An Introduction To The Weakly Relativistic Description Of HF Waves Propagation And Absorption In The Electron Cyclotron Frequency Range

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AN INTRODUCTION TO THE WEAKLY RELATIVISTIC DESCRIPTION OF HF WAVES PROPAGATION AND ABSORPTION IN THE ELECTRON CYCLOTRON FREQUENCY RANGE

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ABSTRACT

The weakly relativistic description of wave propagation and absorption in the electron cyclotron range of frequencies is reviewed. After a brief discussion of the relativistic cyclotron resonance condition, we derive the weakly relativistic dielectric tensor of a Maxwellian plasma in the form which is now 'canonical', in terms of the 'relativistic Plasma Dispersion functions'of Dnestrovsky and Skarofsky. The most useful analytic properties of these functions are summarized. We then obtain the small Larmor radius approximation, which is sufficient for most practical purposes when investigating electron cyclotron heating and current drive in tokamak plasmas. Finally, we present some solution of the finite Larmor radius dispersion relation near the fundamental and its first harmonic, respectively, emphasizing in particular the shape of the cyclotron absorption lines and the differences between the relativistic results and the classical limit.

1 - Introduction. While relativistic effects are usually negligible in laboratory plasmas, an important exception is offered, as well-known, by the interactions of electrons with h.f. waves in the vicinity of the cyclotron resonance and its harmonics. The reason is that in an homogeneous (or nearly homogeneous) magnetic field the relativistic correction to the cyclotron frequency, even if small, is sufficient to shift appreciably the resonance condition, at least for electrons. As a consequence, the investigation of wave propagation and absorption in the electron cyclotron range of frequencies must be done relativistically.

An elegant and compact form of the relativistic plasma dielectric tensor was early obtained by Trubnikov [1]. The analytic structure of this tensor, however, proved very untractable even at temperatures such that $T_e \ll mc^2$. The appropriate expansions for such "weakly relativistic" situation were soon identified by Dnestrovski et al. [2] for perpendicular propagation, and by Skarovsky [3] for small angles of propagation. Only much later, however, the weakly relativistic dielectric tensor of a Maxwellian plasma was put in a form allowing its numerical computation and the solution of the dispersion relation almost as easily as in the classical limit, in particular through the contributions of Airoldi and Orefice [4], Krivensky and Orefice [5], Robinson [6]–[7], and many others.

The present report is an introduction to the weakly relativistic description of wave propagation and absorption in the electron cyclotron range of frequencies. It contains no original contribution (except perhaps a more consistent discussion of the validity domain of the weakly relativistic expansion in section 3), the goal being instead to collect and present in a clearly organized way the basic relevant material, which is scattered in the literature and is not always easy to follow. As an introduction to the underlying physics, in the next section we briefly discuss the relativistic cyclotron resonance condition. In section 3 we derive the weakly relativistic dielectric tensor in the form which is now 'canonical', in terms of the 'relativistic Plasma Dispersion functions'of Dnestrovsky and Skarofsky. The analytic properties of these functions are summarized in section 4 and in the Appendix. In these two sections, apart from some notations and a few minor details, we follow closely [4] and [6]–[7]. In section 5 we obtain the small Larmor radius approximation, which is sufficient for most practical purposes; finally, in section 6 and 7 we present some solution of the finite Larmor radius dispersion relation near the fundamental and its first harmonic, respectively, emphasizing in particular the shape of the cyclotron absorption lines and the differences between the relativistic results and the classical limit.

We assume the reader to have some familiarity with the propagation of the ordinary (OW) and extraordinary (XW) waves in the electron cyclotron (EC) frequency range in the cold plasma limit; a very readable review can be found in [8]. As a preliminary, it is nevertheless useful to recall the order of magnitude of a few dimensionless parameters which characterize the propagation and absorption of such waves in tokamak plasmas. In the first place, the density is always such that the ratio of the plasma to the electron cyclotron frequency is of order unity,

$$\frac{\omega_{pe}^2}{\Omega_{ce}^2} = O(1) \tag{1.1}$$

and typically somewhat smaller than one (this condition is also required for the accessibility of the plasma core to the two cold plasma waves). In the second place, the wavelengths are always much larger than the electron Larmor radius, since when $\omega = O(\Omega_{ce})$ this condition is essentially equivalent to the condition that they must be much larger than the Debye length:

$$\frac{k_\perp v_{the}}{\Omega_{ce}} = \frac{\omega_{pe}}{\Omega_{ce}} k_\perp \lambda_D \ll 1 \tag{1.2}$$

(perpendicular and parallel refer to the direction of the static magnetic field). We therefore exclude the consideration of higher order electron Bernstein waves (EBW), which at tokamak densities satisfy this condition only in extremely narror frequency domains near cyclotron harmonics; the only exception is, as well-known, the first EBW near $\omega = 2\Omega_{ce}$. We next note that, because of the very short vacuum wavelengths, EC waves can be launced only with real angles of propagation, and in most case nearly perpendicularly to the static magnetic field. Thus, even allowing for the fact that $k_\parallel$ is not strictly constant during propagation in a tokamak, we can assume the condition

$$n_\parallel^2 < 1 \tag{1.3}$$

to be always satisfied, and usually by a large margin. Finally, we recall the obvious fact that the so-called 'relativistic parameter'

$$\mu_r = \frac{m_e c^2}{kT_e} = 2\frac{c^2}{v_{the}^2} \simeq \frac{511}{T_e\,(\mathrm{keV})} \tag{1.4}$$

is always much larger than unity even at the highest temperatures which can be reached in a Fusion plasma.

We do not attempt here even a partial review of the relevant literature. A complete bibliography up to 1983 can be found in the review paper by Bornatici et al. [9]; the contributions of different Authors to the understanding of the analytic properties of the relativistic Plasma Dispersion functions are accurately listed by Robinson [6]-[7].

2 – Relativistic effects on electron–cyclotron resonances. The relativistic momentum dependence of the mass modifies the electron cyclotron frequency, so that the resonance condition near the n–th harmonic the resonance condition becomes

$$\omega - k_\| v_\| - n\frac{\Omega}{\gamma} = 0 \tag{2.1}$$

or, in term of momenta,

$$\gamma\omega - \frac{k_\| p_\|}{m_e} - n\Omega = 0 \tag{2.2}$$

(we will preferably use momenta, which are the natural relativistic variables, since with a few exceptions the relativistic equations are simpler when written in terms of momentum components). Here $\Omega = \Omega_{ce} = eB_o/m_e c$ denotes the classical cyclotron frequency, and

$$\gamma = \left(1 + \frac{p_\|^2 + p_\perp^2}{m_e^2 c^2}\right)^{1/2} = \left(1 - \frac{v_\|^2 + v_\perp^2}{c^2}\right)^{-1/2} \tag{2.3}$$

Substituting for γ and eliminating the square root, we can rewrite Eq. (2.1) as

$$\frac{\omega^2}{\Omega^2}\left[(1-n_\|^2)\frac{p_\|^2}{m_e^2 c^2} + \frac{p_\perp^2}{m_e^2 c^2}\right] - 2n\frac{\omega}{\Omega}n_\|\frac{p_\|}{m_e c} + \left(\frac{\omega^2}{\Omega^2} - n^2\right) = 0 \tag{2.4}$$

For $n \geq 1$ this is the equation of an ellipse in the $(p_\|,\ p_\perp)$-plane. A few examples near the fundamental resonance are shown in fig. 1 for the two cases $\omega < \Omega_{ce}$ and $\omega > \Omega_{ce}$, respectively (the ellipticity is barely visible, since the ratio of the two axes is $\omega/\Omega_{ce} \simeq 1$). Since Eq. (1) is even in $p_\perp$, it is not difficult to see that the whole ellipse satisfies the resonance condition. This means that not only the shape, but even the topology of resonant curves in phase space is drastically modified by relativistic effects.

Consider for example electrons with negligible perpendicular momentum, i.e. moving along the static magnetic field. The resonance condition is satisfied for two values of $p_\|$, namely

$$\left(\frac{p_\|}{m_e c}\right)_{res} = \frac{1}{1-n_\|^2}\left\{\frac{n\Omega_{ce}}{\omega}n_\| \pm \left(\frac{n^2\Omega_{ce}^2}{\omega^2} - (1-n_\|^2)\right)^{1/2}\right\} \tag{2.5}$$

One value tends to the classical resonance

$$\left(\frac{p_\|}{m_e c}\right)_{res,\ class} = \frac{1}{n_\|}\left(1 - \frac{n\Omega_{ce}}{\omega}\right) \tag{2.6}$$

when $n_\|$ is large, but is increasingly modified by the relativistic change of the mass as $n_\| \to 0$. The other resonant value, which corresponds always to much larger energies, has no classical equivalent. In the case $\omega < \Omega_{ce}$, this second resonant momentum has even the sign opposite to the classical one, i.e. the same sign as the parallel wavevector,

$$\frac{k_{\parallel} v_{\parallel})_{res}}{\omega - \Omega_{ce}} < 0 \qquad (2.7)$$

This resonance is therefore known as the 'anomalous' or 'inverse' Doppler effect. The values of $p_{\parallel})_{res}$ versus $n_{\parallel}$ are shown in fig. 2 for different values of ω/Ω_{ce}. For clarity, in figs. 2 a and b only half of the curves are actually shown, namely those which connect to the classical case for positive values of $n_{\parallel}$. The complete curves are symmetric with respect to the origin, since the resonance condition is unchanged if the signs of $n_{\parallel}$ and $p_{\parallel}$ are simultaneously changed; the complete plot is shown for the case $\omega < \Omega_{ce}$ in fig. 2 c.

To understand the effects of the relativistic modification of the resonance condition on cyclotron damping, the resonance cureves must be weighted with the number of electrons having the corresponding energy. It is then clear that, under normal conditions, absorption by the inverse cyclotron resonance is negligible in a Fusion plasma. In low density tokamak discharges, however, the parallel steady-state electric field can exceed the Dreicer limit, so that some electrons become collisionally decoupled from the bulk plasma and are accelerated to very large energies; these 'run-away' electrons can then sustain a non-negligible fraction of the total ohmic current. Inverse cyclotron damping by relativistic run-away electrons can occur even at frequencies in the lower hybrid domain.

As can be seen in fig. 2, the relativistic shift of normal cyclotron resonances decreases and finally approaches the classical Doppler effect when $n_{\parallel}$ increases; agreement with the classical limit, however, is reached only at larger values of the parallel index when the frequency is farther away from an exact harmonic. This is because for a given $n_{\parallel}$ the energy of resonant particles increases with $|\omega - n\Omega_{ce}|$. The influence of relativistic effects on the dielectric tensor nevertheless decreases rapidly with $n_{\parallel}$, since only for values of $p_{\parallel})_{res}$ not too far from the thermal domain there is a sufficient number of resonant electrons to contribute appreciably to the total collective response, and in particular to cyclotron absorption.

Figure 1 also shows the characteristic asymmetry of the cyclotron resonance condition with respect to the exact harmonic. The same asymmetry is clear in the plot of the resonant frequency versus parallel momentum for $p_{\perp} = 0$ and different values of $n_{\parallel}$, shown in fig. 3. At frequencies lower than the exact harmonic the resonance condition can be satisfied for any value of $n_{\parallel}$, including zero; as seen above, sufficiently energetic electrons can be in resonance also for values of $n_{\parallel}$ such that $k_{\parallel} v_{\parallel}$ has the sign opposite to $\omega - n\Omega_{ce}$. At frequencies larger than the exact harmonic, on the other hand, it is easily seen that Eq. (1) can be satisfied only if

$$\frac{n\Omega_{ce}}{\omega} \geq (1-n_\parallel^2)\left(1+\frac{p_\perp^2}{m_e^2c^2}\right) \tag{2.8}$$

In particular, for $p_\perp = 0$, the resonance condition can be satisfied only for oblique propagation,

$$n_\parallel \geq n_\parallel)_{crit} = \sqrt{1-\frac{n^2\Omega_{ce}^2}{\omega^2}} \tag{2.9}$$

(fig. 4). The asymmetry of the frequency range of strong absorption with respect to the exact harmonic leads to a down–shift of the optimum resonance frequency for electron cyclotron heating. In the dielectric tensor, however, this effect becomes negligible when $n_\parallel^2 \gg v_{the}^2/c^2$: for such values of $n_\parallel$ it is completely washed out by the normal Doppler broadening of the resonance.

The relativistic resonance condition can be satisfied also for strictly perpendicular propagation, provided $\omega \leq n\Omega_{ce}$. For $n_\parallel = 0$, Eq. (1) reduces to

$$\frac{p_\perp^2 + p_\parallel^2}{m_e^2c^2} = \frac{n^2\Omega_{ce}^2}{\omega^2} - 1 \tag{2.10}$$

Thus electrons with the appropriate momentum can be in resonance with a wave propagating perpendicularly to the static magnetic field at any frequency lower (or static magnetic field higher) than the exact harmonic. This is a purely relativistic effect, which has the consequence that the antihermitian part of the dielectric tensor does not vanish even in the limit of perpendicular propagation. Equation (2.10) describes a sphere in momentum space, whose radius increases with increasing $n\Omega_{ce} - \omega$. The 'width' of the absorption line in $\underline{\underline{\epsilon}}^A$, however, can again be estimated by putting $p/m_ec \simeq v_{the}/c$ in the l.h. side of (2.10): in other words, for perpendicularly propagating waves the range of frequencies for which the number of electrons simultaneously in resonance is sufficient to produce appreciable absorption is of the order of

$$0 \leq n\Omega_{ce} - \omega \lesssim \Omega_{ce}\frac{v_{the}}{c} \tag{2.11}$$

Putting $n = 0$ in Eq. (1), we obtain the relativistic Cerenkoff resonance condition in the form

$$n_\parallel \frac{p_\parallel}{m_ec} \equiv \frac{k_\parallel v_\parallel}{\omega} = \frac{|n_\parallel|}{(n_\parallel^2-1)^{1/2}}\left(1+\frac{p_\perp^2}{m_e^2c^2}\right)^{1/2} \tag{2.12}$$

This condition can be satisfied only if $n_\parallel^2 > 1$: Landau and Transit Time damping of waves with parallel phase velocity greater than the speed of light is strictly impossible, as one would expect (the classical expressions for this damping become exponentially small, but do not vanish for any positive value of $n_\parallel^2$, however small). In the case of Landau damping and Magnetic Pumping, however, the quantitative importance of relativistic corrections is usually so small that this result is of conceptual interest only.

3 – The relativistic dielectric tensor. The relativistic dielectric tensor is obtained by integrating the linearized relativistic Vlasov equation

$$\frac{\partial \tilde{f}_\alpha}{\partial t} + \vec{v}\cdot\frac{\partial \tilde{f}_\alpha}{\partial \vec{r}} + Z_\alpha e\left(\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial \tilde{f}_\alpha}{\partial \vec{p}} = -Z_\alpha e\left(E + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial F_\alpha}{\partial \vec{p}} \tag{3.1}$$

in the presence of a wave fields with harmonic space and time dependence $\exp i(\vec{k}\cdot\vec{r} - \omega t)$, assuming to begin with $\mathrm{Im}\,(\omega) > 0$ to satisfy causality. Expanding the phase factor seen by the particles in a double series of Bessel functions and performing the integrals over the azimuthal angle ϕ in momentum space (this part of the derivation is identical in the relativistic and in the classical case) one obtains:

$$\epsilon_{ij}(\vec{k},\omega) = \delta_{ij} + 2\pi\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int_{-\infty}^{+\infty} dp_\parallel \int_0^{+\infty} p_\perp\, dp_\perp \sum_{n=-\infty}^{+\infty} \frac{\omega}{\gamma\omega - k_\parallel p_\parallel/m - n\Omega_c}\, Q_{ij}^n \tag{3.2}$$

with

$$\begin{aligned}
Q_{xx}^n &= \frac{n^2}{\gamma^2\xi_\perp^2} J_n^2(\gamma\xi_\perp)\, p_\perp \left(\frac{\partial F}{\partial p_\perp} + \frac{k_\parallel}{m\gamma\omega}\Theta_p F\right) \\
Q_{xy}^n &= -Q_{yx}^n = i\frac{n}{\gamma\xi_\perp} J_n(\gamma\xi_\perp) J_n'(\gamma\xi_\perp)\, p_\perp \left(\frac{\partial F}{\partial p_\perp} + \frac{k_\parallel}{m\gamma\omega}\Theta_p F\right) \\
Q_{xz}^n &= \frac{n}{\gamma\xi_\perp} J_n^2(\gamma\xi_\perp)\left(p_\perp \frac{\partial F}{\partial p_\parallel} - \frac{n\Omega_{c\alpha}}{\gamma\omega}\Theta_p F\right) \\
Q_{yy}^n &= J_n'^2(\gamma\xi_\perp)\, p_\perp \left(\frac{\partial F}{\partial p_\perp} + \frac{k_\parallel}{m\gamma\omega}\Theta_p F\right) \\
Q_{yz}^n &= -iJ_n(\gamma\xi_\perp) J_n'(\gamma\xi_\perp)\left(p_\perp \frac{\partial F}{\partial p_\parallel} - \frac{n\Omega_{c\alpha}}{\gamma\omega}\Theta_p F\right) \\
Q_{zx}^n &= \frac{n}{\gamma\xi_\perp} J_n^2(\gamma\xi_\perp)\, p_\parallel \left(\frac{\partial F}{\partial p_\perp} + \frac{k_\parallel}{m\gamma\omega}\Theta_p F\right) \\
Q_{zy}^n &= +iJ_n(\gamma\xi_\perp) J_n'(\gamma\xi_\perp)\, p_\parallel \left(\frac{\partial F}{\partial p_\perp} + \frac{k_\parallel}{m\gamma\omega}\Theta_p F\right) \\
Q_{zz}^n &= J_n^2(\gamma\xi_\perp)\left(p_\parallel \frac{\partial F}{\partial p_\parallel} - \frac{n\Omega_{c\alpha}}{\gamma\omega}\frac{p_\parallel}{p_\perp}\Theta_p F\right)
\end{aligned} \tag{3.3}$$

with

$$\Theta_p F_\alpha = p_\perp \frac{\partial F_\alpha}{\partial p_\parallel} - p_\parallel \frac{\partial F_\alpha}{\partial p_\perp} \tag{3.4}$$

The argument of the Bessel functions has been written $k_\perp p_\perp / m\Omega_c = \gamma\xi_\perp$, so that $\xi_\perp = k_\perp v_\perp/\Omega_c$ has the same meaning as in the classic limit. As a function of $p_\perp$, however, it does not depend explicitly on γ. Note that although in Eq. (3.2) we have considered a sum over all species in the usual way, only the electrons ever need a relativistic treatment.

With the appropriate analytic continuation of the velocity integrals for $\mathrm{Im}\,(\omega) \leq 0$, Eqs. (3)–(4) give the general form of the relativistic dielectric tensor. To perform the analytic continuation explicitly, we must apply the Landau prescription. In particular, in the limit of a real frequency we should interpret the resonant denominators as

$$\frac{\omega}{\gamma\omega - k_\parallel p_\parallel/m - n\Omega_c} = P\left(\frac{1}{\gamma - n_\parallel p_\parallel/mc - n\Omega_c/\omega}\right) + i\pi\,\delta\left(\gamma - n_\parallel \frac{p_\parallel}{mc} - n\frac{\Omega_c}{\omega}\right) \tag{3.5}$$

For real frequency, therefore, the principal parts contribute the hermitean part, and the δ–functions the antihermiteam part of the dielectric tensor, as usual.

If we try to perform the integrals over momenta for a specified distribution function in the relativistic case, we encounter a difficulty which is not present in the classical limit: the convenient factorisation between parallel and perpendicular degrees of freedom of the classical case is now spoiled by the factor γ, so that the singularity in the resonant denominators in Eq. (3.3) depends on both $p_\parallel$ and $p_\perp$; moreover, the two branch points $p = \pm imc$, although far away from the real axis, complicate the analytic behaviour of the integrand in the complex plane. As a consequence, exact expressions in closed forms cannot be written, and even a numerical treatment is often difficult in the general case.

