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On Euler-like discrete models of the logistic differential equation

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Abstract

The stability theory of difference schemes is mostly a linear theory. To understand the behavior of difference schemes on nonlinear differential equations, it seems desirable to extend the stability theory into a nonlinear theory. As a step in that direction, we investigate the stability properties of Euler-related integration algorithms by checking how they preserve and violate the dynamical structure of the logistic differential equation.

We find that partially implicit rational schemes are superior to explicit schemes when they are stable and the blow-up time has not passed. When such a rational scheme turns unstable, however, it has much less desirable properties than explicit schemes.

As a side product of these investigations, we found a map with two branches of stable fixed points. Both of them lose stability to a Feigenbaum sequence of period doubling bifurcations and chaotic trajectories *independently of each other.* To our knowledge, this is the first such example.

1 Introduction and summary of results

Textbooks on the numerical treatment of initial value problems formulate conditions under which the solutions of initial value problems can be approximated arbitrarily well by solutions of difference equations. Also well-known, however, are conditions under which the approximation of continuous dynamical systems by discrete dynamical systems is quite poor. Two examples:

1) Explicit difference schemes can produce spurious solutions [9] and have a tendency to go unstable, i.e. the discrete analog of a stable stationary state of

$$\dot{u} = f(\lambda, u), \qquad \lambda \in \mathbb{R}, \quad u_0 = u(0) \in \mathbb{R}^N \tag{1}$$

is stable only for "sufficiently small" $h \leq h_0$. For $h > h_o$, the discrete trajectory can be chaotic, though the underlying differential equation does not have chaotic solutions ([16, 11, 8] and section 4).

2) Implicit difference schemes are known to have better stability properties. But they can produce non-chaotic discrete images of chaotic solutions of differential equations. This was shown for the backward Euler scheme on the Rössler system [3].

The stability theory of difference schemes is mostly a linear theory, i.e. the stability properties of difference schemes are mostly investigated on linear model problems

$$\dot{u} = \mu u, \qquad \mu \in \mathbb{C}, \quad u(0) = u_0 \in \mathbb{C}. \tag{2}$$

In many applications, the *principle of linearized stability* is valid both for stationary states of (1) and for fixed points of the discrete analog

$$y_{n+1} = g(\lambda, y_n), \qquad \lambda \in \mathbb{R}, \quad y_0 \in \mathbb{R}^N \tag{3}$$

of (1) for almost all values of the parameter λ (see sections 2 and 3 for more details). Most stationary states thus have neighborhoods where the stability analysis of a linear system provides the correct answer. If eq. (1) has only a finite number of stationary states for fixed λ, there is often a common neighborhood of all stationary states in which local linearization is adequate (see [1] and section 5).

In recent years, there have been quite a number of investigations heading towards a nonlinear stability theory, [1, 10, 16, 6, 17, 11, and the references therein]. Nevertheless, it seems that not only have answers to be found for a satisfactory nonlinear theory, but also new questions have to be formulated.

In this paper, we take a close look at some very simple model problems and pursue the following questions:

- Given a dynamical system (1) with several stationary states, what can be said about the fixed points of corresponding discrete systems?
- How large are their neighborhoods in which linearization is adequate?
- How do the domains of attraction compare for the continuous and discrete problems?
- What exactly happens when difference schemes go unstable?

We investigate these questions on several variants of Euler's method for the logistic differential equation

$$\dot{u} = \lambda u(1-u), \qquad u(0) = u_0, \tag{4}$$

which has two stationary states: a stable one and an unstable one. All trajectories are monotonic, some are blow-up solutions (section 2). The stability of both stationary states changes when the parameter λ changes sign. Only for $\lambda = 0$ does the principle of linearized stability not apply to this differential equation.

The difference schemes investigated are:

(a) the explicit forward Euler scheme,

(b) the explicit midpoint Euler scheme, and

(c) two partially implicit rational Euler schemes which are adjoint to each other.

We only consider the fixed-step-length case. It is less favorable than the case with step-size control since step-size control has a stabilizing effect [17, p. 253f].

Briefly, the results are:

(a) With the forward Euler scheme, the discrete analog of the unstable stationary state is an unstable fixed point for all λh. The discrete analog of the stable stationary state is a stable fixed point for $-2 < \lambda h < 2$. For $\lambda > 0$, it turns unstable in a flip bifurcation at $\lambda h = 2$. This flip bifurcation is the beginning of a Feigenbaum cascade of period-doubling bifurcations [19, 11, 8]. Already for $\lambda h > 1$ the discrete scheme is a very poor model: there is no neighborhood of the stable fixed point with correct dynamic behavior. For $0 < \lambda h < 1$ such a neighborhood exists, i.e. it depends on the inital value y_0 whether the dynamic behavior of the discrete solution is qualitatively correct (Fig. 1 and section 4). It should be noted that the curves separating the different regimes for the initial values y_o are either branches of fixed points or closely related to the branches of spurious fixed points for the midpoint Euler scheme (section 5).

(b) With the midpoint Euler scheme, the discrete analog of the unstable stationary state is an unstable fixed point for all λh. The discrete analog of the stable stationary state has a neighborhood with correct dynamic behavior for $-2 < \lambda h < 2$. With $\lambda > 0$, it loses its stability for $\lambda h = 2$ through an exchange of stability with an unstable spurious fixed point. There is another spurious branch of stable fixed points. Both branches of spurious fixed points lose stability to a Feigenbaum cascade of period-doubling bifurcations, *independently of each other.* To our knowledge, this is the first such example (Fig. 2).
This time, the difference equation is a good model up to $\lambda h = 2$, but only in a small domain Ω owing to the spurious fixed points. As a consequence of Beyn's theorem [1], both spurious fixed points become unbounded for $\lambda h \to 0$ (Figs. 2-6 and section 5).

(c) The rational Euler scheme (27) is *globally* stable for $\lambda h < 1$, i.e. it gives the correct dynamic behavior for all initial values $y_o \in I\!R$ for which the blow-up time associated to y_o has not passed: the discrete analog of the stable stationary state is stable, the discrete analog of the unstable stationary state is unstable. and the scheme provides the correct blow-up behavior.
Though the scheme has most desirable properties for $\lambda h < 1$, it has most undesirable properties for $\lambda h > 2$: Both fixed points change their stability for $\lambda h = 2$, blow-up is disguised, and the spuriously stable fixed point is globally

attractive. The dynamics of the scheme is thus completely wrong for $\lambda h > 2$, but 'looks perfectly alright' if blow-up solutions are not expected (section 6). That the dynamics is wrong is much harder to detect for this scheme on a 'real life problem' than for the other two schemes investigated here: in the other two cases, the stable spurious solutions are h-dependent and can thus be revealed by two computations with the same scheme and different h. In this case here, a different scheme should be used for confirmation.

