ON MAGNETIZED PARALLEL FLOWS

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Abstract

In a first part nonlinear stability of dissipative magnetized parallel flows is studied using Lyapunov methods. For 2-dimensional perturbations perpendicular to the direction of the flow, a Lyapunov functional is constructed explicitly by two slightly different methods. This insures the nonlinear unconditional stability of the system. Though the extension of this nonlinear work to 3-dimensional perturbations seems impossible at present, a conjecture concerning linear stability of magnetized Couette flows is stated, whose proof may become a mathematical challenge. In the second part the addition of parallel flows to ideal static equilibria is investigated. It turns out that Palumbo's "isodynamic" equilibrium plays a special role in this problem.

1 Introduction

The nonlinear stability of dissipative magnetohydrodynamic (MHD) flows has been investigated by the author in previous work (see for example Refs. [1, 2, 3]). In Ref. [1] a general sufficient condition is derived for incompressible flows. It leads to unconditional stability if the Reynolds numbers are less than typically $2\pi^2$. Some stability results for decaying force free fields and for 2-dimensional equilibria without flow are reviewed in Ref. [2]. A generalization

of the stability condition to Trkal MHD flows, which are a combination of decaying Beltrami flows and force free fields, is obtained in Ref. [3].

The purpose of this paper is twofold. First, we consider 2-dimensional dissipative flows parallel to a vacuum magnetic field and investigate their nonlinear stability with respect to 2-dimensional perturbations perpendicular to the magnetic field. This is the subject of sections 2 to 4. Second, conditions for the existence of ideal MHD equilibria with parallel flows are derived in section 5.

2   Lyapunov functional

The MHD equations for an incompressible fluid with mass density equal to unity are

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla P + \mu \Delta \mathbf{V}, \tag{1}$$

$$-\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{V} \times \mathbf{B} + \eta \nabla \times \mathbf{J}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \tag{4}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6}$$

where $\mathbf{V}$ and $\mathbf{B}$ are the velocity and the magnetic fields, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current and P, η and μ are the pressure , the resistivity and the viscosity respectively. Let us now split the variables occurring in the previous equations in a zero order solution and a finite perturbation, for instance, $\mathbf{V} = \mathbf{V_0} + \mathbf{v}$ etc.. The zero order solution obeys

$$\mathbf{V_0} \cdot \nabla \mathbf{V_0} = -\nabla \mathbf{P_0} + \mu \Delta \mathbf{V_0} = 0, \tag{7}$$

$$\mathbf{E_0} = \eta \mathbf{J_0} = 0, \tag{8}$$

$$\mathbf{B_0} = \mathbf{e_z} B_0 = \mathbf{ct.}, \tag{9}$$

$$\mathbf{V_0} = \lambda \mathbf{e_z}, \tag{10}$$

$$\mathbf{e_z} \cdot \nabla \lambda = \mathbf{e_z} \cdot \nabla P_0 = 0, \tag{11}$$

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where $\mathbf{e_z}$ is the unit vector in the z direction, while $\lambda, P_0, \mathbf{V_0}$ depend upon x and y only. The equations for the finite perturbations $\mathbf{v}, \mathbf{b}, \mathbf{j}, p$ are given by

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{V_0} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{V_0} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{j} \times \mathbf{B} + \mathbf{j} \times \mathbf{b} - \nabla p + \mu \Delta \mathbf{v}. \quad (12)$$

$$-\frac{\partial \mathbf{b}}{\partial t} + \nabla \times (\mathbf{V_0} \times \mathbf{b} + \mathbf{v} \times \mathbf{B_0} + \mathbf{v} \times \mathbf{b}) = \eta \nabla \times \mathbf{j}. \quad (13)$$

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}, \quad (14)$$

$$\nabla \times \mathbf{b} = \mathbf{j}, \quad (15)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (16)$$

$$\nabla \cdot \mathbf{b} = 0. \quad (17)$$

Take the scalar product of equation (12) with $\mathbf{v}$ and integrate over the volume of the fluid contained in a cylinder oriented along z.

