/\theta} + a_3 F_2 + a_4 F_1 = 0 \, , \tag{15}$$

where

$$a_1 \approx 2 - (Br_o/d_o^2 A')\big((d_o'/d_o) + (B_{/\theta}/B) - 2r_o'/r_o\big) \, ,$$

$$a_2 \approx 2r_o(B'/d_o^2 A') \, ,$$

$$a_3 \equiv r_o B^2/d_o^2 A' - 2(B'/B) - (B/d_o^2 A')\big((B_{/\theta}/B) + r_o(B'_{/\theta}/B) - 2r_o(B' B_{/\theta}/B^2)\big) \, ,$$

$$a_4 \equiv -(B/A')(1/B)'' \, .$$

The approximations in these coefficients consist in having neglected a term proportional to X in a_1, and a term proportional to ϵ' in a_2. It is convenient to introduce the quantity

$$i\nu \equiv 2 - a_1 \rightarrow (Br_o/d_o^2 A')\big((d_o'/d_o) + (B_{/\theta}/B) - 2(r_o'/r_o)\big) \, ,$$

which is real outside the resonance zone because B is imaginary there, and is equal to zero when there is no θ dependence.

2. The case $\epsilon' = 0$

In this section equation (15) is discussed without the non local effect of ICRH, that is without ϵ'; equation (15) thus becomes

$$XF_2'' + a_1F_2' - (B/d_o^2A')\big(XF_{2/\theta}\big)' + a_2F_{2/\theta} + a_3F_2 + a_4F_1 = 0\,. \qquad (16)$$

A consequence of equations (10) and (11) is that the term proportional to F_1 is negligible, so that equation (16) becomes a second order differential equation for F_2. As the coefficient of the second derivative is X, a solution is singular at $X = 0$. An approximation of this solution is easily obtained when inequalities (10) and (11) are satisfied, because then the first two terms of equation (16) dominate over the others, and (with the definition $F_2 = F_{2o} + F_{21} + ...$) one has $F_{2o} = X^{-1+i\nu}$ (the inessential multiplication factor is set equal to unity). The corresponding expression for F_{1o} follows from the first of equations (14):

$$d_o^2BF_{1o} = -d_o^2AF_{2o} + \big(\textstyle\int rBF_{2o}\,dX\big)_{/\theta}\,.$$

With the explicit form of F_{2o} one obtains:

$$F_{1o} \approx -(A'/B)XF_{2o} - i(r_o/d_o^2)\big(X^{i\nu}/\nu\big)_{/\theta} \rightarrow \big(-(A'/B) + r_o(\nu'/d_o^2\nu)\ln X\big)X^{i\nu}\,. \quad (17)$$

It is easy to check that F_{1o} and F_{2o} satisfy inequalities (10) and (11), as they should. The equation for F_{21} is

$$XF_{21}'' + a_1F_{21}' = (B/d_o^2A')\big(XF_{2o/\theta}\big)' - a_2F_{2o/\theta} - a_3F_{2o} - a_4F_{1o}\,. \qquad (18)$$

Here, too, the term proportional to F_{1o} is negligible; the solution of equation (18) is

$$F_{21}' = -i\nu'\big(-(B/d_o^2A') + (i\nu B/d_o^2A' + a_2)/2\ln X\big)X^{-1+i\nu}\ln X - a_3X^{-1+i\nu}\,. \qquad (19)$$

(It follows from equation (19) that F_{21} contains the term $X^{i\nu}/i\nu$; when $\nu = 0$, for example because there is no θ dependence, one has $\lim_{\nu=0}X^{i\nu}/i\nu = \ln X$.) Note that $|F_{21}| \ll |F_{2o}|$, as it should.

A general characteristic of the solution derived is that the projection of the field on the perpendicular to the curve $r = r_o(\theta)$, F_2, diverges for $X = 0$. The projection on the curve $r = r_o(\theta)$, F_1, can remain finite (this happens when there is no θ dependence, as the first of equations (9) already shows); however, it is always singular in $X = 0$ since it contains terms of the kind $X\ln X$.

