AN EXERCISE MOTIVATED BY DISSIPATIVE MAGNETOHYDRDYNAMIC STABILITY

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An Exercise Motivated by Dissipative Magnetohydrodynamic Stability

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Abstract

This exercise should be the first attempt at understanding effects due to the singularity of the "inertial" operator in the general formulation of the linearized stability problem in dissipative magnetohydrodynamics. The analysis shows that the stability conditions are qualitatively similar to the case of nonsingular operators, but suggest more optimism quantitatively. The 2×2 matrices in this exercise do not reflect, however, the huge problems related to a continuous fluid.

A general formulation of the linearized dissipative magnetohydrodynamic stability problem appeared in previous work [1, 2] of the author. The stability criterion obtained from this formulation is given to a good approximation by an Hermitian form [2, 3]. For real situations, however, overstability and Hopf bifurcation are not excluded and are discussed in [4, 5]. The simple examples treated there do not possess the singular character inherent to the "inertial" operator of the general formulation [1]. In fact, in [4, 5] the 'inertial" operator is taken proportional to the identity. A recent mathematical discussion [6] led to the exercise analysed in this paper.

An interesting feature of the general formulation is that it can be designed to describe an ideal or dissipative plasma surrounded by a vacuum and a rotating resistive wall as proposed in [7]. This case is an approximation

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to the more realistic situation of a moderately rotating plasma and a fixed resistive wall.

Let us recall the main equations representing the general formulation [1].

$$\Psi = \begin{pmatrix} \xi \\ \mathbf{A} \end{pmatrix}, \tag{1}$$

$$N\ddot{\Psi} + P\dot{\Psi} + Q\Psi = 0, \tag{2}$$

where N, P and Q are given by, respectively,

$$N = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} \mathbf{B}/\eta_0 \times (\cdots \times \mathbf{B}) & (\cdots \times \mathbf{B}/\eta_0) \\ -(\cdots \times \mathbf{B}/\eta_0 & 1/\eta_0 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} \begin{matrix} \nabla(-\gamma P_0(\nabla \cdot \ldots)) \\ -\nabla(\ldots \cdot \nabla P_0) \end{matrix} & \begin{matrix} -\mathbf{J} \times (\nabla \times \cdots) \\ -1/\eta_0 \nabla P_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \ldots \cdot \nabla \eta_0) \\ +\mathbf{B}/\eta_0 \times (\mathbf{V} \times \nabla \times \cdots) \end{matrix} \\ & \\ 0 & \begin{matrix} \nabla \times \nabla \cdots \\ +\mathbf{J}/\eta_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \ldots \cdot \nabla \eta_0) \\ -\mathbf{V}/\eta_0 \times \nabla \times \cdots \end{matrix} \end{pmatrix}.$$

The first two matrix operators in equation (2) are symmetric and positive while operator Q is obviously not selfadjoint. The notations as well as the derivation can be found in [1] but it is not necessary to enter those details to understand the purpose of the forthcoming exercise. What is important to notice is that the operator N used in the examples of [4, 5] is not singular while the operator N appearing in the "inertial" term of equation (2) is. The question is whether the conclusions of [4, 5] are affected by this singularity. It will be shown in the following exercise that those conclusions remain qualitatively valid.

Our exercise will consist of a particular choice of a 2×2 matrix system represented by a specification of Ψ, N, P, and Q as follows

$$\Psi = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \tag{3}$$

$$N\ddot{\Psi} + P\dot{\Psi} + Q\Psi = 0, \tag{4}$$

where N, P and Q are given by, respectively,

$$N = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} q_1 & -f \\ f & q_2 \end{pmatrix}.$$

In analogy with the physical problem n, p_1 and p_2 are positive. The approximate Hermitean condition mentioned above [1, 2, 3]

$$(\Psi, Q_s\Psi) \geq 0 \tag{5}$$

with Q_s the symmetric part of Q, is fulfilled if q_1 and q_2 are chosen positive as in [4, 5]. In fact, condition (5) is sufficient for stability with respect to purely growing modes [1]. In another context [7], condition (5) corresponds to the stabilization of the "kink" mode or plasma mode by a moving resistive wall. We know from [4, 5] that an overstability can still occur, and from [7] that the "wall" mode can be unstable unless the dissipation is strong enough. This is what we would like to analyse for equations (3) and (4). Make the ansatz $\Psi = \psi e^{\omega t}$ in equation (4) to obtain the characteristic polynomial

$$np_2\omega^3 + (nq_2 + p_1p_2)\omega^2 + (p_1q_2 + p_2q_1)\omega + f^2 + q_1q_2 = 0. \tag{6}$$

