MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Collisional Drift Fluids and Drift Waves

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IPP 6/329 May 1995

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

The usual theoretical description of drift-wave turbulence (considered to be one possible cause of anomalous transport in a plasma), e.g. the Hasegawa-Wakatani theory, makes use of various approximations, the effect of which is extremely difficult to assess. This concerns in particular the conservation laws for energy and momentum. The latter is important as concerns charge separation and resulting electric fields which are possibly related to the L-H transition. Energy conservation is crucial for the stability behaviour; it will be discussed via an example. New collisional multispecies drift-fluid equations were derived by a new method which yields in a transparent way conservation of energy and total angular momentum, and the law for energy dissipation. Both electrostatic and electromagnetic field variations are considered. The method is based primarily on a Lagrangian for dissipationless fluids in drift approximation with isotropic pressures. The dissipative terms are introduced by adding corresponding terms to the ideal equations of motion and of the pressures. The equations of motion , of course, no longer result from a Lagrangian via Hamilton's principle. Their relation to the ideal equations imply, however, also a relation to the ideal Lagrangian of which one can take advantage. Instead of introducing heat conduction one can also assume isothermal behaviour, e.g. $T_\nu(\mathbf{x}) = const.$ Assumptions of this kind are often made in the literature. The new method of introducing dissipation is not restricted to the present kind of theories; it can equally well be applied to theories such as multi-fluid theories without using the drift approximation of the present paper.

Linear instability is investigated via energy considerations and the implications of taking ohmic resitivity into account are discussed. A feature of the results is that for purely electrostatic perturbations the second spatial derivative of the density profile plays a role, contrary to the usual approximations.

For a class of systems with $T_i = 0$, it is shown that linear instability can occur only when the resitivity is sufficiently large, while the Hasegawa-Wakatani theory predicts instability for arbitraryly small nonvanishing resistivity.

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An example which will be treated in a future paper indicates in addition that in systems with vanishing ion temperature electron temperature profiles should strongly influence the stability via resistive effects. This is in addition to effects leading to η_e-modes. It demonstrates also that in general it is not possible to do an expansion with respect to the resistivity η near $\eta = 0$.

The new formalism is interesting not only from a theoretical point of view, but also in particular as a useful tool for numerical calculations.

I. Introduction

Drift-wave turbulence is considered to be one of the possible causes of anomalous transport in a plasma. An exact theory describing these phenomena would be extremely complicated. It is therefore desirable to simplify the theoretical description. An attempt in this direction is the widely used theory of Hasegawa and Wakatani.[1,2] While the steps leading from a two fluid theory to this approximate theory appear to be plausible to a certain extent, there may be problems, as can be illustrated by considering the relation interpreted as the time derivative of the energy. In dimensionless variables the Hasegawa-Wakatani equations, describing quasi-neutral two-fluid plasmas with massless electrons of constant temperature, cold ions, constant magnetic field in the z-direction, unperturbed density gradients in the x-direction and electrostatic perturbations, have the form

$$\begin{aligned} \partial_t \omega + \mathbf{v}\cdot\nabla\omega &= \frac{1}{\nu}\nabla_{\|}^2(\delta n - \varphi) \, , \\ \partial_t \delta n + \partial_y \varphi + \mathbf{v}\cdot\nabla\delta n &= \frac{1}{\nu}\nabla_{\|}^2(\delta n - \varphi) \, , \\ \mathbf{v} = \hat{\mathbf{z}}\times\nabla\varphi \quad , \quad \omega &= \nabla_{\perp}^2\varphi \, . \end{aligned} \tag{1}$$

In contrast to the usual notation we have introduced a dimensionless collision frequency ν in order to indicate which terms relate to dissipation. The right hand sides of the two first equations result from Ohm's law for the parallel current density

$$j_{\|} = \frac{1}{\nu}\nabla_{\|}(\delta n - \varphi) \, . \tag{2}$$

Equations (1) yield the relation

$$\frac{d}{dt}\int \frac{1}{2}\left((\nabla_{\perp}\varphi)^2 + (\delta n)^2\right) dV = -\int \delta n \partial_y \varphi dV - \frac{1}{\nu}\int \left(\nabla_{\|}(\delta n - \varphi)\right)^2 dV \, . \tag{3}$$

The second term on the r.h.s. is ohmic dissipation. The first term on the r.h.s. is nonzero for $\nu \neq 0$. It is the driving term for turbulence. Since it is proportional to the momentum of the plasma in the x-direction, it should be a constant of motion for the system considered, i.e.

$$\int \delta n \hat{\mathbf{z}}\times\nabla\varphi \, dV = const. \tag{4}$$

If this vanishes initially, the above driving term would be zero for all times. This is at variance with numerical solutions.

A dicussion of a class of systems which includes the model corresponding to Eqs. (1) within the framework of the theory presented in this paper is found in Section VII.

The main part of this paper presents a derivation of collisional drift-fluid equations by a new method which guarantees and enables one to prove in a concise and consistent way conservation laws such as for energy and total angular momentum and to obtain the law for energy dissipation for cases in which the dissipated energy does not remain in the plasma as thermal energy. Energy conservation is of relevance for the stability behaviour, in particular for nonlinear instabilities relating to negative-energy perturbations[3–18]. The method consists of three steps:

1. Consideration of single particle motion in drift approximation; description in terms of Littlejohn's Lagrangian for particles in drift approximation[19] in the form given by Wimmel[20];
2. Obtaining, from the single particle Lagrangian, the Lagrangian for dissipationless fluids in drift approximation with isotropic pressures;
3. Introduction of dissipative terms such as resistivity, thermal forces, viscosity, heat conductivity and energy transfer between the different particle species.

Both electrostatic and electromagnetic field variations are considered. The dissipative terms are introduced by adding corresponding terms to the ideal equations of motion and of the pressures. These equations result, of course, no longer from a Lagrangian via Hamilton's principle. Their relation to the ideal equations imply, however, also a relation to the ideal Lagrangian of which one can take advantage. Instead of introducing heat conduction one can also prescribe $T(\mathbf{x}) = const$ or $\mathbf{B} \cdot \nabla T = 0$ initially and take the adiabatic coefficients $\gamma = 1$. This preserves then the initial property of $T(\mathbf{x})$ for all times. Such kind of assumptions are often made in the literature (Hasegawa et al.[1,2,21] and related papers). In this case the dissipated energy is not retained in the plasma as thermal energy, which means that the time derivative of the energy does not vanish and is negative.

This new method is not restricted to the present kind of theories, it can equally well be applied to theories such as multi-fluid theories without using the drift approximation of the present paper.

In order to be flexible in introducing approximations relating to the masses without destroying the Hamiltonian nature of the theory, we replace the masses by mass tensors which distinguish between masses for motions parallel and perpendicular to the magnetic field. This allows different approximations for the parallel and perpendicular motions. We start by making no assumption as to the ratio m_e/m_i of the electron to the ion mass. Later we also let this ratio go to zero. If in addition to this one also requires, as in the Hasegawa-Wakatani theory, $m_{i\|} \to \infty$, the component of the ion velocity parallel to the magnetic field goes to zero, but the corresponding momentum will stay finite and non-zero.

The paper is organized as follows: In Sec. II the single particle Lagrangian for the guiding centers is introduced. In Sec. III we construct the Lagrangian for dissipationless fluids in drift approximation. In Sec. IV, expressions for the total variations of the Lagrangian densities are derived. Then, in Secs. IV A and B, Hamilton's principle is applied to obtain, respectively, implicit and explicit forms of the Euler-Lagrange equations. The conservation laws for energy and momentum in nondissipative systems are derived in Secs. V A and B, respectively, using Noether's formalism, while dissipative systems are treated in Secs. VI A and B. As an illustration, a class of examples, which includes the Hasegawa-Wakatani example presented at the beginning, is considered in Sec. VII. This is done within the frame of a general discussion which includes several problems: 1) To what extent does finite resistivity imply that the systems are not isolated, so that energy is fed into perturbations or removed from them by coupling to an external circuit? 2) To what extent does an electrostatic approximation imply a kind of coupling to something similar to an external circuit? This has to do with the question whether the neglect of magnetic perturbations is justified in systems with $\beta << 1$. 3) Linear instability is investigated via energy considerations and the implications of taking resistivity into account are discussed. The results are then summarized in Sec. VIII. Some useful results are derived in the appendices: in Appendix A the free energy term which appears in the Lagrangian density of an isothermal plasma is derived from thermodynamical considerations. In Appendix B the equations of motion of the quasi-neutral guiding center fluid are used to derive in a way alternative to Noether's formalism the local and global energy conservation laws in nondissipative systems. The properties of symmetry displacements, which are needed to derive momentum conservation laws using the Noether formalism, are derived in Appendix C. Finally, in Appendix D, a simple example is presented to illustrate how a hermitian operator which is made nonhermitian by a widely used approximation can be the source of artificial instabilities.

II. Single particle Lagrangian

Littlejohn's Lagrangian for particles in drift approximation[19] in the form given by Wimmel[4] is

$$L_p = \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}^* - e\Phi^* \tag{5}$$

with

$$\begin{aligned}
\mathbf{A}^* &= \mathbf{A} + \frac{mc}{e}(q_4\mathbf{b} + \mathbf{v}_E) , \\
e\Phi^* &= e\Phi + \mu B + \frac{m}{2}(q_4^2 + \mathbf{v}_E^2) , \\
\mathbf{v}_E &= c\frac{\mathbf{E}\times\mathbf{B}}{B^2} , \\
\mathbf{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} , \quad \mathbf{B} = \nabla\times\mathbf{A} , \quad \mathbf{b} = \frac{\mathbf{B}}{B} , \\
\mu &= \text{magnetic moment.}
\end{aligned} \tag{6}$$

q_4 is an additional variable to be varied indepently of $\mathbf{x}$.

III. Lagrangian for fluids in drift approximation

With indices for particle species suppressed, the following expression, as explained below, should be the Lagrangian of a multi-fluid plasma in drift approximation:

$$L = \int \mathcal{L}_\Sigma d^3x , \quad \mathcal{L}_\Sigma = \sum_{\text{particle species}} \mathcal{L} ,$$

$$
\begin{aligned}
\mathcal{L} &= n(\mathbf{x},t)\hat{L}_{\mathrm{p}} - \frac{p}{\gamma-1} - \frac{1}{8\pi}\left(\mathbf{B}^2 - \mathbf{E}^2\right) \\
n(\mathbf{x},t) &= \text{density of quasiparticles,} \\
\hat{L}_{\mathrm{p}} &= \frac{e}{c}\,\mathbf{v}(\mathbf{x},t)\cdot\hat{\mathbf{A}} - e\hat{\Phi}\ , \\
\mathbf{v}(\mathbf{x},t) &= \text{velocity of quasiparticles,} \\
\hat{\mathbf{A}} &= \mathbf{A}^* = \mathbf{A} + \frac{c}{e}\underline{\underline{m}}\cdot(q_4(\mathbf{x},t)\mathbf{b} + \mathbf{v}_E)\ , \\
&= \mathbf{A} + \frac{c}{e}\left(m_{||}q_4(\mathbf{x},t)\mathbf{b} + m_\perp\mathbf{v}_E\right) \\
e\hat{\Phi} &= e\Phi^* - \mu B = e\Phi + \frac{1}{2}(m_{||}q_4^2 + m_\perp\mathbf{v}_E^2)\ , \qquad (7)
\end{aligned}
$$

where

$$
\underline{\underline{m}} = m_\perp\underline{\underline{I}} + (m_{||} - m_\perp)\mathbf{bb} \quad , \quad I_{ik} = \delta_{ik} \quad . \qquad (8)
$$

Because of the quasi-neutrality condition, the electromagnetic potentials are assumed to be single-valued, such that for instance in a toroidal system the electric field corresponding to a loop voltage must be represented by $-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$. This form of the Lagrangian refers to adiabatic systems. Isothermal systems have a different pressure term. As shown explicitly in appendices A and B, their Lagrangian density is

$$
\mathcal{L} = n(\mathbf{x},t)\hat{L}_{\mathrm{p}} - p\ln\frac{p}{p_c} - \frac{1}{8\pi}\left(\mathbf{B}^2 - \mathbf{E}^2\right)\ , \qquad (9)
$$

where p_c is an arbitrary reference pressure. Most equations below are written with the adiabatic pressure term. For isothermal systems, one then simply replaces this with the proper pressure term. A separate treatment is done only when necessary. Here, the particle mass is expressed as a tensor with different masses for the parallel and perpendicular motion. This is simply *an artificial measure* which facilitates the discussion of different approximations introduced later. The usual representation is obtained by setting $m_{||} = m_\perp = m$.

$p(\mathbf{x},t)$ is the isotropic preassure for which an adiabatic law is assumed, replacing the adiabatic constancy of the magnetic moment μ and a "thermal" energy of the parallel motion. A closer relation to having $\mu =$ const. would

Erratum

Page 7, Eq. (7):

The electromagnetic contribution to the Lagrangian density $\mathcal{L}$ for each particle species must be divided by the number N_s of the particle species. Similar corrections have to be made in a number of other equations, which should be obvious in all cases.

In a few equations one has to replace

$$M \to \mathcal{M} \text{ and } P \to \mathcal{P} \, ,$$

which should also be obvious in all cases.

be a double adiabatic theory like the Chew-Goldberger-Low theory.
When the quasi-neutrality approximation is used, no electric field energy term $\mathbf{E}^2/8\pi$ appears in L; there is only the usual coupling term $en\Phi$ for charged particles in an electric field. This implies that one cannot require within the framework of quasi-neutral theory that the total electric field energy be bounded. Only the thermal and kinetic energy, and also the magnetic energy of systems surrounded somewhere by infinitely conducting walls must be finite.
The notion " quasiparticle " is introduced for entities which perform motions parallel to $\mathbf{B}$ with the velocity $v_{\|}(\mathbf{x},t)$, and perpendicular to $\mathbf{B}$ in the form of drift motions consisting of $\mathbf{E}\times\mathbf{B}$, polarization and centrifugal drifts, the usual additional drifts related to the variation of the direction of $\mathbf{B}$, and a diamagnetic drift. This is in contrast to guiding center motion as concerns the diamagnetic drift, which replaces the ∇B drift resulting from the Lorentz force.
Quantities to be varied are

$$n(\mathbf{x},t)\ ,\ p(\mathbf{x},t)\ ,\ \mathbf{v}(\mathbf{x},t)\ ,$$

and in addition independently

$$q_4(\mathbf{x},t)\ ,\ \Phi(\mathbf{x},t)\ ,\ \mathbf{A}(\mathbf{x},t)\ ,$$

The reasons for the differences compared with the single particle Lagrangian are:

1. q_4 corresponds to $v_{\|}$ of a single particle. For a fluid q_4^2 has to be replaced by its average

$$< q_4^2 >=< (q_4- < q_4 >)^2 > + < q_4 >^2\ ;$$

2. then, the following replacement must be made for an isotropic fluid:

$$n(\mathbf{x},t)\left(\mu B + \frac{m_{\|}}{2} < (q_4- < q_4 >)^2 >\right) \rightarrow \frac{p}{\gamma - 1}\ ;$$

this replacement is analogous to the one leading from a Lagrangian for real particles to the Lagrangian of an ideal normal fluid.

3. the adiabatic constancy of μ has to be replaced by an adiabatic law for the isotropic pressure p;

4. the notation is simplified by

$$< q_4 > \rightarrow q_4(\mathbf{x}, t) \; ;$$

this is done also in the quantity $\hat{\mathbf{A}}$ which contains q_4 linearly.