3. The case $\epsilon' \neq 0$

The non-local effect of ICRH on the singularities is now taken into account by considering $\epsilon' \neq 0$; however, the term proportional to ρ^2 derived in the first section is neglected. The interval of n_z where this is allowed is determined in the next section. In this section the interval considered is $n_z \lesssim \alpha\Theta$, so that outside the resonance zone ϵ (and hence $r_o(\theta)$) is real, whereas it is imaginary (and hence $r_o(\theta)$ is complex) inside it. The quantities r_o, A' and B (and hence the coefficients a_n of equation (15)) depend on ϵ and not on ϵ'; thus r_o, A' and iB are real and do not depend either on v_t or on n_z outside the resonance zone (as in the case $\epsilon' = 0$), whereas inside it they depend on them and are complex.
Since the term proportional to F_1 can again be neglected, (15) is a differential equation for F_2 without singular solutions. The solution is written in the form $F_2 = F_{2o} + F_{21} + ...$, as in the preceding section; F_{2o} is determined by the first three terms of equation (15), which for brevity is written in the form

$$\epsilon' a_5 F_{2o}''' + (X + \epsilon' a_6) F_{2o}'' + a_1 F_{2o}' = 0 \,. \tag{20}$$

(Note that outside the resonance zone $\epsilon' a_5$ and $\epsilon' a_6$ are real.) With

$$F_{2o}' \equiv \exp\left(-(X + \epsilon' a_6)^2 / 4\epsilon'^2 a_5^2\right) h$$

equation (20) becomes

$$h'' + \left(\frac{2a_1 - 1}{2\epsilon' a_5} - \frac{(X + \epsilon' a_6)^2}{4\epsilon'^2 a_5^2} \right) h' = 0 \,. \tag{21}$$

With the definitions

$$2k \equiv a_1 - 1/2 \quad \text{and} \quad z \equiv (X + \epsilon' a_6)/(\epsilon' a_5)^{1/2} \,,$$

equation (21) transforms into the following confluent hypergeometric equation:

$$\frac{d^2 h}{dz^2} + (2k - z^2/4) h = 0 \,. \tag{22}$$

Thus h is a linear combination of the functions

$$z^{-1/2} W_{\pm k, -1/4}(\pm z^2/2) \,,$$

where $W_{k,m}$ is a Whittaker function. The part of the solution that yields the singular solution in the limit $\epsilon' = 0$ is given by the bottom sign. Indeed, outside the strip defined by $X \ll |\epsilon' a_5|^{1/2}$ the asymptotic expansion of $W_{-k,-1/4}$ gives for F_{2o}' the dominant term proportional to $\epsilon'^{-k}(X + \epsilon' a_6)^{-2+i\nu}$. Hence, the required part of the solution is (the inessential multiplication factor independent of ϵ' is set equal to unity)

$$F_{2o}' = (\epsilon')^k z^{-1/2} \exp(-z^2/4)\, W_{-k,-1/4}(-z^2/2) \,. \tag{23}$$

The function defined by equation (23) is not singular in the strip previously introduced. In fact, known properties of the Whittaker functions allow F'_{2o} to be represented by a power series of z^2 valid for all finite values of z, whose coefficients b_n depend on k and m, and therefore (in our case) only on a_1 :

$$F'_{2o} \propto (\epsilon')^k(1 + b_1 z^2 + b_2 z^4 + ...) \to \epsilon'^{(}-1 + i\nu/2) f(z^2) \,.$$

The function $F_{2o} = X^{-1+i\nu}$ outside the strip (and with $X < 0$) is obtained by choosing

$$F_{2o} = (\epsilon')^{(-1+i\nu)/2} \int_{-\infty}^{z} f(z^2)\, dz \,. \tag{24}$$