A comparison of the three schemes shows:
With the two explicit schemes, the discrete analog of an unstable stationary state is an unstable fixed point for all λh. The discrete analog of a stable stationary state is stable only for a limited range of λh-values. 'Stability of the scheme' is thus a *local property* that can differ from fixed point to fixed point at the same λh-value. It has to be verified in the neighborhood of each fixed point separately.
Only in the limit $h \to 0$ (λ fixed) is the domain of attraction of the stable fixed point identical with the domain of attraction of the approximated stable stationary state. Explicit schemes cannot model blow-up.

With the partially implicit schemes, the discrete analog both of the unstable and of the stable stationary state show the correct stability behavior for the same limited range of λh-values, and change stability simultaneously. 'Stability of the scheme' is thus a *global property*.
In the whole λh-range of stability of the scheme, the domain of attraction of the stable fixed point is identical with the domain of attraction of the stable stationary state, as long as the blow-up time has not passed.

Also in their way of going unstable the investigated explicit and implicit schemes differ substantially: With the explicit schemes, the branches of (proper or spurious) stable fixed points eventually undergo a Feigenbaum sequence of period doubling bifurcations and become chaotic lateron. The partially implicit scheme does not feature such period doubling bifurcations or chaotic trajectories on our model problem. This was 'shown by extensive tests' by Twizell et al. [19]. We prove it in section 6 by determining the domains of attraction for all fixed points and for all parameter values with help of inequalities.

It remains to check how these partially implicit schemes act on differential systems with chaotic trajectories: Do they always reproduce chaotic trajectories faithfully, or can they suppress chaos, as the backward Euler scheme sometimes does? We conjecture that they will reproduce chaos correctly for all systems with quadratic nonlinearity (this includes the Lorenz equations and the Rössler system). An answer to this question is beyond the scope of this paper. Further investigations are under progress.

In the next two sections, we recall basic definitions and facts. Some of them have already been used in this introduction. In the other three sections, we treat the three examples in detail.

2 Continuous dynamical systems

In this section we consider continuous dynamical systems and recall basic definitions and facts used in later sections. Consider

$$\dot{u} = f(u), \quad u(0) = u_0 \in \mathbb{R}^N, \tag{5}$$

f continuously differentiable. For such fs eq. (5) has a unique solution $u(t; u_o)$ which exists in some maximum interval $(0, T(u_0))$.

$\bar{u}$ is a *stationary state* of (5) iff[1] $f(\bar{u}) = 0$ for all $t \geq 0$.

$\bar{u}$ is a *stable* stationary state of (5) iff for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $u(t; u_0) \in U_\varepsilon(\bar{u})$ for all $u_0 \in U_\delta(\bar{u})$ and all $t \geq 0$.

$U_\mu(\bar{u}) := \{u \in \mathbb{R}^N : |u - \bar{u}| < \mu\}$.

$\bar{u}$ is an *asymptotically stable* stationary state of (5) iff $\bar{u}$ is stable and

$$\lim_{t\to\infty} |u(t; u_0) - \bar{u}| = 0 \quad \text{for all} \quad u_0 \in U_\delta(\bar{u}) \quad \text{for some} \quad \delta > 0.$$

$\bar{u}$ is a *hyperbolic* stationary state of (5) iff $Re\ \mu \neq 0$ for all eigenvalues μ of the Jacobian $f'(\bar{u})$.

For hyperbolic stationary states a **Principle of Linearized Stability** is valid:

Let $\bar{u}$ be a hyperbolic stationary state of (5) $\dot{u} = f(u)$. Then there are neighborhoods $U(\bar{u})$

[1] 'iff' means 'if and only if'

and $V(0)$ such that the dynamics of $\dot{u} = f(u)$ in $U(\bar{u})$ and of $\dot{v} = f'(\bar{u})v$ in $V(0)$ are equivalent, i.e. there is a homeomorphism between $U(\bar{u})$ and $V(0)$ which preserves the sense of orbits and can also be chosen to preserve parametrization by time.

For more details see Guckenheimer/Holmes, where this is called the *Theorem of Hartman-Grobman* [5, p. 13].

If $\bar{u}$ is a hyperbolic stationary state,
it is thus asymptotically stable if $Re\ \mu_i < 0$ for all eigenvalues μ_i of $f'(\bar{u})$, $i = 1, \ldots, N$;
it is unstable if one μ_{i_0} satisfies $Re\ \mu_{i_0} > 0$.
If $\bar{u}$ is a non-hyperbolic stationary state, it might be a bifurcation point (stationary-stationary or stationary-periodic (Hopf bifurcation)). In this case a nonlinear analysis is necessary to decide on the stability of $\bar{u}$ and on the dynamics of (5) in a neighborhood of $\bar{u}$.

Example: The logistic differential equation

$$\dot{u} = \lambda u(1-u), \qquad u(0) = u_0 \tag{6}$$

has the solution

$$\begin{aligned} u(t) &= \frac{u_o e^{\lambda t}}{1 + u_o(e^{\lambda t} - 1)} \\ &= \frac{u_0}{(1-u_o)e^{-\lambda t} + u_o}. \end{aligned} \tag{7}$$

It has two stationary states for all λ: $\bar{u} = 0$ and $\tilde{u} = 1$.
The principle of linearized stability reveals that
$\bar{u} = 0$ is asymptotically stable for $\lambda < 0$ and unstable for $\lambda > 0$,
$\tilde{u} = 1$ is unstable for $\lambda < 0$ and asymptotically stable for $\lambda > 0$.
For $\lambda = 0$, every constant is a stationary state. They all are non-hyperbolic.
For $\lambda < 0$,
all $u_0 < 1$ lie in the basin of attraction of $\bar{u} = 0$ and convergence is monotonic;
all $u_0 > 1$ lead to trajectories that grow unboundedly in finite time, i.e. to **blow-up solutions**. The **blow-up time** is

$$T = \frac{1}{-\lambda} \ln \frac{u_o}{u_o - 1} = \ln \left(\frac{u_o - 1}{u_o} \right)^{1/\lambda} > 0. \tag{8}$$

For $\lambda > 0$,

all $u_0 > 0$ lie in the basin of attraction of $\tilde{u} = 1$ and convergence is monotonic;

all $u_0 < 0$ lead to trajectories that tend to $-\infty$ in finite time T. The blow-up time is

$$T = \frac{1}{\lambda} \ln \frac{u_0 - 1}{u_0} = \ln \left(\frac{u_0 - 1}{u_0} \right)^{1/\lambda} > 0. \tag{9}$$

The logistic differential equation (and its name) were introduced by Verhulst in 1838 to model the growth of populations in environments with limited resources. Under certain conditions (no major wars, no epidemies (the plague) or other catastrophes inside the country), it is indeed a very good model. See for instance [8, p. 103], where the values computed for the US population by Pearl and Read in 1920 are compared with census data for the years 1790 to 1950.

