

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK  
GARCHING BEI MÜNCHEN

HAMILTONIANS AND FLUCTUATIONS OF  
CONTINUOUS PLASMA MODELS

H. Tasso

IPP 6/326

March 1995

*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

# Hamiltonians and fluctuations of continuous plasma models

H. Tasso

Max-Planck-Institut für Plasmaphysik

Euratom Association

85748 Garching bei München, Federal Republic of Germany

## Abstract

Comments on the nature of turbulence in macroscopically confined plasmas, in contrast to turbulence in hydrodynamics, are made in the introduction. It is suggested that equilibrium statistics may be a reasonable approach for plasma fluctuations. For that purpose, nonlinear drift wave equations are derived from two-fluid theory. This helps to construct continuous plasma models with simple Hamiltonians, which allows canonical distributions to be defined explicitly. Partition functions and correlation functions can be calculated analytically in the one-dimensional case as functional integral averages over canonical distributions. This leads to a Lorentz spectrum in  $k$ -space, which has the observed plateau behaviour for small  $k$ , but disagrees with the  $k^{-3}$ -dependence for large  $k$ . Though explicit calculations of the correlation function are not feasible in the two-dimensional case, the reasonable assumption of its exponential behaviour leads to good agreement with the experiment. In particular, the observed  $k^{-3}$  behaviour for large  $k$  is now confirmed theoretically. The open problem of saturation levels of fluctuations is discussed in the conclusions.

## 1 Introduction

Macroscopically confined plasmas are not quiescent in the small and display a complex structure of fluctuations in space and time. These fluctuations

are the result of small-scale instabilities which saturate due to the effects of sources, nonlinearities and dissipation. The picture is somewhat similar to turbulence in fluids at high Reynolds numbers. It is important to understand turbulence in order to be able to explain observations and improve engineering designs in, for example, astrophysics or fusion plasmas.

Anomalous heat conduction observed in fusion plasma devices limits the energy confinement time. This effect has to be compensated by increasing the size of the plasma and the device itself. This calls for extensive outlay and reduces the economic attractiveness of fusion reactors. Anomalous conduction is caused by the fluctuations mentioned above, which are a manifestation of turbulence. This means that understanding instabilities and turbulence is a crucial question for fusion as an energy source.

There are two basic difficulties in understanding turbulence quantitatively. First, the dynamics is highly nonlinear. Second, the statistics is not at equilibrium as in thermodynamics. More precisely, complex nonlinear and dissipative dynamic systems have, usually, several complex attractors. The statistics on such attractors is not an equilibrium statistics, unless special conditions are met. An instructive example of complex but analytically tractable dynamics is given by a large system of van der Pol-like oscillators interacting through coupling matrices [1]. The location of attractors in a finite region of phase space can be established, the attractors themselves being inaccessible to analysis. When the finite region shrinks to a hypersurface containing the attractor, an equilibrium statistics is then possible. At the same time a Liouville theorem in phase space becomes valid and compensation of driving and damping becomes local.

Usually, turbulence in hydrodynamics is investigated for the case of well-separated sources and sinks. The sources are active at long wavelengths or low  $k$ -vectors, and the sinks are effective at large  $k$ . The main interest is then focussed on the inertial range, which lies in  $k$ -space between the sources and the sinks. This is justified by the fact that, usually, either the boundaries or large-scale instabilities are the cause of sources, and the viscosity is responsible for the sinks. The inertial nonlinearity causes a cascade of energy from the low  $k$  to the large  $k$ .

In macroscopically confined plasmas the situation is quite different. The sources are due to small-scale instabilities perpendicular to the magnetic field, but large scales along the magnetic field, and the sinks are due mainly to the "shear damping" which occurs at low  $k$ -vectors parallel to the magnetic

field. The role of nonlinearities lies in the redistribution of the modes. It is then not too bad an approach to consider the driving and the damping as canceling each other locally, and to try to look for continuous, conservative plasma models and their equilibrium statistics. The paper is structured as follows: In Section 2, continuous plasma models are introduced. Section 3 deals with the construction of Hamiltonians for those models. Correlation functions and spectra are the subject of Sections 4 and 5. The conclusions are presented in section 6.