In order to obtain F_{2o} for $X > 0$ and always outside the strip, it is convenient to write equation (24) in the form

$$F_{2o} = (\epsilon')^{(-1+i\nu)/2} \left(\int_{-\infty}^{\infty} f(z^2)\, dz + \int_{\infty}^{z} f(z^2)\, dz \right) \,. \tag{25}$$

Preceding remarks lead to the conclusion that the first integral in equation (25) does not depend on ϵ', whereas the second is $z^{-1+i\nu}$. Thus, outside the strip the first term is negligible with respect to the second, and the same result as for $\epsilon' = 0$ is obtained. The form of F_{2o} in the strip – where the effect of ϵ' is not negligible – follows from equation (24):

$$F_{2o} = (\epsilon')^{(-1+i\nu)/2} \left(\int_{-\infty}^{0} f(z^2)\, dz + z \right) \,. \tag{26}$$

It thus follows that $F_{2o} \propto (\epsilon')^{(-1+i\nu)/2}$ since the integral does not depend on ϵ', as already stated. This result together with $F'_{2o} \propto (\epsilon')^{-1+i\nu/2}$ shows that condition (11) is satisfied.

4. Discussion of the preceding results

The purpose of the paper is to investigate the non–local effect of ICRH on the singularities of the e.m. field. For comparison, first the situation without this effect was considered. Outside the resonance zone and with $n_z = 0$ (i.e. $\epsilon' = 0$) the region where one singular term dominates over the others is the strip defined by $|X| \ll |d_o^2 A'/4B^2 r_o|$. The dominant singular term of the component perpendicular to the curve $r = r_o(\theta)$ of the electric field is $X^{-1+i\nu}$, with

$$i\nu \equiv (Br_o/d_o^2 A')\big((d_o'/d_o) + (B'/B) - r_o'/r_o\big)\,.$$

Both components F_1 and F_2 diverge on $r_o(\theta)$, except when there is no θ dependence; in this case F_1 remains finite on $r_o(\theta)$ (although being singular).
If outside the resonance zone $\epsilon' \neq 0$, and with the term proportional to ρ^2 being neglected, the solution (for $n_z \lesssim \alpha\Theta$) is the same as for $\epsilon' = 0$ (with X replaced by $X + \epsilon' a_6$) in the region $|\epsilon' a_5|^{1/2} < |X| < |d_o^2 A'/4B^2 r_o|$. In the remaining strip $|X| < |\epsilon' a_5|^{1/2}$, F_{2o}' can be represented by a power series of $z^2 \equiv (X + \epsilon' a_6)^2/(\epsilon' a_5)$ with coefficients independent of ϵ', and valid for all finite values of z. Therefore F_{2o} is regular in the strip.
It is now determined when disregarding the term proportional to ρ^2 is correct, that is, when $\rho^2|\epsilon F_{2o}''| \ll |\epsilon' r_o' F_{2o}'|$. It is again convenient to distinguish the regions outside and inside the strip. Outside it one should have $|X/r| > |\epsilon\rho^2/\epsilon' r^2$, since $F_{2o}'/F_{2o}'' \approx X$. On the other hand, on the border of the strip one has $|X/r| \approx |\epsilon'/\epsilon|^{1/2}$. The solution outside the strip obtained in Section 3 is thus correct if

$$|\epsilon\rho^2/\epsilon' r^2 < |\epsilon'/\epsilon|^{1/2}\,.$$

This condition is equivalent to

$$(\alpha\Theta)^2(\rho/r)^{4/3} < n_z \leq \alpha\Theta\,. \tag{27}$$

There is therefore an interval of n_z where the solution that disregards the term proportional to ρ^2 is correct if $q|\Theta| < (r/\rho)^{1/3}$, which is always verified; a consequence of this inequality is that the allowed n_z interval becomes wider when θ approaches the resonance zone.
Inside the strip and for the allowed n_z interval, the solution of Section 3 is correct for $|X/r| > |\epsilon\rho^2/\epsilon' r^2$. For smaller $|X|$ the term proportional to ρ^2 cannot be neglected; its effect is similar to that of ϵ' in that it makes the solution regular.
In the resonance zone, ν is complex as a consequence of ϵ being imaginary; a further consequence is that r_o is complex too, and thus X cannot be equal to zero.
It has already been shown that the field components satisfy (as they should) the inequalities that were used to deduce and solve the differential equation (15). It remains to be checked whether the the inequality necessary for the validity of equation (2), namely