$$\frac{1}{2} \frac{d}{dt} \int_V \mathbf{v}^2 - \int_S \mathbf{n} \cdot \mathbf{v}(\mathbf{v} \cdot \mathbf{V_0} + \frac{\mathbf{v}^2}{2}) - \int_V \mathbf{v} \cdot \mathbf{V_0} \times \nabla \times \mathbf{v} =$$

$$\int_V \mathbf{v} \cdot (\mathbf{j} \times (\mathbf{B_0} + \mathbf{b}) + + \int_S \mathbf{n} \cdot \mathbf{v}p - \mu \int_S \mathbf{n} \cdot \mathbf{v} \times \nabla \times \mathbf{v} - \mu \int_V (\nabla \times \mathbf{v})^2, \quad (18)$$

where V is the volume of the cylinder and S is its boundary, $\mathbf{n}$ being the normal to S. Note that the incompressiblity conditions (16) and (17) have been used to convert some of the volume integrals.

The same procedure is applied to equation (13) by taking the scalar product with $\mathbf{b}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_V \mathbf{b}^2 - \int_S \mathbf{n} \cdot \mathbf{b} \times (\mathbf{V_0} \times \mathbf{b} + \mathbf{v} \times (\mathbf{B_0} + \mathbf{b})) =$$

$$\int_V \mathbf{j} \cdot (\mathbf{V_0} \times \mathbf{b} + \mathbf{v} \times (\mathbf{B_0} + \mathbf{b})) - \int_S \mathbf{n} \cdot \mathbf{b} \times \mathbf{j} - \eta \int_V \mathbf{j}^2. \quad (19)$$

Let us now assume that the perturbations are 2-dimensional i.e. $\mathbf{v}$ and $\mathbf{b}$ are perpendicular to $\mathbf{e_z}$ and do not depend upon z. This implies that $\nabla \times \mathbf{v}$ and $\nabla \times \mathbf{b}$ lie in the z direction. It follows that the surface integrals vanish if $\mathbf{v} = \mathbf{b} = 0$ at the boundary of the section of the cylinder with the plane perpendicular to z. It follows also that the volume integrals containing $\mathbf{V_0}$ vanish. Adding then equations (18) and (19), one obtains

$$\frac{1}{2} \frac{d}{dt} \int_V (\mathbf{v}^2 + \mathbf{b}^2) = -\mu \int_V (\nabla \times \mathbf{v})^2 - \eta \int_V \mathbf{j}^2. \quad (20)$$

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It is obvious that the integral on the left hand side of equation (20) is a Lyapunov functional. This proves that magnetized parallel flows are nonlinearly stable with respect to 2-dimensional perturbations.

3 Another proof

The solutions of equations (16) and (17) are given in the 2-dimensional case by

$$\mathbf{v} = \mathbf{e}_z \times \nabla U, \tag{21}$$

$$\mathbf{b} = \mathbf{e}_z \times \nabla W. \tag{22}$$

Insert relations (21) and (22) in system (12)-(17) to obtain after crossing with $\mathbf{e}_z$

$$\nabla \dot{U} - \frac{1}{2}\mathbf{e}_z \times \nabla((\nabla U)^2) + (\mathbf{e}_z \times \nabla U)\Delta U = (\mathbf{e}_z \times \nabla W)\Delta W +$$
$$\mathbf{e}_z \times \nabla p + \mu \nabla(\Delta U), \tag{23}$$

$$\nabla \dot{W} + \nabla(\mathbf{e}_z \cdot \nabla U \times \nabla W) = \eta \nabla(\Delta W). \tag{24}$$

Take the scalar product of equation (23) with ∇U and of equation (24) with ∇W to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla U)^2 - \frac{1}{2}\nabla U \cdot \mathbf{e}_z \times \nabla((\nabla U)^2) = \Delta W(\nabla U \cdot \mathbf{e}_z \times \nabla W) +$$
$$\nabla U \cdot \mathbf{e}_z \times \nabla p + \mu \nabla U \cdot \nabla(\Delta U), \tag{25}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla W)^2 + \nabla W \cdot \nabla(\mathbf{e}_z \cdot \nabla U \times \nabla W) = \eta \nabla W \cdot \nabla(\Delta W). \tag{26}$$

Integrating over x and y and assuming $\nabla U = \nabla W = 0$ at the boundary of the fluid region, one finds

$$\frac{1}{2}\frac{d}{dt}\int_V ((\nabla U)^2) + (\nabla W)^2)) = -\mu \int_V (\Delta U)^2 - \eta \int_V (\Delta W)^2. \tag{27}$$

Equation (27) is equivalent to equation (20) and displays the same Lyapunov functional. Again this proves that the system is unconditionally stable with respect to 2-dimensional perturbations.