3 Discrete dynamical systems

In this section we consider discrete dynamical systems and recall basic definitions and facts used in later sections. Consider

$$y_{n+1} = g(y_n), \qquad y_0 \in I\!R^N, \tag{10}$$

with continuously differentiable g. Such difference equations are uniquely solvable.

$\bar{y}$ is a *fixed point* of (10) iff $g(\bar{y}) = \bar{y}$.

$\bar{y}$ is a *periodic point with period* m of (10) iff $\bar{y} = g^m(\bar{y})$.

$\bar{y}$ is a *stable* fixed point of (10) iff for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $g^n(y_0) \in U_\varepsilon(\bar{y})$ for all $y_0 \in U_\delta(\bar{y})$ and all $n \geq 0$.

$\bar{y}$ is an *asymptotically stable fixed point* of (10) iff $\bar{y}$ is a stable fixed point of (10), and

$$\lim_{n\to\infty} |g^n(y_0) - \bar{y}| = 0 \quad \text{for all} \quad y_0 \in U_\delta(\bar{y}) \quad \text{for some} \quad \delta > 0.$$

$\bar{y}$ is an *(asymptotically) stable periodic point* of (10) with period m iff $\bar{y}, g(\bar{y}) \ldots, g^{m-1}(\bar{y})$ are (asymptotically) stable fixed points of $y_{n+1} = g^m(y_n)$.

$\bar{y}$ is a *hyperbolic* fixed point of (10) iff $|\mu| \neq 1$ for all eigenvalues μ of the Jacobian $g'(\bar{y})$.

By the implicit function theorem, hyperbolic fixed points $\bar{y}$ have a neighborhood $U(\bar{y})$ in which $g - id$ is invertible. If the local inverse is differentiable, it is a diffeomorphism. For hyperbolic fixed points and sufficiently smooth g a **Principle of Linearized Stability** is valid:

Let $\bar{y}$ be a hyperbolic fixed point of (10) $y_{n+1} = g(y_n)$ and let g be a diffeomorphism. Then there are neighborhoods $U(\bar{y})$ and $V(0)$ such that the dynamics of $y_{n+1} = g(y_n)$ in $U(\bar{y})$ and of $v_{n+1} = g'(\bar{y})v_n$ in $V(0)$ are equivalent.

For more details see Guckenheimer/Holmes, where this is called the *Theorem of Hartman-Grobman* [5, p. 18].

If $\bar{y}$ is hyperbolic, it is thus asymptotically stable if the spectral radius ρ of the Jacobian $g'(\bar{y})$ satisfies $\rho(g'(\bar{u})) < 1$; it is unstable if $\rho(g'(\bar{y})) > 1$.
If $\bar{y}$ is non-hyperbolic, it might be a bifurcation point (fixed point – fixed point or fixed point – periodic point (flip bifurcation)). In this case, a nonlinear analysis is necessary to decide on the dynamics in a neighborhood of $\bar{y}$.

Example: The logistic difference equation

$$y_{n+1} = \mu y_n(1 - y_n), \qquad y_0 \in [0,1], \qquad 0 < \mu < 4. \tag{11}$$

For $\mu \in [0,4]$, all iterates lie in the interval $[0,1]$ if y_0 does.
$v_1 = 0$ is a fixed point for all $\mu > 0$. It is the only fixed point in $[0,1]$ for $0 < \mu < 1$ and is asymptotically stable for $0 < \mu < 1$. It is unstable for $\mu > 1 =: a_1$.
For $\mu = 1$ there is a bifurcation with exchange of stability. A second branch of fixed points, $v_2(\mu)$, appears in the interval $[0,1]$: $v_2(\mu) = \frac{\mu-1}{\mu} \in [0,1]$ for $\mu \geq 1$. $v_2(\mu)$ is unstable for $\mu < 1$ and asymptotically stable for $1 < \mu < 3$. For $1 < \mu < 2$ convergence to v_2 is monotonic, for $2 < \mu < 3 =: a_2$ it is a damped oscillation.
In $\mu = 3$ this branch of fixed points loses stability in a flip bifurcation:
for $3 < \mu < 1+\sqrt{6} =: a_3$ there is an asymptotically stable 2-cycle $v_3 = g_\mu(v_4)$, $v_4 = g_\mu(v_3)$.
For $\mu = a_3$ there is another flip bifurcation to a 4-cycle. This 4-cycle is asymptotically stable for $a_3 < \mu < a_4$, etc.
The sequence of period-doubling bifurcations accumulates in $a_\infty \approx 3.5699\ldots$ with an aperiodic solution.

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} =: \delta \approx 4.669\ldots \tag{12}$$

is the **Feigenbaum** constant.

For $\mu > a_\infty$ periods other than powers of 2 are possible; first even periods, then also odd periods. For $\mu = 1 + \sqrt{8}$ period 3 occurs. For $\mu \geq 1 + \sqrt{8}$ all periods m are possible, and the iterates are chaotic in the sense of Li and Yorke [13, 5]. For $\mu > 4$, part of the iterates leave the interval $[0, 1]$ and converge to $-\infty$.

4 Forward Euler scheme

We discretize (6) $\dot{u} = \lambda u(1-u)$, $u(0) = u_0$ by Euler's method with fixed time step h and get

$$\begin{aligned} y_{n+1} &= y_n + \lambda\ h\ y_n\ (1 - y_n) \\ &= F_h(y_n), \qquad y_0 = u(0). \end{aligned} \tag{13}$$

The fixed points $\bar{y}$ of (13) satisfy $\lambda h \bar{y}(1-\bar{y}) = 0$ and are thus $\bar{y} = \bar{u} = 0$ and $\tilde{y} = \tilde{u} = 1$ for all λh. The Jacobian is

$$F_h'(y_n) = 1 + \lambda h - 2\lambda h y_n. \tag{14}$$

Let $\lambda = 1$ for the following analysis. Analysis for arbitrary $\lambda > 0$ only requires a rescaling of h. Analysis for $\lambda < 0$ is also similar, but $\bar{y}$ and $\tilde{y}$ then exchange their roles.

