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THREE-DIMENSIONAL CLOSED FIELD LINE MHD EQUILIBRIA WITHOUT SYMMETRIES

A. SALAT AND R. KAISER[a)]

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[a)] Present address: Universität Bayreuth, Lehrstuhl für Angewandte Mathematik, 95440 Bayreuth, Germany.

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Abstract

Exact three-dimensional volume current solutions of the magnetohydrodynamic (MHD) equations are presented. The configurations are infinitely extended along a straight axis and have neither cylindrical nor helical symmetry. All field lines lie in planes orthogonal to the axis and are closed around it. The surfaces of constant pressure have elliptical cross-sections whose ellipticity and orientation are arbitrary functions along the axis.

I. Introduction

In magnetohydrodynamic (MHD) theory plasma equilibria are governed by the equations

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \nabla P , \\ \mathbf{j} &= \nabla \times \mathbf{B} , \\ \nabla \cdot \mathbf{B} &= 0 , \end{aligned} \tag{1}$$

where P is the plasma pressure, and $\mathbf{B}$ and $\mathbf{j}$ are the magnetic field and the current density, respectively.

Of particular interest are configurations which have smooth nested surfaces $F(\mathrm{r})=$ *const* of constant pressure. Equations (1) imply that the magnetic field lines are embedded in these surfaces. If each field line traces out a pressure surface, the pressure surfaces and the magnetic surfaces coincide. In cases where all field lines are closed the definition of a magnetic surface $\hat{F}(\mathrm{r})=$ *const* is somewhat ambiguous [1] and $\hat{F}$ may differ from F. Since we shall focus on the physically relevant pressure surfaces, however, this distinction need not disturb us.

Equations (1) are nonlinear partial differential equations with both real and complex characteristics, and no general theory about the existence of solutions is available. If the equilibrium is independent of at least one coordinate, they can be condensed into a single quasilinear elliptic equation [2, 3, 4]. In this case explicit solutions with smooth nested pressure surfaces are known. This refers to axisymmetric toroidal solutions (no dependence on toroidal angle) [5], cylindrical configurations (no dependence on z-coordinate) [3] and helical configurations (dependence on helical and radial coordinates only) [6]. Many more pertinent references exist.

In the absence of these three (continuous) symmetries numerical evidence [7] indicates that magnetic fields governed by Eqs. (1) are subject to field line chaos and regions of ergodicity. No smooth, nested flux surfaces would then exist. This is found even in the much simpler case of force-free equilibria, with $\nabla P = 0$ and $\mathbf{j} = \lambda \mathbf{B}$ [8]. Theoretical considerations [9] make this behaviour plausible for solutions of Eqs. (1). Recently, it was even proved analytically that toroidal equilibria with smooth pressure surfaces and purely poloidal closed field lines (symmetric to the equatorial plane) cease to exist if axisymmetry is violated [10].

In spite of these negative results it is known that equilibria without the abovementioned three symmetries but with smooth, nested pressure surfaces nevertheless do exist. There are two such cases: First, there are toroidal equilibria with mirror symmetry with respect to a poloidal plane and with small plasma beta [11]. Owing to the symmetry all field lines are toroidally closed. No explicit solution, however, was given. The assumed mirror symmetry was essential in the proof of existence. Second, Woolley [12] derived two classes of *explicit* solutions without continuous symmetries. They are "straight" configurations extending towards infinity. All field lines are closed around a straight axis. In spite of their fundamental importance as regards the existence of three-dimensional solutions of Eqs. (1) Woolley's solutions do not seem to be well known. In connection with work on nonexistence of equilibria [10] one of Woolley's classes was rediscovered recently [10, 13]. Contrary to Woolley's expectation these equilibria, however, have no toroidally closed counterpart, as was shown in [10]. The solutions found in [12], just as those discussed in [11], have mirror symmetry with respect to a plane (more precisely, with respect to two, mutually orthogonal, planes).

Here, we give a new explicit class of solutions of Eqs. (1) with finite pressure gradient and without any of the three continuous symmetries. As a distinctive feature, these new equilibria have no mirror symmetry with respect to one or more planes anymore. This proves that mirror symmetry observed in the previously known classes of solutions is not a prerequisite for the existence of three-dimensional equilibria.

In Section II we present the equilibria found, while in Section III their relation to Woolley's solutions [12] is discussed, a common ansatz for both being used. This discussion brings out more clearly the problems encountered in solving the nonlinear Eqs. (1) than would a mere presentation of the new solutions.