## 2 Continuous Plasma Models

In the search for continuous plasma models, the first idea which comes to mind is to look for the well-known macroscopic descriptions such as ideal magnetohydrodynamics (MHD), the two fluid theory or the Vlasov-Maxwell system. These models are fine but the problem with them is that they cannot be described by a "faithful" Hamiltonian formalism in terms of Euler or Clebsch variables. In short, Euler variables give rise to degenerate Poisson brackets and Clebsch variables are not a single-valued representation of Euler variables. See Refs. [2] and [3] for a discussion of this question.

On the other hand, many scalar equations such as Korteweg-de Vries equation do have essentially faithful Hamiltonians [4]. Fortunately, important fluctuations in plasmas seem to be well approximated by drift wave equations described by a single scalar, the electrostatic potential.

This leads us to concentrate on the latter kind of equations, which can be derived from the two-fluid and Maxwell system. The equations of motion of the two-fluid system read

$$n_i m_i \left( \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i \right) = e n_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i, \quad (1)$$

$$n_e m_e \left( \frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e \right) = -e n_e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e. \quad (2)$$

Let us apply system (1),(2) augmented with continuity, Maxwell equations and appropriate equations of state to the case of a slab of low-pressure plasma immersed in a strong, homogeneous magnetic field  $\mathbf{B}$  pointing in the z-direction. Any time-dependent perturbation about the slab geometry is restricted to being electrostatic,  $\mathbf{E} = \nabla \phi$ , so that Maxwell equations reduce

essentially to quasineutrality if the perturbation wavelength is larger than the Debye length. In addition, we assume low-frequency perturbation, for which the inertia of the ions can be neglected in first approximation, and the electrons can be taken as isothermal along  $\mathbf{B}$ . The relevant equations for a low- $\beta$  ( $\beta$  is the ratio of kinetic pressure to magnetic pressure) plasma are then given by

$$n_i \mathbf{v}_{i\perp} = \frac{n_i \nabla \phi \times \mathbf{B}}{B^2} - \frac{\nabla p_i \times \mathbf{B}}{e B^2}, \quad (3)$$

$$n_e \mathbf{v}_{e\perp} = \frac{n_e \nabla \phi \times \mathbf{B}}{B^2} + \frac{\nabla p_e \times \mathbf{B}}{e B^2}, \quad (4)$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_{i\perp}) = 0, \quad (5)$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0, \quad (6)$$

$$n_e = n_0(x) \exp -\frac{e\phi}{kT_e(x)}, \quad (7)$$

$$n_i = n_e, \quad (8)$$

$$\mathbf{B} = B \mathbf{e}_z, \quad (9)$$

where  $n_e$  and  $n_i$  are the electron and ion densities,  $\mathbf{v}_e$  and  $\mathbf{v}_i$  are the electron and ion macroscopic velocities,  $T_e$  is the electron temperature,  $\phi$  is the electrostatic potential,  $x$  is the coordinate perpendicular to the slab and  $y$  is the coordinate perpendicular to both  $x$  and  $z$ ,  $n_0$  is the unperturbed density,  $e$  is the charge of the proton, and  $k$  is the Boltzmann constant.

Equations (3) and (4) solve the equations of motions for neglected inertia. The parallel motion of the ions is small in view of their large mass. The continuity equations for ions and electrons are expressed by equations (5) and (6), while the electrons behave along  $z$  according to a Boltzmann distribution given by equation (7). Quasineutrality, easily restored by the electrons along the field lines, is ensured by equation (8).

Elimination of  $\mathbf{v}_i$  and  $n_i$  from equation (5) using equations (3) and (8) leads to the following equation for  $\phi$  :

$$\frac{\epsilon B}{kT} \frac{\partial \phi}{\partial t} - \left( \frac{n'}{n} + \frac{T'}{T} \frac{\epsilon \phi}{kT} \right) \frac{\partial \phi}{\partial y} = 0. \quad (10)$$

The subscripts as well as the explicit indication of  $x$ -dependence have been

dropped in equation (10). The prime denotes the derivative with respect to  $x$ .

