MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

TRKAL FLOWS IN MAGNETOHYDRODYNAMICS

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IPP 6/325 January 1995

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

January 18, 1995

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Abstract

Force free fields can be generalized to Trkal flows in magnetohydrodynamics if a special velocity field parallel to the magnetic field is introduced. Such flows decay exponentially in time in case a constant viscosity and a constant resistivity are added. The stability of generalized Trkal flows is investigated and a sufficient condition for nonlinear stability is derived. For low velocities, this condition reduces essentially to the nonlinear sufficient stability condition for force free fields, previously found by the author.

PACS: 52.30.-q , 47.20.-k , 47.65.+a

Trkal flows[1] in hydrodynamics (HD) are decaying Beltrami flows. They decay stably in that the structure of the Beltrami flow is maintained if the Reynolds number is small enough. The analog of a Beltrami flow in magnetohydrodynamics (MHD) is a special force free field. In this note, we generalize the Trkal flows of HD to MHD considering a decaying force free field combined with a similarly decaying flow along the magnetic field lines. We also analyse their nonlinear stability. The basic equations of resistive, viscous and incompressible MHD are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p - \nu \nabla \times \nabla \times \mathbf{v} - \nabla \Phi, \tag{1}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \tag{2}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{5}$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{6}$$

where the mass density in front of the inertial term has been taken equal to 1 for simplicity. The viscosity ν and the resistivity η are material constants.

Φ is an external or gravitational potential, p is the pressure, $\mathbf{v}$ and $\mathbf{B}$ are the velocity and the magnetic field. $\mathbf{E}$ and $\mathbf{j}$ are the electric field and the electric current density.

To construct generalized Trkal flows, we make the following ansatz

$$\mathbf{v} = \mathbf{v}_0 e^{-\nu\lambda^2 t}, \tag{7}$$

$$\mathbf{B} = \mathbf{B}_0 e^{-\eta\lambda^2 t}, \tag{8}$$

$$\nabla \times \mathbf{v}_0 = \lambda \mathbf{v}_0, \tag{9}$$

$$\nabla \times \mathbf{B}_0 = \lambda \mathbf{B}_0, \tag{10}$$

$$\mathbf{v}_0 = \beta \mathbf{B}_0, \tag{11}$$

$$p + \frac{v^2}{2} + \Phi = h(t). \tag{12}$$

with $\mathbf{n} \cdot \mathbf{v}_0 = \mathbf{n} \cdot \mathbf{B}_0 = 0$ at the boundary. λ and β are constants in space and time. $h(t)$ is a function of time only. Equation (12) is essentially an equation for the pressure. The Trkal MHD solution (7)-(12) can be verified by inspection in equations (1)-(6), using the vector formula

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{a} + \mathbf{a} \times \nabla \times \mathbf{b} + \mathbf{b} \times \nabla \times \mathbf{a} \tag{13}$$

in equation (1). We see that the Beltrami flow, the force free field and combination of them can be recovered from the equations (7)-(12) by annihilating ν and η . The Trkal flow of HD is recovered for $\mathbf{B}_0 = 0$.

In previous work of the author[2,3] , the nonlinear stability of decaying force free fields has been analysed by Liapunov methods resulting in an unconditional sufficient stability criterion independent upon the magnetic Reynolds number or Lundquist number. Here, we generalize this stability analysis to the MHD Trkal flows given by (7)-(12). Let us write equations (1)-(6) for finite perturbations about the Trkal solution (7)-(12) after introducing the vector potential perturbation $\mathbf{A}_1$ with the vanishing electrostatic potential gauge

$$\begin{aligned}
\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 &= \mathbf{j}_1 \times \mathbf{B} + \lambda \mathbf{B} \times \mathbf{B}_1 + \\
&\quad \mathbf{j}_1 \times \mathbf{B}_1 - \nabla p_1 - \nabla \Phi_1 - \\
&\quad \nu \nabla \times \nabla \times \mathbf{v}_1, \qquad (14)
\end{aligned}$$

$$-\dot{\mathbf{A}}_1 + \mathbf{v} \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B} + \mathbf{v}_1 \times \mathbf{B}_1 = \eta \mathbf{j}_1. \qquad (15)$$

Equations (3)-(6) for the perturbed quantities are not reproduced here since they would be identical to the nonperturbed equations. We give, however, the curl of equation (2) explicitly since it will be used in the construction of Liapunov functionals

$$\dot{\mathbf{B}}_1 = \nabla \times (\mathbf{v} \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B} + \mathbf{v}_1 \times \mathbf{B}_1) - \eta \nabla \times \mathbf{j}_1. \tag{16}$$