IV. Variation of the Lagrangian

It is straightforward to derive the total variation $\delta\mathcal{L}$ of the Langrangian density $\mathcal{L}$. Since

$$\begin{aligned}
\mathcal{L} &= n\hat{L}_{\mathrm{p}} - \frac{p}{\gamma - 1} - \frac{1}{8\pi}\left(\mathbf{B}^2 - \mathbf{E}^2\right) \\
&= \frac{e}{c}n\mathbf{v}\cdot\mathbf{A} + n\mathbf{v}\cdot\left[m_{||}q_4\mathbf{b} + m_{\perp}\mathbf{v}_E\right] - en\Phi \\
&\quad -\frac{1}{2}n\left[m_{||}q_4^2 + m_{\perp}\mathbf{v}_E^2\right] - \frac{p}{\gamma-1} - \frac{1}{8\pi}\left(\mathbf{B}^2 - \mathbf{E}^2\right) \\
&= \mathcal{L}\left(q_4, n, p, \mathbf{v}, \Phi, \mathbf{A}, \mathbf{E}(\Phi, \mathbf{A}), \mathbf{B}(\mathbf{A})\right) \qquad (10)
\end{aligned}$$

$\delta\mathcal{L}$ is given by

$$\begin{aligned}
\delta\mathcal{L} &= \delta q_4\frac{\partial\mathcal{L}}{\partial q_4} + \delta n\frac{\partial\mathcal{L}}{\partial n} + \delta p\frac{\partial\mathcal{L}}{\partial p} + \delta\mathbf{v}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}} + \delta\Phi\frac{\partial\mathcal{L}}{\partial\Phi} \\
&\quad +\delta\mathbf{A}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{A}} + \delta\mathbf{E}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}} + \delta\mathbf{B}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{B}} \; . \qquad (11)
\end{aligned}$$

Here, all the variations are not completely arbitrary but have to fulfill the following constraints

$$\begin{aligned}
\delta n &= -\nabla\cdot(n\boldsymbol{\zeta}) \; , &(12)\\
\delta p &= -\boldsymbol{\zeta}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\zeta} \;\; \text{(adiabatic systems)} \; , &(13)\\
\delta p &= -\nabla\cdot(p\boldsymbol{\zeta}) \;\; \text{(isothermal systems)} \; , &(14)\\
\delta\mathbf{v} &= \dot{\boldsymbol{\zeta}} + (\mathbf{v}\cdot\nabla)\boldsymbol{\zeta} - (\boldsymbol{\zeta}\cdot\nabla)\mathbf{v} \; , &(15)
\end{aligned}$$

with $\boldsymbol{\zeta}$ an arbitrary virtual displacement for each particle species and

$$\delta\mathbf{E} = -\nabla\delta\Phi - \frac{1}{c}\delta\dot{\mathbf{A}} \;, \tag{16}$$

$$\delta\mathbf{B} = \nabla\times\delta\mathbf{A} \;. \tag{17}$$

For perscribed magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$, i.e. electrostatic perturbations, one has

$$\delta\mathbf{A} = 0 \;. \tag{18}$$

In this case, $\mathbf{A}(\mathbf{x}, t)$ and related quantities entering $\mathcal{L}$ mean an explicit dependence of this quantity on $\mathbf{x}$ and t. Usually, there is no time dependence of the prescribed magnetic field, which then allows to have energy conservation. An $\mathbf{x}$-dependence will influence momentum or angular momentum conservation. In the following $\delta\mathbf{A}$ will be considered arbitrary. For given magnetic field, however, the Euler-Lagrange equation corresponding to $\delta\mathbf{A}$, which is Ampère's law, need not be satisfied.
Upon insertion of Eqs. (12)-(17) into Eq. (11), and performing some minor transformations, one obtains for $\delta\mathcal{L}$ the basic relation

$$\begin{aligned}
\delta\mathcal{L} = {} & \frac{\partial}{\partial t}\left\{\boldsymbol{\zeta}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \frac{1}{c}\delta\mathbf{A}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right\} \\
& +\nabla\cdot\left\{-n\boldsymbol{\zeta}\frac{\partial\mathcal{L}}{\partial n} - \gamma p\boldsymbol{\zeta}\frac{\partial\mathcal{L}}{\partial p} + \left[\boldsymbol{\zeta}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\mathbf{v} - \delta\Phi\frac{\partial\mathcal{L}}{\partial\mathbf{E}} + \delta\mathbf{A}\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right\} \\
& +\delta q_4\frac{\partial\mathcal{L}}{\partial q_4} + \boldsymbol{\zeta}\cdot\left\{n\nabla\left[\frac{\partial\mathcal{L}}{\partial n}\right] + (\gamma-1)\frac{\partial\mathcal{L}}{\partial p}\nabla p + \gamma p\nabla\left[\frac{\partial\mathcal{L}}{\partial p}\right] + \mathbf{v}\times\left[\nabla\times\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\right. \\
& \left. - (\nabla\cdot\mathbf{v})\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \nabla\left[\mathbf{v}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right] - \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\right\} \\
& +\delta\Phi\left\{\frac{\partial\mathcal{L}}{\partial\Phi} + \nabla\cdot\left[\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right]\right\} \\
& +\delta\mathbf{A}\cdot\left\{\frac{\partial\mathcal{L}}{\partial\mathbf{A}} + \frac{\partial}{\partial t}\left[\frac{1}{c}\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right] + \nabla\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right\} \;.
\end{aligned} \tag{19}$$

In the following, the quasi-neutrality approximation will be used so that there is no field energy term $\frac{1}{8\pi}\mathbf{E}^2$ in $\mathcal{L}$.

A. Implicit form of the Euler-Lagrange equations

Hamilton's principle,

$$\delta \int_{t_0}^{t_1} L dt = \delta \int_{t_0}^{t_1} \left[\sum \int \mathcal{L} d^3x \right] dt = 0 = \delta \int_{t_0}^{t_1} \left[\int \mathcal{L}_\Sigma d^3x \right] dt = 0 \ , \tag{20}$$

yields the Euler-Lagrange equations, which are obtained by the vanishing of the factors of δq_4 and $\boldsymbol{\zeta}$ in $\delta\mathcal{L}$, and of $\delta\Phi$ and $\delta\mathbf{A}$ in $\delta\mathcal{L}_\Sigma = \sum \delta\mathcal{L}$.

1. Equation for the additional variable q_4

$$\frac{\partial \mathcal{L}}{\partial q_4} = 0 \ . \tag{21}$$

2. Equations of motion

$$\begin{aligned} n\nabla \left[\frac{\partial \mathcal{L}}{\partial n} \right] + (\gamma - 1) \frac{\partial \mathcal{L}}{\partial p} \nabla p + \gamma p \nabla \left[\frac{\partial \mathcal{L}}{\partial p} \right] + \mathbf{v} \times \left[\nabla \times \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \right] & \\ - (\nabla \cdot \mathbf{v}) \frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \nabla \left[\mathbf{v} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \right] - \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}}{\partial \mathbf{v}} \right] = 0 \ . & \end{aligned} \tag{22}$$

3. Quasi-neutrality condition

$$\sum \left\{ \frac{\partial \mathcal{L}}{\partial \Phi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \right\} = \frac{\partial \mathcal{L}_\Sigma}{\partial \Phi} + \nabla \cdot \frac{\partial \mathcal{L}_\Sigma}{\partial \mathbf{E}} = 0 \ . \tag{23}$$

4. Ampère's law

$$\sum \left\{ c \frac{\partial \mathcal{L}}{\partial \mathbf{A}} + \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}}{\partial \mathbf{E}} \right] + c\nabla \times \frac{\partial \mathcal{L}}{\partial \mathbf{B}} \right\} = c \frac{\partial \mathcal{L}_\Sigma}{\partial \mathbf{A}} + \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}_\Sigma}{\partial \mathbf{E}} \right] + c\nabla \times \frac{\partial \mathcal{L}_\Sigma}{\partial \mathbf{B}} = 0 \ . \tag{24}$$

When these relations are taken into account, one is left with

$$
\begin{aligned}
\delta\mathcal{L}_\Sigma &= \sum\delta\mathcal{L} = \sum\left\{\frac{\partial}{\partial t}\left[\boldsymbol{\zeta}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \frac{1}{c}\delta\mathbf{A}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right]\right. \\
&\left. +\nabla\cdot\left[-n\boldsymbol{\zeta}\frac{\partial\mathcal{L}}{\partial n} - \gamma p\boldsymbol{\zeta}\frac{\partial\mathcal{L}}{\partial p} + \left[\boldsymbol{\zeta}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\mathbf{v} - \delta\Phi\frac{\partial\mathcal{L}}{\partial\mathbf{E}} + \delta\mathbf{A}\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right]\right\} , \qquad (25)
\end{aligned}
$$

valid for all $\boldsymbol{\zeta}$, $\delta\Phi$, $\delta\mathbf{A}$. This relation will later be used to obtain conservation laws.

B. Explicit form of the Euler-Lagrange equations

1. The derivatives of $\mathcal{L}$

The derivatives of $\mathcal{L}$ are, explicitly,

$$\frac{\partial\mathcal{L}}{\partial\mathbf{v}} = \frac{e}{c}n\mathbf{A} + n\left[q_4 m_{\|}\mathbf{b} + m_\perp\mathbf{v}_E\right] , \qquad (26)$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{E}} = \frac{\partial\mathbf{v}_E}{\partial\mathbf{E}}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}_E} = -\frac{cm_\perp n}{B}(\mathbf{v}_\perp - \mathbf{v}_E)\times\mathbf{b} \equiv \boldsymbol{\mathcal{P}} . \qquad (27)$$

Here, $\mathbf{v}_\perp$ is the velocity perpendicular to $\mathbf{B}$, $\mathbf{v}_\perp = \mathbf{v} - v_{\|}\mathbf{b}$ and $\boldsymbol{\mathcal{P}}$ is a polarization vector which appears in the explicit quasi-neutrality condition below, Eq. (58).

$$\frac{\partial\mathcal{L}}{\partial n} = \hat{L}_\mathrm{p} = \frac{e}{c}\mathbf{v}\cdot\mathbf{A} + \mathbf{v}\cdot\left[m_{\|}q_4\mathbf{b} + m_\perp\mathbf{v}_E\right] - e\Phi - \frac{1}{2}\left[m_{\|}q_4^2 + m_\perp\mathbf{v}_E^2\right] , \qquad (28)$$

$$\frac{\partial\mathcal{L}}{\partial p} = -\frac{1}{\gamma - 1} \qquad (29)$$

for an adiabatic system. For an isothermal system with γ replaced by 1, one has

$$\frac{\partial\mathcal{L}}{\partial p} = \frac{\partial}{\partial p}\left[-p\ln\left[\frac{p}{p_c}\right]\right] = -\left[1 + \ln\left[\frac{p}{p_c}\right]\right] , \qquad (30)$$

as shown explicitly in appendices A and B.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\mathbf{B}} &= -\frac{1}{4\pi}\mathbf{B}+\frac{\partial\left[n\hat{L}_{\rm p}\right]}{\partial\mathbf{B}} \\ &= -\frac{1}{4\pi}\mathbf{B}+\frac{\partial\mathbf{b}}{\partial\mathbf{B}}\cdot\mathbf{v}nm_{\|}q_4+\frac{\partial\mathbf{v}_E}{\partial\mathbf{B}}\cdot(\mathbf{v}-\mathbf{v}_E)\,nm_{\perp}\ ,\end{aligned} \tag{31}$$

with

$$\frac{\partial\mathbf{b}}{\partial\mathbf{B}}\cdot\mathbf{v}=\frac{\mathbf{v}}{B}-\frac{(\mathbf{b}\cdot\mathbf{v})}{B}\mathbf{b}=\frac{\mathbf{v}_{\perp}}{B}\ , \tag{32}$$

$$\frac{\partial\mathbf{v}_E}{\partial\mathbf{B}}\cdot(\mathbf{v}-\mathbf{v}_E)=\frac{c}{B^2}(\mathbf{v}-\mathbf{v}_E)\times\mathbf{E}-\frac{2}{B^2}\left[(\mathbf{v}-\mathbf{v}_E)\cdot\mathbf{v}_E\right]\mathbf{B}\ , \tag{33}$$

and, therefore

$$\frac{\partial\mathcal{L}}{\partial\mathbf{B}}=-\frac{1}{4\pi}\mathbf{B}+\boldsymbol{\mathcal{M}}\ , \tag{34}$$

where $\boldsymbol{\mathcal{M}}$ is a magnetization vector defined by the relation

$$\begin{aligned}\boldsymbol{\mathcal{M}} &= \frac{nq_4}{B}(m_{\|}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E)+\frac{cm_{\perp}n}{B^2}(\mathbf{v}_{\perp}-\mathbf{v}_E)\times\mathbf{E} \\ &\quad -\frac{2m_{\perp}n}{B^2}\left[(\mathbf{v}_{\perp}-\mathbf{v}_E)\cdot\mathbf{v}_E\right]\mathbf{B}\ ,\end{aligned} \tag{35}$$

$$\frac{\partial\mathcal{L}}{\partial q_4}=m_{\|}n\,(\mathbf{v}\cdot\mathbf{b}-q_4)=m_{\|}n\left(v_{\|}-q_4\right)\ , \tag{36}$$

$$\frac{\partial\mathcal{L}}{\partial\Phi}=-en\ , \tag{37}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{A}}=\frac{e}{c}n\mathbf{v}\ . \tag{38}$$

Use of these expressions in Eqs. (21)-(22) yields the explicit Euler-Lagrange equations:

2. Equation for the additional variable q_4

$$q_4(\mathbf{x},t)=\mathbf{b}\cdot\mathbf{v}=v_{\|}\ . \tag{39}$$

3. Equations of motion

By taking into account Eqs. (26), (28)-(30) and (39), equation (22) can be written in the neat form

$$en\left(\hat{\mathbf{E}} + \frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right) - \nabla p = 0 \tag{40}$$

with

$$\begin{aligned} e\hat{\mathbf{E}} &= -\frac{e}{c}\dot{\hat{\mathbf{A}}} - e\nabla\hat{\Phi} \\ &= e\mathbf{E} - \frac{\partial}{\partial t}\left(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_E\right) - \frac{1}{2}\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) , \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{e}{c}\hat{\mathbf{B}} &= \frac{e}{c}\nabla\times\hat{\mathbf{A}} \\ &= \frac{e}{c}\mathbf{B} + \nabla\times(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_E) . \end{aligned} \tag{42}$$

The last relation yields

$$\frac{e}{c}\mathbf{v}\times\hat{\mathbf{B}} = \frac{e}{c}\mathbf{v}\times\mathbf{B} + \mathbf{v}\times\left[\nabla\times(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_E)\right] . \tag{43}$$

Insertion of Eq. (41) into Eq. (40) yields

$$n\frac{\partial}{\partial t}\left(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_E\right) + \frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) = ne\mathbf{E} + n\frac{e}{c}\mathbf{v}\times\hat{\mathbf{B}} - \nabla p . \tag{44}$$

Equation for the parallel motion

An explicit expression for $v_{\|}$ results by taking the scalar product of Eq. (44) with $\hat{\mathbf{B}}$, which yields the following equation

$$\hat{\mathbf{B}}\cdot\frac{\partial}{\partial t}\left(v_{\|}\mathbf{b}\right) = \frac{1}{nm_{\|}}\hat{\mathbf{B}}\cdot\left[en\mathbf{E} - nm_{\perp}\frac{\partial\mathbf{v}_E}{\partial t} - \frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) - \nabla p\right] , \tag{45}$$

or also

$$\dot{v}_{\|} = \frac{1}{nm_{\|}} \frac{\hat{\mathbf{B}}}{\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)} \cdot \left[en\mathbf{E} - nm_{\perp}\frac{\partial \mathbf{v}_E}{\partial t} - nm_{\|}v_{\|}\frac{\partial \mathbf{b}}{\partial t} - \frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) - \nabla p\right] . \tag{46}$$

Another more convenient form of the equation for $v_{\|}$ is obtained in the next section, Eq. (56).