Equation (10) is essentially the inviscid Burgers equation in which  $x$  appears as a parameter. It is the simplest model of a nonlinear drift wave equation, and was discovered in 1967 by the author [5]. The nonlinearity is due to the temperature gradient of the electrons, which is present in any confined hot plasma, and is called "scalar nonlinearity" in the literature. In regions of flat temperature profiles, equation (10) becomes linear. In this case, however, higher-order terms due to ion inertia produce a so-called "vector nonlinearity", which is, in essence, two-dimensional and first appeared in Ref. [6].

The solutions of the inviscid Burgers equation are known to develop infinitely steep gradients at finite times, which can be prevented by adding some of the neglected physical terms such as ion inertia or gyroviscosity, thus limiting attention to nondissipative effects.

A first attempt [7] to take such terms into account was to consider the case of cold ions and concentrate on the first inertial term in equation (1). On the assumption of solutions with weak  $x$ -dependence, the correction due to ion inertia is obtained by iteration of equation (1), inserting in the inertial term the approximate solution given by equation (3) for zero ion pressure. This leads to

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} = \frac{\nabla\phi \times \mathbf{B}}{B^2} + \mathbf{e}_y \frac{m_i}{\epsilon B^2} \frac{\partial^2 \phi}{\partial y \partial t}. \quad (11)$$

Let us insert  $\mathbf{v}$  from equation (11) in equation (5), using equations (7) and (8) to obtain

$$\frac{\epsilon B}{kT} \frac{\partial \phi}{\partial t} - \left( \frac{n'}{n} + \frac{T'}{T} \frac{\epsilon \phi}{kT} \right) \frac{\partial \phi}{\partial y} - \frac{m_i}{\epsilon B} \left( \frac{\partial^3 \phi}{\partial t \partial y^2} + \frac{\epsilon}{kT} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial y \partial t} \right) = 0. \quad (12)$$

Equation (12) becomes identical to equation (6) of Ref. [7] if  $\phi$  is replaced by  $-\phi$ , and if solutions of the form  $\phi(y - ut)$  are sought. The discussion of such solutions led to the existence of drift solitons [7] and other nonlinear waves.

To go beyond equation (12), keeping the assumption of cold ions, it will be necessary to go to higher-order terms in the expansion in the inertial terms. In Ref. [6], the so-called vector nonlinearity has been included as

another correction to equation (11) , which yields

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} + \mathbf{v}_{i\perp 2} = \frac{\nabla\phi \times \mathbf{B}}{B^2} + \mathbf{e}_y \frac{m_i}{\epsilon B^2} \frac{\partial^2 \phi}{\partial y \partial t} + \frac{m_i}{\epsilon B^2} \frac{\nabla\phi \times \mathbf{B}}{B^2} \cdot \nabla \left( \frac{\nabla\phi \times \mathbf{B}}{B^2} \right). \quad (13)$$

Similarly to the derivation of equation (12), we insert  $\mathbf{v}$  from equation (13) in equation (5), using equations (7) and (8), and obtain a two-dimensional equation of the type

$$\frac{\partial}{\partial t}(\phi - \nabla^2 \phi) + (\nabla\phi \times \mathbf{e}_z \cdot \nabla) \nabla^2 \phi = 0. \quad (14)$$

The coefficients in equation (14) have been omitted, to give the equation the same form as in Ref. [6]. The scalar nonlinearity of equations (10) and (12) is absent in equation (14) because of problems of ordering, i.e. this nonlinearity is of zero order and would dominate the vector nonlinearity.

Equations (12) and (14) are too simple to describe real situations. Since the ions are not cold, diamagnetic and gyroviscous terms should be taken into account. Also the parallel velocity of the ions could cause acoustic waves along the magnetic field and, if different from the parallel velocity of the electrons, contributes to the creation of electric currents and related magnetic fields. On the other hand, the electrons do not need to behave "adiabatically" as expected in equation (7). Friction between electrons and ions decouples density and potential fluctuations. Finally, the slab geometry and the homogeneous magnetic field are too simple an assumption to represent real toroidal situations. A fair account of sophisticated drift waves models is given in Ref. [8].

All these effects tend to complicate the approximate equations in such a way that they become as difficult to handle as the the original two-fluid-Maxwell system. The Vlasov-Maxwell or the Fokker-Planck-Maxwell system would, of course, be much less tractable either analytically or numerically.