II. Equilibria without symmetries

Our solutions of Eqs. (1) are "straight" configurations extending from, say, $-\infty < z < \infty$, in a Cartesian coordinate system x, y, z. All field lines are closed curves, concentric around a straight axis at $x = y = 0$ and embedded in planes orthogonal to this axis. They are ellipses whose ellipticity and orientation are free functions along the axis.

In detail, the magnetic field is given by

$$\mathbf{B} \; = \; \nabla F \times \nabla G(z) \, , \tag{2}$$

where F, in polar coordinates, with $x = r\cos\theta$, $y = r\sin\theta$, is given by

$$F \; = \; \frac{-c^2}{4P_1} \frac{r^2}{1 - u^2(z)} \{ 1 \; - \; u(z) \, \cos[\, 2\theta - v(z)] \} \, , \tag{3}$$

and G is related to $u(z)$ by

$$\frac{\mathrm{d}G}{\mathrm{d}z} \; = \; \frac{P_1}{c} \sqrt{1 - u^2(z)} \, . \tag{4}$$

The pressure is linear in the surface label F:

$$P(F) \; = \; P_0 + P_1 F \, . \tag{5}$$

In Eqs. (3)-(5) c, P_0 and P_1 are arbitrary constants, while $u(z)$ and $v(z)$, besides $u^2 < 1$, are arbitrary functions of their argument. From Eq. (2) it follows that $\mathbf{B} \cdot \nabla F = 0$, which proves that the surfaces $F = const$ are also magnetic surfaces.

In Cartesian coordinates F is

$$\begin{aligned} F(x, y, z) \; &= \; \frac{-c^2}{4P_1(1 - u^2)} \{ \, [1 - u(z) \cos v(z)] \, x^2 \\ &\quad - 2u(z) \sin v(z) \; x \, y + \, [1 + u(z) \cos v(z)] \, y^2 \, \} \, . \end{aligned} \tag{6}$$

According to Eq. (2) the field lines are determined by the intersection of the magnetic surfaces $F = const$ and $G = const$, where the latter reduce to $z = const$, see Eq. (4). The field lines are thus concentric ellipses, provided that $u^2 < 1$. The latter statement is more evident from Eq. (3). The half-axis ratio $\sqrt{(1+u)/(1-u)}$ of the ellipses is governed by $u(z)$, while their orientation with respect to the x-axis, say, is determined by $v(z)$. Indeed, a rotation of the coordinate system by an angle $\tilde{v}(z) = v(z)/2$ according to

$$\tilde{x} \; = \; x \, \cos\tilde{v}(z) + y \, \sin\tilde{v}(z) \, , \qquad \tilde{y} \; = \; y \, \cos\tilde{v}(z) - x \, \sin\tilde{v}(z) \tag{7}$$

changes F into the nonrotating form

$$F(\tilde{x}, \tilde{y}, z) \; = \; \frac{-c^2}{4P_1(1 - u^2)} \{ \, (1 - u) \, \tilde{x}^{\,2} + \, (1 + u) \, \tilde{y}^{\,2} \, \} \, . \tag{8}$$

The components of the magnetic field are given by

$$
\begin{aligned}
B_x(x,y,z) &= \frac{c}{2\sqrt{1-u^2}}\left[u\sin v\ x-(1+u\cos v)\ y\right]\ , \\
B_y(x,y,z) &= \frac{c}{2\sqrt{1-u^2}}\left[(1-u\cos v)\ x-u\sin v\ y\right]\ , \\
B_z(x,y,z) &= 0\ .
\end{aligned}
\tag{9}
$$

It is elementary to check that $\mathbf{B}=(B_x,B_y,B_z)$, P and F from Eqs. (9), (5) and (6) satisfy Eqs. (1).

The constant c is related to the longitudinal current density j_z as follows:

$$
j_z = \frac{c}{\sqrt{1-u^2}}\ . \tag{10}
$$

In the special case $v(z)=0$ Eq. (6) reduces to

$$
F(x,y,z) = \frac{-c^2}{4P_1}\left[\frac{x^2}{1+u(z)}+\frac{y^2}{1-u(z)}\right]\ . \tag{11}
$$

Together with Eq. (5) this is equivalent to the mirror-symmetric solutions given by Woolley [12] in his Eq. (3.2), with the identification $u=(\zeta^2-1)/(\zeta^2+1)$ and $c_{Woolley}=-c/2$.