Equation for the perpendicular motion

The velocity $\mathbf{v}$ is obtained by crossing Eq. (40) with $\mathbf{b}$ and solving for $\mathbf{v}$

$$\mathbf{v} = v_{\|}\frac{\hat{\mathbf{B}}}{\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)} + \frac{c}{en}\frac{1}{\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)}\left[en\hat{\mathbf{E}} - \nabla p\right]\times\mathbf{b} , \tag{47}$$

or, more explicitly,

$$\begin{aligned} \mathbf{v} \ &= \ v_{\|}\frac{\hat{\mathbf{B}}}{\mathbf{b}\cdot\hat{\mathbf{B}}} + \frac{c}{en}\frac{1}{\mathbf{b}\cdot\hat{\mathbf{B}}}\left[en\mathbf{E} - nm_{\perp}\frac{\partial \mathbf{v}_E}{\partial t} - nm_{\|}v_{\|}\frac{\partial \mathbf{b}}{\partial t}\right. \\ &\quad \left. -\frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) - \nabla p\right]\times\mathbf{b} . \end{aligned} \tag{48}$$

Crossing Eq. (42) twice with $\mathbf{b}$ yields

$$\hat{\mathbf{B}} = \left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)\mathbf{b} - \frac{c}{e}\mathbf{b}\times\left[\mathbf{b}\times\nabla\times\left(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_E\right)\right] . \tag{49}$$

Insertion of Eq. (49) into Eq. (48) and some minor transformations result in the following expression for the velocity $\mathbf{v}$

$$\mathbf{v} = v_{\|}\mathbf{b} + \mathbf{v}_{\perp} , \tag{50}$$

where the perpendicular velocity $\mathbf{v}_{\perp}$ is given explicitly by

$$\begin{aligned} \mathbf{v}_{\perp} \ &= \ \frac{c}{en\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)}\mathbf{b}\times\left[-en\mathbf{E} + nm_{\perp}\frac{\partial \mathbf{v}_E}{\partial t} + \frac{1}{2}nm_{\perp}\nabla\mathbf{v}_E^2 - nm_{\perp}v_{\|}\mathbf{b}\times\left(\nabla\times\mathbf{v}_E\right)\right. \\ &\quad \left. + nm_{\|}v_{\|}\frac{\partial \mathbf{b}}{\partial t} + nm_{\|}v_{\|}^2\left(\mathbf{b}\cdot\nabla\right)\mathbf{b} + \nabla p\right] , \end{aligned} \tag{51}$$

an expression which can also be written as

$$\mathbf{v}_\perp = \frac{c}{en\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)}\mathbf{b}\times\left[-en\mathbf{E}+nm_\perp\frac{\partial\mathbf{v}_E}{\partial t}+\frac{1}{2}n\left[m_\parallel\nabla v_\parallel^2+m_\perp\nabla\mathbf{v}_E^2\right]\right.$$
$$\left.+nm_\parallel v_\parallel\frac{\partial\mathbf{b}}{\partial t}+\nabla p-nv_\parallel\mathbf{b}\times\left[\nabla\times\left[m_\parallel v_\parallel\mathbf{b}+m_\perp\mathbf{v}_E\right]\right]\right] \quad (52)$$

This form is used below to obtain Eq. (56), which is a more compact equation for $v_\parallel$ than Eq. (46). An alternative and useful expression for $\mathbf{v}_\perp$ is

$$\mathbf{v}_\perp = \mathbf{v}_E+\frac{c}{en\left(\mathbf{b}\cdot\hat{\mathbf{B}}\right)}\mathbf{b}\times\left[nm_\perp\left[\frac{\partial}{\partial t}+\mathbf{v}_E\cdot\nabla+v_\parallel\mathbf{b}\cdot\nabla\right]\mathbf{v}_E\right.$$
$$+nm_\parallel v_\parallel^2\mathbf{b}\cdot\nabla\mathbf{b}+nm_\perp v_\parallel(\mathbf{v}_E\cdot\nabla)\mathbf{b}+nm_\parallel v_\parallel\frac{\partial\mathbf{b}}{\partial t}$$
$$\left.+n(m_\perp-m_\parallel)v_\parallel\mathbf{v}_E\times(\nabla\times\mathbf{b})+\nabla p\right] \,. \quad (53)$$

The perpendicular velocity consists of the $\mathbf{E}\times\mathbf{B}$, polarization, centrifugal and diamagnetic drifts and two additional drifts related to the change of the direction of $\mathbf{B}$.

Useful alternative equations for the parallel motion

Using Eq. (48) to express $\hat{\mathbf{B}}$ through $\mathbf{v}$ one obtains an alternative expression for $\dot{v}_\parallel$, namely

$$\dot{v}_\parallel = \frac{\mathbf{v}}{nm_\parallel v_\parallel}\cdot\left[en\mathbf{E}-nm_\perp\frac{\partial\mathbf{v}_E}{\partial t}-nm_\parallel v_\parallel\frac{\partial\mathbf{b}}{\partial t}\right.$$
$$\left.-\frac{1}{2}n\nabla\left(m_\parallel v_\parallel^2+m_\perp\mathbf{v}_E^2\right)-\nabla p\right] \,. \quad (54)$$

The following relation follows from Eq. (52)

$$\mathbf{v}_\perp\cdot\left[-en\mathbf{E}+nm_\perp\frac{\partial\mathbf{v}_E}{\partial t}+\frac{1}{2}n\left[m_\parallel\nabla v_\parallel^2+m_\perp\nabla\mathbf{v}_E^2\right]+nm_\parallel v_\parallel\frac{\partial\mathbf{b}}{\partial t}+\nabla p\right] =$$
$$nv_\parallel\mathbf{v}_\perp\cdot\left[\mathbf{b}\times\left[\nabla\times\left[m_\parallel v_\parallel\mathbf{b}+m_\perp\mathbf{v}_E\right]\right]\right] \,. \quad (55)$$

Taking this relation into account in Eq. (54) yields another useful relation for $\dot{v}_{\|}$

$$\dot{v}_{\|} = \frac{\mathbf{b}}{nm_{\|}} \cdot \left[en\mathbf{E} - nm_{\perp}\frac{\partial \mathbf{v}_E}{\partial t} - \frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) - \nabla p\right]$$

$$-\mathbf{v}_{\perp} \cdot \left[\mathbf{b}\times\left[\nabla\times\left[v_{\|}\mathbf{b} + \frac{m_{\perp}}{m_{\|}}\mathbf{v}_E\right]\right]\right] . \qquad (56)$$

4. Quasi-neutrality condition

Introducing the *space charge density ρ of quasi-particles* of each species,

$$\rho = en - \nabla \cdot \mathcal{P} , \qquad (57)$$

the quasi-neutrality condition, Eq. (23) reads

$$\sum_{\text{particle species}} \rho = \sum [en - \nabla \cdot \mathcal{P}] = \left[\sum en\right] - \nabla \cdot \mathbf{P} = 0 ,$$

$$\text{where} \quad \mathbf{P} = \sum \mathcal{P} . \qquad (58)$$

The term in addition to the particle-like contribution $\sum en$ is a polarisation charge density which does not depend on the charges e of the various particle species. Below a corresponding contribution to the current density is obtained (Eq. (60)).

5. Ampère's law

With Eqs. (27), (31)-(35) and (38) Ampère's law, Eq. (24), can be written as

$$\mathbf{j} = \frac{c}{4\pi}\nabla\times\mathbf{B} , \qquad (59)$$

with the current density $\mathbf{j}$ given by

$$\mathbf{j} = \sum\left[en\mathbf{v} + \frac{\partial \mathcal{P}}{\partial t} + c\nabla\times\mathcal{M}\right]$$

$$= \left[\sum en\mathbf{v}\right] + \frac{\partial \mathbf{P}}{\partial t} + c\nabla\times\mathbf{M} , \qquad (60)$$

where $\mathcal{P}$ and $\mathbf{P}$ are the polarization vectors introduced in Eqs. (27) and (58) and $\mathbf{M}$ is given by

$$\mathbf{M} = \sum \mathcal{M} , \tag{61}$$

with q_4 replaced by $v_{\|}$ in $\mathcal{M}$ after Eq. (39). Besides the particle-like contribution

$$\mathbf{j}_n = \sum en\mathbf{v} , \tag{62}$$

with $\mathbf{v}$ given explicitly by Eqs. (50) and (53), which contains as the main magnetization currents the usual diamagnetic currents $\sim \mathbf{B} \times \nabla p$, there are contributions from polarization currents and additional magnetization currents. The polarization current density together with the polarization charge density in Eq. (58) satisfy the continuity equation for this charge density,

$$\nabla \cdot (\mathbf{j}_P) + \dot{\rho_P} = 0 \quad , \quad \rho_P = -\nabla \cdot \mathbf{P} = \sum \nabla \cdot \left[\frac{cm_{\perp}n}{B} (\mathbf{v}_{\perp} - \mathbf{v}_E) \times \mathbf{b} \right] , \tag{63}$$

but note that the total current density $\mathbf{j}$ is divergence free and the total charge density is zero, corresponding to the quasi-neutrality condition, Eq. (58).

V. Conservation laws

The derivation of the conservation laws for energy and momentum will be based, in the sense of Noether's formalism, on the total variations of the Lagrangian densities, $\delta\mathcal{L}$ and $\delta\mathcal{L}_\Sigma$, as given by Eqs. (19) and (25). This will later prove advantageous also when dissipative effects are introduced.

A. Energy conservation

The local and global energy conservation laws can be obtained from the expressions for $\delta\mathcal{L}$, $\delta\mathcal{L}_\Sigma$ and $\delta L = \int \delta\mathcal{L}_\Sigma d^3x$ using the Noether formalism. To this effect, $\boldsymbol{\zeta}$, $\delta\Phi$ and $\delta\mathbf{A}$ are now taken to result from the dynamical evolution of the system, and *not* as virtual displacements , i.e.

$$\boldsymbol{\zeta} = \mathbf{v}\delta t \ , \quad \delta\Phi = \dot{\Phi}\delta t \ , \quad \delta\mathbf{A} = \dot{\mathbf{A}}\delta t \ . \tag{64}$$

Since $\delta\mathcal{L}$ represents the variation of $\mathcal{L}$ owing to the variations of $n(\mathbf{x},t)$, $p(\mathbf{x},t)$, etc..., it holds that

$$\delta\mathcal{L} = \delta t \frac{\partial \mathcal{L}}{\partial t} \; , \tag{65}$$

where the partial derivative is to be understood at constant $\mathbf{x}$. The equality of $\frac{\delta \mathcal{L}_\Sigma}{\delta t}$ and $\frac{\partial \mathcal{L}_\Sigma}{\partial t}$ resulting from Eqs. (25) and (65) is the local energy conservation law

$$\sum \left\{ \frac{\partial}{\partial t}\left[\mathbf{v}\cdot\frac{\partial\mathcal{L}}{\partial \mathbf{v}} - \frac{1}{c}\dot{\mathbf{A}}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{E}} - \mathcal{L}\right] \right.$$
$$\left. +\nabla\cdot\left[-n\mathbf{v}\frac{\partial\mathcal{L}}{\partial n} - \gamma p\mathbf{v}\frac{\partial \mathcal{L}}{\partial p} + \left[\mathbf{v}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right]\mathbf{v} - \dot{\Phi}\frac{\partial\mathcal{L}}{\partial \mathbf{E}} + \dot{\mathbf{A}}\times\frac{\partial \mathcal{L}}{\partial \mathbf{B}}\right]\right\} . \tag{66}$$

With Eq. (10) and the results of Section IV B 1 this relation is, explicitly,

$$\sum \left\{ \frac{\partial}{\partial t}\left[\frac{n}{2}\left(m_\|v_\|^2 + m_\perp \mathbf{v}_E^2\right) + \frac{p}{\gamma-1} + en\Phi + \frac{1}{8\pi}\mathbf{B}^2 - \frac{1}{c}\mathbf{P}\cdot\frac{\partial \mathbf{A}}{\partial t}\right]\right.$$
$$+\nabla\cdot\left[\frac{n}{2}\left(m_\|v_\|^2 + m_\perp \mathbf{v}_E^2\right)\mathbf{v} + \frac{\gamma}{\gamma-1}p\mathbf{v}\right.$$
$$\left.\left. +en\Phi\mathbf{v} - \dot{\Phi}\mathbf{P} - \frac{1}{4\pi}\frac{\partial\mathbf{A}}{\partial t}\times\left[\mathbf{B} - 4\pi\mathbf{M}\right]\right]\right\} = 0 \; . \tag{67}$$

In appendix B, the same result is obtained from the equations of motion. Eq. (67) yields the following expression for the total energy

$$\mathcal{E} = \sum\int\left[\frac{n}{2}\left(m_\|v_\|^2 + m_\perp \mathbf{v}_E^2\right) + \frac{p}{\gamma-1} + en\Phi - \frac{1}{c}\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial t} + \frac{1}{8\pi}\mathbf{B}^2\right] d^3x \; . \tag{68}$$

If there are no contributions from the boundaries, $\mathcal{E}$ is constant in time. A special boundary contribution could be the Poynting flux owing to $\frac{\partial \mathbf{A}}{\partial t}$ on the surface, corresponding to an externally induced electric field.
With the help of the quasi-neutrality condition (58), one can also write

$$\sum\int\left[en\Phi - \frac{1}{c}\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial t}\right]d^3x \;=\; \int\left[\Phi\nabla\cdot\mathbf{P} - \frac{1}{c}\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial t}\right]d^3x$$

$$
\begin{aligned}
&= -\int \left[\nabla\Phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right]\cdot \mathbf{P} d^3x \\
&= \int \mathbf{E}\cdot \mathbf{P} d^3x \\
&= \sum \int m_\perp n(\mathbf{v}_\perp - \mathbf{v}_E)\cdot \mathbf{v}_E \, d^3x \ . \qquad (69)
\end{aligned}
$$

The velocity combination in this expression is essentially the polarisation drift. With the help of Eq. (69), the energy expression (68) takes the form

$$
\mathcal{E} = \sum \int \left[\frac{n}{2}\left(m_\| v_\|^2 + m_\perp \mathbf{v}_\perp^2 - m_\perp (\mathbf{v}_\perp - \mathbf{v}_E)^2\right) + \frac{p}{\gamma - 1} + \frac{1}{8\pi}\mathbf{B}^2\right] d^3x \ . \tag{70}
$$

Eqs. (68) and (70) are the correct expressions for the energy in adiabatic systems. As mentioned above, and as it is explicitly shown in appendices A and B, in the case of isothermal systems the term $\dfrac{p}{\gamma-1}$ has to be replaced by $p \ln \dfrac{p}{p_c}$, with p_c an arbitrary constant, and the energy is

$$
\mathcal{E} = \sum \int \left[\frac{n}{2}\left(m_\| v_\|^2 + m_\perp \mathbf{v}_\perp^2 - m_\perp (\mathbf{v}_\perp - \mathbf{v}_E)^2\right) + p \ln \frac{p}{p_c} + \frac{1}{8\pi}\mathbf{B}^2\right] d^3x \ . \tag{71}
$$

B. Momentum conservation

Momentum conservation laws are most easily derived considering the change of $\mathcal{L}$ and $\mathcal{L}_\Sigma$ in the form given by Eqs. (19) and (25), $\boldsymbol{\zeta}$ now being again *not* a virtual displacement, but a symmetry displacement having the properties derived in appendix C, i.e.

$$
\boldsymbol{\zeta} = \boldsymbol{\zeta}_s \ , \ \dot{\boldsymbol{\zeta}}_s = 0 \ , \ \nabla\cdot\boldsymbol{\zeta}_s = 0 \ . \tag{72}
$$

Also, $\delta\Phi$ and $\delta\mathbf{A}$ are the corresponding changes related to $\boldsymbol{\zeta}_s$. According to Eqs. (C2) and (C4) they are

$$
\begin{aligned}
\delta\Phi &= -\boldsymbol{\zeta}_s \cdot \nabla\Phi \ , \\
\delta\mathbf{A} &= -\nabla\left[\boldsymbol{\zeta}_s\cdot\mathbf{A}\right] + \boldsymbol{\zeta}_s \times (\nabla\times\mathbf{A}) \ . \qquad (73)
\end{aligned}
$$

Since $\mathcal{L}$ is a scalar quantity, it is transformed also according to Eq. (C2), i.e.

$$\delta\mathcal{L} = -\boldsymbol{\zeta}_s \cdot \nabla\mathcal{L} = -\nabla \cdot [\mathcal{L}\boldsymbol{\zeta}_s] \ . \tag{74}$$

The case where $\mathbf{A}$ is a dynamical quantity (electromagnetic perturbations), and the case where $\mathbf{A}(\mathbf{x}, t)$ is prescribed, i.e. $\delta\mathbf{A} = 0$ (electrostatic perturbations) are now treated separately.