We will now restrict our considerations to the case of Maxwellian electrons. It simplifies the notations to use the normalised momentum $\bar{p} = p/mc$; the relativistic Maxwellian can then be written

$$F_{Mr}(\bar{p}^2) = \frac{\mu_r}{4\pi\,K_2(\mu_r)}\,e^{-\mu_r\sqrt{1+\bar{p}^2}} = \left[\sqrt{\frac{\pi}{2\mu_r}}\frac{e^{-\mu_r}}{K_2(\mu_r)}\right]\left(\frac{\mu_r}{2\pi}\right)^{3/2} e^{-\mu_r(\gamma-1)} \tag{3.6}$$

where K_2 denotes the McDonald function, or modified Bessel function of the second kind, of order 2, and μ_r (Eq. (1.4)) is the parameter which characterises the importance of relativistic corrections (it becomes infinite in the classical limit). The bracket in the last expression tends to unity in the classic limit. A comparison between the classical and relativistic Maxwellians is offered in fig. 5. Expressed in terms of momentum, the relativistic Maxwellian decreases somewhat more slowly than the classical one (the opposite is true if velocity were chosen as variable). Even at thermonuclear temperatures (10 keV), however, the differences are still very small; they are clearly visible, on the other hand, at 100 keV, i.e. $T_e/mc^2 \simeq 0.2$.

Taking into account that

$$\frac{\partial F}{\partial \bar{p}_j} = -\mu_r \frac{\partial \gamma}{\partial \bar{p}_j} F = -\frac{\mu_r}{\gamma} \bar{p}_j F \tag{3.7}$$

equation (3.2) specialised to the distribution function (3.6) becomes

$$\epsilon_{ij}(\vec{k},\omega) = \delta_{ij} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\mu_r^2}{2K_2(\mu_r)} \int_{-\infty}^{+\infty} d\bar{p}_\parallel \int_0^{+\infty} \bar{p}_\perp \, d\bar{p}_\perp \frac{e^{-\mu_r \gamma}}{\gamma} \sum_{n=-\infty}^{+\infty} \frac{P_{ij}^n}{\gamma - n_\parallel \bar{p}_\parallel - n(\Omega_c/\omega)} \tag{3.8}$$

with

$$\begin{aligned}
P_{xx}^n &= \bar{p}_\perp^2 \frac{n^2}{\nu_\perp^2 \bar{p}_\perp^2} J_n^2(\nu_\perp \bar{p}_\perp) \\
P_{xy}^n &= -P_{yx}^n = i\bar{p}_\perp^2 \frac{n}{\nu_\perp \bar{p}_\perp} J_n(\nu_\perp \bar{p}_\perp)\, J_n'(\nu_\perp \bar{p}_\perp) \\
P_{xz}^n &= P_{zx}^n = \bar{p}_\perp \bar{p}_\parallel \frac{n}{\nu_\perp \bar{p}_\perp} J_n^2(\nu_\perp \bar{p}_\perp) \\
P_{yy}^n &= \bar{p}_\perp^2 J_n'^2(\nu_\perp \bar{p}_\perp) \\
P_{yz}^n &= -P_{zy}^n = -i\bar{p}_\perp \bar{p}_\parallel J_n(\nu_\perp \bar{p}_\perp)\, J_n'(\nu_\perp \bar{p}_\perp) \\
P_{zz}^n &= \bar{p}_\parallel^2 J_n^2(\nu_\perp \bar{p}_\perp)
\end{aligned} \tag{3.9}$$

where we have introduced the notation

$$\nu_\perp = \frac{ck_\perp}{\Omega_c} = n_\perp \frac{\omega}{\Omega_c} \tag{3.10}$$

so that $\gamma \xi_\perp = \nu_\perp \bar{p}_\perp$.

Because of the γ in the resonant denominators, the momentum space integrations cannot be performed in closed form. It is however possible to transform (8) into a form which is easier to be evaluated numerically, and more convenient as starting point for analytic approximations appropriate to the weakly relativistic case, when a power expansion of the Bessel functions converges rapidly throughout the range of energies for which the distribution function is not negligibly small. The starting point is the series expansion for products of Bessel functions [11]:

$$J_n^2(\nu_\perp \bar{p}_\perp) = J_{-n}^2(\nu_\perp \bar{p}_\perp) = \frac{(\nu_\perp \bar{p}_\perp)^{2n}}{n!\, 2^{2n}} \sum_{k=0}^{\infty} \frac{C_{n,k}}{(n+k)!} \frac{(\nu_\perp \bar{p}_\perp)^{2k}}{k!\, 2^{2k}} \tag{3.11}$$

with

$$C_{n,k} = \frac{(-1)^k |n|!\,[2(|n|+k)]!}{(2|n|+k)!\,(|n|+k)!}. \tag{3.12}$$

Taking into account that $\nu_\perp^2/\mu_r = k_\perp^2 v_{th}^2/2\Omega_c^2 = \lambda$, we rewrite (8) as

$$\epsilon_{ij}(\vec{k},\omega) = \delta_{ij} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\mu_r}{2\,K_2(\mu_r)} \sum_{n=-\infty}^{+\infty} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^{|n|}$$

$$\sum_{k=0}^{+\infty} \frac{C_{n,k}}{k!}\left(\frac{\lambda}{2}\right)^k \pi_{ij}(n,k)\,\mathcal{J}_n^{(|n|+k,\,r_{ij})}\,(n_\parallel;\, n\Omega_c/\omega) \tag{3.13}$$

with

$$\pi_{xx} = \frac{n^2}{\lambda}$$

$$\pi_{xy} = -\pi_{yx} = i\frac{n(|n|+k)}{\lambda}$$

$$\pi_{xz} = \pi_{zx} = \frac{n}{\sqrt{\lambda}}$$

$$\pi_{yy} = \frac{n^2}{\lambda} + \frac{2k(2|n|+k)(|n|+k-1)}{(2|n|+2k-1)\lambda} \tag{3.14}$$

$$\pi_{yz} = -\pi_{zy}^{(n,k)} = -i\frac{(|n|+k)}{\sqrt{\lambda}}$$

$$\pi_{zz} = 1$$

and

$$\mathcal{J}_n^{(|n|+k,\,r_{ij})}\left(n_\parallel;\, n\frac{\Omega_c}{\omega}\right) = \frac{1}{(|n|+k)!}\left(\frac{\mu_r}{2}\right)^{|n|+k} \int_{-\infty}^{+\infty} d\bar{p}_\parallel$$

$$\mu_r^{r_{ij}/2} \int_0^{+\infty} \bar{p}_\perp\, d\bar{p}_\perp \frac{e^{-\mu_r\gamma}}{\gamma} \frac{\bar{p}_\parallel^{r_{ij}}\, \bar{p}_\perp^{2(|n|+k)}}{\gamma - n_\parallel \bar{p}_\parallel - n(\Omega_c/\omega)} \tag{3.15}$$

Here r_{ij} is equal to the number of times z enters in the indexes of the element ϵ_{ij}, i.e. $r_{xx} = r_{xy} = r_{yx} = r_{yy} = 0$, $r_{xz} = r_{zx} = r_{yz} = r_{zy} = 1$, and $r_{zz} = 2$.

We now turn to the task of evaluating the double integrals $\mathcal{J}^{(s,r)}$, where r can take the values 0, 1 or 2, and $s = |n| + k$ is a non–negative integer. From now on we assume explicitly $T_e \ll mc^2$, and we develop $\mathcal{J}^{(s,r)}$ in powers of the small parameter μ_r^{-1}. It is convenient in the first place to change the integration variables from $(\bar{p}_\parallel, \bar{p}_\perp)$, to $(\bar{p}_\parallel, \gamma)$; in the new variables $\bar{p}_\perp^2 = \gamma^2 - (1 + \bar{p}_\parallel^2)$, and the Jacobian of the transformation is $(\gamma/\bar{p}_\perp)$. With the further change of integration variable $x = \gamma - (1+\bar{p}_\parallel^2)^{1/2}$, the integrals over momenta can be rewritten

$$\int_{-\infty}^{+\infty} d\bar{p}_\parallel\, e^{-\mu_r\sqrt{1+\bar{p}_\parallel^2}} \int_0^\infty dx\, e^{-\mu_r x} \frac{\bar{p}_\parallel^r\, x^s \left(x + 2\sqrt{1+\bar{p}_\parallel^2}\right)^s}{x+\beta_n}$$

$$= \sum_{m=0}^{s} \binom{s}{m} \int_{-\infty}^{+\infty} d\bar{p}_\parallel\, e^{-\mu_r\sqrt{1+\bar{p}_\parallel^2}} \bar{p}_\parallel^r \left(2\sqrt{1+\bar{p}_\parallel^2}\right)^{s-m} \int_0^\infty dx\, e^{-\mu_r x} \frac{x^{s+m}}{x+\beta_n} \tag{3.16}$$

where we have abbreviated $\beta_n = (1+\bar{p}_\parallel^2)^{1/2} - n_\parallel \bar{p}_\parallel - n(\Omega_c/\omega)$, and we have expanded the second factor according to the binomial formula. The inner integrals in the last line can be transformed as follows:

$$\int_0^\infty dx\, e^{-\mu_r x} \frac{x^{s+m}}{x+\beta_n} = \beta_n^{s+m} \int_0^\infty dt\, e^{-\mu_r \beta_n t} \frac{t^{s+m}}{t+1}$$

$$= \frac{(s+m)!}{\mu_r^{s+m}} \int_0^\infty dt\, \frac{e^{-\mu_r\beta_n t}}{(1+t)^{s+m+1}} = -i\frac{(s+m)!}{\mu_r^{s+m}} \int_0^\infty dt\, \frac{e^{i\mu_r\beta_n t}}{(1-it)^{s+m+1}} \tag{3.17}$$

where the third form has been obtained by application of the identity

$$(-\beta_n t)^k\, e^{-\mu_r\beta_n t} = \frac{1}{\mu_r^k} \frac{d^k(e^{-\mu_r\beta_n t})}{dt^k} \tag{3.18}$$

followed by k integrations by parts; moreover for convenience the variable of integration has been changed from t into $-it$, and the integration line has been rotated back to the real axis, taking advantage of the fact that no singularities are crossed in the operation, and the integrand vanishes sufficiently rapidly at infinity. Using (3.17) into (3.16) and interchanging the order of integrations, we thus obtain:

$$\mathcal{J}_n^{(s,r)} = -\frac{i}{s!} \sum_{m=0}^{s} \binom{s}{m} \frac{(s+m)!}{(2\mu_r)^m} \int_0^\infty dt\, \frac{e^{-i\mu_r(n\Omega_c/\omega)t}}{(1-it)^{s+m+1}}$$

$$\mu_r^{r/2} \int_{-\infty}^{+\infty} d\bar{p}_\parallel\, e^{-\mu_r\left\{(1-it)\sqrt{1+\bar{p}_\parallel^2} + in_\parallel \bar{p}_\parallel t\right\}} \bar{p}_\parallel^r \left(\sqrt{1+\bar{p}_\parallel^2}\right)^{(s-m)} \tag{3.19}$$

The Landau pole has now been transformed into the singularity at $t = -i$ in the outer integrand, which originates from the integration over energy (i.e. γ). Therefore we can now develop the factor $(1+\bar{p}_\parallel^2)^{1/2}$ in powers of $\bar{p}_\parallel^2$,

$$\left(1+\bar{p}_\parallel^2\right)^{\lambda/2} = \sum_{k=0}^{\infty} \binom{\lambda/2}{k} \bar{p}_\parallel^{2k} \tag{3.20}$$

without affecting the resonant determinator. For the exponential function, this leads to the expansion

$$e^{-\mu_r(1-it)\sqrt{1+\bar{p}_\parallel^2}} = e^{-\mu_r(1-it)\,(1+\frac{1}{2}\bar{p}_\parallel^2)}\,\Sigma(\mu_r,\bar{p}_\parallel) \tag{3.21}$$

where

$$\Sigma(\mu_r,\bar{p}_\parallel) = 1 + \frac{1}{2\mu_r^2}\left(\frac{\mu_r\bar{p}_\parallel^2}{2}\right)^2 \sum_{l=0}^{\infty}(-1)^l\,\frac{\mathcal{P}_l(\mu_r(1-it))}{\mu_r^l}\left(\frac{\mu_r\bar{p}_\parallel^2}{2}\right)^l \tag{3.22}$$

Here $\mathcal{P}_l$ is a polynomial of order $1+[l/2]$; the first few of these polynomials are

$$\mathcal{P}_0(z) = z \qquad \mathcal{P}_1(z) = z$$

$$\mathcal{P}_2(z) = \frac{z}{4}(5+z) \qquad \mathcal{P}_3(z) = \frac{z}{12}\left(\frac{7}{2}+z\right) \tag{3.23}$$

with $z = \mu_r(1-it)$. Our notations here stress that the development (3.21) is appropriate to the weakly relativistic limit, $\kappa T \ll mc^2$. Indeed, if the electron temperature is not relativistic, the exponential factor makes the integrand negligibly small as soon as $\mu_r\bar{p}_\parallel^2/2 \gtrsim 1$. In other words, the expansion of the exponent of the Maxwellian distribution function around its classical limit does not scramble the ordering of the series (3.19).

Substituting (3.20) and (3.21) into (3.19), we finally obtain

$$\mathcal{J}_n^{(s,r)} = -\frac{i}{s!}\sum_{m=0}^{s}\binom{s}{m}\frac{(s+m)!}{(2\mu_r)^m}\int_0^{\infty} dt\,\frac{e^{-i\mu_r(n\Omega_c/\omega)t}}{(1-it)^{s+m+1}}\,e^{-\mu_r(1-it)}$$

$$\sum_{k=0}^{\infty}\binom{(s-m)/2}{k}\int_{-\infty}^{+\infty} d\bar{p}_\parallel\, e^{-\frac{1}{2}\mu_r(1-it)\bar{p}_\parallel^2 - i\mu_r(n_\parallel\bar{p}_\parallel)t} \tag{3.24}$$

$$\mu_r^{r/2}\,\bar{p}_\parallel^{2k+r}\left\{1+\frac{1}{2\mu_r^2}\sum_{l=0}^{\infty}(-1)^l\,\frac{\mathcal{P}_l(\mu_r(1-it))}{\mu_r^l}\left(\frac{\mu_r\bar{p}_\parallel^2}{2}\right)^{l+2}\right\}$$

All the double integrals occurring in this expansion can be reduced to a single class of functions defined in terms of a simple integral over the variable t. For this purpose, the $\bar{p}_\parallel$–integration can be performed term by term after changing the integration variable to $\xi = \bar{p}_\parallel - in_\parallel t/(1-it)$ and using again the binomial theorem:

$$\int_{-\infty}^{+\infty} d\bar{p}_\parallel\,\bar{p}_\parallel^m\, e^{-\frac{1}{2}\mu_r(1-it)\bar{p}_\parallel^2 - i\mu_r(n_\parallel\bar{p}_\parallel)t}$$

$$= e^{\frac{1}{2}\mu_r n_\parallel^2\frac{t^2}{(1-it)}}\sum_{\lambda=0}^{m}\binom{m}{\lambda}\left(\frac{-in_\parallel t}{1-it}\right)^{m-\lambda}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\mu_r(1-it)\xi^2}\,\xi^\lambda\, d\xi \tag{3.25}$$

$$= e^{\frac{1}{2}\mu_r n_\parallel^2\frac{t^2}{(1-it)}}\sum_{\nu=0}^{[m/2]}\binom{m}{2\nu}\left(\frac{-in_\parallel t}{1-it}\right)^{m-2\nu}\frac{\Gamma(\nu+\frac{1}{2})}{\left[\frac{1}{2}\mu_r(1-it)\right]^{\nu+\frac{1}{2}}}$$

The integrals over t in Eq. (19) are now all of the form

$$\mathcal{F}_q^\lambda(n) = -i \exp\left(-\frac{1}{2}\mu_r n_\parallel^2\right) \int_0^\infty dt \, \frac{(-it)^\lambda}{(1-it)^q} \exp\left\{\mu_r\left[-i\left(\frac{1}{2}n_\parallel^2 - \varpi_n\right)t + \frac{1}{2}\frac{n_\parallel^2}{(1-it)}\right]\right\} \tag{3.26}$$

where λ and $2q$ are integers, and we have abbreviated

$$\varpi_n = \frac{\omega - n\Omega}{\omega} \tag{3.27}$$

Moreover, to simplify the term proportional to $n_\parallel^2$ in the exponential, we have expanded $t^2 = (1+it)(1-it) - 1$. Clearly

$$\mathcal{F}_q^\lambda(n) = \frac{1}{\mu_r^\lambda}\frac{\partial}{\partial(\varpi_n)^\lambda}\mathcal{F}_q^0(n) \tag{3.28}$$

Alternatively, using again the binomial theorem in the form

$$t^\lambda = i^\lambda\left[(1-it)-1\right]^\lambda = i^\lambda \sum_{\nu=0}^{\lambda}(-1)^\nu \binom{\lambda}{\nu}(1-it)^{\lambda-\nu} \tag{3.29}$$

we can further expand these integrals as follows:

$$\mathcal{F}_q^\lambda(n) = \sum_{\nu=0}^{\lambda}(-1)^\nu \binom{\lambda}{\nu}\mathcal{F}_{q-(\lambda-\nu)}^0(n) \tag{3.30}$$

With the help of Eqs. (20) to (30), the integrals (19) have been expressed in terms of one–dimensional integrals of a single type,

$$\mathcal{F}_{p+\frac{1}{2}}(n) = \mathcal{F}_{p+\frac{1}{2}}^0\left(\frac{\omega - n\Omega_c}{\omega}; \mu_r, n_\parallel\right) \tag{3.31}$$

The functions $\mathcal{F}_{p+\frac{1}{2}}$ of half–integer order play in the relativistic dielectric tensor a role similar to the Plasma Dispersion Function in the classical limit. Some of their properties will be discussed in the next paragraph.

In this way the elements of the relativistic dielectric tensor have been expressed as a series in $\lambda_e = k_\perp \rho_e$, each coefficient of which is in turn an asymptotic series in the small quantity $\mu_r^{-1} = v_{th}^2/c^2$. The complete expressions for ϵ_{ij} are rather complicated, and we will not write them in full, since in practice only a few terms are needed. A consistent truncation of the series which results in relatively simple expressions for the dielectric tensor of the weakly relativistic plasma will be discussed in section 5. Here we will only note that although successive terms decrease fastest for almost perpendicular propagation, $n_\parallel^2 \lesssim \mu_r^{-1}$, the ordering is actually respected as long as $n_\parallel \lesssim 1$. Indeed, for oblique propagation, the inner series in Eq. (3.24) (which arises from the expansion of $(1+\bar{p}_\parallel^2)^{\lambda/2}$ performed in Eq. (3.20)) is actually a series in powers of $n_\parallel^2/\mu_r$. This statement is easily proven by noting that in Eq. (3.25) each power of $n_\parallel$ raises by one the upper index in the function $\mathcal{F}_q^k$ which appears in the corresponding term. According to Eq. (3.28), on the other hand, $\mathcal{F}_q^k$ is obtained from $\mathcal{F}_q^0$ by deriving k times with respect to $z = \mu_r \varpi_n$; since ϖ_n is a quantity of order unity, each derivation introduces a factor μ_r^{-1}, as required. The representation of the relativistic dielectric tensor obtained in this section, therefore, can be used for arbitrary real angles of propagation, $n_\parallel = O(1)$. When $n_\parallel^2 \gg \mu_r^{-1}$, on the other hand, the classic limit is rapidly approached.

In concluding this section, it is worth mentioning that Bornatici and Ruffina [10], exploiting the δ-function in Eq. (3.5), have been able to put the antihermitian part of the relativistic dielectric tensor for real frequency in closed form, avoiding the Larmor radius expansion used here.

4 – The relativistic Plasma Dispersion Functions. The relativistic Plasma Dispersion Functions are defined by Eqs. (26) and (31) as

$$\mathcal{F}_q\left(\varpi_n, \mu_r, n_\parallel\right) = -i\, e^{-\frac{1}{2}\mu_r n_\parallel^2} \int_0^\infty dt\, \frac{e^{\mu_r\left[i(\varpi_n - \frac{1}{2}n_\parallel^2)t + \frac{1}{2}\frac{n_\parallel^2}{(1-it)}\right]}}{(1-it)^q} \tag{4.1}$$

Here, as usual, $\mathrm{Im}\,(\omega) > 0$ is assumed; the analytic continuation of $\mathcal{F}_q$ for $\mathrm{Im}\,(\omega) \leq 0$ must be made following the Landau prescription. We will return to this point below.