$$|\epsilon' r_o' F_{2o}'| \ll |\epsilon F_{2o}|\,,$$

is satisfied. Outside the strip $(|\epsilon' r_o'/A'|^{1/2}, X)$, where $F_{20} = X^{-1+i\nu}$, this is obviously true. Inside the strip the verification follows from equation (26), which gives $|\epsilon' r_o' F_{2o}'/F_{2o}| = O(\epsilon'^{1/2})$.

For a final appreciation of the non–local effect of ICRH on the field singularities one should consider that only waves with $n_z \lesssim 1$ can propagate in the vacuum between the coils and plasma. But since the effect of ϵ' dominates over the effect of ρ^2 only for $n_z > (q\Theta)^2(r/\rho)^{2/3}$, the non local effect of ICRH is limited to the very narrow region in the neighbourhood of the resonance zone $(1/\alpha) < \Theta^2 < 1/(q^2\alpha)^{2/3}$ and to a very narrow interval of n_z.

The last conclusion is of course not valid where the field is not singular; indeed, the effect of ϵ' (which is generally neglected) dominates over the effect of ρ^2 when

$$(\epsilon\rho^2/\epsilon' r^2)(rE'/E) \to (q\Theta)^2(rE'/E)/n_z < 1\,,$$

and this condition can be verified by the ICRH field for all θ.

5. Direction of the energy flux

The direction of the energy flux is determined by the ratio of the components of the Poynting vector:

$$D_E \equiv < S_2 > / < S_1 > \to -(E_1 B_3^* + E_1^* B_3)/(E_2 B_3^* + E_2^* B_3).$$

In the neighbourhood of the curve $r = r_o(\theta)$ the singular solution yields the approximation $-iB_3 d_o/\lambda_o \approx d_o^2 F_1' - r_o' F_2 - d_o(r_o F_2/d_o)_{/\theta}$; thus D_E becomes

$$D_E = -\frac{\left[(r_o' F_1 + r_o F_2)(d_o^2 F_1'^* - r_o' F_2^* - d_o(r_o F_2^*/d_o)_{/\theta}\right]_I}{\left[(r_o F_1 - r_o' F_2)(d_o^2 F_1'^* - r_o' F_2^* - d_o(r_o F_2^*/d_o)_{/\theta}\right]_I}\,. \tag{28}$$

Equation (28) is discussed only where ϵ is real (that is, outside the resonance zone), and without the term proportional to ρ^2. The first step is to use the first of equations (14) to obtain

$$-iB_3 d_o/\lambda_o = -\frac{d_o^2}{B}(AF_2)' + (d_o^2 A'/B)(2 - a_1)F_2 + XF_{2/\theta} - d_o^2(B'/B)F_1\,, \tag{29}$$

or (since the term proportional to F_1 is negligible)

$$-iB_3 d_o/\lambda_o = -(d_o^2 A'/B)\big(XF_2' + (a_1 - 1)F_2\big) + XF_{2/\theta}\,. \tag{30}$$

In the strip $|\epsilon' r_o'/A'|^{1/2} \ll |X|$ the dominant term of F_2 makes the first term of the RHS of equation (30) equal to zero; it is therefore convenient to use equation (16) (without the term proportional to F_{1o}) written in the form

$$(XF_{21}')' + (a_1 - 1)F_{21}' - (B/d_o^2 A')\big(XF_{2o/\theta}\big)' = -a_2 F_{2o/\theta} - a_3 F_{2o}\,. \tag{31}$$