1. $\delta\mathbf{A} = \delta\mathbf{A}(\mathbf{x}, t)$, i.e. A is a dynamical variable.

The system considered consists of a plasma surrounded by a vacuum and superconducting coils of some symmetry.

a) Local momentum conservation law

The local momentum conservation law is obtained by setting $\delta\mathcal{L}_\Sigma = \sum \delta\mathcal{L}$, as given by Eq. (74), equal to the expression for $\delta\mathcal{L}_\Sigma$ obtained from Eqs. (25) and (72)-(73). This yields the local momentum conservation law

$$\sum \left\{ \frac{\partial}{\partial t} \left[\boldsymbol{\zeta}_s \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{v}} + \frac{1}{c} [\nabla \cdot (\boldsymbol{\zeta}_s \cdot \mathbf{A}) - \boldsymbol{\zeta}_s \times \mathbf{B}] \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{E}} \right] \right.$$
$$+ \nabla \cdot \left[-n\boldsymbol{\zeta}_s \frac{\partial\mathcal{L}}{\partial n} - \gamma p \boldsymbol{\zeta}_s \frac{\partial\mathcal{L}}{\partial p} + \left[\boldsymbol{\zeta}_s \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \right] \mathbf{v} + [\boldsymbol{\zeta}_s \cdot \nabla\Phi] \frac{\partial\mathcal{L}}{\partial\mathbf{E}} \right.$$
$$\left. \left. + [-\nabla (\boldsymbol{\zeta}_s \cdot \mathbf{A}) + \boldsymbol{\zeta}_s \times \mathbf{B}] \times \frac{\partial\mathcal{L}}{\partial\mathbf{B}} + \mathcal{L}\boldsymbol{\zeta}_s \right] \right\} = 0 \ . \tag{75}$$

Insertion of the relations given in Section IV B 1 for the derivatives of $\mathcal{L}$ yields for this conservation law the form

$$\sum \left\{ \frac{\partial}{\partial t} \left[\boldsymbol{\zeta}_s \cdot n \left[m_\| v_\| \mathbf{b} + m_\perp \mathbf{v}_\perp \right] \right] + \nabla \cdot \left\{ \left[\boldsymbol{\zeta}_s \cdot n \left(m_\| v_\| \mathbf{b} + m_\perp \mathbf{v}_\perp \right) \right] \mathbf{v} \right. \right.$$
$$- (\boldsymbol{\zeta}_s \cdot \mathbf{E}) \mathbf{P} + \frac{1}{c} [\boldsymbol{\zeta}_s \cdot (\mathbf{B} \times \mathbf{P})] \mathbf{v} + p\boldsymbol{\zeta}_s$$
$$\left. \left. + \frac{1}{8\pi} \mathbf{B}^2 \boldsymbol{\zeta}_s - \frac{1}{4\pi} (\boldsymbol{\zeta}_s \cdot \mathbf{B}) \mathbf{B} + (\boldsymbol{\zeta}_s \times \mathbf{B}) \times \mathbf{M} \right\} \right\} = 0 \ . \tag{76}$$

b) Global momentum conservation law

This law is obtained upon integration of Eq. (76) over the plasma volume $V_p(t)$ and the vacuum volume $V_v(t)$. Assuming that the density n vanishes on the plasma surface, then p, $\boldsymbol{\mathcal{P}}$, $\mathbf{P}$ and $\boldsymbol{\mathcal{M}}$, $\mathbf{M}$ vanish there as well. Since the coils are assumed superconducting, there is no tangengial $\mathbf{B}$ at their surface. It is further assumed that the coils have a symmetry, either plane or axial symmetry. Then, $\boldsymbol{\zeta}_s \cdot \delta\mathbf{S}_{\text{coil surface}} = 0$ and one obtains

$$\begin{aligned}
\frac{d}{dt}\int_{V_p(t)} d^3x \left\{\sum \boldsymbol{\zeta}_s \cdot n\left[m_{||}v_{||}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right]\right\} &= \\
&= \int_{V=V_p+V_v} d^3x \nabla \cdot \left[\frac{1}{8\pi}\mathbf{B}^2\boldsymbol{\zeta}_s - \frac{1}{4\pi}\left(\boldsymbol{\zeta}_s \cdot \mathbf{B}\right)\mathbf{B}\right] \\
&= \int_{\text{coil surface}} d\mathbf{S} \cdot \nabla \cdot \left[\frac{1}{8\pi}\mathbf{B}^2\boldsymbol{\zeta}_s - \frac{1}{4\pi}\left(\boldsymbol{\zeta}_s \cdot \mathbf{B}\right)\mathbf{B}\right] \\
&= 0\,. \qquad (77)
\end{aligned}$$

2. Momentum conservation law in a given symmetric magnetic field, $\delta\mathbf{A} = 0$.

When studying *electrostatic* perturbations, the magnetic field is considered as given, and it is assumed here that it depends on $\mathbf{x}$, but not on t. As was previously explained, a prescribed dependence $\mathbf{A}(\mathbf{x})$ and related quantities entering $\mathcal{L}$ means an explicit dependence of this quantity on $\mathbf{x}$, which will, in general, influence momentum or angular momentum conservation. It will be shown here that the variation $\delta\mathcal{L}_{\text{explicit}}$ caused by a symmetry operation and resulting from the explicit $\mathbf{x}$ dependence vanishes when the prescibed field has plane or axial symmetry, and the corresponding momentum conservation laws will be derived.
In the case of a prescribed magnetic field $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$, $\delta\mathbf{A}$ and $\delta\mathbf{B}$ vanish in Eq. (25). Then, instead of Eq. (75), one has

$$\sum \delta\mathcal{L}_{\text{explicit}} + \sum\left\{\frac{\partial}{\partial t}\left[\boldsymbol{\zeta}_s \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\right.$$

$$\left.+\nabla \cdot \left[-n\boldsymbol{\zeta}_s\frac{\partial\mathcal{L}}{\partial n} - \gamma p\boldsymbol{\zeta}_s\frac{\partial\mathcal{L}}{\partial p} + \left[\boldsymbol{\zeta}_s \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\mathbf{v} + \left[\boldsymbol{\zeta}_s \cdot \nabla\Phi\right]\frac{\partial\mathcal{L}}{\partial\mathbf{E}} + \mathcal{L}\boldsymbol{\zeta}_s\right]\right\} = 0\,, \qquad (78)$$

where

$$\delta\mathcal{L}_{\text{explicit}} = \delta\mathbf{A}_{\text{ex}} \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{A}} + \delta\mathbf{B}_{\text{ex}} \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{B}} , \tag{79}$$

with $\delta\mathbf{B}_{\text{ex}} = \nabla \times \delta\mathbf{A}_{\text{ex}}$ and $\delta\mathbf{A}_{\text{ex}}$ given by Eq. (73). Then

$$\delta\mathcal{L}_{\text{explicit}} = -\frac{\partial\mathcal{L}}{\partial\mathbf{A}} \cdot \nabla(\boldsymbol{\zeta}_s \cdot \mathbf{A}) + \frac{\partial\mathcal{L}}{\partial\mathbf{A}} \cdot (\boldsymbol{\zeta}_s \times \mathbf{B}) + \frac{\partial\mathcal{L}}{\partial\mathbf{B}} \cdot [\nabla \times (\boldsymbol{\zeta}_s \times \mathbf{B})] . \tag{80}$$

The symmetry displacement $\boldsymbol{\zeta}_s$ has the properties derived in appendix C. It is now also required that $\boldsymbol{\zeta}_s$ corresponds to the symmetry of the prescribed magnetic field, in particular $\boldsymbol{\zeta}_s \cdot \nabla\mathbf{B}^2 = 0$. This means that the $\frac{\mathbf{B}^2}{8\pi}$ term in $\mathcal{L}$ does not contribute, in agreement with a possible alternative definition of $\mathcal{L}$ which does not contain $\frac{\mathbf{B}^2}{8\pi}$ since $\mathbf{B}$ is not a dynamical variable when the magnetic field is prescribed.

Momentum conservation law in plane symmetry

In Cartesian coordinates x, y, z, a plane symmetric magnetic field is given by

$$\mathbf{A} = A_y(x)\mathbf{e}_y + A_z(x)\mathbf{e}_z \quad , \quad \mathbf{B} = -A_z'(x)\mathbf{e}_y + A_y'(x)\mathbf{e}_z . \tag{81}$$

As a symmetry displacement $\boldsymbol{\zeta}_s$, one can take

$$\boldsymbol{\zeta}_s = const. = c_y\mathbf{e}_y + c_z\mathbf{e}_z \tag{82}$$

Therefore

$$\boldsymbol{\zeta}_s \times \mathbf{B} = \left[c_y A_y' + c_z A_z'\right]\mathbf{e}_x = \nabla(\boldsymbol{\zeta}_s \cdot \mathbf{A}) \quad , \quad \nabla \times (\boldsymbol{\zeta}_s \times \mathbf{B}) = 0 , \tag{83}$$

$$(\boldsymbol{\zeta}_s \cdot \nabla)\mathbf{A} = (\mathbf{A} \cdot \nabla)\boldsymbol{\zeta}_s = 0 , \tag{84}$$

and

$$\delta\mathcal{L}_{\text{explicit}} = 0 . \tag{85}$$

By making use of the relations derived in Section IV B 1 one can write the local momentum conservation law as

$$\sum \left\{ \frac{\partial}{\partial t} n\boldsymbol{\zeta}_s \cdot \left(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right) + \nabla \cdot \left[p\boldsymbol{\zeta}_s + (\boldsymbol{\zeta}_s \cdot \nabla\Phi)\boldsymbol{\mathcal{P}} \right.\right.$$

$$- nm_{\perp}\left[\boldsymbol{\zeta}_s \cdot (\mathbf{v}_{\perp} - \mathbf{v}_E)\right]\mathbf{v} + n\left[\boldsymbol{\zeta}_s \cdot \left(m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right)\right]\mathbf{v}$$

$$\left.\left. + \frac{1}{c}\left[en\mathbf{v} + \frac{\partial\boldsymbol{\mathcal{P}}}{\partial t}\right] \cdot (\boldsymbol{\zeta}_s \cdot \mathbf{A})\right]\right\} = 0 . \tag{86}$$

Integration of this equation over the plasma volume $V(t)$, and again taking the densities n to vanish on the plasma surface, yields the global momentum conservation law

$$\frac{d}{dt}\int d^3x \sum n\mathbf{e}_y\cdot\left[m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right] = \frac{d}{dt}\int d^3x \sum n\mathbf{e}_z\cdot\left[m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right] = 0\,. \tag{87}$$

If the magnetic field does not depend on x, but is constant, $\mathbf{B} = B_0\mathbf{e}_z$, then also the x component of the momentum is conserved:

$$\frac{d}{dt}\int d^3x \sum nm_{\perp}\mathbf{e}_x\cdot\mathbf{v}_{\perp} = 0\,, \tag{88}$$

as can easily be seen choosing $\boldsymbol{\zeta}_s = c_x\mathbf{e}_x$ and $\mathbf{A} = -A_x(y)\mathbf{e}_x$. Then $(\boldsymbol{\zeta}_s\cdot\nabla)\,\mathbf{A} = (\mathbf{A}\cdot\nabla)\,\boldsymbol{\zeta}_s = \nabla\times(\boldsymbol{\zeta}_s\times\mathbf{B}) = 0$ and $\delta\mathcal{L}_{\text{explicit}} = 0$.

Momentum conservation law in axial symmetry

An axisymmetric vector potential $\mathbf{A}$ can be written in the form

$$\mathbf{A} = \Psi\,(R,z)\,\nabla\varphi + F\,(R,z)\,\mathbf{e}_z\,, \tag{89}$$

where R, φ, z are the usual cylindrical coordinates. A symmetry displacement $\boldsymbol{\zeta}_s$ which describes a rigid rotation of the whole system about the z-axis is

$$\boldsymbol{\zeta} = R^2\delta\varphi\nabla\varphi = R\delta\varphi\mathbf{e}_{\varphi}\quad,\quad \boldsymbol{\zeta}_s\times\mathbf{B} = \delta\varphi\nabla\Psi\,. \tag{90}$$

Therefore, one has

$$\boldsymbol{\zeta}_s\cdot\mathbf{A} = \delta\varphi\Psi\quad,\quad \delta\mathcal{L}_{\text{explicit}} = 0\,. \tag{91}$$

The local momentum conservation law is again Eq. (86), but now with $\boldsymbol{\zeta}_s = R^2\delta\varphi\nabla\varphi$. The global momentum conservation law is obtained analogously to the previous case. Explicitly, one has

$$\frac{d}{dt}\int d^3x \sum R\,n\mathbf{e}_{\varphi}\cdot\left[m_{\|}v_{\|}\mathbf{b} + m_{\perp}\mathbf{v}_{\perp}\right] = 0\,. \tag{92}$$

VI. Collisional effects

The starting point is the formal relation for $\delta\mathcal{L}(q_4, n, p, \mathbf{v}, \Phi, \mathbf{A})$, as given by Eq. (11). In this relation we insert the dissipative equations and variations. Of these, the following ones are not modified by collisions: $q_4 = v_{||} \Longrightarrow \dfrac{\partial\mathcal{L}}{\partial q_4} = 0$ and δn, $\delta\mathbf{v}$, $\delta\Phi$, $\delta\mathbf{A}$ as given by Eqs. (12), (15)- (17).

The effecs of collisions on momentum and energy are discussed separately. The first mean a modification of the equations of motion; the second a modification of δp representing the time evolution $\dot{p}\delta t$. The method introduced here makes use of the relations between $\mathcal{L}$ and the dissipationless equations.

A. Equations of motion and momentum conservation law in dissipative systems

The results obtained here will show that the global momentum conservation laws of the systems considered in the preceeding section are unchanged when viscosity, resistivity and thermal collisional effects are introduced.