In Eq. (4.1) $\mathcal{F}_q$ appears to depend on the three variables, ϖ_n, $n_\parallel$, and μ_r only through the combinations $\mu_r \varpi_n$ and

$$\nu_r^2 = \frac{\mu_r}{2} n_\parallel^2 = n_\parallel^2 \frac{c^2}{v_{th}^2} \tag{4.2}$$

A more convenient form is obtained expressing ϖ_n in terms of the argument $x_n = (\omega - n\Omega_{ce})/k_\parallel v_{the}$ of the Plasma dispersion function in the classical limit. Since

$$\mu_r \varpi_n = 2\nu_r\, x_n \tag{4.3}$$

we can rewrite

$$\mathcal{F}_q\left(2\nu_r x_n, \nu_r^2\right) = -i\, e^{2\nu_r(x_n - \nu_r)} \int_0^\infty dt\, \frac{\exp\left[\nu_r(\nu_r - 2x_n)(1-it) + \dfrac{\nu_r^2}{1-it}\right]}{(1-it)^q} \tag{4.4}$$

Although any real index q can be considered in Eqs. (4.1) or (4.4), only semi–integer values $q \geq 5/2$ occur in the dielectric tensor.

From Eq. (4.1) it is possible to obtain a recurrence relation which allows to express all the functions of this family in terms of the first two members, namely those with $q = 1/2$ and $q = 3/2$. As shown in the Appendix,

$$\nu_r^2 \mathcal{F}_{q+\frac{5}{2}}\left(2\nu_r x_n, \nu_r^2\right) = 1 + \nu_r\left(\nu_r - 2x_n\right)\mathcal{F}_{q+\frac{1}{2}}\left(2\nu_r x_n, \nu_r^2\right) - \left(q + \frac{1}{2}\right)\mathcal{F}_{q+\frac{3}{2}}\left(2\nu_r x_n, \nu_r^2\right) \tag{4.5}$$

On the other hand, using the integral representation (4.1), $\mathcal{F}_{1/2}$ and $\mathcal{F}_{3/2}$ can be expressed in terms of the ordinary plasma dispersion function Z: as shown in the Appendix, one finds

$$\mathcal{F}_{\frac{1}{2}} = -\frac{Z^+}{\sqrt{\nu_r\left(\nu_r - 2x_n\right)}} \qquad \mathcal{F}_{\frac{3}{2}} = -\frac{Z^-}{\sigma_{k_\parallel}\nu_r} \tag{4.6}$$

where $\sigma_{k_\parallel} = \mathrm{Sign}(k_\parallel)$, and

$$Z^{\pm} = \frac{1}{2}\left\{Z\left[-\sqrt{\nu_r\left(\nu_r - 2x_n\right)} - \sigma_{k_\parallel}\nu_r\right] \pm Z\left[-\sqrt{\nu_r\left(\nu_r - 2x_n\right)} + \sigma_{k_\parallel}\nu_r\right]\right\} \tag{4.7}$$

Together with the recursion relation (4.5), these equations provide the simplest way of evaluating the relativistic functions $\mathcal{F}_{q+1/2}$ of any order. Unfortunately, Eq. (4.5) is numerically unstable not only for large x_n, but also for small ν_r; in the first case one can use the asymptotic series (A.30), in the latter the Taylor expansion (A.19) around the perpendicular limit.

Equations (4.6) and (4.7) also allow to obtain the analytic continuation of these function for $\mathrm{Im}\,(\omega) \leq 0$, since the r.h. side has a well–know analytic behaviour in the complex plane. In particular, if $\nu_r - 2x_n \geq 0$, the argument of the Z functions is real; on the other hand the Landau prescription implies

$$\sqrt{\nu_r\left(\nu_r - 2x_n\right)} \rightarrow -i\sqrt{\nu_r\left(2x_n - \nu_r\right)} \qquad \text{if } \nu_r/2 - x_n < 0 \tag{4.8}$$

so that the argument of the second Z function is minus the complex conjugate of the first. From the symmetry properties of the Z function in the complex plane, it is easily seen that in this case $\mathcal{F}_{1/2}$ and $\mathcal{F}_{3/2}$ are purely real. If $\nu_r - 2x_n < 0$, therefore, the elements of the dielectric tensor are strictly real even for oblique propagation.

Particularly interesting is the limit of perpendicular propagation. If $n_\parallel \rightarrow 0$, from Eq. (4.1) we obtain

$$\mathcal{F}_{q+\frac{1}{2}}\left(n_\parallel = 0\right) = F_{q+\frac{1}{2}}\left(\mu_r \varpi_n\right) = -i\int_0^\infty dt\, \frac{e^{i\mu_r \varpi_n t}}{(1 - it)^q} \tag{4.9}$$

Thus for perpendicular propagation the relativistic Plasma Dispersion Function reduces to the family of Dnestrovsky functions F_p with half–integer indexes. They satisfy the simpler recurrence relation

$$F_{q+\frac{3}{2}}(z) = \frac{1}{q+\frac{1}{2}}\left(1 - z\,F_{q+\frac{1}{2}}(z)\right) \tag{4.10}$$

which is a particular case of Eq.(4.5). The lowest of these function required to evaluate the dielectric tensor is $F_{3/2}$, which, as shown in the Appendix, is related to the the Plasma dispersion function Z as follows:

$$F_{3/2}(z) = \begin{cases} 2\{1-\sqrt{\pi z}\,e^{z}\,\mathrm{Erfc}\,(\sqrt{z})\} & \text{if } |\mathrm{Arg}\,(z)| < \pi \\ 2\{1-\sqrt{-x}\,Z(-\sqrt{-x})\} = -Z'(-\sqrt{-x}) & \text{if } x < 0 \end{cases} \tag{4.11}$$

In particular, for real negative argument,

$$\mathrm{Im}\,F_{3/2}(-|x|) = -2\sqrt{\pi|x|}\,e^{-|x|} \tag{4.12}$$

As a consequence, for frequencies lower than harmonics of the cyclotron frequency, the antihermitean part of the dielectric tensor is non vanishing even for perpendicular propagation, in obvious contrast to the classical behaviour.

On the other hand, for $|x_n| \gg \nu_r/2$ and $\nu_r \gtrsim 1$, Eqs. (4.5) and (4.7) can be used to show that

$$\mu_r\mathcal{F}_q\,(2\nu_r x_n, \nu_r) \simeq -x_o Z(x_n) \tag{4.13}$$

for all q. When this approximation is used in the dielectric tensor, and only the lowest order terms in μ_r^{-1} are retained, one recovers the classic dielectric tensor. Moreover, for $\mu_r|\varpi_n| \gg 1$,

$$\mu_r\mathcal{F}_q\,(2\nu_r x_n, \nu_r) \simeq \frac{1}{\varpi_n - n_\parallel^2/2} \tag{4.14}$$

which is again independent on q (the r.h. side is the first term of the asymptotic series (A.30)). When this condition is satisfied, one recovers the classical behaviour in the limit of negligible parallel dispersion ($|x_n| \gg 1$ for all n), and this holds also for perpendicular propagation. The domain in which relativistic effects are dominant can thus be identified as follows:

$$\left|\frac{\omega - n\Omega_c}{\omega}\right| \lesssim \begin{cases} n_\parallel^2 & \text{if } n_\parallel^2 \gtrsim \mu_r^{-1} \\ \mu_r^{-1} & \text{if } n_\parallel^2 \ll \mu_r^{-1} \end{cases} \tag{4.15}$$

A qualitative estimate of the value of $n_\parallel$ above which relativistic effects become unimportant can be obtained as follows. Because of the relativistic change of the electron mass, we have seen that cyclotron absorption on the high frequency side of the n-th harmonics ceases at $2x_n = \nu_r$. If we ask classic absorption at this Doppler shift to be anyhow smaller than a fraction $\epsilon \ll 1$ of its peak value at exact resonance, we obtain the condition

$$|n_\parallel| > -\frac{8}{\mu_r}\ln(\epsilon) \tag{4.16}$$

The relative large factor $-8\ln(\epsilon)$ shows that the transition to the classical limit is relatively slow, and confirms the importance of relativistic effects on electron cyclotron absorption.

The first four relativistic Plasma Dispersion functions for an electron plasma with $T_e = 2$ keV ($\mu_r = 255.5$) are shown in fig. 6 for $n_\parallel^2 = 0$, and Fig. 7 for $n_\parallel^2 = 0.1$, 0.2 and 0.5 (the corresponding values of ν_r are $\nu_r = 1.13$, 2.26, and 5.65; the latter is already in the domain of validity of the classical approximation). Figure 8 shows the first few Dnestrovsky functions of integer order.

5 – The Finite Larmor radius approximation. The weakly relativistic approximation developed in section 3 automatically expresses the elements of the dielectric tensor as a series in the electron Larmor radius. In terms of the integrals defined by Eqs. (3.24) all terms of order zero and one in the Larmor radius are

$$\epsilon_{xx} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2\,K_2(\mu_r)}\left\{\frac{1}{2}\left(\mathcal{J}_1^{1,0} + \mathcal{J}_{-1}^{1,0}\right) - \frac{\lambda_e}{2}\left(\mathcal{J}_1^{2,0} + \mathcal{J}_{-1}^{2,0}\right)\right.$$
$$\left. + \frac{\lambda_e}{2}\left(\mathcal{J}_2^{2,0} + \mathcal{J}_{-2}^{2,0}\right)\right\}$$

$$\epsilon_{xy} = -\epsilon_{yx} = -i\frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2\,K_2(\mu_r)}\left\{\frac{1}{2}\left(\mathcal{J}_1^{1,0} - \mathcal{J}_{-1}^{1,0}\right) - \lambda_e\left(\mathcal{J}_1^{2,0} + \mathcal{J}_{-1}^{2,0}\right)\right.$$
$$\left. + \frac{\lambda_e}{2}\left(\mathcal{J}_2^{2,0} - \mathcal{J}_{-2}^{2,0}\right)\right\}$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2\,K_2(\mu_r)}\frac{\lambda_e^{1/2}}{2}\left(\mathcal{J}_1^{1,1} - \mathcal{J}_{-1}^{1,1}\right)$$

$$\epsilon_{yy} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2\,K_2(\mu_r)}\left\{2\lambda_e\,\mathcal{J}_0^{2,0} + \frac{1}{2}\left(\mathcal{J}_1^{1,0} + \mathcal{J}_{-1}^{1,0}\right) - \frac{3\lambda_e}{2}\left(\mathcal{J}_1^{2,0} + \mathcal{J}_{-1}^{2,0}\right)\right.$$
$$\left. + \frac{\lambda_e}{2}\left(\mathcal{J}_2^{2,0} + \mathcal{J}_{-2}^{2,0}\right)\right\}$$

$$\epsilon_{yz} = -\epsilon_{zy} = -i\frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2K_2(\mu_r)}\frac{\lambda_e^{1/2}}{2}\Big\{-2\mathcal{J}_0^{1,1} + \Big(\mathcal{J}_1^{1,1} + \mathcal{J}_{-1}^{1,1}\Big)\Big\}$$

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\mu_r}{2K_2(\mu_r)}\Big\{\mathcal{J}_0^{0,2} - \lambda_e\mathcal{J}_0^{1,2} + \frac{\lambda_e}{2}\Big(\mathcal{J}_1^{1,2} + \mathcal{J}_{-1}^{1,2}\Big)\Big\} \tag{5.1}$$

(here, as in Eqs. (3.24), the superscripts give the powers of $p_\perp$ and $p_\parallel$ in the integrand, respectively, while the subscript denotes the cyclotron harmonics).

Using the results of section 3 and of the appendix, the integrals $\mathcal{J}_n^{s,r}$ can be expressed in terms of the relativistic Plasma Dispersion Functions $\mathcal{F}_q(n)$ and its derivatives. Strictly speaking, for waves in the electron cyclotron range (electron Bernstein waves excepted) λ_e and μ_r^{-1} are of comparable magnitude. Thus in an expansion to first order in λ_e we should for consistency also keep all contributions of order μ_r^{-1} at least in the the zero Larmor radius terms. As discussed at the end of section 3, moreover, for oblique propagation the coefficient of each power of λ_e is itself a series in $n_\parallel^2$. To identify correctly the order of each term in the μ_r^{-1} and $n_\parallel^2$ expansions, we recall our remark that $\mathcal{F}_{q+1/2}^r = O(\mu_r^{-(r+1)})$. Thus although ν_r^2 can become of order $\mu_r \gg 1$ within the range in which relativistic effects are important, the terms proportional to ν_r^2 which occur in $\mathcal{J}_n^{s,2}$ are nevertheless small corrections of order μ_r^{-1} or higher. The integrals required in the weakly relativistic FLR dielectric tensor are then:

$$\frac{\mu_r}{2K_2(\mu_r)}\mathcal{J}_n^{1,0} \simeq \mu_r\Big(\mathcal{F}_{5/2}(n) + \frac{3}{2\mu_r}\mathcal{F}_{7/2}(n)\Big)$$

$$\frac{\mu_r}{2K_2(\mu_r)}\mathcal{J}_n^{2,0} \simeq \mu_r\Big(\mathcal{F}_{7/2}(n) + \frac{4}{\mu_r}\mathcal{F}_{9/2}(n)\Big)$$

$$\frac{\mu_r}{2K_2(\mu_r)}\mathcal{J}_n^{1,1} \simeq \nu_r\,\mu_r\Big(\mathcal{F}_{7/2}^1(n) + \frac{5}{2\mu_r}\mathcal{F}_{9/2}^1(n)\Big) \tag{5.2}$$

$$\frac{\mu_r}{2K_2(\mu_r)}\mathcal{J}_n^{0,2} = \mu_r\Big(\mathcal{F}_{5/2}(n) + \nu_r^2\,\mathcal{F}_{7/2}^2(n)\Big)$$

$$\frac{\mu_r}{2K_2(\mu_r)}\mathcal{J}_n^{1,2} \simeq \mu_r\Big(\mathcal{F}_{7/2}(n) + \frac{5}{2\mu_r}\mathcal{F}_{9/2}(n) + \nu_r^2\mathcal{F}_{9/2}^2(n)\Big)$$

With these results, and taking the classic limit for non-resonant terms, we finally obtain the relativistic dielectric tensor in the form

$$\begin{array}{lll}
\epsilon_{xx} \simeq S - \sigma n_\perp^2 & \epsilon_{xy} \simeq -i(D - \delta n_\perp^2) & \epsilon_{xz} \simeq n_\parallel n_\perp \eta \\
\epsilon_{yx} \simeq +i(D - \delta n_\perp^2) & \epsilon_{yy} \simeq S - (\sigma + 2\tau)n_\perp^2 & \epsilon_{yz} \simeq +i n_\parallel n_\perp \xi \\
\epsilon_{zx} \simeq n_\parallel n_\perp \eta & \epsilon_{zy} \simeq -i n_\parallel n_\perp \xi & \epsilon_{zz} \simeq P - \pi n_\perp^2
\end{array} \tag{5.3}$$

where

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2}\left\{\frac{\mu_r}{2}\left[\mathcal{F}_{5/2}(1) + \frac{3}{2\mu_r}\mathcal{F}_{7/2}(1)\right] + \frac{\omega}{\omega + \Omega_{ce}}\right\}$$

$$D = -\frac{\omega_{pe}^2}{\omega^2}\left\{\frac{\mu_r}{2}\left[\mathcal{F}_{5/2}(1) + \frac{3}{2\mu_r}\mathcal{F}_{7/2}(1)\right] - \frac{\omega}{\omega + \Omega_{ce}}\right\} \tag{5.4}$$

$$P = 1 - \frac{\omega_{pe}^2}{\omega^2}\mu_r\left[\mathcal{F}_{5/2}(0) + \nu_r^2\mathcal{F}_{7/2}^2(0)\right]$$

are the zero Larmor radius contributions, and

$$\sigma = \frac{1}{4}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\frac{v_{the}^2}{c^2}\left\{-\left(\mu_r\mathcal{F}_{7/2}(1) + \frac{\omega}{\omega+\Omega_{ce}}\right) + \mu_r\mathcal{F}_{7/2}(2) + \frac{\omega}{\omega + 2\Omega_{ce}}\right\}$$

$$\delta = \frac{1}{4}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\frac{v_{the}^2}{c^2}\left\{-2\left(\mu_r\mathcal{F}_{7/2}(1) + \frac{\omega}{\omega+\Omega_{ce}}\right) + \mu_r\mathcal{F}_{7/2}(2) - \frac{\omega}{\omega + 2\Omega_{ce}}\right\}$$

$$\tau = \frac{1}{2}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\frac{v_{the}^2}{c^2}\left\{\mu_r\mathcal{F}_{7/2}(0) - \frac{1}{2}\left(\mu_r\mathcal{F}_{7/2}(1) + \frac{\omega}{\omega+\Omega_{ce}}\right)\right\}$$

$$\eta = \frac{1}{4}\frac{\omega_{pe}^2}{\omega\Omega_{ce}}\frac{v_{the}^2}{c^2}\left\{\mu_r^2\mathcal{F}_{7/2}^1(1) - \left(\frac{\omega}{\omega+\Omega_{ce}}\right)^2\right\} \tag{5.5}$$

$$\xi = \frac{1}{4}\frac{\omega_{pe}^2}{\omega\Omega_{ce}}\frac{v_{the}^2}{c^2}\left\{2\mu_r^2\mathcal{F}_{7/2}^1(0) + \mu_r^2\mathcal{F}_{7/2}^1(1) + \left(\frac{\omega}{\omega+\Omega_{ce}}\right)^2\right\}$$

$$\pi = \frac{1}{4}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\frac{v_{the}^2}{c^2}\left\{-2\mu_r\mathcal{F}_{7/2}(0) + \mu_r\mathcal{F}_{7/2}(1) + \frac{\omega}{\omega+\Omega_{ce}}\right\}$$

the finite Larmor radius (FLR) contributions. It is interesting to note that the expression for P, in contrast to all other elements, is exact for all values of $n_\parallel^2$.

To the same approximation, the FLR dispersion relation is a cubic equation for $n_\perp^2$:

$$-\sigma n_\perp^6 + An_\perp^4 - Bn_\perp^2 + C = 0 \tag{5.6}$$

In the coefficients if this equation

$$A = S + \alpha$$

$$B = RL + PS - n_\parallel^2(P+S) + \beta \tag{5.27}$$

$$C = P(n_\parallel^2 - R)(n_\parallel^2 - L)$$

the zero order terms are formally the same as in the cold plasma approximation, while the FLR conrrections are given by

$$\alpha = P\sigma - (n_\parallel^2 - S)(2\sigma + 2\tau + \pi) - 2D\delta + n_\parallel^2(\sigma + 2\tau + 2\eta)$$

$$\beta = -2P\left[(n_\parallel^2 - S)(\sigma + \tau) + D\delta\right] + \pi\left[(n_\parallel^2 - S)^2 - D^2\right] - 2n_\parallel^2\left[\eta(n_\parallel^2 - S) + D\xi\right] \tag{5.8}$$

This approximation is sufficient for the ordinary and extraordinary waves, for electrostatic waves of the Langmuir type, and for the lowest order electron Bernstein wave near $\omega = 2\Omega_{ce}$.

6 – Fundamental electron cyclotron damping. In a Maxwellian plasma in the classic limit electron cyclotron damping can be expected in a frequency range

$$\Delta\omega \simeq q|k_\parallel|\frac{v_{the}}{c}\Omega_{ce} \qquad \Delta f = q\frac{1.76\,10^{11}}{255}n_\parallel T_e B_o \text{ hz} \tag{6.1}$$

symmetric around the cyclotron resonance, where T_e is in keV, B_o in Tesla, and q is a number somewhat larger than unity (typically $q \simeq 3$). The most remarkable features of the relativistic "cyclotron absorption lines", as already mentioned, are that they do not disappear in the limit of perpendicular propagation, and that they are asymmetric with respect to the exact cyclotron resonance. In this domain, on the other hand, relativistic corrections to the real part of the index are relatively small, and do not alter the main features predicted by the cold-plasma dispersion relation.

In fig. 9, in which the imaginary part of $n_\perp^2$ of both the extraordinary and the ordinary wave at perpendicular propagation are plotted versus ω/Ω_{ce} near the fundamental cylotron resonance, assuming $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$ and varying the electron temperature. In the temperature range typical of fusion plasmas, damping of the ordinary wave is appreciably stronger than that of the extraordinary wave, although the latter increases much faster with increasing temperature. This at first sight surprising result is a polarization effect. Cyclotron absorption is due to the acceleration of charged particles by the perpendicular component of the wave electric field gyrating in the same direction as the particle gyromotion. In the cold limit the electric field of the ordinary wave has no component perpendicular to the static magnetic field; it is therefore clear that cyclotron damping of this wave is a finite temperature effect. The electric field of extraordinary wave, by contrast, is purely perpendicular. In the cold limit, however, the component gyrating with the electrons (E_-) vanishes exactly at $\omega = \Omega_{ce}$, since

$$\frac{E_-}{E_+} = \frac{L}{R} \tag{6.2}$$

and R diverges in the cold limit at $\omega = \Omega_{ce}$. It follows that fundamental cyclotron damping of the extraordinary wave is a finite temperature effect as well. We will see below that in the classical limit, when finite temperature effects are taken into account, $\mathrm{Im}(n_\perp^2)$ vanishes for $n_\parallel \to 0$ in the case of the extraordinary wave, while it behaves as a δ-function in the case of the ordinary wave (cfr. fig. 11). Even when relativistic effects are taken into account, damping at perpendicular propagation turns out to be weaker for the former than for latter.

