MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK
GARCHING BEI MÜNCHEN

ANALYTIC THREE-DIMENSIONAL SOLUTIONS
OF THE MAGNETOHYDROSTATIC EQUATIONS
WITH TWISTED FIELD LINES

R. KAISER* AND A. SALAT

IPP 6/341

April 1996

* Universität Bayreuth, Lehrstuhl für Angewandte Mathematik, 95440 Bayreuth, Germany.

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

Two types of exact solutions of the MHD equilibrium equations are presented. The equilibria exhibit neither continuous nor mirror symmetry. The configurations are infinitely extended along a straight axis with surfaces of constant pressure closed around the axis. The cross sections are elliptic for equilibria of the first type. For the second type they are elliptic only close to the axis. Field and current lines turn around the axis and extend from minus to plus infinity in the axial direction. In these limits the configurations become singular. The rotational transform and the local shear are discussed.

The most promising way to stably confine a plasma sufficiently long in order to achieve thermonuclear fusion is magnetic confinement in a toroidal configuration. Two such configurations have been investigated especially carefully. In the tokamak a toroidal magnetic field generated by external coils together with an externally driven toroidal current sustain an, in principle, axially symmetric equilibrium. The discrete set of coils, however, leads to an unavoidable "rippling" of the magnetic field and thus to a certain amount of nonsymmetry. A stellarator, in contrast, is an inherently three-dimensional (3-d) configuration far from axisymmetry, whose magnetic field is generated completely by external coils. In both configurations the magnetic field lines twist around the axis with finite pitch. The equilibria are supposed to be well described by the equations of ideal MHD,

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \nabla P, \\ \mathbf{j} &= \text{curl} \mathbf{B}, \\ \text{div} \mathbf{B} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{B}$, $\mathbf{j}$ and P denote the magnetic field, current density and pressure, respectively.

To find 3-d solutions of Eqs. (1) is, however, notoriously difficult. This is because Eqs. (1) represent a system of partial differential equations of mixed elliptic-hyperbolic type, for which there is no general theory on existence and uniqueness of solutions. Only with additional assumptions can the problem be rendered tractable. For plane, axi- or helical symmetry Eqs. (1) can be reduced to a single quasilinear elliptic equation for which an elaborate existence theory as well as many explicit solutions are known [1, 2, 3]. The axisymmetric case, for example, is governed by the well-known Grad-Shafranov equation [4].

Discrete symmetries seem to facilitate the problem, too. In [5] equilibria with a purely toroidal magnetic field, which is mirror symmetric with respect to a poloidal plane have been considered. In that particular case the so-called β -iteration [6] was proved to converge for low plasma pressure. The pertinent pressure surfaces, however, are not poloidally closed in general, the closure being plausible only for configurations far from axisymmetry [7].

If symmetries are no longer present in the system numerical work shows that magnetic surfaces tend to disrupt and magnetic islands and ergodic regions appear in Poincaré-plots of the field lines (see e.g. [8]). A well-defined smooth pressure function may not be expected in these cases. This view is corroborated by theoretical analysis of Eqs. (1) for nonaxisymmetric equilibria with field line

shear [2], [9]. It indicates that finite jumps in the pressure distribution and the formation of current sheets at each rational surface are to be expected generically. Grad conjectured for that reason that nonsymmetric equilibria in the sense of classical solutions do not exist at all [9].

It is due to these intricacies that the present-day available exact results on existence and nonexistence of equilibria refer to quite special configurations. *Nonexistence* of configurations with purely poloidal magnetic fields, for example, has been investigated in [10], [11]. Other nonexistence results refer to the isodynamic [12] or the “quasihelical” stellarator [13].

As to the *existence* of 3-d equilibria all known results [14, 15, 16, 17] refer to straight-axis configurations (with the exception of the mirror-symmetric configuration mentioned above). If such configurations are poloidally closed and periodic in the axial direction they are still topologically equivalent to a torus. The solutions obtained in [14], [16] are of this type, in the sense that free functions along the axis can be chosen to be periodic. The field lines, however, are confined there to planes orthogonal to the axis. The solutions in [17], on the other hand, have field lines purely along the axis. (In addition, periodicity along it is violated.) Thus, in neither of these cases do the field lines twist around the axis with a finite slope.