Our aim is to extract statistical information about the system, but this does not seem possible to achieve on such difficult equations, especially if dissipative terms are included [1]. On the other hand, nondissipative equations do not always have faithful Hamiltonians in useful variables. This leads us to look for models containing an essential part of the physics and possessing

faithful Hamiltonians simple enough to be able to carry, on canonical distributions constructed with them, calculations of statistical averages such as correlation functions. Note that such correlations are functional integrals, which can be very difficult to evaluate.

### 3 Hamiltonians

To construct canonical Gibbs distributions, first we need to have Hamiltonians. Usually Hamiltonians are derived from Lagrangians using Legendre transformations. Lagrangian formulation has the advantage of ensuring invariance properties such as Galilean invariance of the original exact equations. The penalty, however, lies in the complexity, if not chaoticness, of Lagrangian variables. This makes the corresponding Hamiltonian highly non-linear and useless for evaluating functional integrals. It may be of a great analytical advantage to remain in Eulerian variables [2], [4] and elaborate drift wave approximations as we did in the previous section. The penalty is now that some of the model equations obtained may, for instance, not be Galilei invariant. As long as we remain in the reference system in which the approximations made are physically valid, we are safe.

Equation (10), as mentioned above, is the inviscid Burgers equation. In reduced form it reads

$$\frac{\partial u}{\partial t} - (a + bu) \frac{\partial u}{\partial y} = 0. \quad (15)$$

Its Hamiltonian in terms of Eulerian variables is given by the functional

$$H = \int (a \frac{u^2}{2} + b \frac{u^3}{6}) dy, \quad (16)$$

so that equation (15) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \frac{\delta H}{\delta u} = [u, H], \quad (17)$$

where the brackets denote the generalized Poisson brackets of two functionals and are defined as [11]

$$[F, G] = \int \frac{\delta F}{\delta u} \frac{\partial}{\partial y} \frac{\delta G}{\delta u} dy. \quad (18)$$



Brackets (18), though noncanonical, have the property that their symplectic operator, i.e.  $\frac{\partial}{\partial y}$ , is independent of the dynamic variable  $u$ . This property is important for defining simple volume elements in phase space and for satisfying the Liouville theorem despite the noncanonical formulation [9]. The symplectic operator of brackets (18) is essentially nondegenerate, i.e. the brackets possess few or none Casimir invariants [10].

As discussed above, the solutions of equation (10) may contain infinite gradients, which led us to introduce equation (12). Unfortunately, equation (12) has mixed time and space derivatives together with nonlinearities, which makes the introduction of a Hamiltonian formulation very difficult. In order to proceed, we have to make assumptions in addition to the expansion in the strength of the inertial terms, assumptions which cannot be justified by expansions in small parameters.

We now state that equation (12) can be modelled by Korteweg-de Vries equation for the following reasons: First, both equations have solitary wave solutions. Second, the time derivative in the two terms within the parentheses of the last part of equation (12) can be approximated by the space derivative for long wavelengths. Third, the second of these two terms is smaller than the first for small amplitudes. Our model equation is then of the form [12]

$$u_t - C_1 u u_y + C_2 u_{yyy} = 0. \quad (19)$$

For the coefficients and further calculations see Ref. [13].

Equation (19) has the following Hamiltonian [11] :

$$H = \int \left( \frac{C_1}{6} u^3 + \frac{C_2}{2} u_y^2 \right) dy, \quad (20)$$

together with bracket (18), so that

$$u_t = \frac{\partial}{\partial y} \frac{\delta H}{\delta u}. \quad (21)$$

Hamiltonian (20) is not bounded because of the cubic term and the fact that  $u$  can have both signs. Note that, in some cases, unbounded Hamiltonians can be used in equilibrium statistics, but at the expense of a rather difficult analysis [14].