As in the preceding section, a symmetry displacement $\boldsymbol{\zeta}_s$ is considered. The corresponding change in $\mathcal{L}$ is given by

$$\delta\mathcal{L} = -\boldsymbol{\zeta}_s \cdot \nabla\mathcal{L} = -\nabla\cdot(\mathcal{L}\boldsymbol{\zeta}_s) \ . \tag{93}$$

The changes in n and p can be written as

$$\delta n = -\boldsymbol{\zeta}_s \cdot \nabla n = -\nabla\cdot(n\boldsymbol{\zeta}_s) \quad , \quad \delta p = -\boldsymbol{\zeta}_s \cdot \nabla p = -\boldsymbol{\zeta}_s \cdot \nabla p - \gamma p \nabla\cdot\boldsymbol{\zeta}_s \ . \tag{94}$$

The change in $\mathbf{v}$ is, according to Eq. (C4),

$$\delta\mathbf{v} = -(\boldsymbol{\zeta}_s\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\boldsymbol{\zeta}_s - \mathbf{v}\times[\nabla\times\boldsymbol{\zeta}_s] \ , \tag{95}$$

which with $\dot{\boldsymbol{\zeta}}_s = 0$ and Eq. (C19) can be written as

$$\delta\mathbf{v} = \dot{\boldsymbol{\zeta}}_s + (\mathbf{v}\cdot\nabla)\boldsymbol{\zeta}_s - (\boldsymbol{\zeta}_s\cdot\nabla)\mathbf{v} \ . \tag{96}$$

These changes are altogether the same relations as in the ideal case. Inserting Eqs. (94) and (97) into Eq. (11), and proceeding as is Section IV, one obtains the same expression for $\delta\mathcal{L}$ as in Eq. (19), but now with a symmetry displacement $\boldsymbol{\zeta}_s$ instead of $\boldsymbol{\zeta}$, i.e. $\delta\mathcal{L} = \delta\mathcal{L}(\boldsymbol{\zeta}_s)$. Setting this expression equal to $\delta\mathcal{L}$ as given by Eq. (93) yields

$$\delta\mathcal{L}(\boldsymbol{\zeta}_s, \delta\mathbf{A}, \delta\Phi, \delta n(\boldsymbol{\zeta}_s), \ldots) = -\nabla\cdot(\mathcal{L}\boldsymbol{\zeta}_s) \ . \tag{97}$$

In the expression on the l.h.s. of this equation (which is valid also in the isothermal case with γ set equal to 1) $\boldsymbol{\zeta}_s$ appears, in particular, multiplied by the l.h.s. of Eqs. (22) and (40), i.e.

$$en\left(\hat{\mathbf{E}} + \frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right) - \nabla p \ . \tag{98}$$

Without dissipation, the vanishing of this factor yields the equations of motion. With dissipation, however, one has

$$en\left(\hat{\mathbf{E}} + \frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right) - \nabla p - \nabla\cdot\underline{\underline{\Pi}} + \mathbf{R} = 0 \ , \tag{99}$$

where $\underline{\underline{\Pi}}$ is the (symmetric) stress tensor of each species and $\mathbf{R}$ is the momentum gain of the species considered by collisions with the other species. The equations of motion are now given by Eq. (99) instead of Eq. (40). *This means that in* Eqs. (45)-(48) *and* (51)-(56) $-\nabla p$ *has to be replaced by* $-\nabla p - \nabla\cdot\underline{\underline{\Pi}} + \mathbf{R}$.
Obviously, if one requires

$$\sum_{\text{all species}} \mathbf{R} = 0 \ , \tag{100}$$

then $\mathbf{R}$ does not contribute after summing Eq. (97) over all species. Similarly to Eq. (75), one obtains

$$\begin{aligned}\sum\Bigg\{&\frac{\partial}{\partial t}\left[\boldsymbol{\zeta}_s\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}} + \frac{1}{c}\left[\nabla\cdot(\boldsymbol{\zeta}_s\cdot\mathbf{A}) - \boldsymbol{\zeta}_s\times\mathbf{B}\right]\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right]\\ &+\nabla\cdot\left[-n\boldsymbol{\zeta}_s\frac{\partial\mathcal{L}}{\partial n} - \gamma p\boldsymbol{\zeta}_s\frac{\partial\mathcal{L}}{\partial p} + \left[\boldsymbol{\zeta}_s\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right]\mathbf{v} + \left[\boldsymbol{\zeta}_s\cdot\nabla\Phi\right]\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right.\\ &\left.+\left[-\nabla(\boldsymbol{\zeta}_s\cdot\mathbf{A}) + \boldsymbol{\zeta}_s\times\mathbf{B}\right]\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}} + \mathcal{L}\boldsymbol{\zeta}_s + \underline{\underline{\Pi}}\cdot\boldsymbol{\zeta}_s\right]\Bigg\} = 0 \ ,\end{aligned} \tag{101}$$

where Eq. (C20) has been used. The *local* momentum conservation law in dissipative systems thus differs from that in collisionless ones through the viscosity term $\nabla \cdot (\underline{\underline{\Pi}} \cdot \zeta_s)$. Including this term in Eqs. (76), (78) and (86) yields the corresponding expressions for dissipative systems. Integrating these expressions in a way similar to that of Section V B eliminates the viscosity term, and *the global momentum conservation laws in the collisionless and dissipative systems considered are the same.*

The stress tensor $\underline{\underline{\Pi}}$ can be written as

$$\Pi_{ik} = -\mu_{ik,lm} W_{lm} \ , \tag{102}$$

where $\mu_{ik,lm}$ is the viscosity tensor and W_{lm} the strain tensor,

$$W_{lm} = \frac{\partial v_l}{\partial x_m} + \frac{\partial v_m}{\partial x_l} - \frac{2}{3}\delta_{lm} \nabla \cdot \mathbf{v} \ . \tag{103}$$

Both tensors $\underline{\underline{\Pi}}$ and $\underline{\underline{W}}$ are symmetric and have zero trace. More details about this quantities in a plasma can be found in Ref.[22] .

The friction term $\mathbf{R}$ of each species is assumed to split into two parts

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 \ , \tag{104}$$

where $\mathbf{R}_1$ represents ohmic friction, and $\mathbf{R}_2$ thermal forces which arise from a gradient in the electron temperature.

Explicit expressions for R

A plasma consisting of electrons and one species of positively charged ions is now considered. Resistivity is taken to be a tensor, $\underline{\underline{\eta}}$.
According to Eq. (58), the total charge density of each species is $\rho = en - \nabla \cdot \mathcal{P}$, with $\mathcal{P} = -\frac{cm_\perp n}{B}(\mathbf{v}_\perp - \mathbf{v}_E) \times \mathbf{b}$, and the quasi-neutrality condition is $\sum \rho = 0$. If one introduces the *quasi-particle current density* $\mathbf{j}_\rho$,

$$\mathbf{j}_\rho = \sum \rho \mathbf{v} \ , \tag{105}$$

then the ohmic friction term $\mathbf{R}_1$ of each species can be written as

$$\mathbf{R}_1 = -\rho \underline{\underline{\eta}} \cdot \mathbf{j}_\rho \ , \tag{106}$$

and, because of quasi-neutrality,

$$\sum \mathbf{R}_1 = 0 \; . \tag{107}$$

The thermal friction terms $\mathbf{R}_2$ for the electrons and ions are taken as

$$\begin{aligned} \mathbf{R}_{2e} &= -c_1 n_e \left(\mathbf{b} \cdot \nabla T_e\right) \mathbf{b} - c_2 \frac{n_e}{\omega_e \tau_e} \mathbf{b} \times \left(\nabla T_e\right) \; , \\ \mathbf{R}_{2i} &= -\mathbf{R}_{2e} \; , \end{aligned} \tag{108}$$

and therefore

$$\sum \mathbf{R}_2 = 0 \; . \tag{109}$$

ω_e is the electron gyrofrequency, τ_e the electron collision time and T_e the electron temperature. c_1, c_2 are two constants (e.g. $c_1 = 0.71$, $c_2 = \frac{3}{2}$ for a hydrogen plasma)[22] .

B. Energy balance in dissipative systems

In a way similar to that of the collisionless case, displacements $\boldsymbol{\zeta}$ and changes $\delta\mathcal{L}$, δn, δp, etc..., which result from the dynamical evolution of the system are now considered.

$$\boldsymbol{\zeta} = \mathbf{v}\delta t \quad \Longrightarrow \quad \dot{\boldsymbol{\zeta}} = \dot{\mathbf{v}}\delta t \; , \tag{110}$$

$$\delta\mathbf{v} = \dot{\mathbf{v}}\delta t \; , \; \delta n = \dot{n}\delta t = -\nabla \cdot (n\mathbf{v})\,\delta t = -\nabla \cdot (n\boldsymbol{\zeta}) \; , \tag{111}$$

which are of course the same as δn and $\delta\mathbf{v}$ from Eq. (12) and (15) when $\boldsymbol{\zeta} = \mathbf{v}\delta t$ is inserted there.
For an isothermal plasma, with $\dot{T} + \mathbf{v} \cdot \nabla T = 0$, the generally valid relation $\dot{p} = -\nabla \cdot [nT\mathbf{v}] + n[\dot{T} + \mathbf{v} \cdot \nabla T]$ yields

$$\delta p = \dot{p}\delta t = -\nabla \cdot [nT\mathbf{v}]\,\delta t \; , \tag{112}$$

which is the same as in the ideal case. For an adiabatic plasma, however, δp is collision-dependent. It can be broken down into two parts, separating the ideal contributions $\dot{p}_{\text{id}}$ from the purely collisional ones, $\dot{p}_{\text{diss}}$:

$$\delta p = \dot{p}\delta t = (\dot{p}_{\text{id}} + \dot{p}_{\text{diss}})\,\delta t \; , \tag{113}$$

$$\text{where } \dot{p}_{\rm id} = -\mathbf{v}\cdot\nabla p - \gamma p \nabla\cdot\mathbf{v} \, . \tag{114}$$

and

$$\dot{p}_{\rm diss} = (\gamma-1)\, p\nabla\cdot\mathbf{v} + n\left[\dot{T} + \mathbf{v}\cdot\nabla T\right] \, . \tag{115}$$

Inserting Eqs. (110)-(114) into Eq. (11), proceeding as in the derivation of Eq. (19), and using Eq. (99), one obtains

$$\frac{\delta \mathcal{L}}{\delta t} = \left(\frac{\delta \mathcal{L}}{\delta t}\right)_{\rm collisionless} + \mathbf{v}\cdot\left[\nabla\cdot\underline{\underline{\Pi}} - \mathbf{R}\right] + \dot{p}_{\rm diss}\frac{\delta\mathcal{L}}{\delta p} \, , \tag{116}$$

where $\delta\mathcal{L}_{\rm collisionless}$ is given by Eq. (19) with $\frac{\delta\mathcal{L}}{q_4} = 0$, i.e. $q_4 = v_{\|}$ and Eqs. (110) and (111) taken into account. The same procedure as that of Section V A, applied to Eq. (116), yields the local energy balance:

$$\begin{aligned} &\sum\left\{\frac{\partial}{\partial t}\left[\frac{n}{2}\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right) - \mathcal{L}_p + en\Phi + \frac{1}{8\pi}\mathbf{B}^2 - \frac{1}{c}\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial t}\right]\right. \\ &+\nabla\cdot\left[\frac{n}{2}\left(m_{\|}v_{\|}^2 + m_{\perp}\mathbf{v}_E^2\right)\mathbf{v} - \gamma p\frac{\partial\mathcal{L}_p}{\partial p}\mathbf{v}\right. \\ &\left.\left. + en\Phi\mathbf{v} - \dot{\Phi}\mathbf{P} - \frac{1}{4\pi}\frac{\partial\mathbf{A}}{\partial t}\times\left[\mathbf{B} - 4\pi\mathbf{M}\right]\right]\right\} \\ = \; &\sum\left\{\mathbf{v}\cdot\left[-\nabla\cdot\underline{\underline{\Pi}} + \mathbf{R}\right] - \dot{p}_{\rm diss}\frac{\partial\mathcal{L}_p}{\partial p}\right\} \, , \end{aligned} \tag{117}$$

where $\mathcal{L}_p = -p/(\gamma-1)$ in the adiabatic case. In the isothermal case, $\mathcal{L}_p = -p\ln(p/p_c)$, γ is set equal to 1 and $\dot{p}_{\rm diss} = 0$.
Writing now the indexes ν, μ of the different particle species where it appears appropriate to avoid confusion, the $\dot{p}_{\rm diss}$ term for an adiabatic plasma is, more explicitly,

$$\begin{aligned} -\dot{p}_{{\rm diss}\,\nu}\frac{\partial\mathcal{L}_{p_\nu}}{\partial p_\nu} &= \frac{\dot{p}_{{\rm diss}\ \nu}}{(\gamma_\nu - 1)} = \frac{n_\nu}{(\gamma_\nu-1)}\left[\dot{T}_\nu + \mathbf{v}_\nu\cdot\nabla T_\nu\right] + p_\nu\nabla\cdot\mathbf{v}_\nu \\ &= -\nabla\cdot\mathbf{q}_\nu - \underline{\underline{\Pi_\nu}} : \nabla\mathbf{v}_\nu - \sigma_\nu\frac{\mathbf{j}_\rho}{\rho_\nu}\cdot\mathbf{R}_{1\nu} - \tilde{\sigma}_\nu\frac{\mathbf{j}_\rho}{\rho_\nu}\cdot\mathbf{R}_{2\nu} - \sum_\mu\frac{T_\nu - T_\mu}{\alpha_{\nu\mu}} \end{aligned} \tag{118}$$

with $\mathbf{q}_\nu$ the heat flux of the species ν and

$$\sum_\nu \sigma_\nu = 1 \ , \ \sum_\nu \tilde{\sigma}_\nu = 1 \tag{119}$$

$$\text{and} \ \ \alpha_{\nu\mu} = \alpha_{\mu\nu} \ \ \Longrightarrow \ \ \sum_{\nu\mu} \frac{T_\nu - T_\mu}{\alpha_{\nu\mu}} = 0 \ . \tag{120}$$

σ_ν and $\tilde{\sigma}_\nu$ are the fractions of the ohmic and thermal friction energies, respectively, gained by each particle species ν. Usually, because $m_e << m_i$, $\sigma_{\text{electron}} = 1$, $\tilde{\sigma}_{\text{el.}} = 1$, $\sigma_{\text{ion}} = 0$ and $\tilde{\sigma}_{\text{i}} = 0$. Again, more details can be found in Ref.[22].

The thermal friction heating term in $\dot{p}_{\text{diss}}$ summed over all species, $\sum \tilde{\sigma}_\nu \frac{\mathbf{j}_\rho}{\rho_\nu} \cdot \mathbf{R}_{2\nu}$, must be equal to the corresponding thermal friction term $\sum \mathbf{v}_\nu \cdot \mathbf{R}_{2\nu}$ from the equations of motion, and one has

$$\sum_\nu \left[\mathbf{v}_\nu \cdot \mathbf{R}_{2\nu} - \tilde{\sigma}_\nu \frac{\mathbf{j}_\rho}{\rho_\nu} \cdot \mathbf{R}_{2\nu} \right] = 0 \ . \tag{121}$$

Since for an isothermal plasma $\dot{p}_{\text{diss}} = 0$, the terms on the r.h.s. of Eq. (117) are

$$\begin{aligned} &-\mathbf{j}_\rho \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_\rho + \sum \left[-\mathbf{v} \cdot \nabla \cdot \underline{\underline{\Pi}} + \mathbf{v} \cdot \mathbf{R}_2 \right] \\ &-\mathbf{j}_\rho \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_\rho + \sum \left[-\nabla \cdot \left[\underline{\underline{\Pi}} \cdot \mathbf{v} \right] + \underline{\underline{\Pi}} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{R}_2 \right] \ , \end{aligned} \tag{122}$$

while for the adiabtic case there is also the contribution

$$\sum \frac{\dot{p}_{\text{diss}}}{(\gamma - 1)} = \mathbf{j}_\rho \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_\rho - \sum \left[\nabla \cdot \mathbf{q} + \underline{\underline{\Pi}} : \nabla \mathbf{v} + \tilde{\sigma} \frac{\mathbf{j}_\rho}{\rho} \mathbf{R}_2 \right] \tag{123}$$

so that the ohmic heating terms cancel, likewise the thermal heating terms, and the contributions from the viscosity add to $-\sum \nabla \cdot \left[\underline{\underline{\Pi}} \cdot \mathbf{v} \right]$. Integrating Eq. (117) over the plasma volume and assuming that there are no contributions from the boundaries, one obtains the same expression for the conserved energy as in the ideal case, Eqs. (68) and (70).