In contrast to these previous results, the equilibria presented here are – to our knowledge – the first examples of analytic 3-d equilibria with twisted field lines. They demonstrate that field line twist need not be detrimental to 3-d configurations as was suspected earlier [9].

Two types of solution of Eqs. (1) were obtained. In both cases the magnetic field has the representation

$$\mathbf{B}(x, y, z) = \text{curl}[H(x, y, z)\nabla z] + \text{curl curl}[K(x, y, z)\nabla z]. \quad (2)$$

For the first type the functions H and K are given by

$$\begin{aligned} H(x, y, z) &= \frac{-h}{2} [a(z)x^2 + 2b(z)xy + c(z)y^2], \\ K(x, y, z) &= \frac{-k}{2\kappa} [a(z)x^2 + 2b(z)xy + c(z)y^2], \end{aligned} \quad (3)$$

while the pressure P is determined by

$$\begin{aligned} P^*(x, y, z) &\equiv -2 \frac{P - P_0}{h^2 + k^2} = \\ &= [a(z) + c(z)][a(z)x^2 + 2b(z)xy + c(z)y^2]. \end{aligned} \quad (4)$$

Here, h and k are two arbitrary constants which determine the weight of the contributions from H and K , respectively, to the magnetic field. P_0 is the pressure on the z axis. The functions $a(z)$, $b(z)$, $c(z)$ are as follows:

$$\begin{aligned} a(z) &= a_1 e^{\kappa z} + a_2 e^{-\kappa z}, \\ b(z) &= b_1 e^{\kappa z} + b_2 e^{-\kappa z}, \\ c(z) &= c_1 e^{\kappa z} + c_2 e^{-\kappa z}, \end{aligned} \quad (5)$$

with arbitrary constants κ , a_1 , a_2 , c_1 , c_2 . The coefficients b_1 , b_2 are related to the other ones by

$$b_1^2 = a_1 c_1, \quad b_2^2 = a_2 c_2. \quad (6)$$

A consequence thereof is the relation

$$a(z)c(z) - b^2(z) = (\sqrt{a_1 c_2} - \sqrt{a_2 c_1})^2 = \text{const} \geq 0. \quad (7)$$

The functions a , b , c can be normalized by setting the constant in the equation above to one (which excludes the singular case $c_2/c_1 = a_2/a_1$).

The surfaces of constant pressure are closed surfaces, composed of ellipses nested around the z axis. At fixed P their width, axis ratio and orientation varies along the z axis.

In component notation the magnetic field is

$$\begin{aligned} B_x &= -h [b(z)x + c(z)y] - \frac{k}{\kappa} [a'(z)x + b'(z)y], \\ B_y &= h [a(z)x + b(z)y] - \frac{k}{\kappa} [b'(z)x + c'(z)y], \\ B_z &= \frac{k}{\kappa} [a(z) + c(z)], \end{aligned} \quad (8)$$

where $a'(z) = da(z)/dz$, etc.. The contribution from H is a purely poloidal field which was derived as a solution of Eqs. (1) previously in [16]. The axial component of the field is contributed by the K term only. The fact that the fields originating from H and K can be superposed independently in the *nonlinear* equations (1) is astonishing indeed.

The current density components are

$$\begin{aligned} j_x &= -h [a'(z)x + b'(z)y] + \frac{k}{\kappa} [b''(z)x + c''(z)y], \\ j_y &= -h [b'(z)x + c'(z)y] - \frac{k}{\kappa} [a''(z)x + b''(z)y], \\ j_z &= h [a(z) + c(z)]. \end{aligned} \quad (9)$$

In the limit $z \rightarrow \pm\infty$ the configuration becomes singular because the current density and the magnetic field grow without bounds. This squeezes the cross section of each pressure surface down to a line of finite length. For $\kappa z \gg 1$ the inclination $dy/dx = \sqrt{a_1/c_1}$ of the line is evident from $P^* \approx (a_1 + c_1)(\sqrt{a_1}x + \sqrt{c_1}y)^2 \exp 2\kappa z$. The maximal extensions of the line, $x^2 \leq P^*c_1/(a_1 + c_1)$ and $y^2 \leq P^*a_1/(a_1 + c_1)$, follow from treating Eq. (4) as a quadratic form exactly, and going to the limit only afterwards. For $\kappa z \ll -1$ the situation is analogous, with the index 1 replaced by 2. Thus, the pressure surfaces consist of a middle region of finite thickness which gets progressively thinner towards both sides, with inclinations determined asymptotically by the coefficients a_1, a_2, c_1, c_2 .