Figures 10 a) and b) show $\mathrm{Im}(n_\perp^2)$ versus ω/Ω_{ce} for different (real) angles of propagation, again for $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$, and for $\beta_e = 0.002$ ($T_e = 1.02$ keV), and $\beta_e = 0.01$ ($T_e = 5.1$ keV), respectively. While absorption of the extraordinary wave increases steadily with increasing $n_\parallel$, the peak value of $\mathrm{Im}(n_\perp)$ for the ordinary rapidly decreases, while the frequency range in which absorption occurs widens in such a way that the area under the curve remains roughly constant. Again, this can be understood qualitatively in purely classical terms. First, we note that if in Eq. (6.2) we substitute for R its hot plasma expression including finite temperature effects in the Plasma Dispersion function, and take into account that $-x_o\, Z(x_1) = O(\omega/k_\parallel v_{the})$ when $|x_1| \lesssim 1$, the ratio E_-/E_+ for the extraordinary wave does not vanish at exact resonance; instead, its minimum value is found to be of the order of $n_\parallel v_{the}/c$. This explains why cyclotron damping of the X wave increases roughly as $n_\parallel^2\, T_e$ for small angles of propagation (the fact that it does not vanish exactly even for $n_\parallel^2 = 0$, of course, is a purely relativistic effect). The behaviour of the ordinary wave is more difficult to explain with elementary considerations, because it turns out that in the range of strong damping around the cyclotron frequency the FLR dispersion relation (5.6) is inadequate in the classical limit. This is due to the fact that the out of diagonal FLR elements ξ and η are particularly large near Ω_{ce}, so that in the classical limit the profile of electron cyclotron damping for the ordinary mode is the result of complicated internal cancellations between different terms. It is therefore necessary to turn to the complete classical hot plasma dispersion relation, which predicts an absorption peak centered on the resonance: as anticipated above, the imaginary part of $n_\perp$ behaves classically as a δ-function in the limit $n_\parallel \to 0$, the area under the curve tending to a finite constant. In the relativistic case such complications do not arise; if Eq. (5.6) is perturbed around the cold-plasma solution $n_\perp^2 = P$, to lowest order in both $n_\parallel^2 \ll 1$ and λ_e one obtains

$$n_\perp^2)_O \simeq P - \pi \tag{6.3}$$

which is already sufficiently accurate for small $n_\parallel^2$. Relativistic effects are responsible for the finite width (and height) of the absorption line in the limit of perpendicular propagation. For $\omega > \Omega_{ce}$ the behaviour of both waves at larger propagation angles is also in agreement with the fact that, as $n_\parallel^2$ increases, the O mode goes over into the L-whistler, which is purely left-handedly polarized, hence unaffected by electron cyclotron damping; the X-mode, on the other hand, belongs to the same branch as the heavily damped R-whistler, although in this frequency range it is separated from it by the resonance cone (divergence of the cold plasma refractive index in the direction $\tan\vartheta = -P/S$, approaching which the oscillations become essentially electrostatic).

If the dispersion relation is solved for $n_\perp$ while $n_\parallel$ is kept fixed, as it is natural when considering bounded plasmas, the resonance cone is unaccessible; if the frequency is increased sufficiently above the cyclotron frequency, on the other hand, one finally encounters its perpendicular limit, i.e. the upper hybrid resonance, or, more precisely in a finite temperature plasma, the confluence with the Upper Hybrid hot plasma wave which occur at a slightly lower frequency. Because of the large thermal speed of the electrons due to their small mass, at finite angle angles of propagation Doppler broadening is sufficient to extend electron cyclotron damping of the extraordinary wave up to the upper hybrid resonance already at moderate temperatures. This is clearly visible in fig. 10 b: at $T_e = 5$ keV, for $n_\parallel^2 \gtrsim 0.5$ the absorption line merges with the evanescence domain (frequency gap) of the extrordinary wave beyond this confluence.

Finally, the importance of relativistic effects on electron cyclotron damping can be easily appreciated from these figures. Even at $T_e = 1$ keV the asymmetry of the absorption lines is clearly visible at least up to $n_\parallel \simeq 0.2$; at $T_e = 5$ keV the classic approximation is inaccurate even at $n_\parallel \simeq 0.6$, particularly for the extraordinary wave. This is more explicitly illustrated in fig. 11, which shows the relativistic solutions and the classic approximations together for various values of $n_\parallel$. Although quantitatively inaccurate, however, the classic dispersion relation reproduces all important features of electron cyclotron damping in a qualitatively correct way, except for the relativistic downshift of the absorption frequencies and the finite width of the absorption lines at perpendicular propagation. This is clearly seen also from fig. 12, in which $\mathrm{Im}(n_\perp^2)$ is plotted versus $n_\parallel$ at two different frequencies, one just below and one just above the electron cyclotron resonance.

7 – First harmonic electron cyclotron damping. The occurrence of damping near the first harmonic of the cyclotron frequency is manifestly a finite Larmor radius effect, since all contributions to the dielectric tensor resonant at $\omega = 2\Omega_{ce}$ vanish in the cold limit. In this domain relativistic corrections to the index of the ordinary wave appear only in terms of higher order in the Larmor radius and in μ_r^{-1}; harmonic cyclotron damping of this wave is very weak for all angles of propagation. We will therefore concentrate our attention to the extrordinary mode.

For perpendicular propagation, the classical hot plasma dispersion relation predicts a confluence between the X wave and the lowest electron Bernstein wave at $\omega = 2\Omega_{ce}$; this confluence is of the 'mode transformation' type, i.e. without frequency gap, since both waves are forward waves (formally, the Bernstein wave has also a backward branch with much larger perpendicular index, which makes a second confluence with the forward branch at a frequency slightly above the harmonic; beyond this confluence the BW root is evanescent. At the densities typical of tokamak plasmas, however, the backward branch has $k_\perp \lambda_D > 1$ throughout its existence domain, and therefore it is not physically meaningful). The existence of damping just below $\omega = 2\Omega_{ce}$ due to relativistic effects, on the other hand, strongly affects the dispersion curves of the extraordinary and Bernstein waves even at perpendicular propagation. The smooth mode transformation predicted by the classical dispersion relation across the harmonic disappears (fig. 13): if the two roots are followed in the complex plane while ω/Ω_{ce} is varied, they miss each other, since at the point where the real part of the two index are equal their imaginary part is non-zero and very different for the two waves. As a consequence, the extraordinary wave mantains its identity while the frequency crosses the resonance, showing instead a domain of dispersion and absorption. Its dispersion relation in this range, for nearly perpendicular propagation, can be approximately written

$$n_\perp^2)_X \simeq \frac{RL}{S}\left(1 + \frac{L}{S}\rho\right) \qquad (n_\parallel^2 \ll 1) \tag{7.1}$$

where R, L and S can be taken in the cold plasma approximation, while

$$\rho \simeq \frac{1}{4}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\frac{v_{the}^2}{c^2}\,\mu_r\,\mathcal{F}_{7/2}(2) \tag{7.2}$$

is the resonant term of σ and δ in Eq. (5.5). The confluence between the two waves occurs again at larger densities, $\omega_{pe}^2 \gtrsim \Omega_{ce}^2$ [12].

At the densities considered here the Bernstein wave is also heavily absorbed at frequencies below $2\Omega_{ce}$; when damping disappears above the resonance this wave is already propagative with a very short wavelength (note, however, that the relativistic dispersion curves of fig. 13 b for the Bernstein wave are not very accurate, since the finite Larmor radius expansion has always at best marginal validity for this root). The situation is the same for all temperatures: as T_e increases the absorption line simply expands, without changing shape or height (fig. 14; this implies, however, that the optical thickness of the absorption line increases linearly with temperature). Thus at low temperatures the intensity of first harmonic damping of the extraordinary wave at perpendicular propagation is appreciably stronger than near the fundamental; at the density of the figures, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$, the latter becomes larger only at $\beta_e \gtrsim 0.02$.

At oblique propagation (fig. 15) the peak value of $\mathrm{Im}(n_\perp)$ for the extraordinary wave rapidly decreases, while the frequency range in which absorption occurs widens in such a way that the area under the curve remains roulghy constant. Again, this occurs also in the classical limit: the imaginary part of the index behaves as a δ-function when $n_\parallel \to 0$, as can be easily seen from Eq. (8) using the classical approximation for ρ.

Appendix – The Relativistic Plasma Dispersion Functions.

In this Appendix we collect a number of formulas relevant to the Relativistic Plasma Dispersion Function and related functions. A more extensive list, with appropriate references, can be found in [6]–[7].

A.1 – Dnestrovsky functions. The functions

$$F_q(z) = -i\int_0^\infty dt\,\frac{e^{izt}}{(1-it)^q} = \frac{1}{\Gamma(q)}\int_0^\infty \frac{x^{q-1}e^{-x}}{x+z}\,dx \tag{A.1}$$

are known as (generalised) Dnestrovsky functions. They satisfy the recurrence relations

$$F_{q+1}(z) = \frac{1}{q}\left(1 - z\,F_q(z)\right) \tag{A.2}$$

which is easily obtained from (1) integrating by parts for any $q \neq 0$. It holds by continuity also for $q = 0$ if we put

$$F_o(z) = \frac{1}{z} \tag{A.3}$$

which is also consistent with (1). It is also easily checked that

$$\frac{dF_{q+1}(z)}{dz} = F_{q+1}(z) - F_q(z) \tag{A.4}$$

and more generally

$$\frac{d^n}{dz^n}\left(e^{-z}F_q(z)\right) = (-1)^n e^{-z}F_{q-n}(z) \tag{A.5}$$

A.2 – Dnestrovsky functions of integer order. The Dnestrovsky functions of integer order appear as integrand in the integral representation of the Relativistic Plasma Dispersion Function for oblique propagation, as shown below. From the definition (1) with $q = 1$ it follows immediately that

$$F_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt = e^z\,E_1(z) \qquad |\mathrm{Arg}\,(z)| < \pi \tag{A.6}$$

In particular, therefore, for real argument

$$F_1(x) = \begin{cases} e^x\,E_1(x) & \text{if } x \geq 0 \\ -\,e^x\left(Ei\,(-x) + i\pi\right) & \text{if } x < 0 \end{cases} \tag{A.7}$$

where $Ei\,(x)$ is the exponential integral

$$Ei\,(x) = -P\int_{-x}^{\infty} \frac{e^{-t}}{t}\,dt \qquad x > 0 \qquad (A.8)$$

The lowest integer order Dnestrovsky function which appears in the dielectric tensor written using the integral representation(A.17) below is

$$F_2(x) = \begin{cases} 1 + x\,e^x\,E_1(x) & \text{if } x \geq 0 \\ 1 - x\,e^x\,(Ei\,(-x) + i\pi) & \text{if } x < 0 \end{cases} \qquad (A.9)$$

Note that $-(Ei\,(-x) + i\pi)$ is the limit of $E_1(z)$ as $\mathrm{Arg}\,(z) \to \pi$ from above: this is the Landau prescription in the relativistic case.

The exponential integrals have the well–known power expansions

$$Ei\,(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{1}{n}\frac{x^n}{n!} \qquad x > 0$$

$$E_1(z) = -(\gamma + \ln x) + \sum_{n=1}^{\infty} \frac{1}{n}\frac{(-z)^n}{n!} \qquad \mathrm{Arg}\,(z) < \pi \qquad (A.10)$$

where $\gamma = 0.57722\ldots$ is Euler's constant. These expansions are always convergent, but are useful for practical evaluations when the argument is small or of order unity. This is however also the domain in which relatvistic effects are most important. According to this expansion, $F_2(x)$ is finite at the origin ($F_2(0) = 1$), but has a logarithmic singulariry in the first derivative ($F_2'(0) = -\infty$). More generally, $F_n(x)$ has a logarithmic singularity in the $(n-1)$–th derivative at $x = 0$.

For large argument, the recursion formula (3) must be used with care because of internal cancellations, as it is easily apparent from the asymptotic expansion of E_1,

$$E_1(z) \sim \frac{e^z}{z}\left\{1 - \frac{1}{z} + \frac{1\cdot 2}{z^2} - \frac{1\cdot 2\cdot 3}{z^3} + \ldots\right\} \qquad (A.11)$$

Thus the lowest order function needed in the dielectric tensor, namely F_2, has the asymptotic expansion

$$F_2(z) \sim \frac{1}{z} - \frac{1\cdot 2}{z^2} + \frac{1\cdot 2\cdot 3}{z^3} + \ldots \qquad (A.12)$$

from which the lowest order term has dropped out. Similarly, an elementary exercise shows that the asymptotic expansion of F_n begins always with with z^{-1}:

$$F_n(z) \sim \frac{1}{z} - \frac{n}{z^2} + \frac{n(n+1)}{z^3} - \ldots - (-1)^k \frac{n(n+1)\ldots(n+k-2)}{z^k} + \ldots \qquad (A.13)$$

These expansion are useful to investigate the transition to the classical limit.

A.2 – The relativistic Plasma Dispersion Functions. The relativistic Plasma Dispersion Functions, or Skarofsky functions, are defined by Eqs. (3.26),

$$\mathcal{F}_q^r(\mu_r \varpi_n, \mu_r n_\parallel^2/2) = -i\, e^{-\frac{1}{2}\mu_r n_\parallel^2} \int_0^\infty dt\, (it)^r \frac{e^{\mu_r \left[i\left(\varpi_n - \frac{1}{2}n_\parallel^2\right)t + \frac{1}{2}\frac{n_\parallel^2}{(1-it)} \right]}}{(1-it)^q} \tag{A.14}$$

with $\mathcal{F}_q = \mathcal{F}_q^0$. It is easy to recognize that

$$\mathcal{F}_q^r(z, \nu_r^2) = \frac{\partial^r}{\partial z^r} \mathcal{F}_q(z, \nu_r^2) \tag{A.15}$$

Moreover, using the binomial theorem as in Eq. (3.29) we get

$$\mathcal{F}_q^r(z, \nu_r^2) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \mathcal{F}_{q-j}(z, \nu_r^2) \tag{A.16}$$

We recall that here $z = \mu_r \varpi_n = 2\nu_r x_n$.

Using the intermediate results of section 3, one can alternatively express $\mathcal{F}_q$ as a convolution of the (classical) Maxwellian distribution with the generalized Dnestrovsky function $F_q(z)$ of integer index,

$$\mathcal{F}_{q+\frac{1}{2}}(\varpi_n, n_\parallel) = \sqrt{\frac{\mu_r}{2\pi}} \int_{-\infty}^{+\infty} d\bar{p}_\parallel\, e^{-\mu_r \bar{p}_\parallel^2/2}\, F_q(\mu_r \beta_n) \tag{A.17}$$

where

$$\beta_n = \frac{\omega - n\Omega_c}{\omega} - n_\parallel \bar{p}_\parallel + \frac{1}{2}\bar{p}_\parallel^2 = \frac{1}{2}\left(\bar{p}_\parallel^2 - n_\parallel^2\right)^2 - \alpha_n$$
$$\alpha_n = \frac{1}{2}n_\parallel^2 - \varpi_n = \frac{1}{\mu_r}(\nu_r^2 - 2\,x_n) \tag{A.18}$$

To prove the equivalence between (A.17) and (A.14), exchange the order of the integrations in the latter, and perform the $\bar{p}_\parallel$–integration as in Eq. (3.25). The actual range contributing to the integration over $\bar{p}_\parallel$ is restricted to a few times the thermal momentum; together with the difference $\omega - n\Omega_c$ and the value of $n_\parallel$, this defines the relevant range of values of the argument $\mu_r \beta_n$ of the function F_q. Note also that the r.h. side of (A.17) becomes negative for some t if and only if $\alpha_n > 0$; if this condition is not satisfied, $\mathcal{F}_{q+\frac{1}{2}}$ is strictly real. From its definition it is also possible to express $\mathcal{F}_q$ as a series over Dnestrovsky functions:

$$\mathcal{F}_q(z, \nu_r^2) = e^{-\nu_r^2} \sum_{k=0}^{\infty} \frac{\nu_r^{2k}}{k!} F_{q+k}(z) \tag{A.19}$$

which can be useful when investigating quasi-perpendicular propagation, $\nu_r^2 \ll 1$. The series on the r.h. side of Eq. (A.19) converges if $\nu_r^2 < |z|$.

A recursion relation for these functions can be obtained starting from the identity

$$1 = -e^{z-2a} \int_1^\infty d(1-it) \frac{\exp\left[(a-z)(1-it) + \frac{a}{1-it}\right]}{(1-it)^q} \left(a - z - \frac{q}{1-it} - \frac{a}{(1-it)^2}\right) \tag{A.20}$$

which is easily proved by noting that the integrand is a total differential. Comparing the r.h. side with $a = \mu_r n_\parallel^2/2$ and $z = \mu_r \varpi_n$ with the integrand of Eq. (15) we get

$$\nu_r^2 \mathcal{F}_{q+2}(\mu_r \varpi_n, \nu_r^2) = 1 + \left(\nu_r^2 - \mu_r \varpi_n\right) \mathcal{F}_q(\mu_r \varpi_n, \nu_r^2) - q \mathcal{F}_{q+1}(\mu_r \varpi_n, \nu_r^2) \tag{A.21}$$

Apart for the notations, this coincides with Eq. (4.5).

To express $\mathcal{F}_{1/2}$ in terms of the ordinary plasma dispersion function Z, considering first the case $\alpha_n > 0$, put $\bar{p}_\parallel = n_\parallel + \sqrt{2\alpha_n}\, t$ in Eq.(A.17): this gives

$$\mathcal{F}_{q+\frac{1}{2}} = \sqrt{\frac{\mu_r \alpha_n}{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{\mu_r}{2}(n_\parallel + \sqrt{2\alpha_n}\, t)} F_q\left(\mu_r \alpha_n (t^2 - 1)\right) \tag{A.22}$$

But $F_o(z) = 1/z$; hence

$$\mathcal{F}_{\frac{1}{2}} = \frac{1}{\sqrt{\pi \mu_r \alpha_n}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{\mu_r}{2}(n_\parallel + \sqrt{2\alpha_n}\, t)^2}}{t^2 - 1} \tag{A.23}$$

Decomposing the fraction and changing again the variable of integration, we easily obtain

$$\mathcal{F}_{\frac{1}{2}}(\mu_r \varpi_n, n_\parallel) = -\frac{1}{2\phi} \{Z(\psi - \phi) + Z(-\psi - \phi)\} \tag{A.24}$$

where

$$\psi = \sqrt{\mu_r/2}\, n_\parallel \qquad \phi = \sqrt{\mu_r \alpha_n} = \sqrt{\mu_r (n_\parallel^2/2 - \varpi_n)} \tag{A.25}$$

Next, integrating over t the identity

$$\frac{d}{dt}\left\{e^{-(\psi+\phi t)^2} F_{q+1}(\phi^2(t^2-1))\right\} = - 2\phi \left\{\psi F_{q+1}(\phi^2(t^2-1)) - \phi t F_q(\phi^2(t^2-1))\right\} \tag{A.26}$$

where in the r.h. side we have used Eq. (A.4) to eliminate the derivative of F_{q+1}, we obtain

$$\mathcal{F}_{q+\frac{3}{2}} = -\frac{\alpha_n}{n_\parallel}\sqrt{\frac{2\mu_r}{\pi}}\int_{-\infty}^{+\infty} e^{-\frac{\mu_r}{2}(n_\parallel+\sqrt{2\alpha_n}\,t)}\, t\, F_q\left(\mu_r\alpha_n(t^2-1)\right) \tag{A.27}$$

i.e. for $q=0$

$$\mathcal{F}_{\frac{3}{2}} = -\frac{\alpha_n}{n_\parallel}\sqrt{\frac{2\mu_r}{\pi}}\int_{-\infty}^{+\infty} e^{-\frac{\mu_r}{2}(n_\parallel+\sqrt{2\alpha_n}\,t)^2}\frac{t}{t^2-1} \tag{A.28}$$

from which, proceeding as above, we finally obtain

$$\mathcal{F}_{\frac{3}{2}}(\varpi_n, n_\parallel) = -\frac{1}{2\psi}\left\{Z(\psi-\phi) - Z(-\psi-\phi)\right\} \tag{A.29}$$

From (A.24), (A.29), and the definition of ϕ in Eq. (A.25), it is apparent that the functions $\mathcal{F}_{q+1/2}$ have a branch point at $\varpi_n = n_\parallel^2/2$. If a cut in the complex plane is drawn along the negative real axis, the argument ϖ_n of $\mathcal{F}$ tends to the lower lid of the cut as $\mathrm{Im}\,(\omega) \to 0+$ according to the Landau prescription. Thus Eqs. (A.24) and (A.25) still hold if $\alpha_n < 0$, with the substitution $\sqrt{\alpha_n} \to -i\sqrt{-\alpha_n}$. Again, with a change in the notations, these equations give (4.6) and (4.7).