An integration of equation (31) gives

$$XF_2'+(a_1-1)F_2-(B/d_o^2A')\left(XF_{2o/\theta}\right) = -a_2(\nu'/\nu)\left(i/\nu+\ln X\right)X^{i\nu}+i(a_3/\nu)X^{i\nu}\,. \quad (32)$$

With this result equation (30) yields

$$-iB_3d_o/\lambda_o = (d_o^2A'/B)\,(a_2(\nu'/\nu)(i/\nu+\ln X) - i(a_3/\nu))\,X^{i\nu}\,. \quad (33)$$

The contribution of F_2 to the Poynting vector is obtained by multiplying the RHS of equation (33) by $X^{-1-i\nu}$ and taking the imaginary part of the product. Since B (and therefore a_2) is imaginary, the result for $\nu \neq 0$ is

$$\left(2r_o(B'/B)(\nu'/\nu^2) - (d_o^2A'/B_I)(a_{3I}/\nu)\right)X^{-1}\,, \quad (34)$$

with $a_{3I} = -(B_I/d_o^2A')\left((B_{/\theta}/B) + r_o(B'_{/\theta}/B) - 2r_o(B'B_{/\theta}/B^2)\right)$. The limit $\nu = 0$ (independence of θ) is obtained by taking the limit of equation (34) before forming the product; one thus obtains

$$-(d_o^2A'/B_I)a_{3R}X^{-1}\ln X\,, \quad (35)$$

with $a_{3R} = r_oB^2/d_o^2A' - 2(B'/B)$. If the region considered contains $X=0$ (i.e. if $n_z = 0$) the energy flux diverges for $X=0$, but its integral over X is finite, as it should be.

The contribution of F_1 to the Poynting vector is easily obtained by using equation (17); the result (not given here) confirms the expectation that the ratio of the contributions of F_1 and of F_2 is zero at $X=0$. For $X \to 0$ the numerator of the RHS of equation (28) is therefore $-r_o/r_o'$ times the denominator and one has $r_o'D_E/r_o = 1$; the energy flux thus becomes parallel to the curve $r = r_o(\theta)$. It is easy to derive the correction of the energy flux direction where $X \neq 0$ by using the expressions given in the paper.

Inside the strip $|X| < |\epsilon' r_o'/A'|^{1/2}$ the inequality $|F_1| \ll |F_2|$ is still valid, and the energy flux is parallel to the curve $r = r_o(\theta)$ at $X=0$. In this region, however, the energy flux tends to a finite value at $X=0$.

Conclusion

In order to study the effect of ICRH on the singularities of the e.m. field of a tokamak (Vlasov) plasma, first we derived the position and form of the singularities (at zero temperature). For this purpose the electric field was determined in the relevant two–dimensional strip (with the assumption of toroidal periodicity) by introducing the components of the electric field parallel and perpendicular to the curve on which the solution is singular, and by requiring that these components satisfy some inequalities. The effect of ICRH was first deduced without the effect of the finite Larmor radius; from a comparison of the two it is concluded that the non–local effect of ICRH on the field singularities is limited to a very narrow region about the resonance zone and to a very narrow interval of n_z. On the other hand, the non–local effect of ICRH dominates where the field is not singular (and should not be neglected, as is generally done). Finally, the direction of the energy flux in the neighbourhood of the resonance region (without and with the effect of ICRH) was deduced. It becomes parallel to the curve where the field is singular for $v_t = 0$ when it is exactly on it; this is true not only when the field is singular, but also when the non–local ICRH effect is taken into account. Obviously, the energy flux diverges in the first case and is finite in the second one.

R E F E R E N C E S

Cattanei, G. & Croci, R., 1977 *Nucl. Fusion* **17**, 239

Croci, R., 1995 *IPP 6/331*