4 Discussion and Conjecture

This first part shows that the presence of a constant vacuum magnetic field does not deteriorate the stability of parallel flows. The proofs given are not restricted to small perturbations, they are valid even for unconditional perturbations. The 2-dimensionality of the perturbations is, however, a severe restriction, which is presumably very difficult to overcome. The magnitude of this difficulty can be seen from the case of Couette flows, which are a special 1-dimensional parallel flow. It is a big challenge, indeed, to prove even their linear stability with respect to 3-dimensional perturbations, though it is the general belief [4] that they are linearly stable for all Reynolds numbers.

This last remark and the main result of this first part of the paper lead us to postulate a statement on linear stability of Couette flows : If the Couette flow is linearly stable with respect to 3-dimensional perturbations and for all Reynolds numbers, the magnetized Couette flow will also be linearly stable with respect to 3-dimensional perturbations and for all Reynolds numbers. This can be understood from the fact that a vacuum magnetic field is expected to constrain the system without adding a new free energy reservoir. It may be, however, difficult to prove this conjecture.

Let us now proceed to the second part of the paper.

5 Ideal MHD equilibria with parallel flows

In contrast with the first part of the paper we use now the equations for ideal MHD equilibria with flows

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla P, \tag{28}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \tag{29}$$

$$\nabla \cdot \rho \mathbf{V} = 0, \tag{30}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{31}$$

where ρ is the space-dependent mass density. System (28)-(31) can be obtained from equations Equations (1), (4)-(6), if the time derivatives are cancelled and if the mass density ρ is inserted in front of $\mathbf{V}$. Assume a parallel flow given by

$$\mathbf{V} = \lambda \mathbf{B}. \tag{32}$$

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Then, it follows from equation (30) and (31)

$$\mathbf{B} \cdot \nabla(\rho\lambda) = 0 \qquad (33)$$

or

$$\rho\lambda = f(\psi), \qquad (34)$$

where ψ labels the magnetic surfaces. Insert now $\mathbf{V}$ from equation (32) into equation (28) to obtain

$$\rho[\frac{1}{2}\nabla(\lambda^2 B^2) - \lambda\mathbf{B} \times (\lambda\nabla \times \mathbf{B} + \nabla\lambda \times \mathbf{B})] = (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla P \qquad (35)$$

or

$$\rho[\frac{\lambda^2}{2}\nabla B^2 + \lambda(\mathbf{B} \cdot \nabla\lambda)\mathbf{B}] + \nabla P = (1 - \rho\lambda^2)(\nabla \times \mathbf{B}) \times \mathbf{B}. \qquad (36)$$

Assuming the ideal gas law for the pressure $P = \alpha\rho T$, where the temperature T is taken as function of ψ or $T = T(\psi)$, take the scalar product of equation (36) with $\mathbf{B}$ to obtain

$$\mathbf{B} \cdot \nabla(\frac{\lambda^2 \mathbf{B}^2}{2} + \alpha T \log\rho) = 0 \qquad (37)$$

or

$$\frac{\lambda^2 \mathbf{B}^2}{2} + \alpha T \log\rho = F(\psi). \qquad (38)$$

Note that the assumption $T = T(\psi)$ is best fulfilled in hot plasmas.

An important simplification of equation (36) is obtained by making a further assumption on the ψ functional dependence of either ρ or λ. Equations (34) and (38) lead then to a ψ dependence of P and B^2, so that equation (36) becomes

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla H(\psi), \qquad (39)$$

where $H(\psi)$ is a function constructed with $P(\psi)$, $f(\psi)$, $F(\psi)$ and $T(\psi)$. Equation (39) is formally equivalent to the equation for static equilibria, well known in MHD. Combined with the ψ dependence of B^2, equation (39) has only one solution: the Palumbo solution [5]. It is proved in this section that this solution can contain incompressible parallel flows.

Palumbo's equilibrium is known to be unstable with respect to Mercier localized modes [6] in the static case. Though no general stability criterion exist for the extended solution with finite flow, it is easy to prove instability by expanding in the strength of the flow, following Ref. [7].

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