We get $F_h'(0) = 1 + h > 1$. Thus $\bar{y} = 0$ is unstable for all h, as is $\bar{u} = 0$.
Because $\bar{u} = 0$ is unstable, it has a neighborhood where trajectories sensitively depend on the initial value $u(0) = u_0$: If $0 < u_0 < 1$, then $\lim_{t\to\infty} u(t; u_0) = 1$ and $\lim_{t\to\infty} u(t; -u_0) = -\infty$, no matter how small $|u_0|$ is.
$\bar{y} = 0$ has a neighborhood with the same sensitive dependence on the initial value. *Because* the two problems behave qualitatively in the same way, there is a neighborhood of 0 in which rounding errors can produce completely wrong trajectories. This is another example of "**inherent instability** of difference schemes" .

We get $F_h'(1) = 1 - h$. Thus we get $|F_h'(1)| < 1$ for $0 < h < 2$ and $\tilde{u} = 1$ is stable for $0 < h < 2$.
1) For $0 < h < 1$ we get $0 < F_h'(1) < 1$.
If $y_o < 0$, the iterates tend monotonically to $-\infty$ for $n \to \infty$. Though the continuous solution exists only for $t < T$ as given by (9), the discrete iterates exist for all

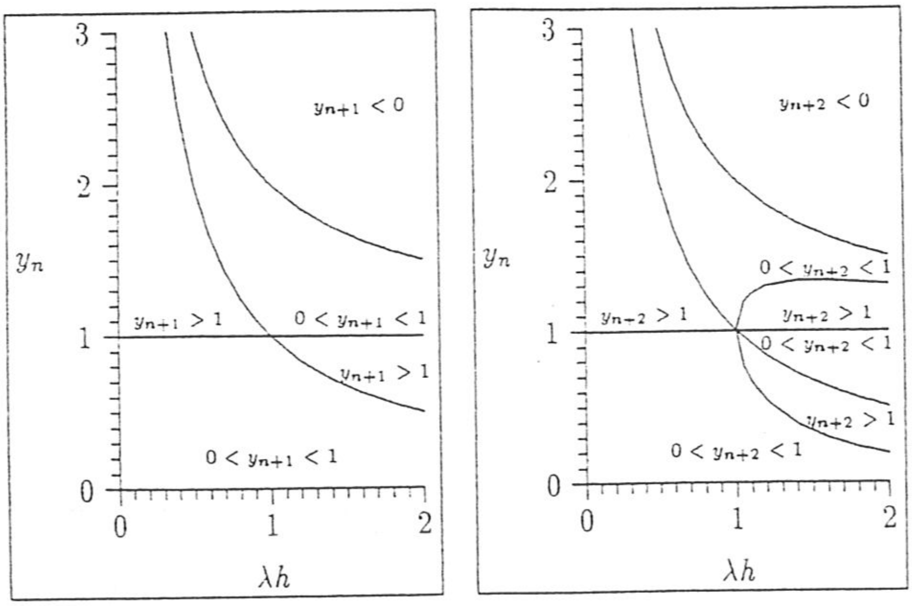

Fig. 1. Mapping properties of forward Euler in the $(y_n, \lambda h)$ plane:
Figure 1a shows to where y_n is mapped after one iteration. The limiting curves are $y_n = 0$; $y_n = 1$; $y_n = \frac{1}{\lambda h}$, $y_n = 1 + \frac{1}{\lambda h}$.
Figure 1b shows to where it is mapped after two iterations. The additional borders are given by $y_n = \frac{1}{2\lambda h}\left(1 + \lambda h \pm \sqrt{(-1 + \lambda h)(3 + \lambda h)}\right)$.

$t_n = n\,h$, $n \to \infty$. This was already noticed by Dahlquist not later than 1959 [16].
If $0 < y_0 < 1$, all trajectories $\{y_n\}_{n \in \mathbb{N}}$ grow monotonically to $\tilde{y} = 1$ and thus behave qualitatively correctly.
If $y_0 > 1$, some iterates "overshoot" and the trajectories tend to $-\infty$ for $n \to \infty$ [7]. This is the case for all (y_o, h) values above the curve $y_o = 1 + \frac{1}{h}$. Or they enter the region $0 < y < 1$ and continue monotonically. This is the case for all (y_o, h) values satisfying $\frac{1}{h} < y_o < 1 + \frac{1}{h}$. But $y_o = 1$ does have a neighborhood in which the discrete trajectories tend monotonically to $\tilde{y} = 1$ and thus behave qualitatively similarly to the continuous trajectories (see Fig. 1).
2) For $1 < h < 2$ we get $-1 < F_h'(1) < 0$, and the iterates oscillate in all neighborhoods of $\tilde{y} = 1$. Hence $\tilde{y} = 1$ does not have a neighborhood where trajectories behave qualitatively correctly. But they still converge to the correct limit for certain initial values. The dependence of the limit on the initial value y_0 is illustrated in Fig. 1. The curves were computed using Mathematica [12].

3) For $h = 2$ there is a flip bifurcation to the 2-cycle

$$\bar{y}_{3,4} = \frac{h + 2 \pm \sqrt{h^2 - 4}}{2h} \in \mathbb{R}, \tag{15}$$

which is stable for $2 < h < \sqrt{6}$. What happens for larger h can best be seen from the map [19]

$$v_n = \frac{h}{1+h}\, y_n, \tag{16}$$

which is a homeomorphism for $h > 0$ and maps

$$y_{n+1} = y_n + hy_n(1 - y_n) \tag{17}$$

to

$$v_{n+1} = (1+h)v_n(1 - v_n), \tag{18}$$

i.e. to the logistic map (11) with $\mu = 1 + h$. The numerical parameter h can thus produce all the peculiar behavior which is known for the logistic map, and which was briefly described in section 3. For $h > \sqrt{8}$ we get chaotic trajectories. A Feigenbaum diagram of (17) is shown in [11, Fig. 3].

Note that the homeomorphism (16) *must* break down for $h = 0$: the fixed points 0 and 1 of (17) are different from each other for all h, but the fixed points 0 and $\frac{h}{1+h}$ of (18) meet in a bifurcation point for $h = 0$.

5 Midpoint Euler scheme

For smooth one-step methods Beyn proved the following

Theorem [1]: *Let $\Omega \subset \mathbb{R}^N$ be compact and assume that*

$$\dot{u} = f(u), \qquad u(0) = u_0 \in \mathbb{R}^N \tag{19}$$

has finitely many stationary solutions v_i, $i = 1, \ldots, K$ in the interior of Ω, and that all v_i are regular, i.e. $f'(v_i)$ is invertible for $i = 1, \ldots, K$. Let ϕ be a smooth one-step method of order $p \geq 1$. Then there exists an $h_0 > 0$ such that the discrete system

$$y_{n+1} = \phi(h, y_n), \tag{20}$$

$h \le h_0$, has exactly K fixed points $v_i(h)$, $i = 1, \ldots, K$ in Ω, and these satisfy

$$v_i(h) = v_i + \mathcal{O}(h^p), \qquad i = 1, \ldots, K. \tag{21}$$

Moreover, if $Re\ \mu > 0$ for some eigenvalue μ of $f'(v_i)$, then $v_i(h)$ is an unstable fixed point of (20); and if $Re\ \mu < 0$ for all eigenvalues μ of $f'(v_i)$ then it is an asymptotically stable fixed point.