There is another [15] symplectic representation of equation (19) with Hamiltonian

$$H_E = \frac{1}{2} \int u^2 dy \quad (22)$$

and bracket

$$[F, G] = - \int \frac{\delta F}{\delta u} \left( C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y \right) \frac{\delta G}{\delta u} dy. \quad (23)$$

so that equation (19) can be written as

$$u_t = [u, H_E] = - \left( C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y \right) \frac{\delta H_E}{\delta u}. \quad (24)$$

Representation (22)-(24) has a bounded Hamiltonian (22), but a rather complicated symplectic operator  $-(C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y)$ , which depends upon  $u$ , so that nice properties, such as the Liouville theorem and the simple volume element in phase space, are lost. Fortunately, a transformation [16]

$$u = \frac{C_1}{6} v^2 + C_2^{\frac{1}{2}} v_y \quad (25)$$

exists which induces a new phase space  $v$  with  $\frac{\partial}{\partial y}$  as symplectic operator.  $v$  obeys the modified Korteweg-de Vries equation, which can be written as

$$v_t = \frac{C_1^2}{6} v^2 v_y - C_2 v_{yyy} = \frac{\partial}{\partial y} \frac{\delta H_E}{\delta v}, \quad (26)$$

with

$$H_E(v) = \frac{1}{2} \int \left( \frac{C_1^2}{36} v^4 + C_2 v_y^2 \right) dy, \quad (27)$$

which is formally identical to the one-dimensional Ginzburg-Landau potential [17]. Note, however, that any solution of equation (26) is a solution of (19), but the opposite is not true, which means that solutions of the Korteweg-de Vries equation may be lost through transformation (25). This will have to be remembered when averages over phase space  $v$  are taken later.

The last equation for which we would like to have a Hamiltonian is equation (14). In view of the problems already mentioned, concerning the Hamiltonian formulation of equation (12), one is discouraged from looking for a simple canonical Hamiltonian. There is, however, a noncanonical formulation for equation (14) [18]. The brackets in this formulation are noncanonical and depend upon the dynamic variable, so that they are not very useful for the calculation of averages.

Instead of equation (14), it is more practical to use an ad hoc model containing terms reminiscent of the scalar nonlinearity of equation (10) as well as two-dimensional dispersive terms reminiscent of the one-dimensional dispersive term of equation (12) or (19). This ad hoc equation has, at the same time, a simple Hamiltonian formulation and appeared for the first time in Ref. [19]. It reads

$$u_t = \frac{\partial}{\partial y} \frac{\delta H}{\delta u}, \quad (28)$$

with a Hamiltonian of the type

$$H = \int [\gamma u^4 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2] dx dy \quad (29)$$

and Poisson brackets identical to those given by equation (18). Hamiltonian (29) is this time formally identical to the two-dimensional Ginzburg-Landau potential [17].

## 4 Correlation Functions and Spectra in One Dimension

As stated above, statistical averages for continuous systems necessitate the calculation of functional integrals. Functional integrals are the limit of multiple integrals when the number of integrations goes to infinity in a proper way. It is known [20] that such integrations can usually be done explicitly for one-dimensional problems and only occasionally for two-dimensional systems. Though observations suggest two-dimensional behaviour of drift waves, for technical reasons we would like to start with one-dimensional systems.

Following Ref. [21] and Ref. [13], we calculate first the partition function  $Z$  for a system described by equations (26) and (27),

$$Z = \int D(v) \exp(-\beta H_E(v)), \quad (30)$$

where  $\beta^{-1}$  is the "temperature" or energy of the fluctuations. The Hamiltonian  $H_E(v)$  from equation (27) is rewritten as

$$H_E = \int \xi^{-1} (bv^4 + cv_y^2). \quad (31)$$

with

$$\frac{b}{\xi} = \frac{C_1^2}{72}, \frac{c}{\xi} = \frac{C_2}{2}. \quad (32)$$

and the length  $\xi$  will later be related to the correlation length. Expression (30) is written first in discretized form as

$$Z = \lim_{N \rightarrow \infty} D^{-N} \int \prod_{i=-N}^N dv_i \exp\left\{-\frac{\beta \Delta y}{\xi} [bv_{i+1}^4 + c\left(\frac{v_{i+1} - v_i}{\Delta y}\right)^2]\right\}, \quad (33)$$

where  $N$  is defined by  $N = \frac{L}{\Delta y}$ ,  $L$  being the periodicity length of the function  $v$ . When  $N$  goes to infinity,  $\Delta y$  goes to zero with  $L$  fixed. The integration can be reduced to a product of single integrals by using the eigenvalues of the transfer integral operator [22] :

$$D^{-1} \int dv_{i-1} \exp[-\beta f(v_i, v_{i-1})] \psi_n(v_{i-1}) = \exp(-\beta \epsilon_n \frac{\Delta y}{\xi}) \psi_n(v_i). \quad (34)$$