In the isothermal case, assuming again that there are no contributions from the boundaries, the integration of Eq. (117) yields the global energy dissipation law:

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \frac{d}{dt}\sum\int\left[\frac{n}{2}\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2\right)+p\ln\frac{p}{p_c}+en\Phi-\frac{1}{c}\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{8\pi}\mathbf{B}^2\right]d^3x\\ &= \sum\int\left[-\mathbf{j}_\rho\cdot\underline{\underline{\eta}}\cdot\mathbf{j}_\rho+\Pi_{ik}\frac{\partial v_i}{\partial x_k}+\mathbf{v}\cdot\mathbf{R}_2\right]d^3x\ ,\end{aligned} \tag{124}$$

or also

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \frac{d}{dt}\sum\int\left[\frac{n}{2}\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2-m_{\perp}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)^2\right)+p\ln\frac{p}{p_c}+\frac{1}{8\pi}\mathbf{B}^2\right]d^3x\\ &= \sum\int\left[-\mathbf{j}_\rho\cdot\underline{\underline{\eta}}\cdot\mathbf{j}_\rho+\Pi_{ik}\frac{\partial v_i}{\partial x_k}+\mathbf{v}\cdot\mathbf{R}_2\right]d^3x\ .\end{aligned} \tag{125}$$

The dissipated energy on the r.h.s. of these last two equations has to be absorbed by the " heat reservoir " which is necessary to keep $\frac{dT}{dt}=0$. The viscosity term $\Pi_{ik}\frac{\partial v_i}{\partial x_k}$ is negative definite: one can write

$$\Pi_{ik}\frac{\partial v_i}{\partial x_k}=\frac{1}{2}\left[\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\nabla\cdot\mathbf{v}\delta_{ik}\right]\Pi_{ik}\ . \tag{126}$$

Here, the first two terms are equal contributions because of the symmetry of $\underline{\underline{\Pi}}$, and the third term vanishes because $\underline{\underline{\Pi}}$ has zero trace. Since the quantity in the brackets is W_{ik}, one obtains

$$\begin{aligned}\Pi_{ik}\frac{\partial v_i}{\partial x_k} &= \frac{1}{2}W_{ik}\Pi_{ik}\\ &= \frac{1}{2}W_{11}\Pi_{11}+\frac{1}{2}\left[W_{11}+W_{33}\right]\left[\Pi_{11}+\Pi_{33}\right]+\frac{1}{2}W_{33}\Pi_{33}\\ &\quad +W_{12}\Pi_{12}+W_{13}\Pi_{13}+W_{23}\Pi_{23}\ ,\end{aligned} \tag{127}$$

where it has again been used that $\underline{\underline{\Pi}}$ and $\underline{\underline{W}}$ are symmetric and have zero trace. In a coordinate system with the magnetic field in the direction of $\mathbf{e}_3$, the components of $\underline{\underline{\Pi}}$ are, expicitly[22]

$$\Pi_{11} = \frac{1}{2}\mu_0W_{33}-\frac{1}{2}\mu_1\left[2W_{11}+W_{33}\right]-\mu_3W_{12}\ ,$$

$$
\begin{aligned}
\Pi_{33} &= -\mu_0 W_{33} , \\
\Pi_{12} &= -\mu_1 W_{12} + \frac{1}{2}\mu_3 \left[2W_{11} + W_{33}\right] , \\
\Pi_{13} &= -\mu_2 W_{13} - \mu_4 W_{23} , \\
\Pi_{23} &= -\mu_2 W_{23} + \mu_4 W_{13} .
\end{aligned} \tag{128}
$$

and, therefore

$$
\frac{1}{2} W_{ik}\Pi_{ik} = -\frac{3}{4}\mu_0 W_{33}^2 - \frac{1}{4}\mu_1 \left[2W_{11} + W_{33}\right]^2 - \mu_1 W_{12}^2 - \mu_2 \left[W_{13}^2 + W_{23}^2\right] . \tag{129}
$$

The thermal friction term $\sim \mathbf{R}_2$ in Eqs. (124) and (125) can have either sign; it represents a reversible generation of heat (see[22], p. 233).

VII. Consistency Problems - Stability

The results obtained in the preceding sections are valid also for systems which are coupled to an external circuit, e.g. via an electric field driving the equilibrium current in a resistive plasma. As concerns the stability properties of such systems there is a problem that energy could be fed into the perturbations or removed from them by this coupling. In an isolated resistive plasma the driving electric field is generated by the decaying corresponding flux, which does not eliminate the problem. This means that the investigation of the stability properties of resistive plasmas must take into account these circumstances.

Another point concerns the electrostatic approximation. This approximation also implies a coupling to an - artificial - external circuit: $\delta\mathbf{B} \equiv 0$ requires, strictly speaking, that at least the currents corresponding to the perturbations must be compensated locally by currents flowing in an artificial medium. Since there are also electric fields with components parallel to these currents, the currents in the artificial medium must be driven against these fields, and this means that energy is fed into the plasma or removed from it. In order to find out whether this process could be relevant, let us look into some details of the energy relation (125). For the present purpose it is sufficient to consider resistive plasmas with no other dissipative effects and neglect Joule heating such that the energy relation (125) for isothermal plasmas is valid also for plasmas with the adiabatic coefficient different from 1 when the thermal energy term is replaced by the corresponding adiabatic one. For small perturbations the RHS of Eq. (125) is

$$-\sum \int \left[\mathbf{j}_{\rho 0} \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_{\rho 0} + 2\mathbf{j}_{\rho 1} \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_{\rho 0} + 2\mathbf{j}_{\rho 2} \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_{\rho 0} + \mathbf{j}_{\rho 1} \cdot \underline{\underline{\eta}} \cdot \mathbf{j}_{\rho 1} + \cdots \right] d^3x \ . \tag{130}$$

For the discussion of the stability properties the second-order quantities are of interest. The term which is bilinear in the first-order perturbation $\mathbf{j}_{\rho 1}$ is negative semi-definite, whereas the term linear in the second-order perturbation $\mathbf{j}_{\rho 2}$ is not. This latter term contributes, however, only if $\mathbf{j}_{\rho 2}$ possesses a nonvanishing average - say in a straight tokamak with $\mathbf{j}_{\rho 0} = const$ - parallel to $\mathbf{j}_{\rho 0}$. Since to such an average current corresponds a global perturbation of the magnetic field, the magnitude of this average current depends on boundary conditions determining the inductivity of the system. In a straight tokamak with walls only at infinity the average of $\mathbf{j}_{\rho 2}$ must vanish and the second-order dissipative term is negative. If one neglected the magnetic field perturbation, the term with $\mathbf{j}_{\rho 2}$ could become an instability driving term overcompensating the damping term bilinear in $\mathbf{j}_{\rho 1}$.

A second point, where neglect of the magnetic perturbation can lead to wrong results - independent of the strength of the unperturbed magnetic field - is in the energy expression itself: For an illustration let us assume, as often done,

$$m_e = 0 \ , \ T_e = const \ , \ m_{i\|} = \infty \ , \ T_i = 0 \ , \ \mathbf{E}_0 \propto \underline{\underline{\eta}} \ . \tag{131}$$

In this case the kinetic energy is the ion kinetic energy of the motion perpendicular to $\mathbf{B}$ only. Its zero-order contribution is proportional to $\underline{\underline{\eta}}^2$. The potential energy is the thermal energy of the electrons and the magnetic energy. Their second-order expressions in terms of displacements $\boldsymbol{\xi}_e$ for the electrons can be obtained with the help of one of the methods introduced by Pfirsch and Sudan[23] in the following way:

The second-order contribution to the thermal energy results, with $\delta^{(1)}n_e$, $\delta^{(2)}n_e$ being the first and second variation of the electron density, from

$$(n_{e0}+\delta^{(1)}n_e+\frac{1}{2}\delta^{(2)}n_e+\cdots)\left(\ln n_{e0} + \frac{\delta^{(1)}n_e}{n_{e0}} - \frac{1}{2}\left(\frac{\delta^{(1)}n_e}{n_{e0}}\right)^2 + \frac{1}{2}\frac{\delta^{(2)}n_e}{n_{e0}} + \cdots\right) . \tag{132}$$

The second-order thermal energy of the electrons is then

$$W_e^{(2)} = \frac{T_e}{2} \int \left\{ \delta^{(2)} n_e \ln n_{e0} + \frac{(\delta^{(1)} n_e)^2}{n_{e0}} \right\} d^3x \ . \tag{133}$$

The mentioned method of Pfirsch and Sudan[23] consists in the following steps:

$$\delta^{(1)} n_e = -\nabla \cdot (n_{e0} \boldsymbol{\xi}_e) \ , \ \delta^{(2)} n_e = -\nabla \cdot (\boldsymbol{\xi}_e \delta^{(1)} n_e) - \nabla \cdot (n_{e0} \delta^{(1)} \boldsymbol{\xi}_e) \ . \tag{134}$$

The variation of $\boldsymbol{\xi}_e$ is obtained from representing the shifted position $\hat{\mathbf{x}}$ of an electron fluid element on the one hand by referring to the unperturbed position $\mathbf{x}$ in the unperturbed system,

$$\hat{\mathbf{x}} = \mathbf{x} + \boldsymbol{\xi}_e(\mathbf{x}, t) \ , \tag{135}$$

and on the other hand by referring to the perturbed position in the perturbed system,

$$\hat{\mathbf{x}} = \mathbf{x} + \hat{\boldsymbol{\xi}}_e(\hat{\mathbf{x}}, t) \ . \tag{136}$$

Hence,

$$\hat{\boldsymbol{\xi}}_e(\hat{\mathbf{x}}, t) = \boldsymbol{\xi}_e(\mathbf{x}, t) \tag{137}$$

The Eulerian variation of $\boldsymbol{\xi}_e(\mathbf{x}, t)$ in Eq. (134) is given by

$$\delta^{(1)} \boldsymbol{\xi}_e(\mathbf{x}, t) = \hat{\boldsymbol{\xi}}_e(\mathbf{x}, t) - \boldsymbol{\xi}_e(\mathbf{x}, t) = \boldsymbol{\xi}_e(\mathbf{x} - \boldsymbol{\xi}_e(\mathbf{x}, t), t) - \boldsymbol{\xi}_e(\mathbf{x}, t) = -\boldsymbol{\xi}_e \cdot \nabla \boldsymbol{\xi}_e \ . \tag{138}$$

With this result the second variation of the electron density becomes

$$\delta^{(2)} n_e = \nabla \cdot ((\nabla \cdot n_e \boldsymbol{\xi}_e) \boldsymbol{\xi}_e) + \nabla \cdot (n_{e0} \boldsymbol{\xi}_e \cdot \nabla \boldsymbol{\xi}_e) \ . \tag{139}$$

Twice the $\delta^{(2)} n_e$ contribution in $W_e^{(2)}$ is then

$$\begin{aligned} \int \delta^{(2)} n_e \ln n_{e0} \, d^3x &= \int \left[\delta^{(1)} n_e \boldsymbol{\xi}_e - n_{e0} \boldsymbol{\xi}_e \cdot \nabla \boldsymbol{\xi}_e \right] \cdot \nabla \ln n_{e0} \, d^3x \\ &= -\int \left[\nabla \cdot (n_{e0} \boldsymbol{\xi}_e) \boldsymbol{\xi}_e + n_{e0} \boldsymbol{\xi}_e \cdot \nabla \boldsymbol{\xi}_e \right] \cdot \nabla \ln n_{e0} \, d^3x \\ &= -\int \nabla \cdot (n_{e0} \boldsymbol{\xi}_e \boldsymbol{\xi}_e) \cdot \nabla \ln n_{e0} \, d^3x \\ &= \int \boldsymbol{\xi}_e \boldsymbol{\xi}_e : n_{e0} \nabla \nabla \ln n_{e0} \, d^3x \ . \end{aligned} \tag{140}$$

The general expression for the second-order thermal energy of the electrons is therefore

$$W_e^{(2)} = \int \frac{T_e}{2} \left\{ \frac{(\delta^{(1)} n_e)^2}{n_{e0}} + \boldsymbol{\xi}_e \boldsymbol{\xi}_e : n_{e0} \nabla\nabla \ln n_{e0} \right\} d^3x \; . \tag{141}$$

The second-order magnetic energy can be obtained easily for the limit $\underline{\underline{\eta}} \to 0$. In this case the magnetic field is frozen into the electrons. One can then again apply the method of Pfirsch and Sudan, which yields

$$\begin{aligned}
\delta^{(1)}\mathbf{B} &= \nabla\times(\boldsymbol{\xi}_e\times\mathbf{B}) \, , \\
\delta^{(2)}\mathbf{B} &= \nabla\times[(\boldsymbol{\xi}_e\cdot\nabla\boldsymbol{\xi}_e)\times\mathbf{B})] + \nabla\times[\boldsymbol{\xi}_e\times(\nabla\times(\boldsymbol{\xi}_e\times\mathbf{B}))] \, , \\
\delta^{(2)}\frac{\mathbf{B}^2}{8\pi} &= \frac{1}{4\pi}\left(\delta^{(1)}\mathbf{B}\right)^2 + \frac{1}{4\pi}\mathbf{B}\cdot\delta^{(2)}\mathbf{B} \, , \\
\left(\frac{\mathbf{B}^2}{8\pi}\right)^{(2)} &= \frac{1}{2}\delta^{(2)}\frac{\mathbf{B}^2}{8\pi} \, .
\end{aligned} \tag{142}$$

When the last term is integrated in space, integration by parts leads to the replacement

$$\begin{aligned}
\frac{1}{8\pi}\mathbf{B}\cdot\delta^{(2)}\mathbf{B} &\to \frac{1}{8\pi}(\nabla\times\mathbf{B})\cdot[-(\boldsymbol{\xi}_e\cdot\nabla\boldsymbol{\xi}_e)\times\mathbf{B} + \boldsymbol{\xi}_e\times(\nabla\times(\boldsymbol{\xi}_e\times\mathbf{B}))] \\
&= \frac{1}{2c}(\mathbf{j}\times\mathbf{B})\cdot(\boldsymbol{\xi}_e\cdot\nabla\boldsymbol{\xi}_e) + \frac{1}{2c}(\boldsymbol{\xi}_e\times\mathbf{j})\cdot\delta^{(1)}\mathbf{B} \\
&= \frac{1}{2}(\boldsymbol{\xi}_e\cdot\nabla\boldsymbol{\xi}_e)\cdot\nabla p + \frac{1}{2c}(\boldsymbol{\xi}_e\times\mathbf{j})\cdot\delta^{(1)}\mathbf{B} \, .
\end{aligned} \tag{143}$$

This yields for the second-order magnetic energy

$$W_B^{(2)} = \int \left\{ \frac{1}{8\pi}\left(\delta^{(1)}\mathbf{B}\right)^2 + (\boldsymbol{\xi}_e\cdot\nabla\boldsymbol{\xi}_e)\cdot\nabla p + \frac{1}{c}(\boldsymbol{\xi}_e\times\mathbf{j})\cdot\delta^{(1)}\mathbf{B} \right\} d^3x \tag{144}$$

It is important to note that this relation obtains only when the equilibrium equations are satisfied exactly. The interesting point with this result is the term with ∇p. It is independent of the strength of the unperturbed magnetic field and combines directly with the thermal energy of the electrons. This "combined" thermal energy $W_c^{(2)}$ is

$$\begin{aligned}
W_c^{(2)} &= \int \frac{T_e}{2} \left\{ \frac{(\delta^{(1)} n_e)^2}{n_{e0}} - \frac{1}{n_{e0}}(\boldsymbol{\xi}_e\cdot\nabla n_{e0})^2 - (\nabla\cdot\boldsymbol{\xi}_e)(\boldsymbol{\xi}_e\cdot\nabla n_{e0}) \right\} d^3x \\
&= \int \frac{T_e}{2n_{e0}} \left\{ \left[n_{e0}\nabla\cdot\boldsymbol{\xi}_e + \frac{1}{2}(\boldsymbol{\xi}_e\cdot\nabla n_{e0}) \right]^2 - \frac{1}{4}(\boldsymbol{\xi}_e\cdot\nabla n_{e0})^2 \right\} d^3x \, .
\end{aligned} \tag{145}$$

It is remarkable that the term resulting from the magnetic field perturbation canceld the term with the second derivative of the density profile. Only first derivatives are left.