For Runge-Kutta schemes, (21) is too pessimistic: Runge-Kutta schemes exactly reproduce all stationary states of the differential equation (i.e. $v_i(h) = v_i$) [1, 9], but they often add some spurious fixed points. Bifurcation points between branches of proper fixed points and branches of spurious fixed points are characterized by Iserles et al. [9].

We shall apply these results to the scheme

$$\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f(y_n + \frac{h}{2}k_1), \\ y_{n+1} &= y_n + hk_2 \end{aligned} \tag{22}$$

for equation (6). It is a Runge-Kutta scheme sometimes called 'midpoint Euler scheme' [8] since it is derived by using the midpoint rule (or first Gauss formula) for integration [6, Chap. II, (1.4)].

Another formulation of (22) is

$$\begin{aligned} y_{n+1} &= y_n + hf(y_n + \frac{h}{2}f(y_n)) \\ &=: F_h(y_n). \end{aligned} \tag{23}$$

Since $f(u) = \lambda u(1-u)$ is a polynomial of 2nd order, $F_h(y_n)$ is a polynomial of 4th order. The equation $F_h(y) - y = 0$ thus always has four complex solutions. These turn out to be real for all h. They are [12]

$$0, \quad \frac{2}{\lambda h}, \quad 1, \quad 1 + \frac{2}{\lambda h}. \tag{24}$$

The spurious fixed points $\frac{2}{\lambda h}$, $1+\frac{2}{\lambda h}$ converge to the proper fixed points 0, 1 for $h \to \infty$. For $h \to 0$, both of them become unbounded. Note the connection between these spurious fixed points and the spurious curves governing the convergence for forward Euler (Fig. 1a): the factor 2 is due to the factor $\frac{h}{2}$ in the middle line of (22).

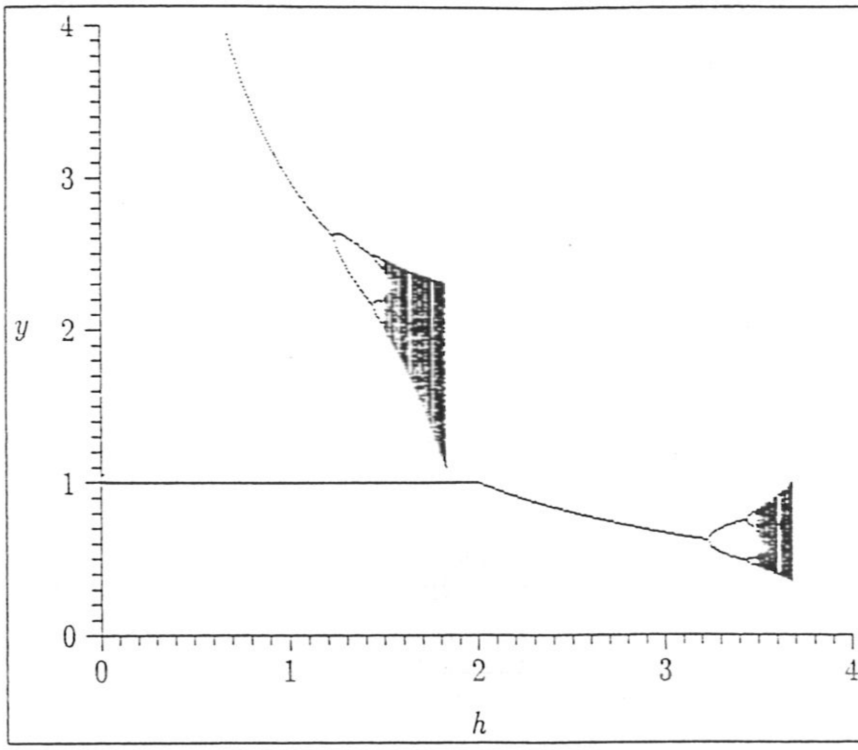

Fig. 2. Feigenbaum diagram for midpoint Euler in the (y, h)-plane, $\lambda = 1$. The 200th to 700th iterates are shown. The lower part of this figure is already given in [11, Fig. 4]. The upper part is missing there.

Applying the principle of linearized stability in the case $\lambda = 1$ gives:
$\bar{y} = 0$ is unstable for all h.
$\tilde{y} = 1$ is stable for $0 < h < 2$, and convergence is monotonic for $0 < y_o < \frac{2}{h}$. $\tilde{y}$ loses its stability to $\bar{y}_3 = \frac{2}{h}$. This happens in a bifurcation point [11]: the two stationary states $\tilde{y}(h) \equiv 1$ and $\bar{y}_3(h) = \frac{2}{h}$ meet for $h = 2$ and exchange stability there.
$\bar{y}_3 = \frac{2}{h}$ is stable for $2 < h < 1 + \sqrt{5} \approx 3.24$ and loses stability to a Feigenbaum cascade of period-doubling bifurcations.
$\bar{y}_4 = 1 + \frac{2}{h}$ is stable for $0 < h < -1 + \sqrt{5} \approx 1.24$ and loses stability to a Feigenbaum cascade of period-doubling bifurcations.

This example demonstrates how closely related are the size of the compact domain Ω and the step size h_0 in Beyn's theorem: if we choose $\Omega = [-\omega, 1 + \varepsilon]$, then $h_0 < \frac{2}{1+\varepsilon}$, in order to exclude the spurious unstable fixed point $\bar{y}_3(h) = \frac{2}{h}$. Thus for small $\varepsilon > 0$, h_0 is nearly given by the stability limit of the method. If we choose $\Omega = [-\omega, 3]$, then $h_0 < \frac{2}{3}$. Figure 2 shows the stable fixed points of (22) with $f(y) = y(1 - y)$, and their transition to