As mentioned in section 3, these identities together with the recurrence relation (21) offer the easiest way to evaluate the relativistic dispersion function numerically. The recurrence relation (21), however, is plagued by internal cancellations when the argument of the Z functions is large. In this situation, one can instead use the asymptotic expansion (which we cite without deriving it)

$$\mathcal{F}_q(\varpi_n, n_\parallel) = -\sum_{k=0}^{\infty}\frac{C_{q,k}}{\left[\mu_r\left(\frac{1}{2}n_\parallel^2 - \varpi_n\right)\right]^{k+1}} \tag{A.30}$$

with

$$\begin{aligned} C_{q,0} &= 1 \quad \text{for all } q \\ C_{q,k} &= (k+q-1)C_{q,k-1} + \frac{1}{2}\mu_r n_\parallel^2 C_{q+1,k-1} \end{aligned} \tag{A.31}$$

which is valid for $\arg(\varpi_n) \le 3\pi/2$.

From (30) we can deduce that when $|\varpi_n| \gg n_\parallel^2$ and $\mu_r\varpi_n \gg 1$

$$\mu_r\mathcal{F}_{q+1/2}(\nu_r x_n, \nu_r^2) \sim -x_o Z(x_n) \tag{A.32}$$

and

$$\begin{aligned} &\mu_r^2\left[\mathcal{F}_{q+1/2}(2\nu_r x_n, \nu_r^2) - \mathcal{F}_{q+3/2}(2\nu_r x_n, \nu_r^2)\right] \sim x_o^2 Z'(x_n) \\ &\mu_r^3\left[\mathcal{F}_{q+1/2}(2\nu_r x_n, \nu_r) - 2\mathcal{F}_{q+3/2}(2\nu_r x_n, \nu_r^2) + \mathcal{F}_{q+3/2}(2\nu_r x_n, \nu_r^2)\right] \\ &\qquad\qquad \sim -x_o^3 Z''(x_n) \end{aligned} \tag{A.33}$$

which are useful when investigating the transition to the classic limit. Remarkably, these relatins do not depend on the order q. The conditions under which they have been derived, however, are rather restrictive, since they imply that both the classic and the relativistic dispersion functions are asymptotic, i.e. that spatial dispersion is weak (for sufficiently large $n_\parallel^2$ the classic limit can of course be approached even in the domain of strong spatial dispersion, but Eqs. (32)–(33) are not sufficient to prove it).

A.4 – The limit of perpendicular propagation. For $n_\parallel^2 \to 0$ the relativistic Plasma Dispersion Function reduces to the simpler family of Dnestrovsky functions F_p with half–integer index:

$$\mathcal{F}_{q+\frac{1}{2}}\,(n_\parallel = 0) = F_{q+\frac{1}{2}}\left(\frac{\omega - n\Omega_c}{\omega}\right) \tag{A.34}$$

The following integral relation holds between the two families of Dnestrovsky functions with integer and half-integer index:

$$F_{q+\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dx\, e^{-x^2}\, F_q(z + x^2) \tag{A.35}$$

We also note that in the limit $n_\parallel^2 \to 0$ Eq. (A.21) reduces to the appropriate recurrence relation Eq. (A.2) for F_q.

The Dnestrovsky functions of half–integer index are related to the Error function, hence also with the Plasma dispersion function Z, as expected from the results of the previous paragraph. Indeed, for $q = 0$ and 1 we find

$$\begin{aligned} F_{1/2}(z) &= \sqrt{\frac{\pi}{z}}\, e^z\, \mathrm{Erfc}\,(\sqrt{z}) \\ F_{3/2}(z) &= 2\left\{1 - \sqrt{\pi z}\, e^z\, \mathrm{Erfc}\,(\sqrt{z})\right\} \end{aligned} \qquad |\mathrm{Arg}\,(z)| < \pi \tag{A.36}$$

This function is real for real positive z; on the negative real axis, where $\mathrm{Arg}\,(z) = +\pi$ according to the Landau prescription, we have

$$\begin{aligned} F_{1/2}(-x) &= -i\sqrt{\frac{\pi}{x}}\, e^{-x}\, \mathrm{Erfc}\,(i\sqrt{x}) = -\frac{1}{\sqrt{x}}\, Z(-\sqrt{x}) \\ F_{3/2}(-x) &= 2\left\{1 - i\sqrt{\pi x}\, e^{-x}\, \mathrm{Erfc}\,(i\sqrt{x})\right\} \qquad (x > 0) \\ &= 2\left\{1 - \sqrt{x}\, Z(-\sqrt{x})\right\} = -Z'(-\sqrt{x}) \end{aligned} \tag{A.37}$$

The function $F_{1/2}(z)$ has a singularity with branch point at $z = 0$. All higher–order functions of this family have the same branch point, but are finite at the origin: the q–th derivative of $F_{q+1/2}(z)$ is however infinite there.

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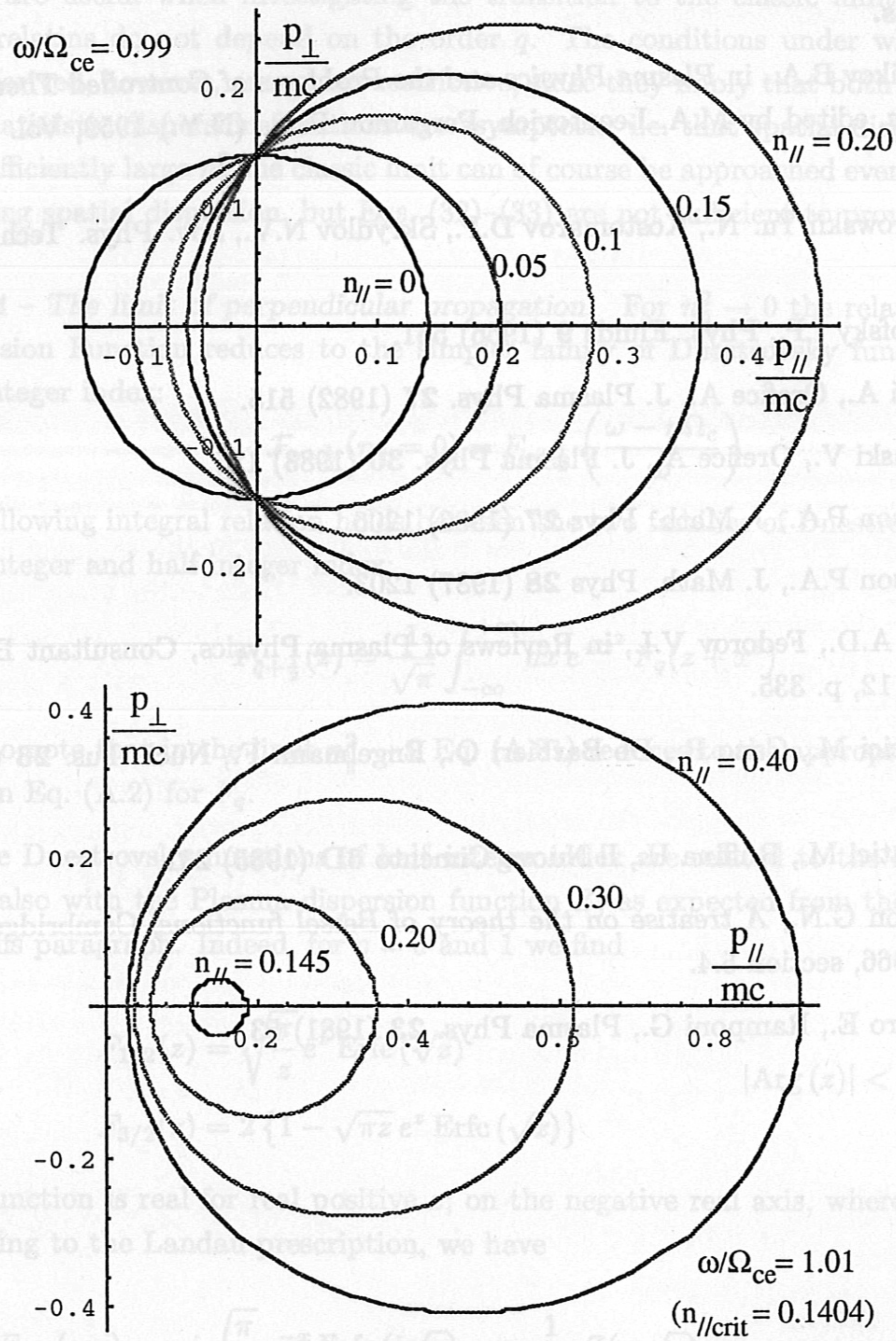


Fig. 1 - Relativistic resonance ellipses in momentum space:
a) $\omega/\Omega_{ce} < 1$; b) $\omega/\Omega_{ce} > 1$

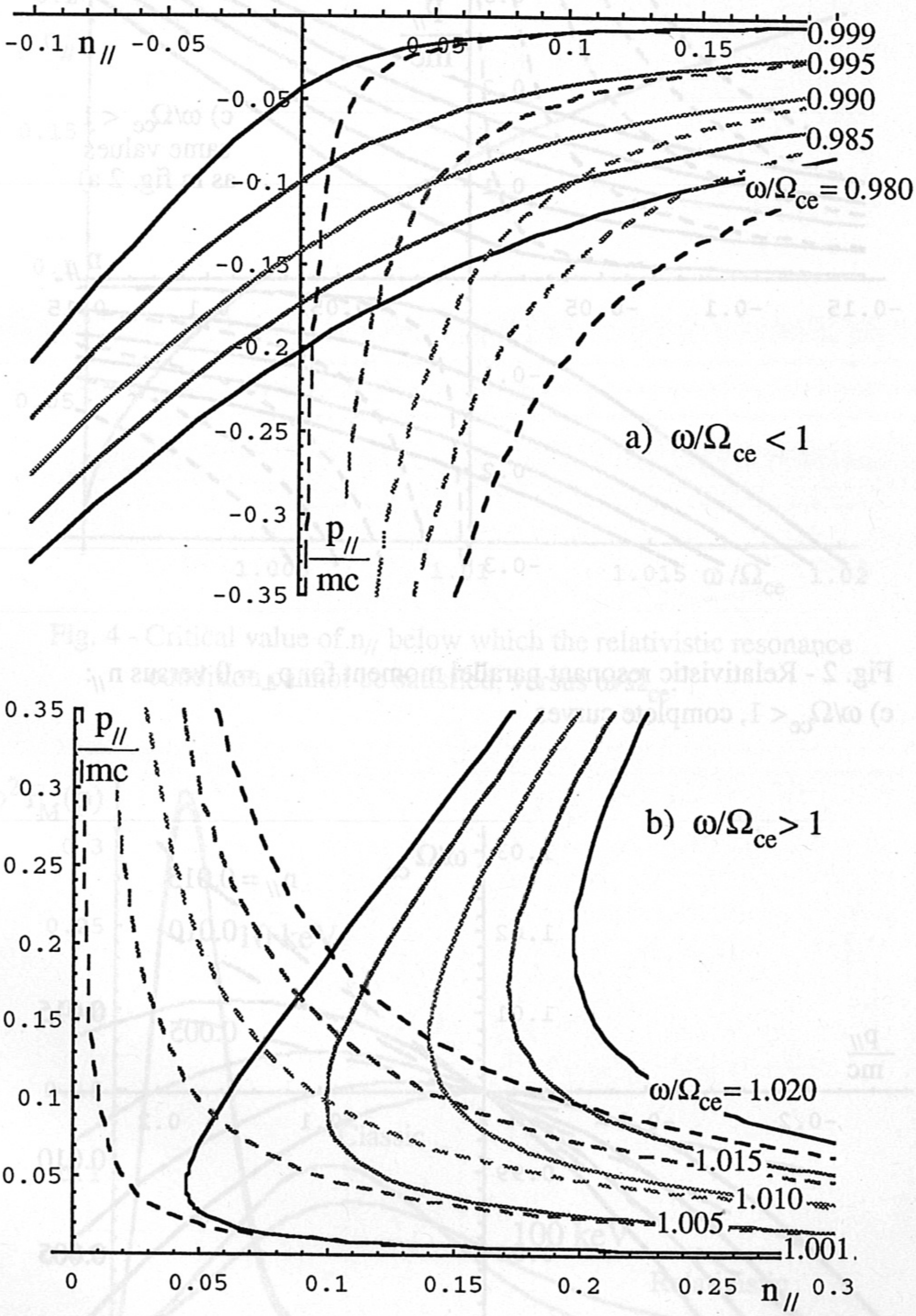


Fig. 2 - Relativistic resonant parallel moment for $p_\perp = 0$ versus $n_{//}$:
a) $\omega/\Omega_{ce} < 1$; b) $\omega/\Omega_{ce} > 1$; the classical values are the dotted lines.

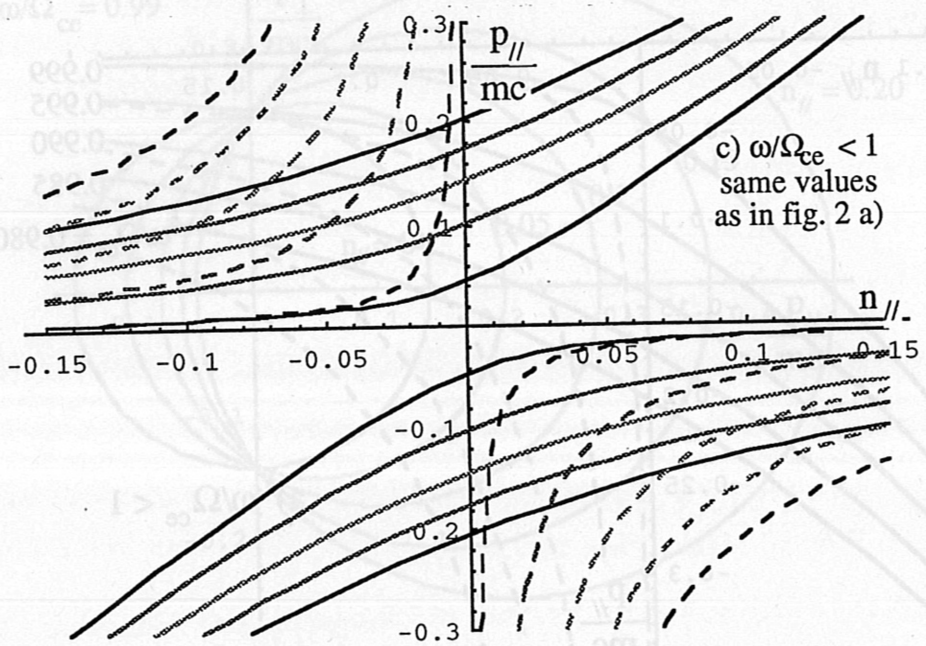


Fig. 2 - Relativistic resonant parallel moment for $p_{\perp} = 0$ versus $n_{//}$:
c) $\omega/\Omega_{ce} < 1$, complete curves.

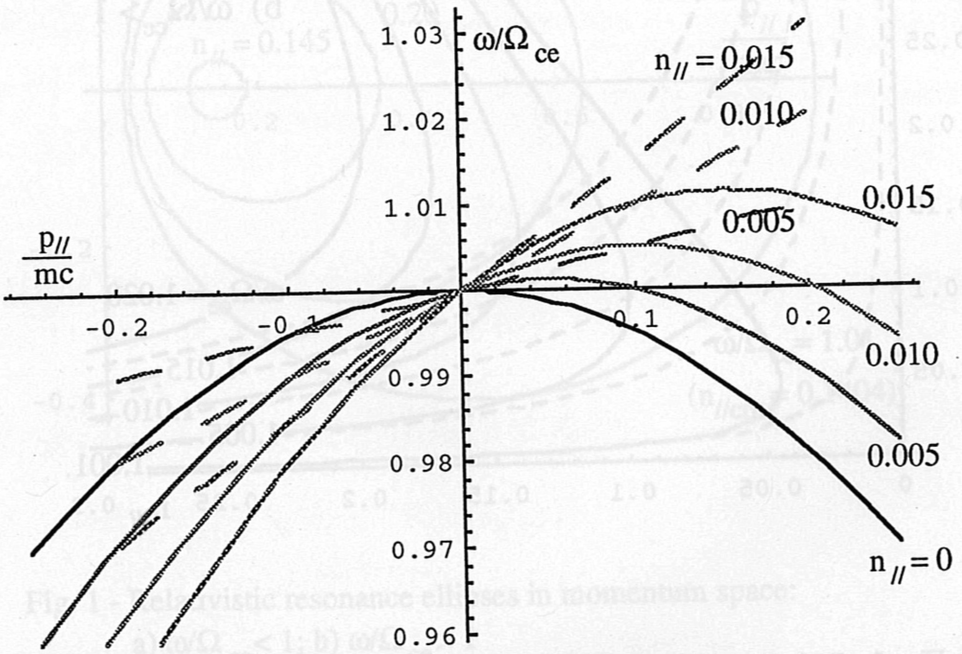


Fig. 3 - $\omega/\Omega_{ce})_{res}$ versus $p_{//}/mc$ for different values of $n_{//}$ near the fundamental cyclotron resonance. The dashed curves give the classical limit.

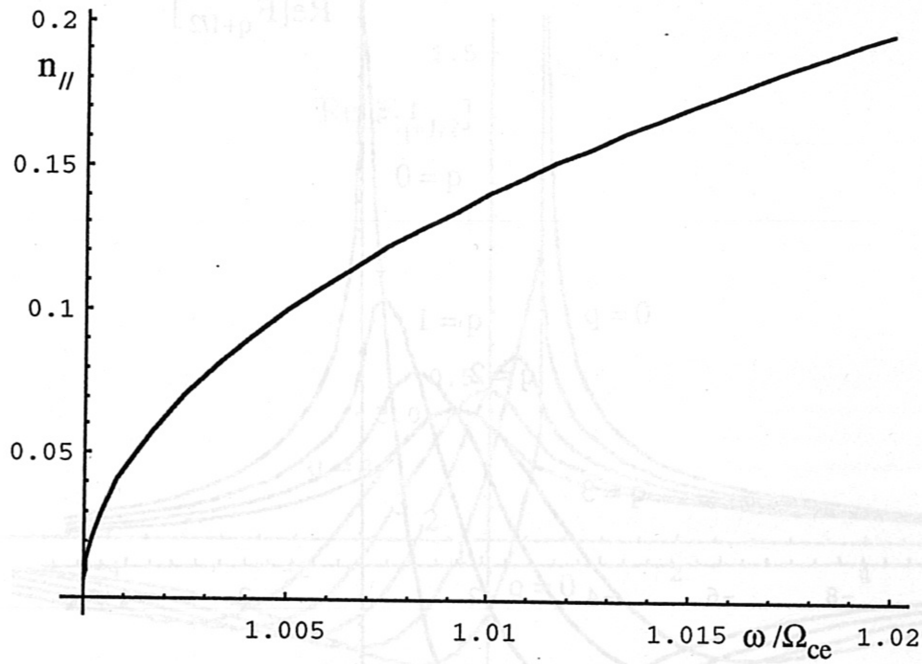


Fig. 4 - Critical value of $n_{//}$ below which the relativistic resonance condition cannot be satisfied, versus ω/Ω_{ce}.