The fact that the solution (6) for $F(x,y,z)$ is related by a simple rotation to the less general solution (11), valid for $v(z)=0$, is not trivial. Woolley's second class of solutions, for example, *cannot* be transformed into a more general one by replacing x, y in $F(x,y,z)$, i.e. in the pressure $P(x,y,z)$, with $\tilde{x}$, $\tilde{y}$, Eq. (7), no matter how $G(z)$ is chosen.

An MHD equilibrium of finite radial extent can of course be obtained by bounding the plasma with a conducting wall at some $F=F_0=const$. This also prevents the pressure from becoming negative at large r. According to Eqs. (5) and (3) the pressure always decreases with growing distance from the axis. For $|F|\leq|F_0|$ nonnegativity of the pressure is assured by taking the central pressure P_0 large enough.

A necessary condition for the existence of equilibria is that $I=\oint \mathrm{d}l/B$ be the same for all closed field lines on a given flux surface, $I=I(F)$, ($B=|\mathbf{B}|$, $\mathrm{d}l=$ length element along $\mathbf{B}$) [14]. Evaluation of I with Eqs. (9) and (3) yields $I=4\pi/c$, which is constant, not only on $F=const$, but also absolutely, for all F.

Some examples of pressure surfaces (3) are presented in Figures (1a) - (2c). The z-axis is in the vertical direction. Two solutions which incorporate nonconstant functions $u(z)$ and $v(z)$ and which are therefore devoid of symmetries are shown in Figures 1a and 2a. In Figure 1a $u(z)$ is a periodic function:

$$u(z) \;=\; 0.3 - 0.1 \sin z \;, \quad v(z) \;=\; z \;, \tag{12}$$

while in Figure 2a $u(z)$ is localized to a small region around the origin:

$$u = -0.4 e^{-0.3 z^2} \;, \quad v(z) \;=\; 0.25 z \;. \tag{13}$$

From the continuum of nested surfaces $F = const$ two surfaces are presented in each case.

The equilibrium surfaces shown in Figures 1a, 2a are, so to speak, nonlinear compositions of solutions which exploit either $u(z)$ alone or $v(z)$ alone. This composition is shown in Figures 1b, 1c, 2b, 2c. Figures 1b, 2b differ from Figures 1a, 2a in that they have $v(z) = const = 0$. They are thus examples of Woolley-type solutions. They are still three-dimensional solutions, i.e. they depend on x, y and z, but they have the double mirror symmetry discussed above. In Figures 1c, 2c, finally, $u(z)$ is replaced by a constant, -0.4 and 0.3, respectively, while $v(z)$ is left unchanged in relation to Figures 1a, 2a. These solutions are helically symmetric. They are shown here in order to illustrate the contribution of the functions $v(z)$ to the full solutions (6) with $u(z)$ and $v(z)$ given by Eqs. (12), (13).

III. Relation to other three-dimensional equilibria

In this section a unified description of the known three-dimensional "straight" equilibria with plane closed field lines is given. This puts the solution discussed in the preceding section into perspective, and the method may also be helpful in finding other three-dimensional equilibria.

With the ansatz

$$\mathbf{B} \;=\; \nabla H \times \nabla z \tag{14}$$

for the magnetic field Eqs. (1) reduce to

$$\begin{aligned}(\Delta_2 H)\,\partial_x H + \partial_x P = 0\ ,\\ (\Delta_2 H)\,\partial_y H + \partial_y P = 0\ ,\end{aligned} \tag{15}$$

$$\partial_z\left[(\partial_x H)^2 + (\partial_y H)^2 + 2P\right] = 0\ . \tag{16}$$

Here, we are working with Cartesian coordinates, $H = H(x, y, z)$, $P = P(x, y, z)$, and Δ_2 denotes the two-dimensional Laplacian in the x, y-plane. Note that the pressure is now assumed as a function of the variables x, y, z, and that, therefore, the simplified ansatz (14) (in comparison with (2)) is sufficient [1]. From Eq. (16) one has the representation for P,

$$P(x, y, z) = \tilde{p}(x, y) - \frac{1}{2}\left[(\partial_x H)^2 + (\partial_y H)^2\right]\ , \tag{17}$$

which, inserted in Eqs. (15), yields

$$\begin{aligned}\partial_x \tilde{p} &= (\partial_y H)\,\partial_x\partial_y H - (\partial_x H)\,\partial_y^2 H\ ,\\ \partial_y \tilde{p} &= (\partial_x H)\,\partial_x\partial_y H - (\partial_y H)\,\partial_x^2 H\ .\end{aligned} \tag{18}$$