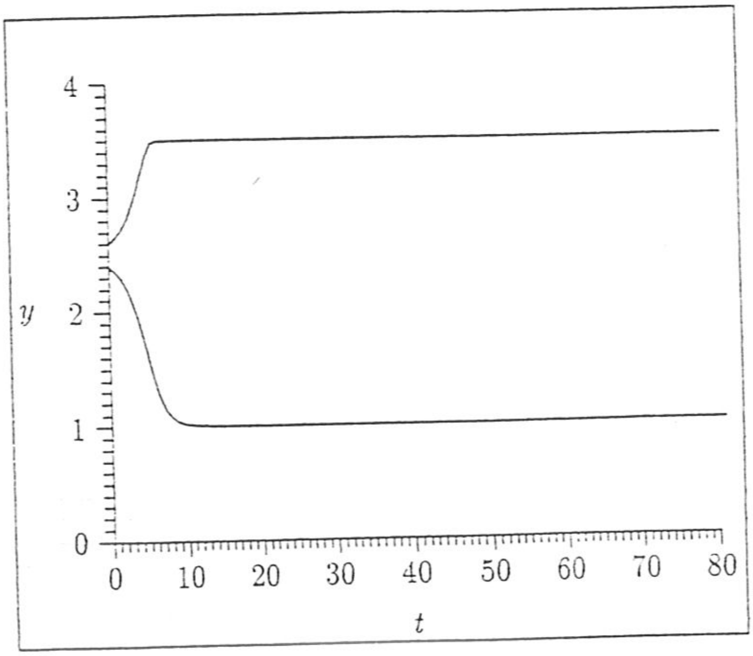

Fig. 3. Midpoint Euler, $\lambda = 1, \ h = 0.8$.
For $y_o = 2.4 < 2/h = 2.5$ the trajectory converges to the proper fixed point $\tilde{y} = 1$.
For $y_o = 2.6 > 2/h$ it converges to the spurious stable fixed point $\bar{y}_3 = 1 + 2/h = 3.5$.

chaos. The lower part of this figure was already given in [11, Fig. 4]. Iserles has already pointed out very clearly that spurious fixed points are unwelcome. It requires at least two runs with different h to detect their h-dependence and thus the fact that they are spurious.

Figures 3 – 6 comment on Fig. 2. They show: in practical computations it depends on the initial value, by which state the trajectories are attracted.

6 Rational Euler schemes

In the numerical treatment of systems of partial differential equations, it is common practice to solve initial boundary value problems for $t \to \infty$ in search of steady-states. These indirect methods need less storage but much more computing time than methods which solve the steady state equations directly. Reasonable spatial discretization of spatially 3-dimensional problems is very often only feasible by time-dependent methods. It is thus

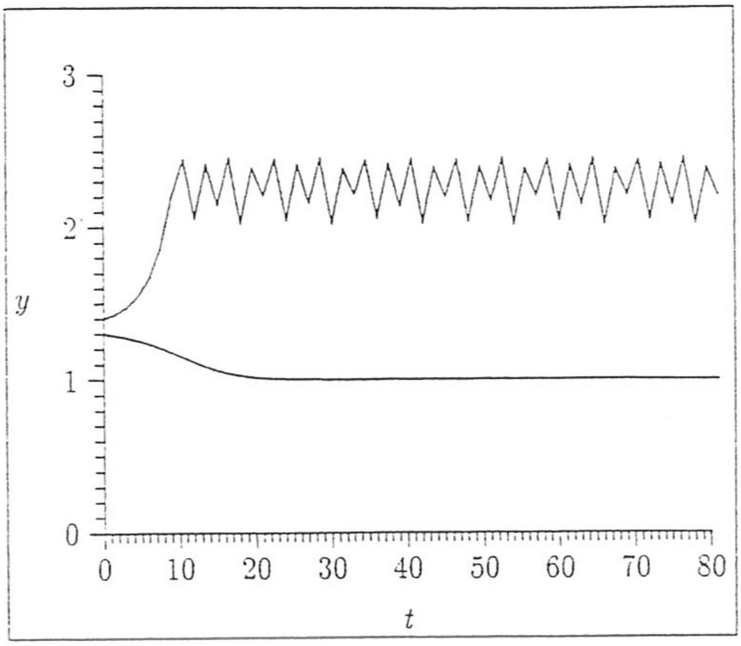

Fig. 4. Midpoint Euler, $\lambda = 1$, $h = 1.5 > \sqrt{5} - 1 \approx 1.24$:
For $y_o = 1.3 < 2/h = 4/3$, the trajectory converges to the proper solution $\tilde{y} = 1$.
For $y_o = 1.4 > 2/h$, it converges to the stable spurious solution of period 4.

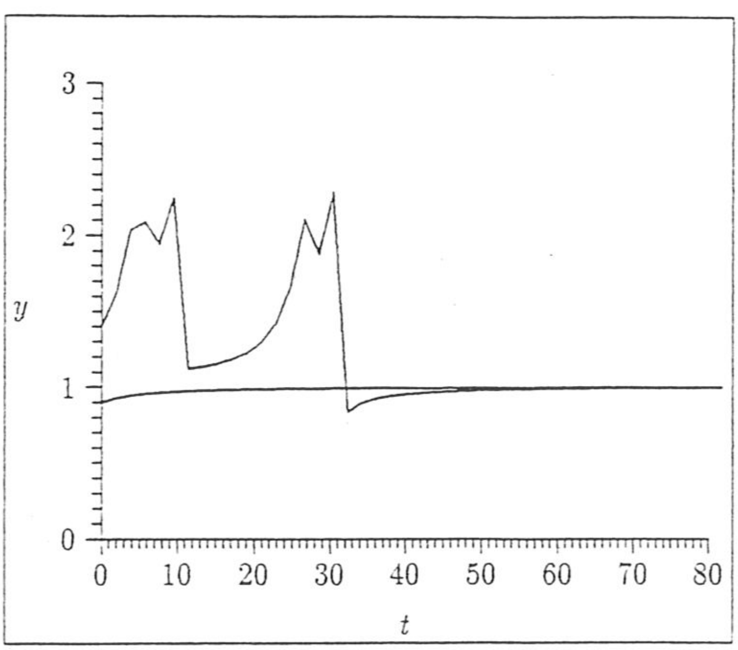

Fig. 5. Midpoint Euler, $\lambda = 1$, $h = 1.9$:
For $y_o = 1.4 > 2/h$, the iterates first wander in the chaotic regime of the spurious stable branch, then they enter the basin of attraction of $\tilde{y} = 1$.
For $y_o = 0.9$, they converge monotonically to $\tilde{y} = 1$.