For the second type of solutions the functions H and K are given by

$$\begin{aligned} H(x, y, z) &= \frac{h\eta}{\kappa} [a(z) \cos \lambda x + c(z) \cos \lambda y], \\ K(x, y, z) &= \frac{k}{\lambda\kappa} [a(z) \cos \lambda x + c(z) \cos \lambda y], \end{aligned} \quad (10)$$

while the pressure P is determined by

$$\begin{aligned} P^*(x, y, z) &\equiv -2 \frac{P - P_0}{(h^2 + k^2)\eta^2} \\ &= a^2(z) \sin^2 \lambda x + 2(1 - \cos \lambda x \cos \lambda y) + c^2(z) \sin^2 \lambda y. \end{aligned} \quad (11)$$

Here λ , with $\lambda^2 \leq \kappa^2$, is an arbitrary constant and η is defined by

$$\eta = \sqrt{\kappa^2 - \lambda^2} / \kappa. \quad (12)$$

The equations (5) – (7) are again valid, with the restriction $b_1 = b_2 = b(z) = 0$. A nontrivial solution is then possible for $a_2 = c_1 = 0$ only. With the normalization mentioned there results $a(z) = a_1 e^{\kappa z} = 1/c(z)$.

For $|\lambda x|, |\lambda y| \ll 1$, replacing $\sin \lambda x, \cos \lambda x$ by $\lambda x, 1 - (\lambda x)^2/2$, respectively, etc., the type-II pressure (11) takes on the same form as the type-I pressure (4) in the case $b(z) = 0$. Close to the axis, therefore, the surfaces also consist of ellipses, with major axis in the x or y direction. Farther away, with P^* increasing from its on-axis value zero, they are progressively deformed into a more rhomboidal shape.

In the limit $\kappa z \gg 1$, the pressure surfaces (11) approximately take on the form $c(z)P^*/4 = a(z) \sin^2(\lambda x/2) + c(z) \sin^2(\lambda y/2)$. This implies again that for $\kappa z \rightarrow \infty$ the pressure surfaces shrink down into a line, this time at $x = 0$ and with an extension $y^2 \leq y_\infty^2$, where $\sin^2(\lambda y_\infty/2) = P^*/4$. For $\kappa z \ll -1$ the situation is analogous, with the roles of x and y interchanged.

The magnetic field of the type-II solution is

$$\begin{aligned} B_x &= -h\eta c(z) \sin \lambda y - ka(z) \sin \lambda x, \\ B_y &= h\eta a(z) \sin \lambda x + kc(z) \sin \lambda y, \\ B_z &= k\lambda [a(z) \cos \lambda x + c(z) \cos \lambda y] / \kappa. \end{aligned} \quad (13)$$

Again, the contribution from H is a purely poloidal field. It was derived as a solution of Eqs. (1) in [14]. The axial component of the field is contributed by the K term only which, as an independent solution was found in [17]. Again, these two independent solutions can be superposed to give a new solution of the nonlinear equilibrium problem.

The current density components for the type-II solutions are

$$\begin{aligned} j_x &= \eta\kappa [-ha(z) \sin \lambda x + k\eta c(z) \sin \lambda y], \\ j_y &= \eta\kappa [+hc(z) \sin \lambda y - k\eta a(z) \sin \lambda x], \\ j_z &= \eta\lambda h [a(z) \cos \lambda x + c(z) \cos \lambda y]. \end{aligned} \quad (14)$$

If the perpendicular and the axial scale lengths κ and λ coincide η and therefore the current density $\mathbf{j}$ vanishes.

An example of a type-II solution is given in Fig. 1. It shows two nested pressure surfaces, with $P^* = 0.7$ (inner surface) and $P^* = 2.0$ (outer surface). Further parameters are $\kappa = \sqrt{2}$, $\lambda = 1$ and $a_1 = 1$. The axial region presented extends from $z = -1.4$ to $z = +1.4$. The figure might also serve for a qualitative representation of a type-I equilibrium (with asymptotic inclinations of zero and ninety degrees). In Fig. 2, which is a (enlarged) continuation of the outer surface of Fig. 1 to the left, and extends from $z = -1.4$ to $z = -4.4$, the asymptotic behaviour for $z \rightarrow -\infty$ is rather evident.