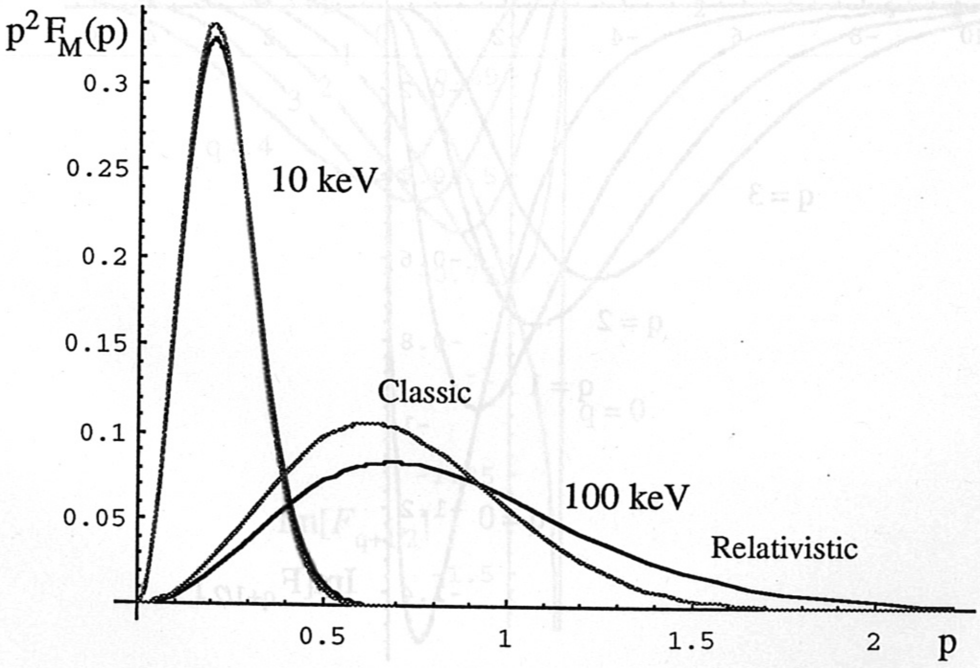


Fig. 5 - Comparison of the relativistic (heavy lines) and classic (light lines) Maxwellians at 10 and 100 keV (momentum in units of mc).

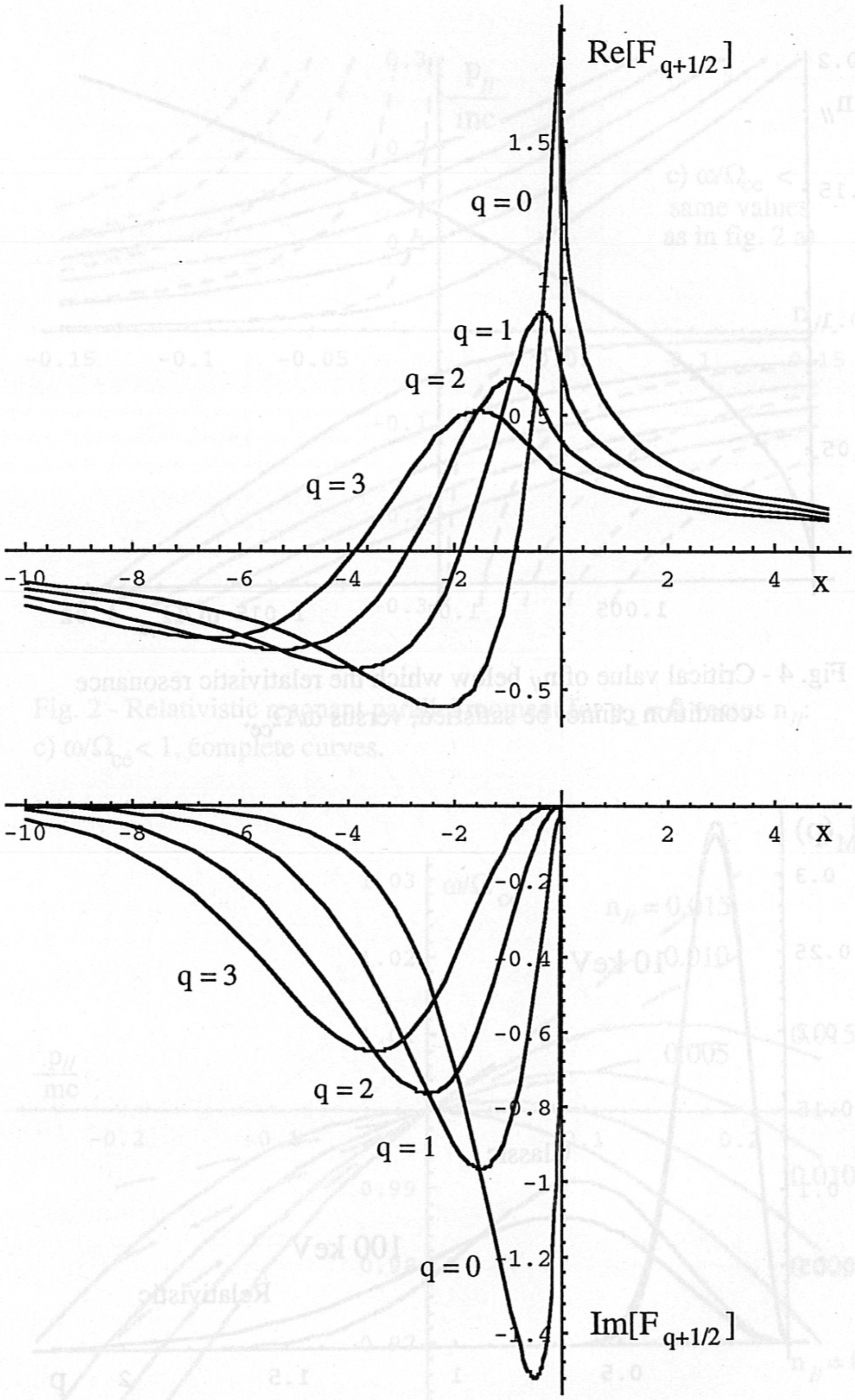


Fig. 6- Relativistic Plasma Dispersion functions for perpendicular propagation (Dnestrovsky functions of half-integer order).

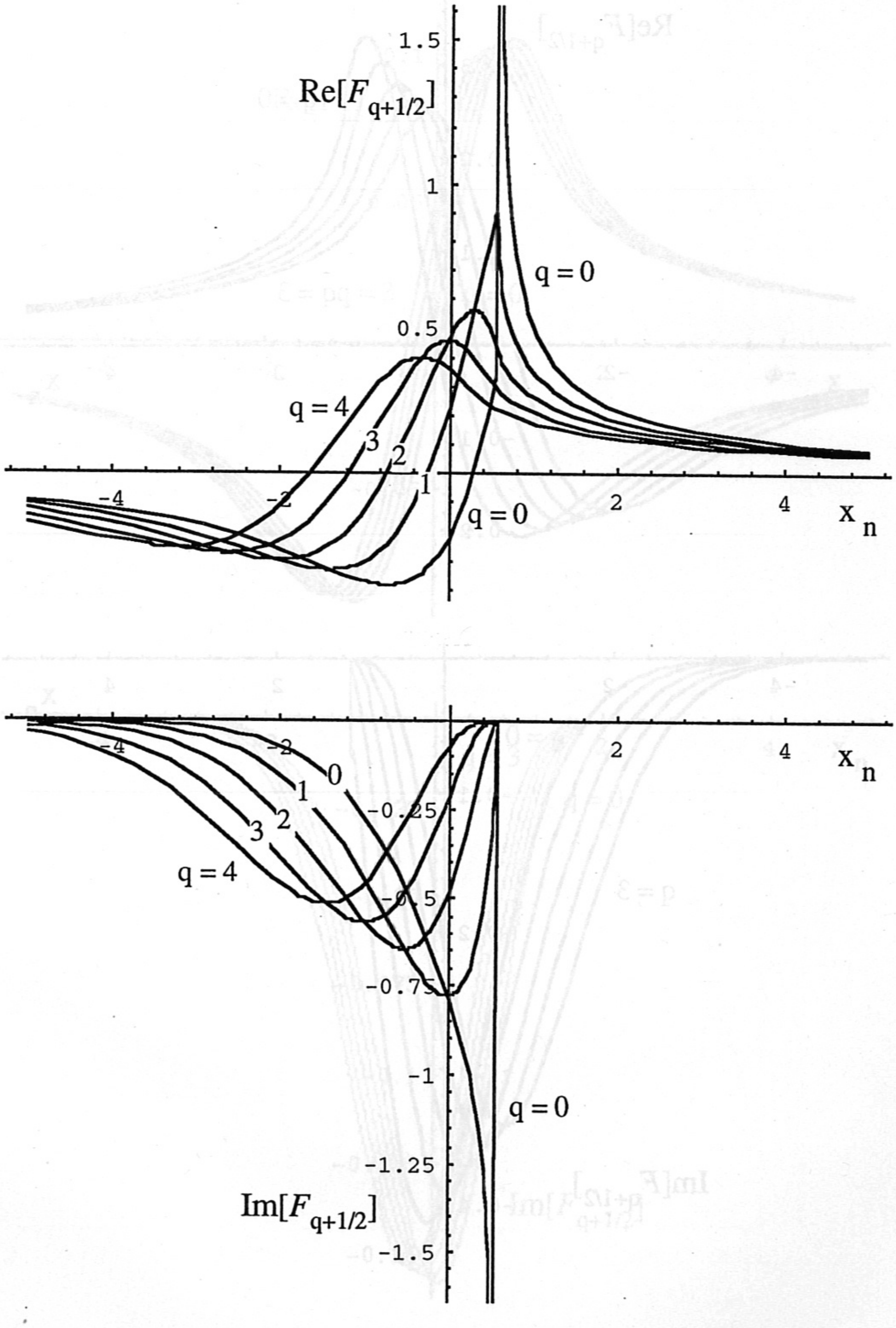


Fig. 7 - Relativistic Plasma Dispersion Functions for oblique propagation (Skarofsky functions of half-integer order):

a) $T_e = 2$ keV, $n_{//} = 0.1$ ($\nu_r = 1.23$); $x_n = (\omega - n\Omega)/k_{//} v_{th}$

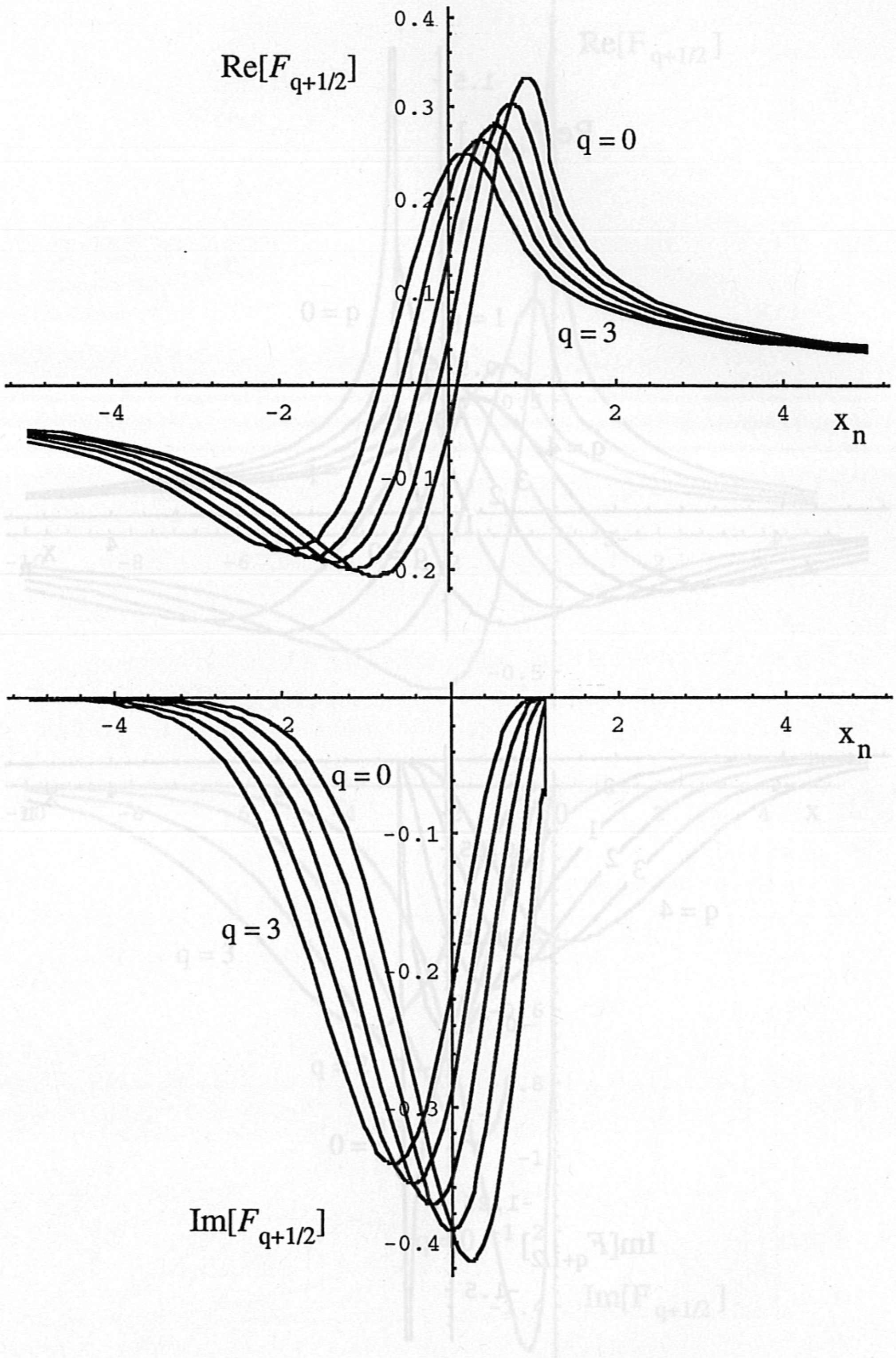


Fig. 7 - Relativistic Plasma Dispersion Functions for oblique propagation:
b) T_e = 2 keV, $n_{//}$ = 0.2 (ν_r = 2.26); $x_n = (\omega - n\Omega)/k_{//}v_{th}$

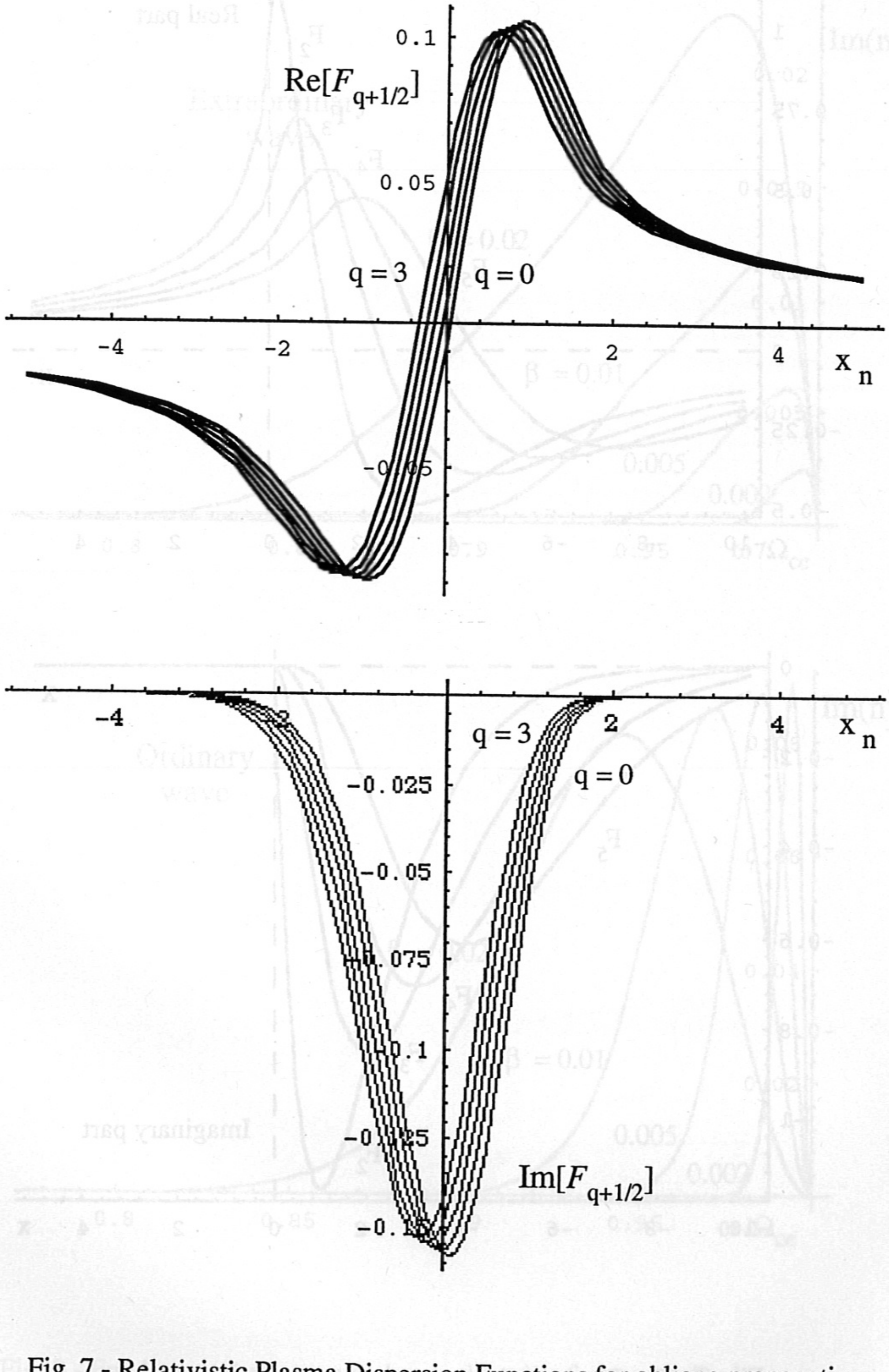
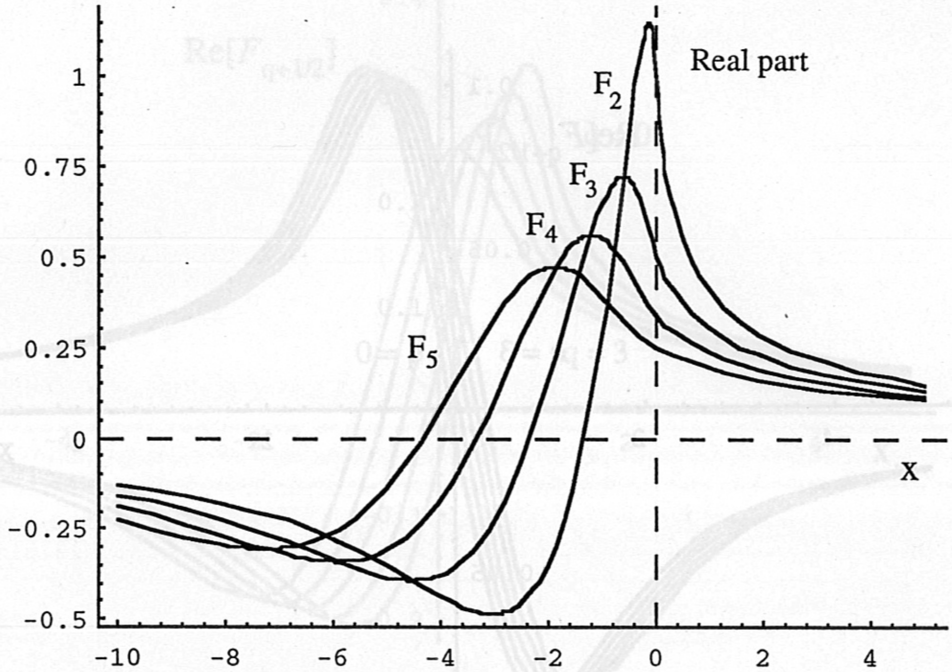


Fig. 7 - Relativistic Plasma Dispersion Functions for oblique propagation:

c) $T_e = 2$ keV, $n_{//} = 0.5$ ($\nu_r = 5.65$); $x_n = (\omega - n\Omega)/k_{//}v_{th}$