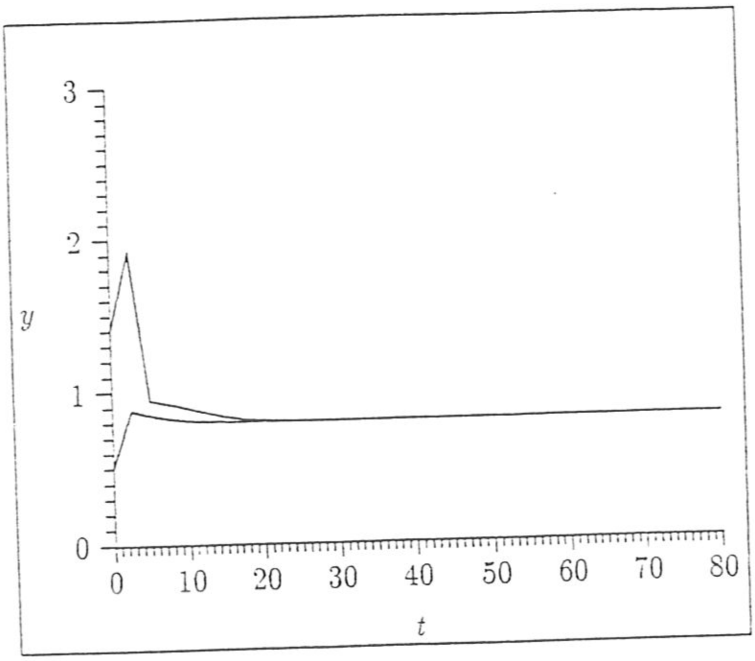

Fig. 6. Midpoint Euler, $\lambda = 1$, $h = 2.5$: For both initial values $y_o = 0.5$ and $y_o = 1.4$ the trajectory converges to the stable spurious solution $\bar{y} = 0.8$.

desirable to compute with the largest possible temporal step size h. The stability of the method is the most severe bound on h. Also in the numerical modeling of magneto-hydrodynamic turbulence it is desirable to compute with step sizes as large as possible. In this case, the step size is limited by two considerations: the step size must be short enough to model the phenomena of interest accurately, and the scheme must be stable. A big advantage of implicit methods is that they usually allow much larger step sizes than explicit methods without going unstable.

From our point of view, using the fully implicit backward Euler scheme does not make sense for the model problem (6): The equation

$$y_{n+1} = y_n + \lambda h y_{n+1}(1 - y_{n+1})$$

has two real (or two complex) solutions y_{n+1} for any given y_n. Therefore each time step would require a decision which one of the two possible y_{n+1}-values should be chosen. We think that such a multiply solvable discrete initial value problem is not an acceptable discrete model of a uniquely solvable continuous initial value problem.
The Rössler system treated by Corless et al. [3] by the backward Euler scheme does contain quadratic terms and it is likely that the nonlinear discrete equations do have

more than one solution in most time steps for most initial values in the parameter range in which chaos occurs. The authors used Newton's method and continuation in h as their strategy for picking the values in the $n+1$st time step. It might have been this strategy that suppressed chaos.

Another reason why fully implicit methods are not very practical is the following: If the amount of work needed for solving the nonlinear system in each time step is so large that the whole computation with the implicit scheme is as time-consuming as computations with an explicit scheme of same accuracy with smaller time steps, it might not be worth the effort. Hence partially implicit schemes are used a lot in the numerical solution of systems of ordinary differential equations derived from systems of partial differential equations [2, 4].

For our model problem (6), Twizell et. al. [18] introduced and analyzed the partially implicit schemes

$$y_{n+1} = y_n + \lambda h y_{n+1}(1 - y_n) \tag{25}$$

and

$$y_{n+1} = y_n + \lambda h y_n(1 - y_{n+1}). \tag{26}$$

Both can be transformed into rational schemes

$$y_{n+1} = \frac{y_n}{1 - \lambda h(1 - y_n)} =: g_o(y_n; \lambda h) \tag{27}$$

and

$$y_{n+1} = \frac{(1 + \lambda h)y_n}{1 - \lambda h(1 - y_n)} =: g_1(y_n; \lambda h). \tag{28}$$

They are related to each other in an obvious way: each of them treats one of the two stationary states implicitly, the other one explicitly. They are **adjoint** to each other in the sense of Definition 8.2 of [6, Chap. II]: the map $h \mapsto -h$; $y_{n+1} \mapsto y_n$; $y_n \mapsto y_{n+1}$ replaces scheme (25) by scheme (26), and scheme (26) by scheme (25).

A substantial part of the analysis by Twizell et al. was done by "extensive tests carried out in [18] for many values of λh and y_o"[19, p. 514]. Also, their way of considering vanishing or sign-changing denominators was not adequate [14]. We thus fully describe here the stability properties of schemes (27) and (28).

As was shown in [14], the difference equation (27) has the solution

$$y_n = \frac{y_o}{(1-\lambda h)^n(1-y_o)+y_o}, \tag{29}$$

and the difference equation (28) has the solution

$$y_n = \frac{(1+\lambda h)^n y_o}{1+y_o((1+\lambda h)^n-1)}. \tag{30}$$

Both (29) and (30) are approximations to (7), with $e^{\pm\lambda h}$ replaced by the first two terms of their Taylor expansion. $1 \pm \lambda h$ is a qualitatively correct approximation to $e^{\pm\lambda h}$ for those λh for which $1\pm\lambda h > 0$, i.e. for $\pm\lambda h > -1$. From linear stability theory and formulas (29) and (30) it could thus be expected that scheme (27) converges monotonically for $\lambda h < 1$, converges oscillatorily for $1 < \lambda h < 2$, and is unstable for $\lambda h > 2$ (analogously for scheme (28)). As was explained earlier, equation (6) has several different types of trajectories for each λ. The performance of the scheme should hence be considered separately for each type of trajectory.

We look at scheme (27) and give a review of all findings first.
For $\lambda < 0$, $\bar{y} = 0$ is stable for all h, and $\tilde{y} = 1$ is unstable for all h. Trajectories with initial value $y_o \leq 1$ behave qualitatively correctly for all h. Trajectories with initial value $y_o > 1$ behave qualitatively correctly as long as the blow-up time T has not passed, i.e. as long as

$$t_N := \sum_{n=1}^{N} nh < T = \ln\frac{y_o-1}{y_o}. \tag{31}$$

For $\lambda > 0$ and $\lambda h < 2$, $\bar{y} = 0$ is unstable and $\tilde{y} = 1$ is stable. Trajectories with arbitrary initial value y_o behave qualitatively correctly for $\lambda h < 1$ (as long as the blow-up time has not passed in the blow-up case). For $1 < \lambda h < 2$, convergence to the correct limit is oscillatory. For $\lambda h > 2$, both fixed points have the wrong stability, blow-up is disguised, and the spuriously stable fixed point $\bar{y} = 0$ is globally attracting. Hence the whole dynamics is wrong for $\lambda h > 2$, but 'looks perfectly alright' if there is no pre-knowledge of the behavior of trajectories and if blow-up solutions are not expected. That the dynamics is wrong is much harder to detect for this scheme on a 'real life problem' than for the other two schemes investigated here: in the other two cases, the stable spurious solutions are h-dependent (see (15), (24), ...) and can thus be revealed by two computations with the same scheme and different h. In this case here, a different scheme should be used for comparison. D. Düchs reported that, using a partially implicit method, he found an h-independent solution which he did not expect from the physics of the problem treated. This solution disappeared when he used a different difference method [4].