Equation (6) is cubic and could be solved explicitly. Since all the coefficients are positive, it is easier to exploit the relationships between roots and coefficients to check for stability. The three relations are

$$\omega_1 + \omega_2 + \omega_3 = -\frac{nq_2 + p_1p_2}{np_2}, \tag{7}$$

$$\omega_2\omega_3 + \omega_1\omega_3 + \omega_1\omega_2 = \frac{p_1q_2 + p_2q_1}{np_2}, \tag{8}$$

$$\omega_1\omega_2\omega_3 = -\frac{f^2 + q_1q_2}{np_2}, \tag{9}$$

where ω_1, ω_2 and ω_3 are the roots of equation (6).

As expected from the Hermitian condition (5) any real root of equation (6) has to be negative which means stability. Only complex roots could become unstable. Since we cannot have more than two complex roots, let us assume that ω_3 is the negative real root and set

$$\omega_1 = \alpha + i\beta, \tag{10}$$

$$\omega_2 = \alpha - i\beta. \tag{11}$$

Equations (7)-(9) become

$$2\alpha + \omega_3 = -\frac{nq_2 + p_1 p_2}{np_2}, \tag{12}$$

$$2\alpha\omega_3 + \alpha^2 + \beta^2 = \frac{p_1 q_2 + p_2 q_1}{np_2}, \tag{13}$$

$$(\alpha^2 + \beta^2)\omega_3 = -\frac{f^2 + q_1 q_2}{np_2}. \tag{14}$$

Eliminating $\alpha^2 + \beta^2$ in equations (13) and (14), we have

$$2\alpha\omega_3^2 = \omega_3 \frac{p_1 q_2 + p_2 q_1}{np_2} + \frac{f^2 + q_1 q_2}{np_2}. \tag{15}$$

Insert ω_3 from equation (12) into the righthand side of equation (15) to obtain

$$2\alpha(1 + np_2\omega_3^2) = f^2 + q_1 q_2 - \frac{(nq_2 + p_1 p_2)(p_1 q_2 + p_2 q_1)}{np_2}. \tag{16}$$

Instability occurs if α can be made positive, which is possible if

$$np_2 f^2 > np_1 q_2^2 + p_1 p_2 (p_1 q_2 + p_2 q_1). \tag{17}$$

Let us now consider several limiting cases in condition (17)

$\mathbf{f = 0}$ Condition (17) is never verified and the system is stable. In fact, equation (4) reduces to two decoupled scalar equations and the characteristic polynomial becomes

$$(n\omega^2 + p_1\omega + q_1)(p_2\omega + q_2) = 0, \tag{18}$$

from which the roots can be easily extracted

$$\omega_1 = \frac{-p_1 + \sqrt{p_1^2 - 4nq_1}}{2n}, \tag{19}$$

$$\omega_2 = \frac{-p_1 - \sqrt{p_1^2 - 4nq_1}}{2n}, \tag{20}$$

$$\omega_3 = -\frac{q_2}{p_2}. \tag{21}$$

All roots are stable.

n = 0 Condition (17) is never verified and the system is stable. In this case equation (6) reduces to a quadratic equation with stable roots.

f large Condition (17) can be verified and the system is unstable.

n large Condition (17) is verified and the system is unstable if

$$f^2 > q_2^2 \frac{p_1}{p_2}. \tag{22}$$

This exercise shows that, once the purely growing modes have been stabilized by fulfilling condition (5), the overstable modes and the related Hopf bifurcations behave qualitatively in agreement with [4, 5] though the "inertial" operator is singular. Physically, it means that a strong viscosity in a real plasma may stabilize the overstabilities due to the antisymmetric part of the Q operator (f in the exercise) and the "inertial" operator (n in the exercise), once the "kink" and "tearing" modes have been stabilized by a proper current distribution or by a moving wall [7] (positive q_1 and q_2 in the exercise).

The singular "inertial" operator leads, however, to more optimistic quantitative stability results than the nonsingular one as we can see from the case **n large** and condition (22) : A large n alone does not produce overstability as in [4, 5], it must be accompanied by a sufficiently large f.

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