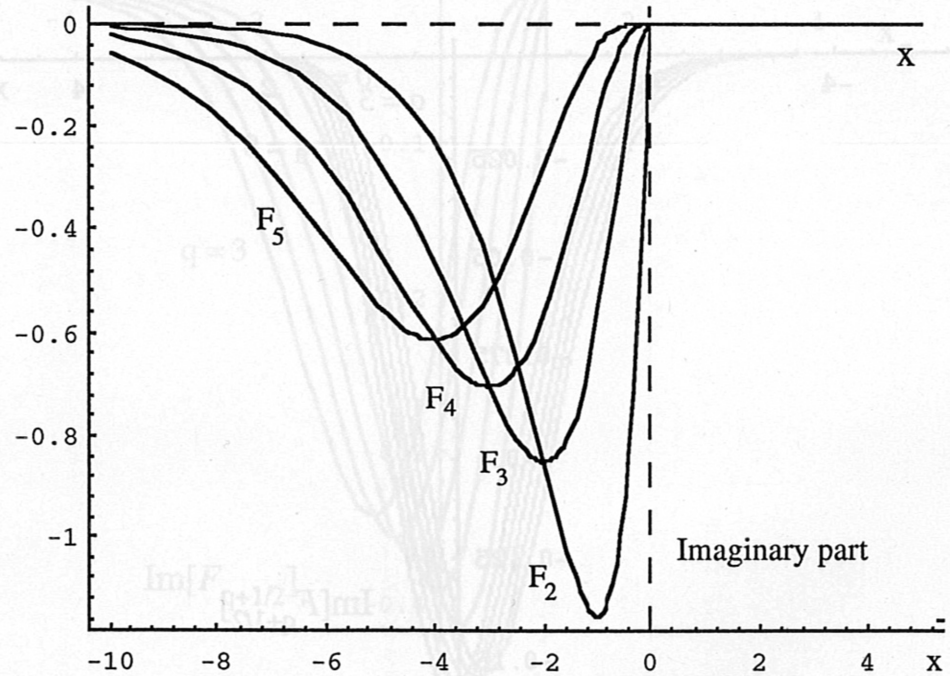
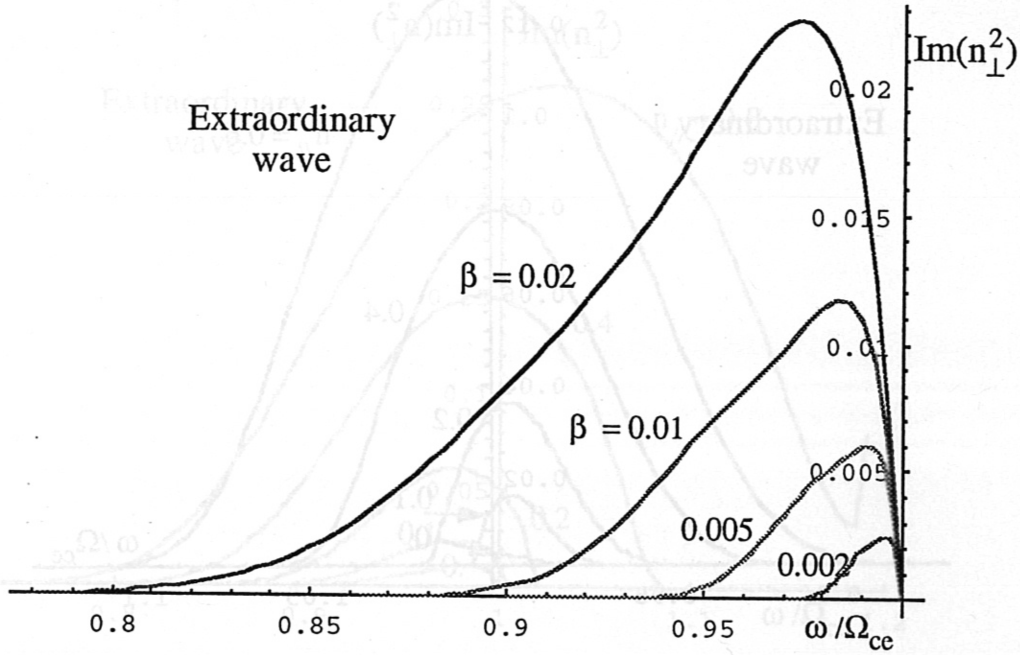


Fig. 8 - The Dnestrovsky functions of integer order (q = 2 to 5).



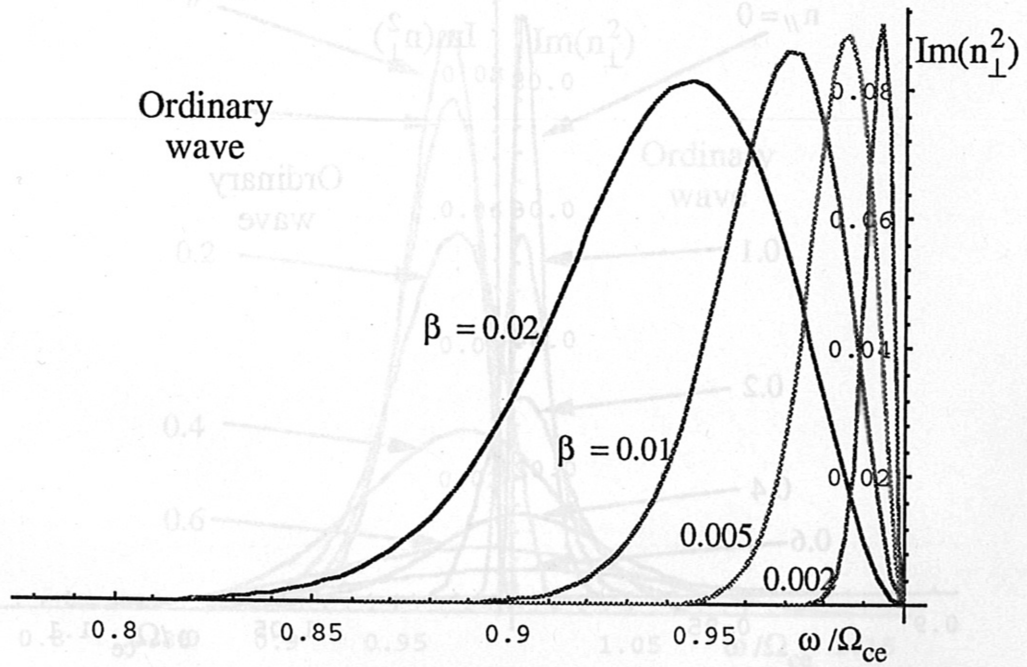
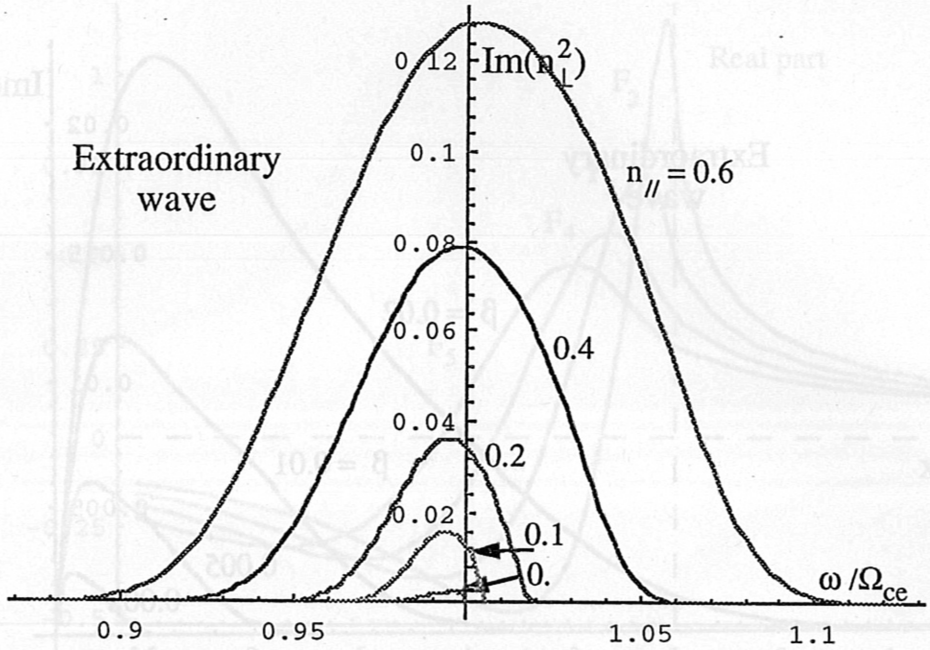


Fig. 9 - Fundamental electron cyclotron damping for perpendicular propagation, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$ (T_e = 1.02, 2.55, 5.1 and 10.2 keV)



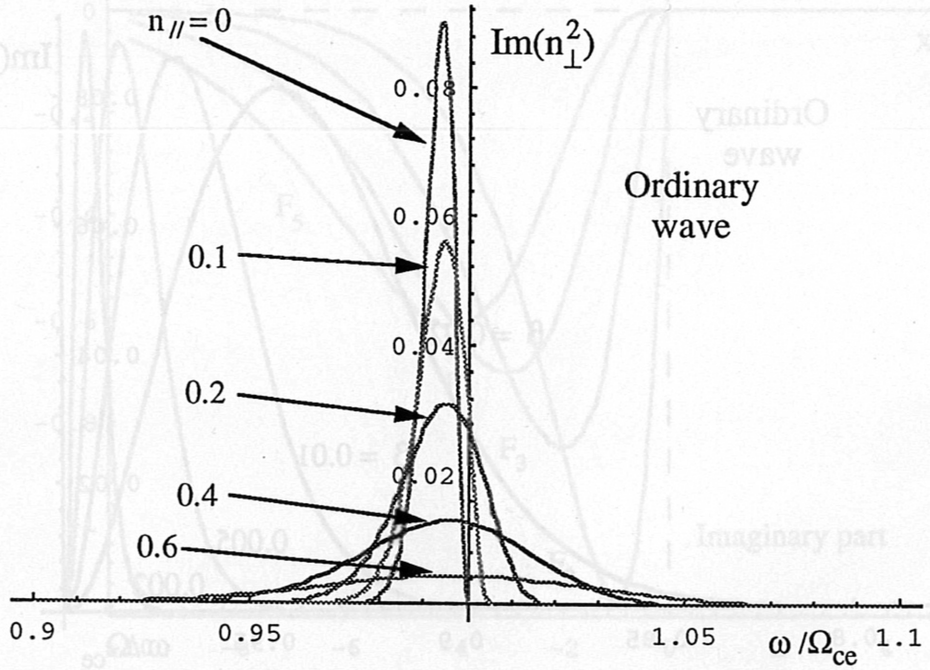
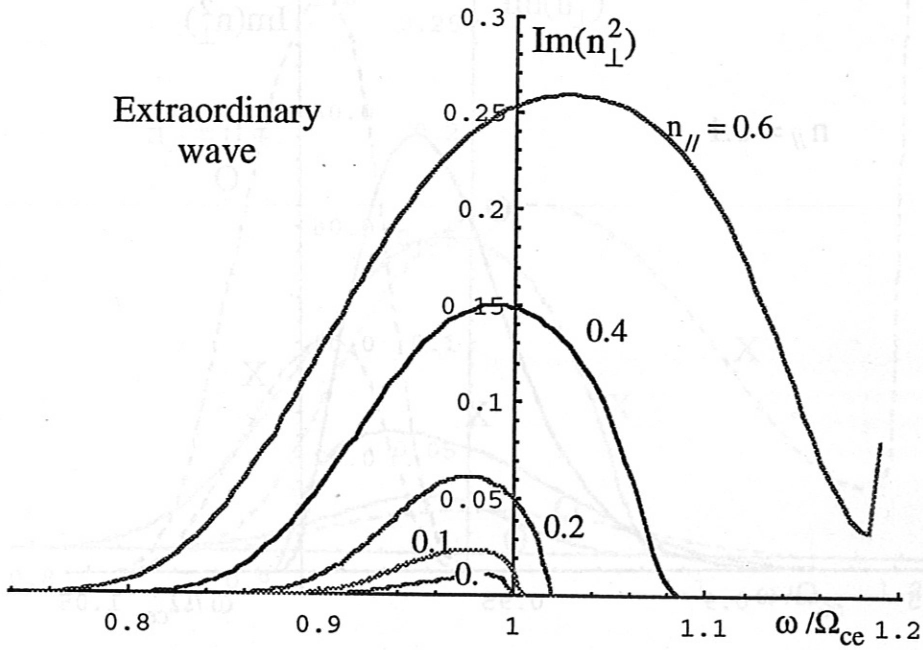


Fig. 10 - Fundamental electron cyclotron damping (relativistic) for different angles of propagation. $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$,

a) $\beta_e = 0.002$ ($T_e = 1.02$ keV).



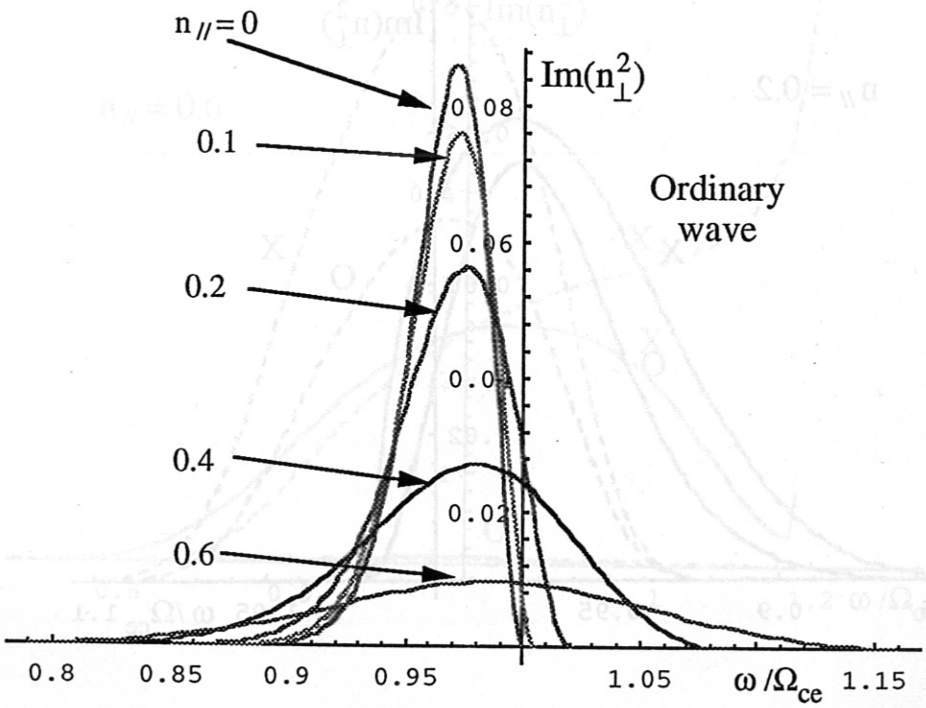
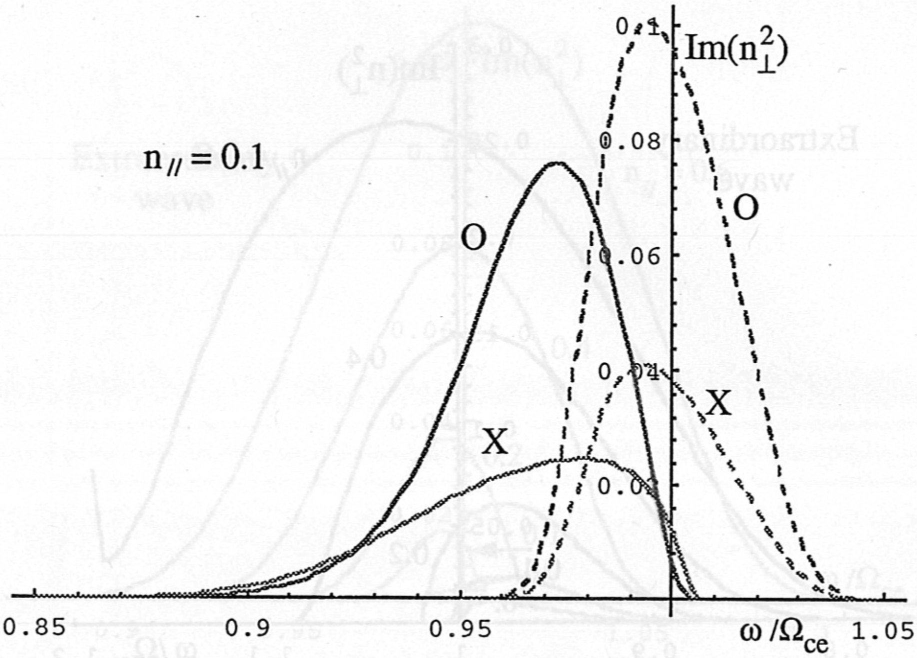


Fig. 10 - Fundamental electron cyclotron damping (relativistic) for different angles of propagation. $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$,

b) $\beta_e = 0.01$ ($T_e = 5.1$keV).



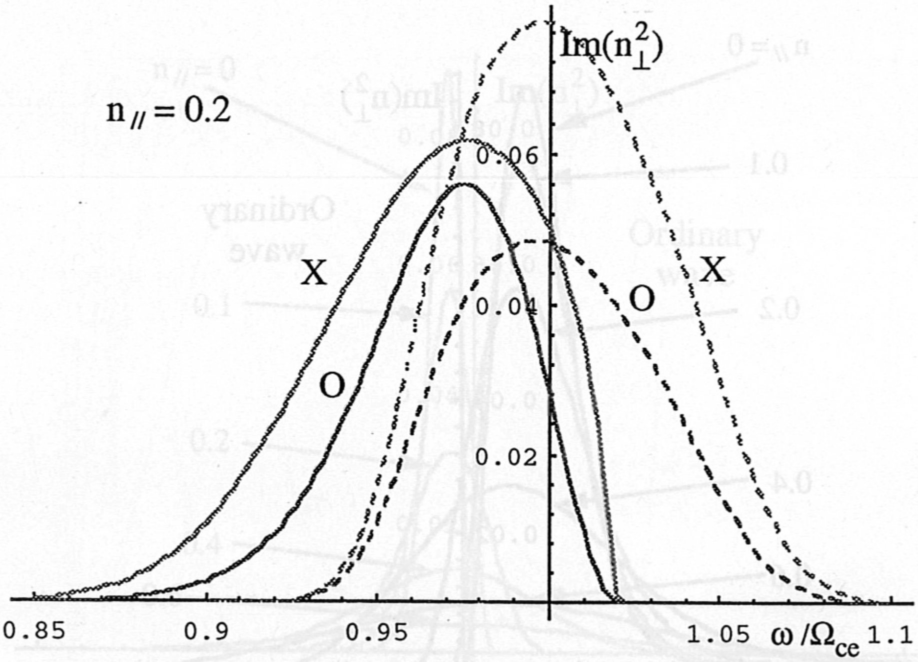


Fig. 11 - Fundamental electron cyclotron damping: relativistic (full lines) and classic approximation (dashed lines). $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$ $\beta_e = 0.01$ ($T_e = 5.1$ keV).