In Figs. 1 and 2 several field lines are also shown as thick lines. $h = 5$, $k = 1$ is used for these orbits. In polar coordinates r , θ , defined by $x = r \cos \theta$, $y = r \sin \theta$, the field line equations are

$$\frac{d\theta}{dz} = \frac{1}{rB_z} (B_y \cos \theta - B_x \sin \theta) \quad (15)$$

and an analogous equation for dr/dz .

The asymptotic behaviour of the field lines can also be discussed analytically. In the limit $\kappa z \gg 1$ the field line twist turns out to vanish. All field lines eventually end up at a height y which depends on the starting point but does not change with z anymore. Together with $x = 0$ this implies $\theta = \pm\pi/2$, modulo π .

The analogous behaviour results for $z \rightarrow -\infty$. In Fig. 2 this asymptotic behaviour of the field lines is also evident.

In toroidal geometry the rotational transform ι , for unclosed field lines, is defined as $\iota = \lim_{\phi \rightarrow \infty} M(\phi)/N(\phi)$, where ϕ is some toroidal coordinate and $M(\phi)$, $N(\phi)$ are the number of poloidal and toroidal turns, respectively, of a field line around the axis. This limit is equivalent to averaging over the total surface, and in consequence ι and the shear $s = d\iota/dV$ are constant on pressure surfaces (V is a flux label of the surface, e.g. the pressure itself). Since the present equilibria, however, do not have an axial periodicity this averaging property does not take place, and ι differs, in principle, from field line to field line.

A natural definition of the rotational transform $\iota(x_0)$ in the present case would seem to be the limit of the number of turns from $z = -\infty$ to $+\infty$ as a function of the starting coordinate x_0 , with $N = 1$. According to the discussion of $\theta(z)$ above, however, $\iota(x_0)$ then takes on only half-integer values, $\iota(x_0) = \pm m/2$, with at most two consecutive integer values m , for a given equilibrium. A reasonable definition of the shear with this ι is not possible.

Global definitions of ι , thus, do not seem appropriate here. In many respects, however, *local* values of the field line slope $\theta' = d\theta/dz$ are also relevant, anyway. θ' not only varies along z and from field line to field line but, for type-II solutions, also from pressure surface to pressure surface. This is evident from Figure 3 where the angle $\theta(z)$ is plotted for two field lines for a type-II solution in the region, $z \in [-0.5, 1.8]$. The field lines start at $\theta = 0$ and $\theta = \pi/2$, respectively. (These two values are selected for ease of identification.) The solid curves correspond to $P^* = 0.5$ and the dashed curves to $P^* = 2.0$, with h , k , κ , λ and a_1 as in Figs. 1 and 2. The difference between the solid and the dashed curves confirms the afore-mentioned existence of a “radial” derivative of $d\theta/dz$ which could serve as a local substitute for the shear.

In type-I solutions the shear is absent: B_x , B_y are linear in r , while B_z is independent of r . In consequence, the variable r completely drops out of Eq. (15). The field line trajectories on different pressure surfaces are all equivalent. The asymptotic behaviour of the field lines is the same as for type-II solutions.

The ratio of plasma pressure to magnetic field energy, $\beta = (P_0 - P)/(B^2/2)$, is determined by Eqs. (4), (8) and Eqs. (11), (13), for type-I and type-II solutions, respectively. The β values depend on h/k , κx_0 or κ/λ , and the position vector. (x_0 is an arbitrary fixed value of x .) On a surface of constant pressure, labelled by P^* , β still varies as a function of the surface coordinates. While β_I is bounded

everywhere, β_{II} may become arbitrarily large for $P^* \rightarrow 4$, e.g. for $a_1 e^{\kappa z} = 1$, $\lambda x = m\pi$, $\lambda y = n\pi$, with $m, n = 0, \pm 1, \pm 2, \dots$, such that $\cos(\lambda x) + \cos(\lambda y) = 0$.