Operator (34) reduces to the one-dimensional Schrödinger operator in the limit of  $N \rightarrow \infty$  and  $\Delta y \rightarrow 0$  [22] . It reads

$$\left(-\frac{1}{4} \frac{d^2}{dv^2} + v^4\right) \psi_n(v) = \epsilon_n \beta^{\frac{4}{3}} \beta_0^{-\frac{1}{3}} \psi_n(v) \equiv E_n \psi_n(v), \quad (35)$$

where  $\beta_0^{-1} = b$  and  $\xi$  has been chosen as  $\frac{\xi}{b}$ . Obviously, it holds that  $E_n = \epsilon_n \beta^{\frac{4}{3}} \beta_0^{-\frac{1}{3}}$ . A good approximation to  $Z$  is

$$Z \approx \exp\left(-\frac{L}{\xi} \beta \epsilon_0\right), \quad (36)$$

where  $\epsilon_0$  is the lowest eigenvalue of the anharmonic oscillator.

We can now proceed to the evaluation of the space (equal time) correlation function. The definition is given by

$$C(y) = \langle \delta u(y) \delta u(0) \rangle = \int D(v) \delta u(y) \delta u(0) \frac{\exp(-\beta H_E)}{Z}, \quad (37)$$

where

$$\delta u = u - \langle u \rangle, \langle u \rangle = \int D(v) \delta u \frac{\exp(-\beta H_E)}{Z}, \quad (38)$$

and  $u$  is expressed in terms of  $v$  through equation (25). The explicit calculation of correlation function (37) closely follows the evaluation of partition function (30) using the transfer integral operator technique (34). It closely follows the method given in Ref. [22]. One obtains

$$C(y) = \frac{C_1^2}{36} \sum_{n=1}^{\infty} [\langle \psi_n | v^2 | \psi_0 \rangle - (\frac{\beta_0}{\beta})^{\frac{2}{3}} (E_n - E_0)^2 \langle \psi_n | v | \psi_0 \rangle] \times \exp[-\frac{y}{\xi} (\frac{\beta_0}{\beta})^{\frac{1}{3}} (E_n - E_0)]. \quad (39)$$

where the Dirac brackets denote scalar products in the Hilbert space of eigenvectors.

We are now in a position to calculate the  $k$ -spectrum of the fluctuations by taking the Fourier transform of correlation function (39) :

$$S(k) = \int dy \exp^{iky} C(y) = 2 \sum_{n=1}^{\infty} \frac{q_n}{k^2 + p_n}, \quad (40)$$

where

$$p_n = \xi^{-1} (\frac{\beta_0}{\beta})^{\frac{1}{3}} (E_n - E_0), \quad (41)$$

$$q_n = \frac{C_1^2}{36} [\langle \psi_n | v^2 | \psi_0 \rangle - \langle \psi_n | v | \psi_0 \rangle (p_n \xi)^2]. \quad (42)$$

To subject spectrum (40) to observation, it is first necessary to relate the "constants"  $C_1$  and  $C_2$  of equation (19) to the experiment. For  $C_1$  we use the coefficient in front of the steepening term of equation (10), originally discovered in [5]. For  $C_2$  we can use the coefficient of the linear "dispersive" term of equation (12) or think of some gyroviscous effect due to the finite gyroradii of the ions [13]. The choices of  $\beta$  and  $L$  are related to the observed level of fluctuations and to the large radius of the toroidal tokamak experiment [13], respectively. This procedure yields essentially a Lorentz spectrum in  $k$  since the terms for  $n > 1$  are negligible in equation (40). For more details and a plot see Ref. [13].

In the meantime, recent experiments [23] confirm the plateau behaviour of the spectrum for small  $k$  but give a  $k^{-3}$  behaviour for large  $k$ , in disagreement with the  $k^{-2}$  behaviour of spectrum (40). In the next section it will be

seen that the disagreement is due to the one-dimensional calculation, which contradicts the two-dimensionality of the observed turbulence.