The total second-order potential energy is then

$$W_{pot}^{(2)} = \int \left\{ \frac{1}{8\pi} \left(\delta^{(1)}\mathbf{B}\right)^2 + \frac{1}{c} (\boldsymbol{\xi}_e \times \mathbf{j}) \cdot \delta^{(1)}\mathbf{B} + \frac{T_e}{2n_{e0}} \left[\left[n_{e0} \nabla \cdot \boldsymbol{\xi}_e + \frac{1}{2} (\boldsymbol{\xi}_e \cdot \nabla n_{e0}) \right]^2 - \frac{1}{4} (\boldsymbol{\xi}_e \cdot \nabla n_{e0})^2 \right] \right\} d^3x \,. \quad (146)$$

The residual magnetic terms with $\delta^{(1)}\mathbf{B}$ are the same ones as in ideal MHD with $\boldsymbol{\xi}$ replaced by $\boldsymbol{\xi}_e$. It is again emphasized that relation (146) is, as mentioned before, valid only for small resistivity.

For $\underline{\underline{\eta}} = 0$ the electrons behave "adiabatically" if in addition the effect of the the electric field corresponding to the magnetic field perturbation is neglected. This has the consequence that, for modes, $\boldsymbol{\xi}_e \cdot \nabla n_{e0} = 0$ and the "thermal" term resulting from the magnetic perturbation does not enter. The kinetic energy is now directly of second order and the potential energy is positive semi-definite and in fact positive for drift-waves for which $\delta^{(1)} n_e \neq 0$. This result is in agreement with the physical mechanism behind drift-waves: it consists in an oscillation of the second-order energy between the thermal energy of the electrons and the kinetic energy of the ions. For $\underline{\underline{\eta}} \neq 0$ the electron displacements can possess components in the direction of the density gradient and allow therefore expansion with corresponding lowering of the thermal energy. The total second-order energy to become negative requires, however, a minimum resistivity. This would be necessary for instability if the average of $\mathbf{j}_{\rho 2}$ vanishes, a result which contradicts the Hasegawa-Wakatani theory.

VIII. Summary

Ideal and collisional drift-fluid theories were obtained starting with a Lagrangian for the drift motion of particles. The adiabatic invariant magnetic moment and the "thermal" parallel energy are incorporated in a pressure for which an adiabatic law combined with dissipative terms or an isothermal law can be prescribed. Resistivity and thermal forces can be added in a transparent way such that energy laws and conservation laws for momenta are immediately obtained. The final equations for quasi-neutral electrostatic perturbations are similar to those, for instance, of the Hasegawa-Wakatani theory and tractable numerically like these. Also electromagnetic perturbations are possible. The only restriction involved is the validity of the drift approximation. The new theory avoids problems relating to conservation laws and boundary conditions. This is of importance in several respects: energy conservation is important for stability, in particular for nonlinear instabilities relating to negative-energy perturbations. Momentum conservation, for instance, is important in discussing the generation of radial electric fields which are presently considered to be relevant for the L-H-transition. Boundary conditions which do not introduce unclear "external" influences can be imposed only when treating a complete system without approximating unperturbed density profiles and density gradients as contants. The latter is dangerous also, because it can create artificial non-hermitian parts of an in reality hermitian operator. Although such a non-hermitian part is possibly small compared with the correct hermitian operator, it can lead to artificial instabilities as shown via a simple model equation in Appendix D. The class of systems considered in Section VII shows that for resistive quasi-neutral electrostatic instabilities to occur the second spatial derivative of the equilibrium density profile plays a role, a quantity which is neglected in the usual drift-wave theories. It is shown that for essentially electrostatic instabilities, magnetic perturbations in resistive systems may not be negligible even for $\beta << 1$. An example which will be treated in a future paper indicates, in addition, that in systems with $T_i = 0$ electron temperature profiles should strongly influence the stability via resistive effects. This is in addition to effects leading to η_e-modes. It demonstrates also that in general it is not possible to do an expansion with respect to the resistivity η near $\eta = 0$.

APPENDIX A

THE FREE ENERGY OF AN ISOTHERMAL PLASMA

The expression for the thermal energy of an isothermal plasma is derived in this appendix from thermodynamical considerations. For isothermal processes, the part of the thermal energy of the system which can be converted into work is the free energy. The free energy density f of each particle species is given by

$$f = u - Ts \ , \tag{A1}$$

with u and s the internal energy and entropy densities, respectively. Explicitly, these are

$$u = \frac{p}{\gamma - 1} \ , \tag{A2}$$

$$s = n \left[\ln \frac{n_c}{n} + \frac{1}{\gamma - 1} \ln \frac{T}{T_c} \right] \ , \tag{A3}$$

which, with $\dfrac{n_c}{n} = \dfrac{p_c T}{p T_c}$, where n_c, p_c are reference constant values, yields

$$s = n \left[- \ln \frac{p}{p_c} + \frac{\gamma}{\gamma - 1} \ln \frac{T}{T_c} \right] \ . \tag{A4}$$

Therefore, the free energy density is

$$f = p \ln \frac{p}{p_c} + \frac{p}{\gamma - 1} \left[1 - \gamma \ln \frac{T}{T_c} \right] \ . \tag{A5}$$

For an isothermal plasma, i.e. a plasma in which the temperature of the individual plasma elements is time independent, one has

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathrm{v} \cdot \nabla T = 0 \ , \quad \frac{\partial p}{\partial t} + \nabla \cdot (p \mathrm{v}) = 0 \ . \tag{A6}$$

In this case, the second term in Eq. (A5) plays no role in the dynamics since its integral is a constant:

$$\frac{d}{dt} \int_{V(t)} p \left(1 - \gamma \ln \frac{T}{T_c} \right) d^3x \ = \ \int_{V(t)} \left[\frac{\partial}{\partial t} \left[p \left(1 - \gamma \ln \frac{T}{T_c} \right) \right] + \nabla \cdot \left[p \left(1 - \gamma \ln \frac{T}{T_c} \right) \mathrm{v} \right] \right] d^3x$$

$$= \int_{V(t)} \left[\left(1 - \gamma \ln \frac{T}{T_c} \right) [\dot{p} + \nabla \cdot (p\mathbf{v})] - \gamma n \left[\dot{T} + \mathbf{v} \cdot \nabla T \right] \right] d^3x$$

$$= 0 \; . \tag{A7}$$

Adding this term to the Lagrangian would give no contribution when Hamilton's principle is evaluated. Therefore, only the term

$$p \ln \frac{p}{p_c} \tag{A8}$$

has to be included in the Lagrangian density. The same result can be obtained by direct integration of the corresponding term in the equations of motion. This is explicitly done in appendix B.

APPENDIX B

ENERGY CONSERVATION

In this appendix, energy conservation is derived from the equations of motion for adiabatic as well as for isothermal systems.
The equation of motion for each particle species can be taken in the form given by Eq. (44), namely

$$n\frac{\partial}{\partial t}\left(m_{\|}v_{\|}\mathbf{b}+m_{\perp}\mathbf{v}_E\right)+\frac{1}{2}n\nabla\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2\right)=ne\mathbf{E}+n\frac{e}{c}\mathbf{v}\times\hat{\mathbf{B}}-\nabla p\,. \quad \text{(B1)}$$

One can write

$$\begin{aligned}\mathbf{v}\cdot\frac{\partial}{\partial t}(m_{\|}v_{\|}\mathbf{b}+m_{\perp}\mathbf{v}_E) &= \frac{1}{2}\frac{\partial}{\partial t}\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2\right)+\frac{v_{\|}}{B}\left(m_{\|}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E\right)\frac{\partial\mathbf{B}}{\partial t}\\ &\quad +m_{\perp}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{v}_E}{\partial t}\,. \qquad \text{(B2)}\end{aligned}$$

Scalar multiplication of Eq. (B1) with $\mathbf{v}=v_{\|}\mathbf{b}+\mathbf{v}_{\perp}$ then yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{n}{2}\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2\right)\right]+\nabla\cdot\left[\frac{n}{2}\left(m_{\|}v_{\|}^2+m_{\perp}\mathbf{v}_E^2\right)\mathbf{v}\right]\\ &+n\frac{v_{\|}}{B}\left(m_{\|}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{B}}{\partial t}+nm_{\perp}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{v}_E}{\partial t}\\ &=en\mathbf{v}\cdot\mathbf{E}-\mathbf{v}\cdot\nabla p\,,\end{aligned} \qquad \text{(B3)}$$

where the continuity equation $\dot{n}+\nabla\cdot(n\mathbf{v})=0$ has been used.
From the equation $\mathbf{v}_E=c(\mathbf{E}\times\mathbf{B})/B^2$, one obtains

$$\frac{\partial\mathbf{v}_E}{\partial t}=-\frac{1}{B^2}\frac{\partial B^2}{\partial t}\mathbf{v}_E+\frac{c}{B^2}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}+\frac{c}{B^2}\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\,. \qquad \text{(B4)}$$

Therefore

$$n\frac{v_{\parallel}}{B}\left(m_{\parallel}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{B}}{\partial t}+nm_{\perp}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{v}_E}{\partial t}=$$

$$-\left[\frac{cnm_{\perp}}{B^2}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\times\mathbf{B}\right]\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{nv_{\parallel}}{B}\left(m_{\parallel}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{B}}{\partial t}$$

$$+\frac{cm_{\perp}n}{B^2}\left[\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\times\mathbf{E}\right]\cdot\frac{\partial\mathbf{B}}{\partial t}-\frac{2m_{\perp}n}{B^2}\left[\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\cdot\mathbf{v}_E\right]\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

$$=\boldsymbol{\mathcal{P}}\cdot\frac{\partial\mathbf{E}}{\partial t}+\boldsymbol{\mathcal{M}}\cdot\frac{\partial\mathbf{B}}{\partial t}\,. \tag{B5}$$

When summed over all particle species, the term $en\mathbf{v}\cdot\mathbf{E}$ can be transformed with the help of Eqs. (59) and (60):

$$\begin{aligned}\sum en\mathbf{v}\cdot\mathbf{E} &= \left[-\frac{\partial\mathbf{P}}{\partial t}+\frac{c}{4\pi}\nabla\times\left[\mathbf{B}-4\pi\mathbf{M}\right]\right]\cdot\mathbf{E}\\ &= -\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}+\frac{c}{4\pi}\left[\mathbf{B}-4\pi\mathbf{M}\right]\cdot\left[\nabla\times\mathbf{E}\right]-\nabla\cdot\left[\mathbf{E}\times\frac{c}{4\pi}\left[\mathbf{B}-4\pi\mathbf{M}\right]\right]\\ &= -\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}-\frac{1}{4\pi}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial t}-\nabla\cdot\left[\mathbf{E}\times\frac{c}{4\pi}\left[\mathbf{B}-4\pi\mathbf{M}\right]\right]\,.\end{aligned} \tag{B6}$$

Therefore, one has

$$\sum\left[n\frac{v_{\parallel}}{B}\left(m_{\parallel}\mathbf{v}_{\perp}-m_{\perp}\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{B}}{\partial t}+nm_{\perp}\left(\mathbf{v}_{\perp}-\mathbf{v}_E\right)\cdot\frac{\partial\mathbf{v}_E}{\partial t}-en\mathbf{v}\cdot\mathbf{E}\right]=$$

$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}\mathbf{B}^2+\mathbf{E}\cdot\mathbf{P}\right]+\nabla\cdot\left[\mathbf{E}\times\frac{c}{4\pi}\left[\mathbf{B}-4\pi\mathbf{M}\right]\right]\,. \tag{B7}$$

This equation can be cast into a different form which is useful when studying electrostatic perturbations. If $\mathbf{E}$ is replaced by $-\nabla\Phi-\dot{\mathbf{A}}/c$ in Eq. (B7), the terms involving $\nabla\Phi$ yield

$$\frac{\partial}{\partial t}\left[-\nabla\Phi\cdot\mathbf{P}\right]+\nabla\cdot\left[-\nabla\Phi\times\frac{c}{4\pi}\left[\mathbf{B}-4\pi\mathbf{M}\right]\right]$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}\left[\Phi\nabla\cdot\mathbf{P} - \nabla\cdot(\Phi\mathbf{P})\right] + \nabla\Phi\cdot\left[\nabla\times\frac{c}{4\pi}\left[\mathbf{B} - 4\pi\mathbf{M}\right]\right] \\
&= \frac{\partial}{\partial t}\left[\Phi\nabla\cdot\mathbf{P} - \nabla\cdot(\Phi\mathbf{P})\right] + \nabla\cdot\left[\Phi\nabla\times\frac{c}{4\pi}\left[\mathbf{B} - 4\pi\mathbf{M}\right]\right] \\
&= \frac{\partial}{\partial t}\left[\Phi\nabla\cdot\mathbf{P}\right] + \nabla\cdot\left[-\dot{\phi}\mathbf{P} + en\Phi\mathbf{v}\right] ,
\end{aligned}
\tag{B8}
$$

where Eqs. (59) and (60) have been used.
In order to transform the term $\mathbf{v}\cdot\nabla p$, the adiabatic and the isothermal cases are considered separately.