For all toroidal MHD equilibria with closed field lines the so-called current-closure condition [2] for $F := \oint dl/|\mathbf{B}|$ has to hold, namely that F be the same for all closed field lines on a fixed pressure surface. Although the present equilibria are not topologically toroidal and the field lines are not closed, except for $k = 0$, it is nevertheless found that F is constant on $P = \text{const}$ if the integration along the unclosed field lines is extended from $z = -\infty$ and $+\infty$. For type-I solutions it is found that

$$F = \frac{\pi}{2|k|} \frac{1}{\sqrt{(a_1 + c_1)(a_2 + c_2)}}, \text{ for } k \neq 0; \quad F = \frac{2\pi}{|h|}, \text{ for } k = 0, \quad (16)$$

i.e. F does not even depend on P at all. For type-II solutions there results

$$\begin{aligned} F &= \frac{1}{|k\lambda|} \mathbf{K} \left(\sqrt{P^*/4} \right), \quad \text{for } k \neq 0, \\ F &= \frac{4}{|h\lambda\eta|} \mathbf{K} \left(\sqrt{P^*/4} \right), \quad \text{for } k = 0, \end{aligned} \quad (17)$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind.

The equilibria just described have, of course, the serious drawback that toroidal curvature is missing and that they are not even periodic in the axial direction. An answer to the question whether *periodic* 3-d equilibria with straight axis and twisted field lines exist or not would be especially interesting since the potential for island formation and ergodization of field lines is then the same as in toroidal geometry but complications from toroidal curvature are still absent.

Fig. 1: Nested pressure surfaces of type-II MHD equilibria. Field lines rotate around the axis.

Fig. 2: Continuation of the outer surface of Fig. 1 to the left (enlarged). Asymptotically, the surface becomes squeezed flat and the field lines approach a constant distance to the axis.

Fig. 3: Poloidal angle θ of field lines for low pressure, $P^* = 0.5$, (solid lines) and high pressure, $P^* = 2$, (dashed lines). Starting values are $\theta_0 = 0$ (lower pair) and $\theta_0 = \pi/2$ (upper pair). Difference between solid and dashed curves indicates field line shear.

References

- [1] A. I. Morozov, L. S. Solov'ev, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2.
- [2] L. S. Solov'ev, V. D. Shafranov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. 5, p. 1.
- [3] J. Edenstrasser, *J. Plasma Phys.* **24**, 299 (1980).
- [4] R. Lüst, A. Schlüter, *Z. Naturforsch.* **12a**, 850 (1957),
H. Grad, H. Rubin, *Proc. of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958*, edited by United Nations, (United Nations Publication, Geneva, 1958), Vol. **31**, p. 190,
V. D. Shafranov, *Zh. Exp. Teor. Fiz.* **33**, 710 (1957) [*Sov. Phys. JETP* **6**, 545 (1958)].
- [5] D. Lortz, *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, **21**, 196 (1970).
- [6] L. Spitzer, *Phys. Fluids* **1**, 253 (1958).

- [7] D. Lortz, *Contributions to the Fourth European Conference on Controlled Fusion and Plasma Physics, Rome, 1970*, edited by Comitato Nazionale Energia Nucleare, (Ufficio Edizioni Scientifiche, Rome, 1970), p. 13.
- [8] T. Hayashi, T. Sato, A. Takei, *Phys. Fluids* **B2**, 329 (1990).
- [9] H. Grad, *Phys. Fluids* **10**, 137 (1967), *Int. J. Fusion Energy* **3**, 33 (1985).
- [10] R. Kaiser, A. Salat, J. Tataronis, *Phys. Plasmas* **2**, 1599 (1995).
- [11] A. Salat, *Phys. Plasmas* **2**, 1652 (1995).
- [12] D. Palumbo, *Proc. Acad. Sci. Arts Palermo, Ser. 5, Pt. 1: Sci* **IV**, 475 (1984).
- [13] D. A. Garren, A. H. Boozer, *Phys. Fluids* **B3**, 2822 (1991).
- [14] M. L. Woolley, *J. Plasma Phys.* **17**, 259 (1977).
- [15] B. K. Shivamoggi, *Quarterly Appl. Math.* **XLIV**, 487 (1986).
- [16] A. Salat, R. Kaiser, *Phys. Plasmas* **2**, 3777 (1995).
- [17] R. Kaiser, A. Salat, *J. Plasma Phys.*, to be published.