The integrability condition for $\tilde{p}$ reads

$$\partial_x\partial_y \tilde{p} - \partial_y\partial_x \tilde{p} = (\partial_x H)\,\partial_y \Delta_2 H - (\partial_y H)\,\partial_x \Delta_2 H = 0\ , \tag{19}$$

which, in turn, is the condition for the following (local) representation of $\Delta_2 H$:

$$\Delta_2 H = f(H, z) \tag{20}$$

with arbitrary profile function f. In order to ensure that $\tilde{p}$ does not depend on z, H has to satisfy in addition the relations following from Eqs. (18):

$$\begin{aligned}\partial_z[\,(\partial_x H)\,\partial_x\partial_y H - (\partial_y H)\,\partial_x^2 H\,] &= 0\ ,\\ \partial_z[\,(\partial_y H)\,\partial_x\partial_y H - (\partial_x H)\,\partial_y^2 H\,] &= 0\ .\end{aligned} \tag{21}$$

So, instead of solving Eqs. (15), (16) for H and P direct, one can solve the two-dimensional problem (20) with arbitrarily given function f first and then try to satisfy the compatibility conditions (21) for the z-dependence of H.

The solutions of Ref. [12] and the foregoing section fit easily into this scheme: consider first the case with constant (with respect to H) profile function $f = \tilde{f}(z)$. If H is taken of the form

$$H = a(z)\, x^2 + c(z)\, y^2 \,, \qquad a, c > 0 \,, \tag{22}$$

with z-dependent coefficients a and c, Eqs. (20) and (21) reduce to the equations

$$\tilde{f} = 2(a + c) \,, \qquad \partial_z (ac) = 0 \,, \tag{23}$$

which leave one function of z free. These solutions correspond to the first class in Ref. [12]. If the more general ansatz

$$H = a(z)\, x^2 + b(z)\, x\, y + c(z)\, y^2 \,, \qquad a, c > 0 \tag{24}$$

is tried, Eqs. (20) and (21) furnish the conditions

$$\tilde{f} = 2(a + c) \,, \qquad \partial_z (4ac - b^2) = 0 \,, \tag{25}$$

which leave even two functions of z free. Noting that due to the different representations (2) and (14) F and H are related by

$$H = F \frac{P_1}{c} \sqrt{1 - u^2(z)} \,, \tag{26}$$

one can easily check that the solution of the foregoing section satisfies the conditions (25).

Consider second the case with linear profile function $f = \kappa H$, $\kappa = const.$ An ansatz for H of the form

$$H = a(z) g(x) + c(z) h(y) \tag{27}$$

here leads to the conditions

$$\partial_x^2 g = \kappa g \,, \qquad \partial_y^2 h = \kappa h \,, \qquad \partial_z (ac) = 0 \,. \tag{28}$$

For positive as well as negative κ solutions of Eq. (28) with closed poloidal contours of H and with one free function of z exist; these solutions correspond to the second class in [12].

Some further remarks are in order. In the case with constant profile function the most general second-order polynomial (in x and y) for H contains first-order terms, too. These terms lead, however, only to a z-independent shift of the center of the poloidal section and no further free functions appear. On the other hand, if higher than second-order polynomials are used no solutions with nontrivial z-dependence exist at all. The reason is that for higher orders Eqs. (21) furnish too many independent conditions for the coefficients of the polynomial. In the case with linear profile function a generalization of the ansatz (27) with mixed terms did not prove to be successful. Finally, one should mention that analytic solutions of Eq. (20) with closed poloidal sections exist also for nonlinear profile functions, e.g. $f = c\,e^{-H}$ [15]; solutions of this type with nontrivial z-dependence, however, could not yet be found.