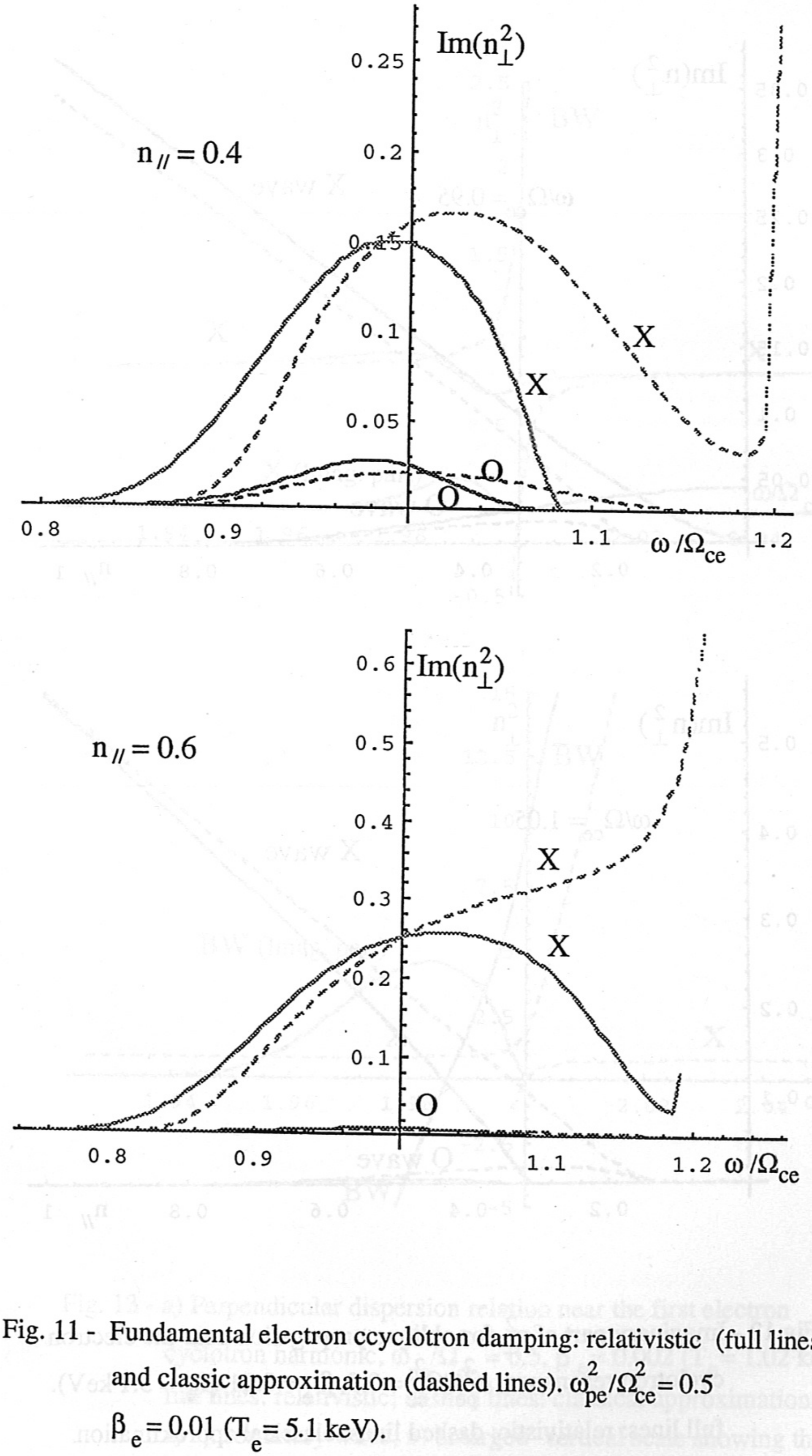


Fig. 11 - Fundamental electron ccyclotron damping: relativistic (full lines) and classic approximation (dashed lines). $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$ $\beta_e = 0.01$ ($T_e = 5.1$ keV).

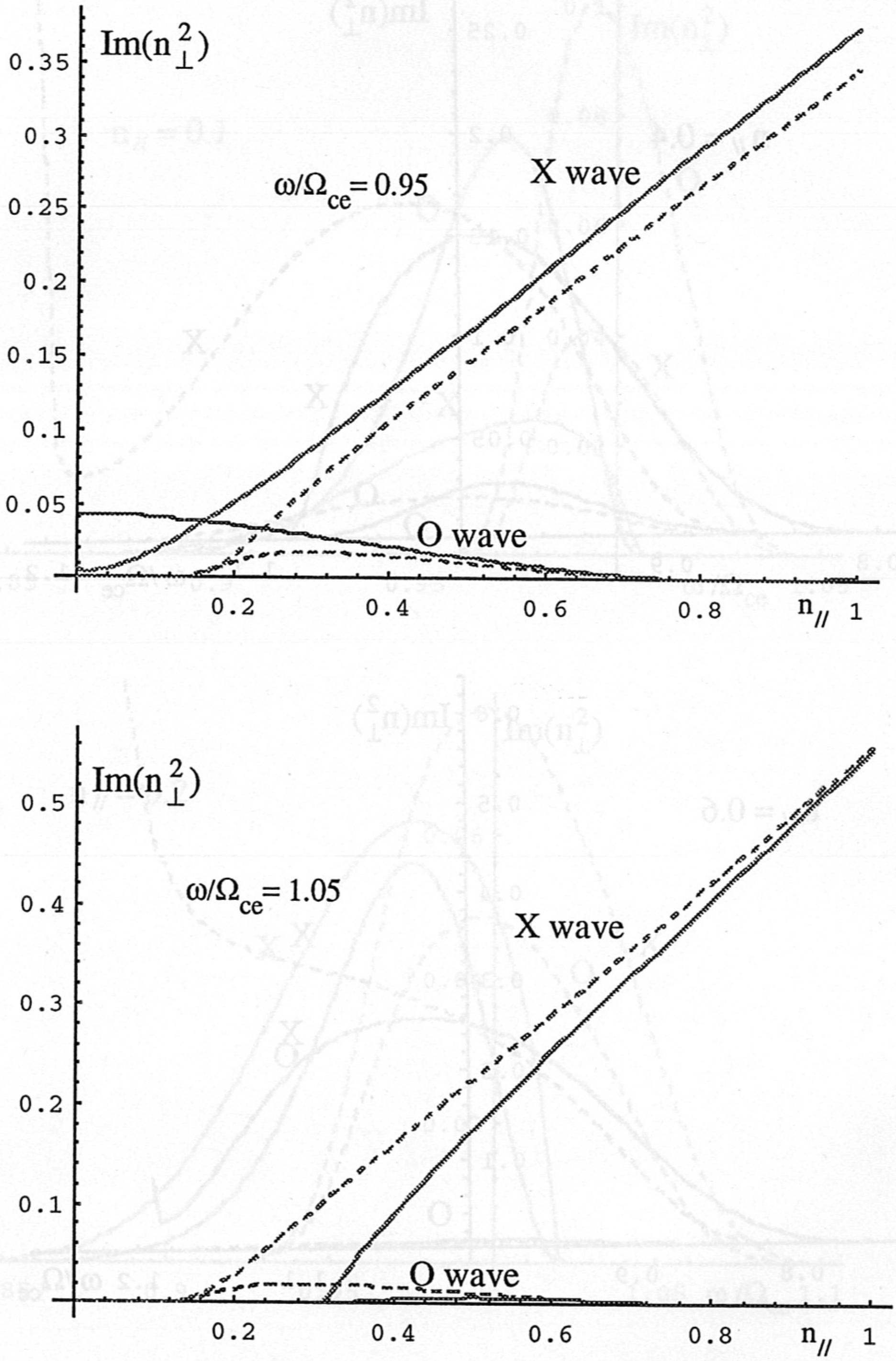


Fig.12 - Imaginary part of $n_\perp^2$ for oblique propagation near the electron cyclotron resonance, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$, $\beta_e = 0.01$ ($T_e = 5.1$ keV). full lines: relativistic; dashed lines: classical approximation.

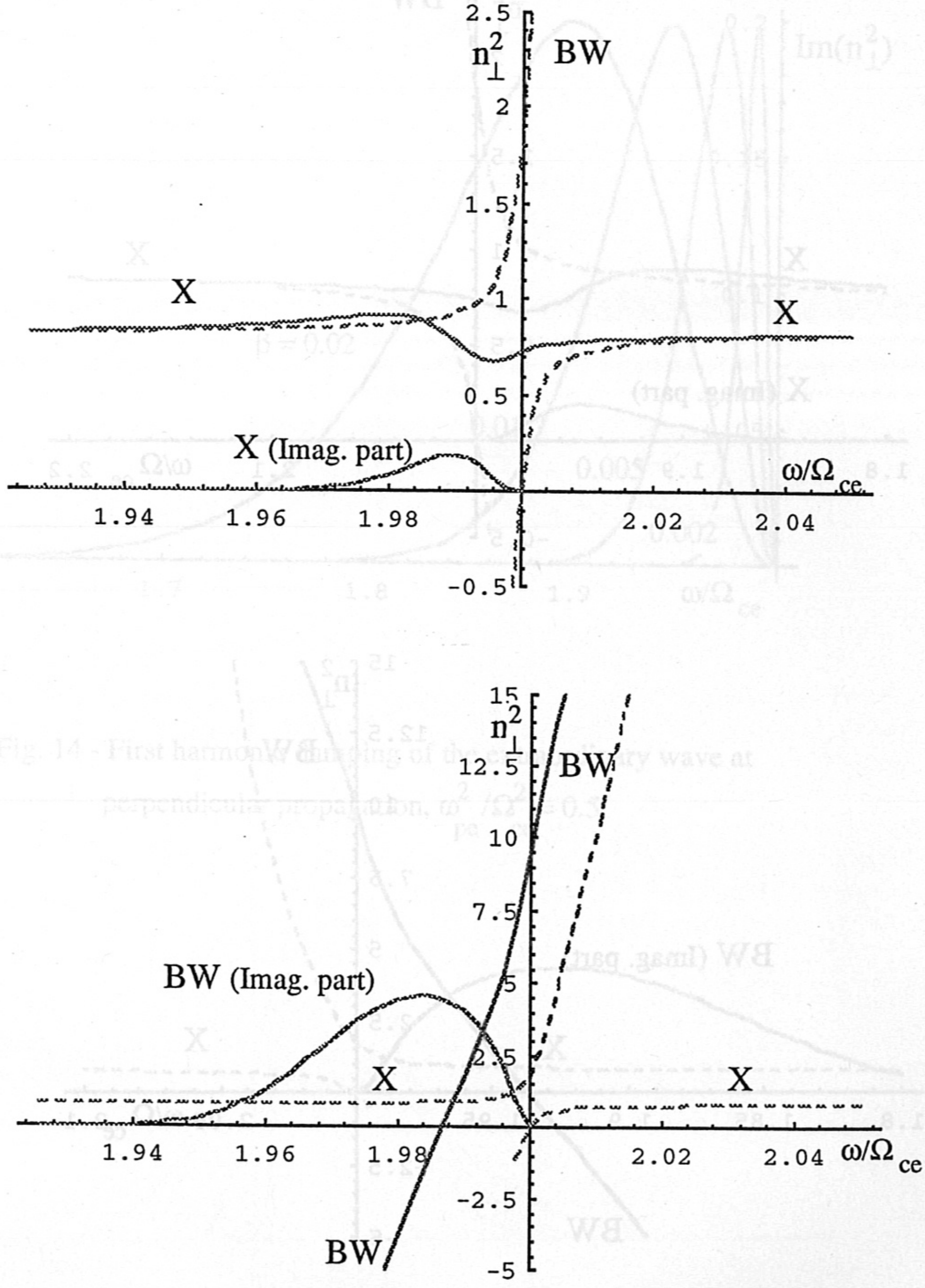
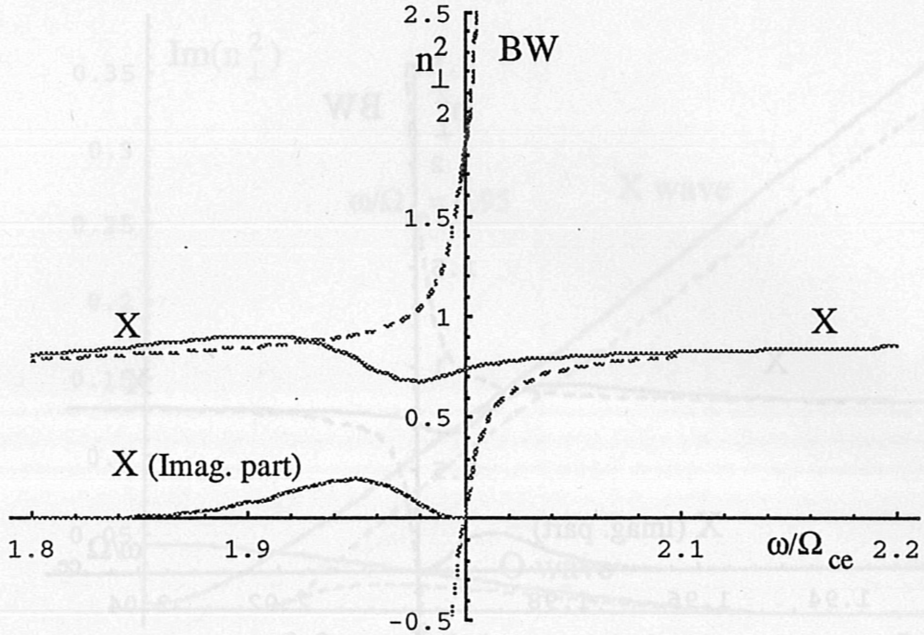


Fig. 13 - a) Perpendicular dispersion relation near the first electron cyclotron harmonic, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$, $\beta_e = 0.002$ ($T_e = 1.02$ keV) full lines: relativistic; dashed lines: classical approximation. a) extrordinary wave; b) enlarged vertical scale showing the first Bernstein wave.



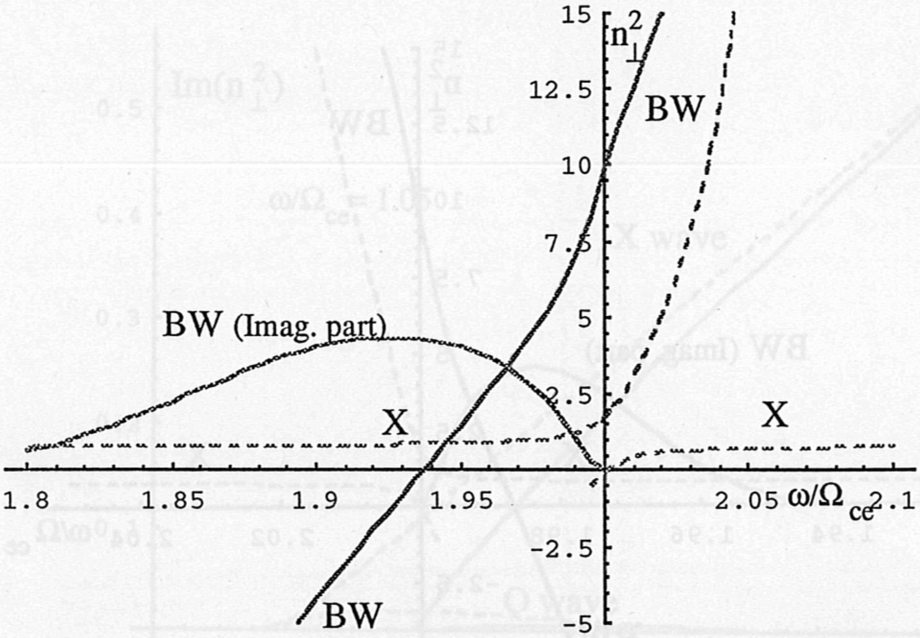


Fig. 13 - b) Perpendicular dispersion relation near the first electron cyclotron harmonic, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$, $\beta_e = 0.01$ ($T_e = 5.1$ keV) full lines: relativistic; dashed lines: classical approximation. a) extrordinary wave; b) enlarged vertical scale showing the first Bernstein wave.

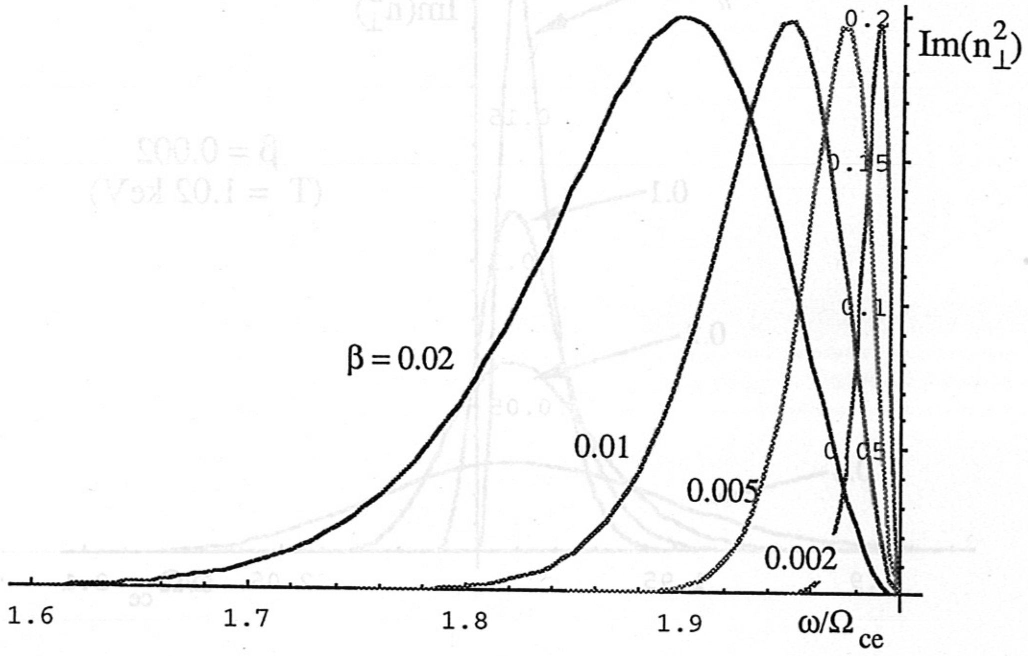
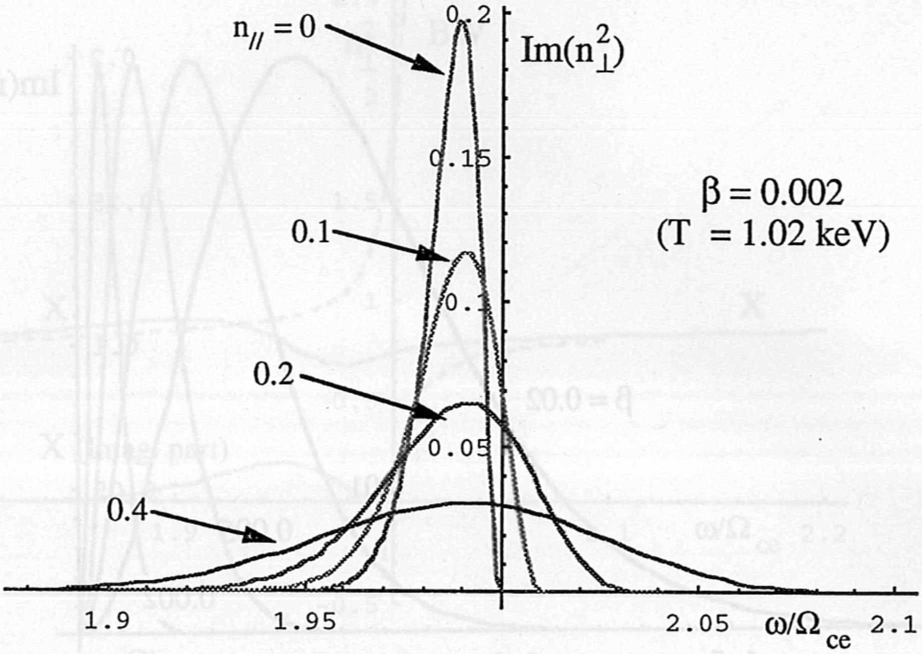


Fig. 14 - First harmonic damping of the extraordinary wave at perpendicular propagation, $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$.



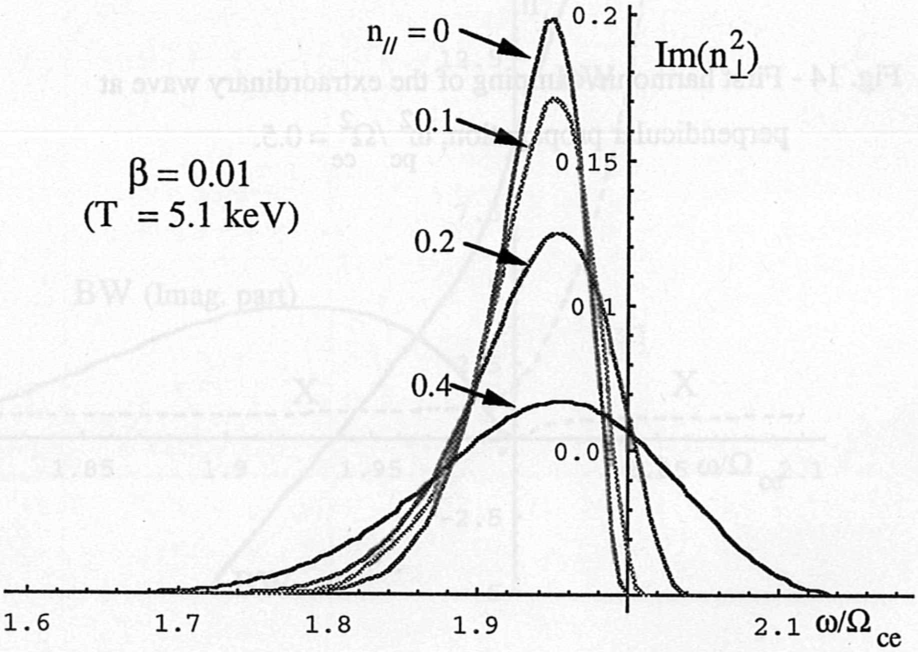


Fig. 15 - Imaginary part of n near the first electron cyclotron harmonic for the extraordinary wave at oblique propagation. $\omega_{pe}^2/\Omega_{ce}^2 = 0.5$.