We now consider scheme (27) in detail. We shall first discuss the case $\lambda < 0$ and then the case $\lambda > 0$.

Let $\lambda < 0$. Then $\bar{y} = 0$ is a stable fixed point of scheme (27) for all h and $\bar{y} = 1$ is an unstable fixed point of (27) for all h:

From (27) we get

$$g_o'(y_n; \lambda h) = \frac{1 - \lambda h}{(1 - \lambda h + \lambda h y_n)^2} \tag{32}$$

and thus

$$0 < g_0'(0; \lambda h) = \frac{1}{1 - \lambda h} < 1 \quad \text{for all } \lambda h < 0 \tag{33}$$

and

$$g_o'(1; \lambda h) = 1 - \lambda h > 1 \qquad \text{for all } \lambda h < 0. \tag{34}$$

Trajectories with initial value $y_o < 1$ converge monotonically to $\bar{y} = 0$:

$$y_n < 1 \Rightarrow 1 - y_n > 0 \Rightarrow -\lambda h(1 - y_n) > 0 \Rightarrow 1 - \lambda h(1 - y_n) > 1 \Rightarrow$$

$$|y_{n+1}| = \frac{|y_n|}{1 - \lambda h(1 - y_n)} < |y_n|. \tag{35}$$

For $y_o > 1$, the qualitatively correct behavior of the iterates depends on the size of $|\lambda h|$ and of the iteration index n: If $-\lambda h > 0$ is small enough, it follows that $0 < 1 - \lambda h(1 - y_o) < 1$, and thus $y_1 > y_o > 1$. For all λh and n with $0 < 1 - \lambda h(1 - y_n) < 1$ we thus get $y_{n+1} > y_n$ and $1 - \lambda h(1 - y_n) > 1 - \lambda h(1 - y_{n+1})$. For computations with fixed step size h, either there is an N with

$$1 - \lambda h(1 - y_N) > 0 \quad \text{and} \quad 1 - \lambda h(1 - y_{N+1}) < 0, \tag{36}$$

or it happens that

$$1 - \lambda h(1 - y_N) = 0. \tag{37}$$

In the case of eq. (37), the iteration comes to a stop, blow-up has happened. In the case of (36), the denominator changes sign without vanishing. The following iterates are negative and approach $\bar{y} = 0$ from below. This is a discrete analog of "a rational function passes a pole and returns from $-\infty$". In the case considered here, however, iteration for $n > N$ does not make sense. The iterates do not approximate the solution $u(t; y_o)$ of the differential equation anymore. They do approximate the solution $u(t; y_{N+2})$ with initial value $y_{N+2} < 0$ for $n \geq N+2$.

In [14], discrete and continuous blow-up times were briefly compared. The general formula for case (37) is

$$(1-\lambda h)^N = \frac{u_o}{u_o - 1} = e^{-\lambda T}.$$

If we choose h so large that one single step causes blow-up, we get an upper bound for the continuous blow-up time T, and the error is of order $h^2 < 1$ [14]:
In the special case $\lambda = -1,\ N = 1,\ u_o > 2$ this results in $T = h - \frac{h^2}{2} + \frac{h^3}{3} - + \cdots$.

Let $\lambda > 0$. Then $\bar{y} = 0$ is unstable and $\tilde{y} = 1$ is stable for $\lambda h < 2$.
For $\lambda h > 2$, both fixed points of (27) show the wrong stability properties: $\bar{y} = 0$ is stable and $\tilde{y} = 1$ is unstable.
From (32) we get

$$g_o'(0; \lambda h) = \frac{1}{1-\lambda h},$$

and this satisfies
$g_o'(0; \lambda h) > 1$ for $0 < \lambda h < 1$,
$g_o'(0; \lambda h) < -1$ for $1 < \lambda h < 2$, and
$g_o'(0; \lambda h) > -1$ for $\lambda h > 2$.
Note that $g_o'(0; \lambda h)$ is singular for $\lambda h = 1$.
For $\tilde{y} = 1$ we get

$$g_o'(1; \lambda h) = 1 - \lambda h \tag{38}$$

and this satisfies
$0 < g_o'(1; \lambda h) < 1$ for $0 < \lambda h < 1$,
$-1 < g_o'(1; \lambda h) < 0$ for $1 < \lambda h < 2$,
$g_o'(1; \lambda h) < -1$ for $2 < \lambda h$.

We now show that convergence is monotonic for $0 < \lambda h < 1$ and *all initial values* y_o, as long as the blow-up time has not passed. This readily follows by using formula (27)

$$y_{n+1} = \frac{y_n}{1 - \lambda h(1-y_n)} \tag{39}$$

and

$$1 - y_{n+1} = \frac{(1-\lambda h)(1-y_n)}{1-\lambda h(1-y_n)}. \qquad (40)$$

Let $0 < y_n < 1$. Then $y_n < y_{n+1} < 1$:

$0 < 1 - y_n < 1 \Rightarrow 0 < 1 - \lambda h(1-y_n) < 1$ and $y_{n+1} > y_n$ from (39).

From (40) it follows that $y_{n+1} < 1$.

Let $1 < y_n$. Then $1 < y_{n+1} < y_n$:

$y_n > 1 \Rightarrow 1 - \lambda h(1-y_n) > 1 \Rightarrow y_{n+1} < y_n$ by using eq. (39).

From (40) we now get $1 - y_{n+1} < 0$.

Thus the stable fixed point $\tilde{y} = 1$ attracts all trajectories with initial value $y_o > 0$. If $y_n < 0$, it follows from (39) and (40) that

$y_{n+1} < y_n$ if $1 - \lambda h(1-y_n) > 0$ and $y_{n+1} > 0$ if $1 - \lambda h(1-y_n) < 0$.

What has been said earlier about approximation of blow-up solutions applies here analogously.

As far as scheme (28) is concerned, everything is very similar. This follows from the fact that both schemes are adjoint to each other. Scheme (28) thus converges monotonically to the correct fixed points for $\lambda h > -1$ and all initial values y_o, as long as the blow-up time has not passed. For $-2 < \lambda h < -1$ the stable fixed point $\tilde{y} = 0$ has a neighborhood of oscillating convergence. For $\lambda h < -2$ the stability of both fixed points disagrees with the stability of the stationary states of the differential equation, blow-up is disguised, and the whole dynamics is wrong.

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