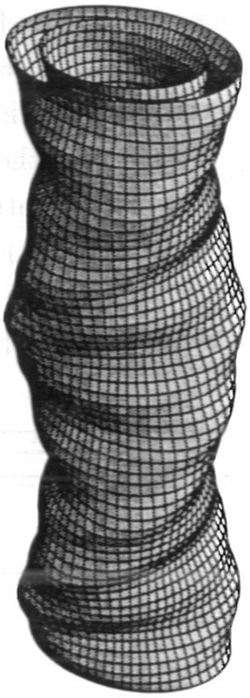

Fig. 1a

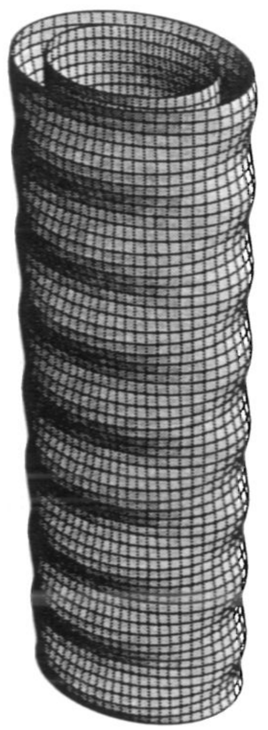

Fig. 1b

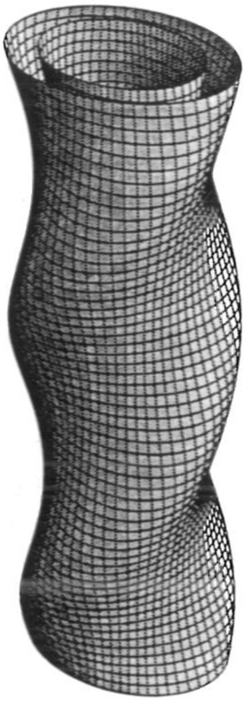

Fig. 1c

Fig. 1a: Pressure surfaces $F = const$ without symmetries. Deviation from circular cross-section: $u(z) = 0.3 - 0.1 \sin z$. Helical-pitch function: $v(z) = z$. Vertical interval: $z \in [0,\, 6\pi]$.

Fig. 1b: Same as Fig. 1a, except that $v(z) = const = 0$. Surfaces with mirror symmetry.

Fig. 1c: Same as Fig. 1a, except that $u(z) = const = 0.3$. Surfaces with helical symmetry.

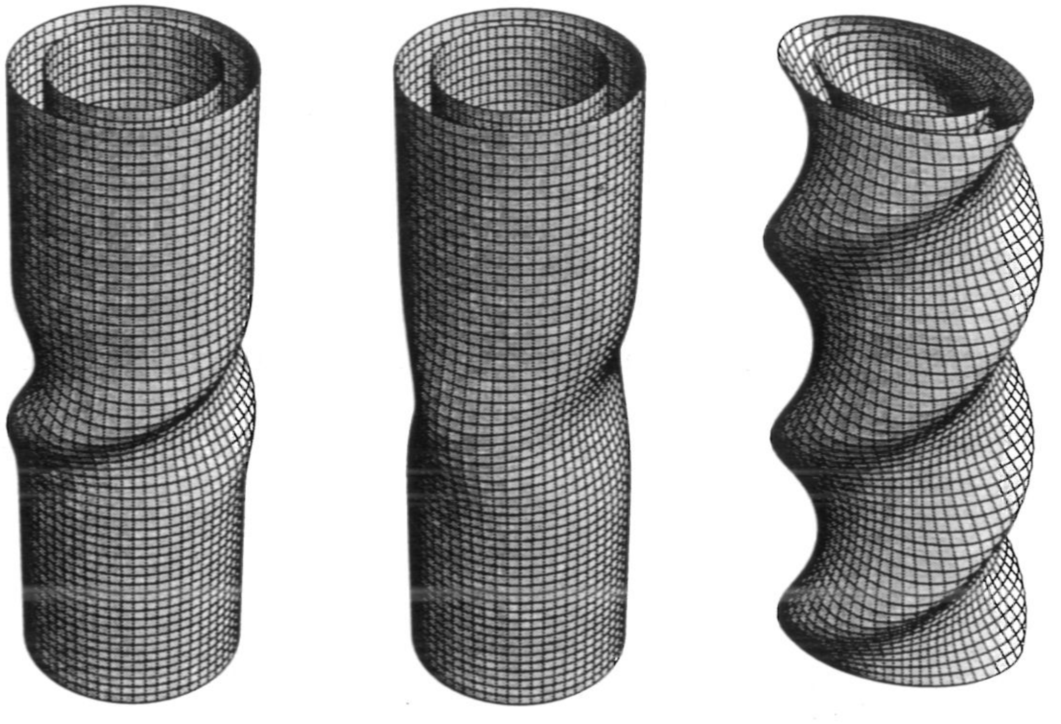

Fig. 2a Fig. 2b Fig. 2c

Fig. 2a: Pressure surfaces $F = const$ without symmetries. Deviation from circular cross-section: $u(z) = -0.4\exp(-0.3z^2)$. Helical-pitch function: $v(z) = z/4$. Vertical interval: $z \in [-9, 9]$.

Fig. 2b: Same as Fig. 2a, except that $v(z) = const = 0$. Surfaces with mirror symmetry.

Fig. 2c: Same as Fig. 2a, except that $u(z) = const = -0.4$. Surfaces with helical symmetry.

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