Taking the scalar product of equation (14) with $\mathbf{v}_1$ and of equation (16) with $\mathbf{B}_1$, integrating over the volume of the fluid and adding we obtain, omitting the volume element in the integrals

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int (v_1^2 + B_1^2) &= \int \mathbf{v}_1 \cdot \mathbf{v} \times \nabla \times \mathbf{v}_1 + \int \nabla \times \mathbf{B}_1 \cdot \mathbf{v}_1 \times \mathbf{B}_1 + \\ &\lambda \int \mathbf{v}_1 \times \mathbf{B} \cdot \mathbf{B}_1 \\ &-\nu \int (\nabla \times \mathbf{v}_1)^2 - \eta \int j_1^2. \end{aligned} \tag{17}$$

To cancel the surface integrals in the derivation of equation (17) we have used equations (3)-(6), (10) and (13) and assumed that $\mathbf{v}_1 = \mathbf{B}_1 = 0$ at the boundary of the fluid. The "nonslip" condition for $\mathbf{B}_1$ is rather unusual and needs for its realization a good conducting boundary but, at the same time, a high contact resistance normal to the boundary.

The third integral on the right hand side of (17) can be evaluated by taking the scalar product of (15) with $\mathbf{B}_1$ and making use of the boundary conditions to give

$$\lambda \int \mathbf{v}_1 \times \mathbf{B} \cdot \mathbf{B}_1 = \frac{\lambda}{2} \frac{\partial}{\partial t} \int \mathbf{A}_1 \cdot \nabla \times \mathbf{A}_1 + \lambda\eta \int \mathbf{j}_1 \cdot \mathbf{B}_1. \tag{18}$$

Inserting the left hand side of (18) in (17) leads to

$$\frac{1}{2}\frac{d}{dt}\int (v_1^2 + \alpha(\nabla \times \mathbf{A}_1)^2 + (1-\alpha)(\nabla \times \mathbf{A}_1)^2 - \lambda \mathbf{A}_1 \cdot \nabla \times \mathbf{A}_1) =$$

$$(-\nu \int (\nabla \times \mathbf{v}_1)^2 + \int \mathbf{v}_1 \cdot \mathbf{v} \times \mathbf{v}_1) +$$

$$(-\alpha\eta \int (\nabla \times \mathbf{B}_1)^2 + \int \nabla \times \mathbf{B}_1 \cdot \mathbf{v} \times \mathbf{B}_1)$$

$$-\eta \int ((1-\alpha)(\nabla \times \mathbf{B}_1)^2 - \lambda \mathbf{B}_1 \cdot \nabla \times \mathbf{B}_1), \tag{19}$$

where α is some positive number between 0 and 1 .

The Reynolds number being defined as

$$R_\nu = \frac{Lv}{\nu}. \tag{20}$$

let us introduce

$$R_\eta = \frac{Lv}{\alpha\eta}, \tag{21}$$

in which ν has been replaced by $\alpha\eta$. We can guess that if R_ν and R_η are small enough the two first brackets on the right hand side of (19) can be made negative. This is, in fact, proved for the first bracket in Serrin's paper[4] for general bounded flows in HD. Since $\mathbf{B}_1$ obeys the nonslip condition, the proof automatically extends to the second bracket also. Note that if α is chosen small, v will have to be accordingly small as well.

Now, assume that v is small enough so that the two first brackets on the right hand side of (19) are negative, we can state that a sufficient condition for nonlinear stability at all levels of perturbations is given by

$$\int((\nabla \times \mathbf{A}_1)^2 - \frac{\lambda}{(1-\alpha)}\mathbf{A}_1 \cdot \nabla \times \mathbf{A}_1) \geq 0 \qquad (22)$$

for all $\mathbf{A}_1$ and $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ satisfying the boundary conditions. The proof of that statement is essentially given in Ref.[5] . Note that if the Trkal flow becomes a force free field, α can be chosen null and condition (22) is the same as the one found in Ref.[5] and Ref.[3] . As long as α can be chosen small compared to 1, condition (22) gives a geometric limitation for λ but no limitation on the magnetic Reynolds number or the Lundquist number. This proof goes beyond the arguments given in Ref.[6] and Ref.,[7] which are valid for vanishing flows and in the limit of small resistivity only.

In summary, it is found that certain force free magnetic fields with parallel flows proportional to the magnetic field exist and that the field and the flows decay exponentially due to finite resistivity and viscosity while the spatial structure is maintained.

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