The Lorentz spectrum and the related exponential decay of correlation function (39) seem to have a kind of universal character. Hamiltonians of the form

$$H = \int [g(u) + \alpha u_y^2] dy, \quad (43)$$

with  $g(u)$  higher than quadratic and  $H$  bounded or  $H > 0$  give rise to an exponential shape for the correlation function and to a Lorentz shape for the spectrum [24, 10]. For example, equations of the type

$$u_t - u^{2n} u_y + u_{yyy} = 0, \quad (44)$$

with  $n$  an integer, would belong to the "universality" class. This fact gives us some freedom for modeling physical systems in two dimensions, as demonstrated in the next section.

## 5 Spectra in Two Dimensions

As already mentioned above, it is important to have a model containing the main physical effects and possessing a simple Hamiltonian formulation. It turns out that the simplest two-dimensional extension is given by Hamiltonian (29) [19]. Unfortunately, functional integrals with Hamiltonian (29) in the canonical distribution like

$$C(x, y) = \langle \delta u(x, y) \delta u(0, 0) \rangle = \int D(u) \delta u(x, y) \delta u(0, 0) \frac{\exp(-\beta H)}{Z} \quad (45)$$

are not tractable analytically for two-dimensional models. Note that fluctuation  $\delta u$  is defined similarly to the one-dimensional case as in equation (38).

At this point, we have to guess the right behaviour for the correlation function. Inspired from two-dimensional calculations in spin systems [25] and from the one-dimensional result (39), we assume

$$C(r) \approx \exp(-\mu r), \quad (46)$$

where  $r = \sqrt{x^2 + y^2}$ . To obtain the spectrum we take the two-dimensional Fourier transform of correlation (46), which according to theorem (56) of

Ref. [26] is

$$S(k) = \int_0^\infty \exp(-\mu r) r J_0(kr) dr, \quad (47)$$

where  $J_0$  is the zeroth-order Bessel function and  $k = \sqrt{k_x^2 + k_y^2}$ . Integral (47) is known [26], which gives

$$S(k) = \frac{\mu}{[\sqrt{\mu^2 + k^2}]^3}. \quad (48)$$

Spectrum (48) has the observed  $k^{-3}$  behaviour for large  $k$  with a plateau for small  $k$ , in agreement with the experiment [23]. This result is quite encouraging for the pursuit of a macroscopic modeling of drift wave turbulence. Though equilibrium statistics cannot deliver the height of the spectrum, which is related to the saturation level of the turbulence and is responsible for the observed anomaly in diffusion, it gives a consistent picture of the spectrum, which may increase our understanding of the scalings to large plasmas.

## 6 Conclusions

Because of the special nature of drift wave turbulence, as discussed in the introduction, it is possible to apply the equilibrium statistical approach successfully. The introduction of approximate nonlinear drift wave equations together with adequate model equations having simple Hamiltonians is a basic starting element of this approach. Basic difficulties relating to the explicit analytic calculation of functional integrals as averages over canonical distributions can only be overcome for one-dimensional continuous systems. The Lorentz spectrum obtained in this case disagrees with the  $k^{-3}$  fluctuation spectrum observed in large tokamaks [23]. Though the two-dimensional case is not completely tractable analytically, the reasonable assumption of exponential behaviour of the space correlation function leads to a spectrum in excellent agreement with observation.

This approach ignores the difficult problem of the saturation level of fluctuations in assuming a local balance between driving and damping [1]. This makes the problem analytically solvable, but gives a partial answer, which is the  $k$ -spectrum of fluctuations. The determination of the saturation level remains an important open problem whose solution is crucial for a quantitative

estimate of the observed anomaly in diffusion. Its solution must involve the sources and sinks of turbulence. The statistical part of the problem will have to be done out of equilibrium. This is a major question in turbulence which seems to resist all attempts at solution, including the "Maximum Entropy Principle" advocated in Ref. [27]. The application of this principle implies the introduction of side-conditions compatible with the dynamics [1]. The side-conditions are, in general, impossible to find and formulate for nonlinear dissipative systems. The saturation level of drift wave fluctuations and the related anomalous diffusion will have to be extracted from the subtleties accompanying the application of such a principle or any other nonequilibrium approach.



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