Adiabatic case

The adiabatic law

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= -\mathbf{v}\cdot\nabla p - \gamma p\nabla\cdot\mathbf{v} \\
&= (\gamma - 1)\mathbf{v}\cdot\nabla p - \gamma\nabla\cdot(p\mathbf{v})
\end{aligned}
\tag{B9}
$$

yields

$$
\mathbf{v}\cdot\nabla p = \frac{\partial}{\partial t}\left[\frac{p}{\gamma - 1}\right] + \nabla\cdot\left[\frac{\gamma}{\gamma - 1}p\mathbf{v}\right] . \tag{B10}
$$

Isothermal case

In this case $\gamma = 1$, corresponding to infinitely many degrees of freedom, and Eq. (B10) cannot be used to express $\mathbf{v}\cdot\nabla p$. Eq. (B9), however, is still valid

$$
\frac{\partial p}{\partial t} = -\nabla\cdot(p\mathbf{v}) \tag{B11}
$$

and one obtains

$$
\begin{aligned}
\mathbf{v}\cdot\nabla p &= p\frac{\mathbf{v}\cdot\nabla p}{p} \\
&= p\mathbf{v}\cdot\nabla\ln\frac{p}{p_c} \\
&= -\left[\nabla\cdot(p\mathbf{v})\right]\ln\frac{p}{p_c} + \nabla\cdot\left[p\ln\frac{p}{p_c}\mathbf{v}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial p}{\partial t} \ln \frac{p}{p_c} + \nabla \cdot \left[p \ln \frac{p}{p_c} \mathbf{v} \right] \\
&= \frac{\partial}{\partial t} \left[p \ln \frac{p}{p_c} \right] - \frac{\partial p}{\partial t} + \nabla \cdot \left[p \ln \frac{p}{p_c} \mathbf{v} \right] \\
&= \frac{\partial}{\partial t} \left[p \ln \frac{p}{p_c} \right] + \nabla \cdot \left[p \left[1 + \ln \frac{p}{p_c} \right] \mathbf{v} \right] . \qquad \text{(B12)}
\end{aligned}
$$

The first term is just the time derivative of the thermal free energy, as obtained in appendix A. Eqs. (B3), (B7), (B8), (B10), (B12) and the quasi-neutrality condition
$[\sum en] - \nabla \cdot \mathbf{P} = 0$ then yield the

Local energy conservation law for adiabatic systems

$$
\begin{aligned}
&\sum \left\{ \frac{\partial}{\partial t} \left[\frac{n}{2} \left(m_{\|} v_{\|}^2 + m_{\perp} \mathbf{v}_E^2 \right) + \frac{p}{\gamma - 1} + \frac{1}{8\pi} \mathbf{B}^2 + \mathbf{E} \cdot \mathbf{P} \right] \right. \\
&\left. + \nabla \cdot \left[\frac{n}{2} \left(m_{\|} v_{\|}^2 + m_{\perp} \mathbf{v}_E^2 \right) \mathbf{v} + \frac{\gamma}{\gamma - 1} p \mathbf{v} + \mathbf{E} \times \frac{c}{4\pi} \left[\mathbf{B} - 4\pi \mathbf{M} \right] \right] \right\} = 0 , \qquad \text{(B13)}
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
&\sum \left\{ \frac{\partial}{\partial t} \left[\frac{n}{2} \left(m_{\|} v_{\|}^2 + m_{\perp} \mathbf{v}_E^2 \right) + \frac{p}{\gamma - 1} + en\Phi + \frac{1}{8\pi} \mathbf{B}^2 - \frac{1}{c} \mathbf{P} \cdot \frac{\partial \mathbf{A}}{\partial t} \right] \right. \\
&+ \nabla \cdot \left[\frac{n}{2} \left(m_{\|} v_{\|}^2 + m_{\perp} \mathbf{v}_E^2 \right) \mathbf{v} + \frac{\gamma}{\gamma - 1} p \mathbf{v} \right. \\
&\left. \left. + en\Phi \mathbf{v} - \dot{\Phi} \mathbf{P} - \frac{1}{4\pi} \frac{\partial \mathbf{A}}{\partial t} \times \left[\mathbf{B} - 4\pi \mathbf{M} \right] \right] \right\} = 0 , \qquad \text{(B14)}
\end{aligned}
$$

and the

Local energy conservation law for isothermal systems

$$
\sum \left\{ \frac{\partial}{\partial t} \left[\frac{n}{2} \left(m_{\|} v_{\|}^2 + m_{\perp} \mathbf{v}_E^2 \right) + p \ln \frac{p}{p_c} + \frac{1}{8\pi} \mathbf{B}^2 + \mathbf{E} \cdot \mathbf{P} \right] \right.
$$

$$+\nabla \cdot \left[\frac{n}{2} \left(m_\| v_\|^2 + m_\perp \mathbf{v}_E^2 \right) \mathbf{v} + p \left[1 + \ln \frac{p}{p_c} \right] \mathbf{v} + \mathbf{E} \times \frac{c}{4\pi} \left[\mathbf{B} - 4\pi \mathbf{M} \right] \right] \Bigg\} = 0 \; , \tag{B15}$$

which can also be written as

$$\sum \Bigg\{ \frac{\partial}{\partial t} \left[\frac{n}{2} \left(m_\| v_\|^2 + m_\perp \mathbf{v}_E^2 \right) + p \ln \frac{p}{p_c} + en\Phi + \frac{1}{8\pi} \mathbf{B}^2 - \frac{1}{c} \mathbf{P} \cdot \frac{\partial \mathbf{A}}{\partial t} \right]$$
$$+\nabla \cdot \left[\frac{n}{2} \left(m_\| v_\|^2 + m_\perp \mathbf{v}_E^2 \right) \mathbf{v} + p \left[1 + \ln \frac{p}{p_c} \right] \mathbf{v} \right.$$
$$\left. + en\Phi \mathbf{v} - \dot{\Phi} \mathbf{P} - \frac{1}{4\pi} \frac{\partial \mathbf{A}}{\partial t} \times \left[\mathbf{B} - 4\pi \mathbf{M} \right] \right] \Bigg\} = 0 \; . \tag{B16}$$

If there are no contributions from the boundaries, the terms which are written as divergenges do not contribute to the *global* energy conservation laws.

APPENDIX C

PROPERTIES OF SYMMETRY DISPLACEMENTS

It is now investigated, how scalar and vector quantities are transformed under symmetry operations corresponding to displacement or rotation of the system as a whole. The transformations of interest here are described by a symmetry displacement, $\boldsymbol{\zeta}_s$, e.g. a parallel displacement of the whole system, or a rotation of the whole system about the z-axis by an angle $\delta\varphi$. The properties of the symmetry displacements $\boldsymbol{\zeta}_s$ and the transformation properties of the physical quantities are obtained from infinitesimal translational and rotational invariance.
Let $\mathbf{x}_o$ be the original position of a point of the system. Under a displacement $\boldsymbol{\zeta}_s$, the point originally at $\mathbf{x}_o$ has the new position $\mathbf{x}_n = \mathbf{x}_o + \boldsymbol{\zeta}_s(\mathbf{x}_o)$. An scalar quantity Ψ, such as the density or the temperature, remains unchanged under such transformation:

$$\Psi_n\left(\mathbf{x}_n = \mathbf{x}_o + \boldsymbol{\zeta}_s(\mathbf{x}_o)\right) = \Psi_o\left(\mathbf{x}_o\right) \ , \tag{C1}$$

To first order in $\boldsymbol{\zeta}_s$, this yields

$$\delta\Psi \equiv \Psi_n\left(\mathbf{x}_o\right) - \Psi_o\left(\mathbf{x}_o\right) = -\boldsymbol{\zeta}_s \cdot \nabla\Psi_o \ . \tag{C2}$$

The transformation properties of a gradient follow from Eq. (C2)

$$\begin{aligned} \delta\nabla\Psi &\equiv \nabla\Psi_n\left(\mathbf{x}_o\right) - \nabla\Psi_o\left(\mathbf{x}_o\right) \\ &= -\left(\boldsymbol{\zeta}_s\cdot\nabla\right)\nabla\Psi - \left(\nabla\boldsymbol{\zeta}_s\right)\cdot\nabla\Psi \\ &= -\left(\boldsymbol{\zeta}_s\cdot\nabla\right)\nabla\Psi - \left(\nabla\Psi\cdot\nabla\right)\boldsymbol{\zeta}_s - \nabla\Psi\times\left[\nabla\times\boldsymbol{\zeta}_s\right] \ . \end{aligned} \tag{C3}$$

A vector $\mathbf{w}$ is transformed in the same way as a gradient

$$\begin{aligned} \delta\mathbf{w} &= -\left(\boldsymbol{\zeta}_s\cdot\nabla\right)\mathbf{w} - \left(\nabla\boldsymbol{\zeta}_s\right)\cdot\mathbf{w} \\ &= -\left(\boldsymbol{\zeta}_s\cdot\nabla\right)\mathbf{w} - \left(\mathbf{w}\cdot\nabla\right)\boldsymbol{\zeta}_s - \mathbf{w}\times\left[\nabla\times\boldsymbol{\zeta}_s\right] \\ &= -\nabla\left[\boldsymbol{\zeta}_s\cdot\mathbf{w}\right] + \boldsymbol{\zeta}_s\times\left(\nabla\times\mathbf{w}\right) \ , \end{aligned} \tag{C4}$$

from which

$$
\begin{aligned}
\delta\,\mathbf{w}^2 &= 2\mathbf{w}\cdot\delta\mathbf{w} \\
&= -(\boldsymbol{\zeta}_s\cdot\nabla)\,\mathbf{w}^2 - 2\mathbf{w}\cdot[(\mathbf{w}\cdot\nabla)\,\boldsymbol{\zeta}_s]
\end{aligned}
\tag{C5}
$$

follows. On the other hand, the symmetry transformations considered here (a displacement or a rotation of the whole system) do not change *the absolute value* of $\mathbf{w}$, and $\mathbf{w}^2$ must transform itself according to Eq. (C2), i.e.

$$
\delta\,\mathbf{w}^2 = -(\boldsymbol{\zeta}_s\cdot\nabla)\,\mathbf{w}^2\ . \tag{C6}
$$

Equations (C5) and (C6) are simultaneously valid only if

$$
\mathbf{w}\cdot[(\mathbf{w}\cdot\nabla)\,\boldsymbol{\zeta}_s] = 0 \tag{C7}
$$

for all possible vectors $\mathbf{w}$. This implies a condition on the acceptable $\boldsymbol{\zeta}_s$'s. From Eq. (C7), it follows that

$$
(\mathbf{w}\cdot\nabla)\,\boldsymbol{\zeta}_s = \boldsymbol{\alpha}\,(\mathbf{x})\times\mathbf{w}\ . \tag{C8}
$$

Scalar multiplication of this equation with a *constant vector* $\boldsymbol{\beta}$ yields

$$
\boldsymbol{\beta}\cdot[(\mathbf{w}\cdot\nabla)\,\boldsymbol{\zeta}_s] = \mathbf{w}\cdot\nabla\,(\boldsymbol{\beta}\cdot\boldsymbol{\zeta}_s) = \mathbf{w}\cdot[\boldsymbol{\beta}\times\boldsymbol{\alpha}(\mathbf{x})]\ . \tag{C9}
$$

Since this relation must be valid for all $\mathbf{w}$, it implies

$$
\nabla\,(\boldsymbol{\beta}\cdot\boldsymbol{\zeta}_s) = \boldsymbol{\beta}\times\boldsymbol{\alpha}(\mathbf{x}) \tag{C10}
$$

and also

$$
\begin{aligned}
0 = \nabla\times[\nabla\,(\boldsymbol{\beta}\cdot\boldsymbol{\zeta}_s)] &= \nabla\times(\boldsymbol{\beta}\times\boldsymbol{\alpha}(\mathbf{x})) \\
&= \boldsymbol{\beta}\,(\nabla\cdot\boldsymbol{\alpha}) - (\boldsymbol{\beta}\cdot\nabla)\,\boldsymbol{\alpha}\ .
\end{aligned}
\tag{C11}
$$

This must be valid for all $\boldsymbol{\beta}$ = const., and therefore

$$
\boldsymbol{\alpha} = \text{const.} \tag{C12}
$$

Equation (C10) can now be integrated:

$$
\begin{aligned}
\boldsymbol{\beta}\cdot\boldsymbol{\zeta}_s &= (\boldsymbol{\beta}\times\boldsymbol{\alpha})\cdot\mathbf{x} + \text{const.} \\
&= \boldsymbol{\beta}\cdot[\boldsymbol{\alpha}\times\mathbf{x} + \mathbf{c}]\ ,
\end{aligned}
\tag{C13}
$$

where the constant has been written as $\boldsymbol{\beta} \cdot \mathbf{c}$. The possible symmetry displacements are then given by

$$\boldsymbol{\zeta}_s = \boldsymbol{\alpha} \times \mathbf{x} + \mathbf{c} \ , \tag{C14}$$

where $\boldsymbol{\alpha}$ and $\mathbf{c}$ are arbitrary constant vectors to be chosen accordingly to the problem of interest. Eq (C14) implies the following relations

$$\nabla \cdot \boldsymbol{\zeta}_s = 0 \ , \tag{C15}$$

$$\nabla \times \boldsymbol{\zeta}_s = \boldsymbol{\alpha} \nabla \cdot \mathbf{x} - (\boldsymbol{\alpha} \cdot \nabla) \mathbf{x} = 2\boldsymbol{\alpha} \tag{C16}$$

and

$$(\mathbf{w} \cdot \nabla) \boldsymbol{\zeta}_s = \boldsymbol{\alpha} \times \mathbf{w} \quad , \quad \mathbf{w} \cdot [(\mathbf{w} \cdot \nabla) \boldsymbol{\zeta}_s] = 0 \ , \tag{C17}$$

$$\mathbf{w} \times (\nabla \times \boldsymbol{\zeta}_s) = -2\boldsymbol{\alpha} \times \mathbf{w} \ , \tag{C18}$$

$$\text{i.e. } 2(\mathbf{w} \cdot \nabla) \boldsymbol{\zeta}_s + \mathbf{w} \times [\nabla \times \boldsymbol{\zeta}_s] = 0 \ \text{ for any } \ \mathbf{w} \ . \tag{C19}$$

Also, if $\underline{\underline{\Pi}}$ is a symmetric tensor, then

$$\boldsymbol{\zeta}_s \cdot \left[\nabla \cdot \underline{\underline{\Pi}}\right] = \nabla \cdot \left[\underline{\underline{\Pi}} \cdot \boldsymbol{\zeta}_s\right] \tag{C20}$$

since

$$\begin{aligned} \boldsymbol{\zeta}_s \cdot \left[\nabla \cdot \underline{\underline{\Pi}}\right] &= \frac{\partial}{\partial x_j} [\Pi_{ji} \boldsymbol{\zeta}_s \cdot \mathbf{e}_i] - \Pi_{ji} \frac{\partial}{\partial x_j} [\alpha \cdot (\mathbf{x} \times \mathbf{e}_i)] \\ &= \nabla \cdot \left[\underline{\underline{\Pi}} \cdot \boldsymbol{\zeta}_s\right] - \alpha \Pi_{ji} \mathbf{e}_j \times \mathbf{e}_i \\ &= \nabla \cdot \left[\underline{\underline{\Pi}} \cdot \boldsymbol{\zeta}_s\right] \ . \end{aligned} \tag{C21}$$

For instance, to describe an infinitesimal rotation about the z-axis of the usual cylindrical coordinates system R, φ, z, one can choose

$$\boldsymbol{\alpha} = \delta\varphi \mathbf{e}_z \quad , \quad \mathbf{c} = 0 \ , \tag{C22}$$

Equation (C14) then yields

$$\begin{aligned} \boldsymbol{\zeta}_s &= \delta\varphi \mathbf{e}_z \times [R\mathbf{e}_r + x\mathbf{e}_z] \\ &= \delta\varphi R \mathbf{e}_\varphi = \delta\varphi R^2 \nabla\varphi \ . \end{aligned} \tag{C23}$$

APPENDIX D

ARTIFICIAL NONHERMITIAN OPERATOR AND INSTABILITY

Is it allowed to take $n_0(x) = const.$ together with $n_0'(x) = const \neq 0$? Consequences of such approximations are discussed via the following example equation

$$\ddot{\xi} = \frac{\partial}{\partial x} p(x) \frac{\partial \xi}{\partial x}, \; p > 0. \tag{D1}$$

This yields

$$\frac{d}{dt} \int \frac{1}{2} \left\{ \dot{\xi}^2 + p \left(\frac{\partial \xi}{\partial x} \right)^2 \right\} dx = 0 \; . \tag{D2}$$

The equation corresponding to the above approximations is

$$\ddot{\eta} = p' \frac{\partial \eta}{\partial x} + p \frac{\partial^2 \eta}{\partial x^2} \; , \; p' = const \; , \; p = const \; . \tag{D3}$$

This yields

$$\frac{d}{dt} \int \frac{1}{2} \left\{ \dot{\eta}^2 + p \left(\frac{\partial \eta}{\partial x} \right)^2 \right\} dx = p' \int \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial t} \, dx = \text{ “driving term”}. \tag{D4}$$

That the r.h.s. can in fact drive an -artificial- instability follows from the eigenmodes:

$$\xi \propto e^{-i\omega_\xi t} \; , \; \eta \propto e^{-i\omega_\eta t + ikx} \; . \tag{D5}$$

$$\omega_\xi^2 \text{ real and } > 0 \text{ from the energy expression, Eq. (D2);} \tag{D6}$$

$$\omega_\eta = \pm \sqrt{pk^2 - ip'k} \approx \pm \sqrt{p} k \mp i \frac{1}{2} \frac{p'}{\sqrt{p}} \; . \tag{D7}$$

Hence, p' “drives the instabilty”.

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