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PROPERTIES OF IDEAL FERMION SYSTEMS

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Abstract:

Several properties of the perfect fermion gas in thermal equilibrium are compiled, especially the mean energy per Fermi particle, partly also the energy of the mean momentum and of the mean square momentum, the most probable quantities (maxima of the distribution function), pressure and chemical potential. The limiting cases, i.e. 1) non-degenerated - completely degenerated, 2) non-relativistic - ultra-relativistic, are considered as well as the intermediate states. With the help of series expansions and interpolating formulae approximative and easily evaluable results are given, moreover numerical data (in tables) of high precision and figures for a rapid survey. In most of the relations dimensionless quantities are used. These results refer to the ideal state only: the influence of interactions between the particles is neglected. Approximately, the compiled results are applicable e.g. to the electron gas in low-density plasmas and in solid conductors, to the theory of atomic nuclei, to liquid He^3, to white dwarfs and to neutronic matter in neutron stars. They may be used also as a basis for investigations regarding the influence of all kinds of interactions on the statistical properties.

PROPERTIES OF IDEAL FERMION SYSTEMS

Contents

1. Introduction

Any ideal (or perfect) gas of Fermi-particles in thermal equilibrium, in the limit of very large numbers and of negligible mutual interactions, enclosed by perfectly reflecting walls: That is one of the simplest and most basic examples of a statistical ensemble. Its averaged properties are of basic interest. A few special cases (the most common ones) are usually treated in many textbooks of Statistical Physics (for instance, [1 - 10], [29]), but the whole range of states is hardly considered. Further investigations regarding as well the basic theory as the numerical evaluation of ideal Fermion systems in the (more or less) general case are published, for instance, in [30 - 43]. Moreover, the properties of such systems are extended to special conditions or are used in many different applications, such as: in curved space-time or with gravitational interactions [44 - 46], in neutron stars [47 - 50], in nuclear matter [51 - 53], in (relativistic) plasmas [54 - 58], in magnetic fields [59]; the dependence on the dimensions of space is considered in [60 - 65], on the size of the system in [66 - 67], density profiles are calculated in [68 - 69]; the connection with suprafluidity and supraconductivity is discussed in [70 - 75], [84], and with radiation properties in [76 - 77]. Finally, for comparison the corresponding qualities of Bose-Einstein gases are investigated in [78 - 83]. These are only examples of wide-spread applications, indicating the importance of ideal fermion systems as basis to further studies.

Therefore, a review on some essential properties of such systems is presented in this study. We will not discuss the general Statistical Theory of such systems or repeat the basic thermodynamic relations; in this regard see the mentioned textbooks.

In order to simplify the calculation of the mean energy per particle and of some other averaged quantities, expansions and approximate formulae are compiled, covering as well the different special cases as the general state of the system. Moreover, sets of almost exact numerical data and figures are given, representing the intermediate states, for comparison and interpolation. The results are applicable e.g. to an electron gas, on the idealized condition that it is embedded in a completely neutralizing positive background space charge. Such conditions are fulfilled (at least approximately) in dilute classical gas plasmas, but also in the largely degenerated electron gas of metallic conductors. Another simple example is a gas of neutrons, especially in the degenerate (and partly relativistic) state as it exists in neutron stars and pulsars, or in nuclear matter. Approximately, the derived formulae can be used also in many other cases, if the deviations from the ideal state are small.

A guide to the contents: After a short remembrance of the basic equations (section 3) the averaged properties (essentially: energies) are treated systematically in steps with increasing generality: At first (section 4) the four simple limiting cases of fermion systems, after that the transitions between these special cases (sections 5 - 6) and the fixation of limits between them (section 7); and finally the most general case of such systems (sections 8 - 9). The used symbols are explained in section 2. For simplicity, most of the results are given in dimensionless units, which are defined in section 5; the transformation back to physical properties is shown in section 9. In the appendix (11) the contents of this study (derived formulae, tables, figures) are registered once more.

2. Symbols

f	distribution function
$\mathbf{p} = (p_x, p_y, p_z)$	particle momentum ($\lvert\mathbf{p}\rvert = p$)
ε	particle (kinetic) energy
n	density
T	temperature
m	particle mass
A, α, μ	parameters of the distribution function
V	volume
P	pressure
E	ensemble (kinetic) energy (=internal energy)
N	number of particles
H	enthalpy of the ensemble (heat content)
χ	heat content per particle

Indices:

F	Fermi
T	thermal
nd	non-degenerated (classical)
cd	completely (extremely) degenerated
nr	non-relativistic
er	extremely relativistic (ultra-relativistic)
R	relativistic (generally)
N	norm
S	standard
m	maximum (=most probable)

Each quantity u: $u' = \langle u \rangle$ = mean (averaged) value of u,
and $u'' = ((u^2)')^{1/2}$.

As far as they are non-evident, the explanations to these designations are given below. Definitions of dimensionless quantities are added in section 5.

3. Basics

A homogeneous ensemble of Fermi particles (spin s, mass m) in thermal equilibrium (but without mutual non-pointlike interactions) is characterized by two parameters only: the density n and the temperature T (and, of course, by the total number of particles N = nV, with the volume V). As is well known (see textbooks on Statistical Physics, e.g. [1 - 10]), the distribution function f of the individual momentums **p** in phase space - after the trivial integration over the unit volume - is given by

$$d^3n = \frac{g\,f}{h^3}\,dp_x dp_y dp_z \qquad (1)$$

with the statistical weight $g = 2s + 1$ and the Fermi-Dirac distribution function

$$f = [\,1 + A\,\exp(\frac{\epsilon}{kT})\,]^{-1} \qquad (2)$$

where $A = e^{-\alpha} = \exp(-\mu/kT) = \text{const}$, ($\mu$ = chemical potential),
while the kinetic energy ϵ becomes

$$\epsilon = [\,(cp)^2 + (mc^2)^2\,]^{1/2} - mc^2 \quad . \qquad (3)$$

The parameter A (or μ, respectively), which may be considered also as normalization constant in equ. (2), 1/A describing the degree of degeneracy, is determined by integration of (1). With $dp_x dp_y dp_z = p^2\,dp\,d\Omega$ (assuming isotropy) we have

$$n = \frac{4\pi g}{h^3}\int_0^\infty f\,p^2\,dp \qquad (4)$$

resulting in $A = A(n,T)$. Accordingly, the averaged momentum p' of a particle is given by

$$p' = \frac{4\pi g}{nh^3}\int_0^\infty f\,p^3\,dp \qquad (5)$$

and the averaged particle energy

$$\epsilon' = \frac{4\pi g}{nh^3} \int_0^\infty f \epsilon p^2 \, dp \quad . \qquad (6)$$

The total (kinetic) energy of the particle ensemble becomes simply

$$E = V n \epsilon' = N \epsilon' \quad , \qquad (6a)$$

and its enthalpy $H = E + PV$, while $\chi' = H/nV$. In connection with the equation of state further macroscopic quantities of the ensemble may be derived, such as the free energy $F = E - TS$, with the entropy $S = \int(dE + PdV)/T = -(\partial F/\partial T)_V$ (reversible case), the heat capacity $C_V = T(\partial S/\partial T)_V$, while the mean (thermal) wave number of the particles becomes $k_w = p'/\hbar$ and the reciprocal thermal wave length $\lambda = \hbar/p'$, a.s.o.

In all numerical applications, we use $m = m_e$ (mass of the electron) and $g = g_e = 2$.

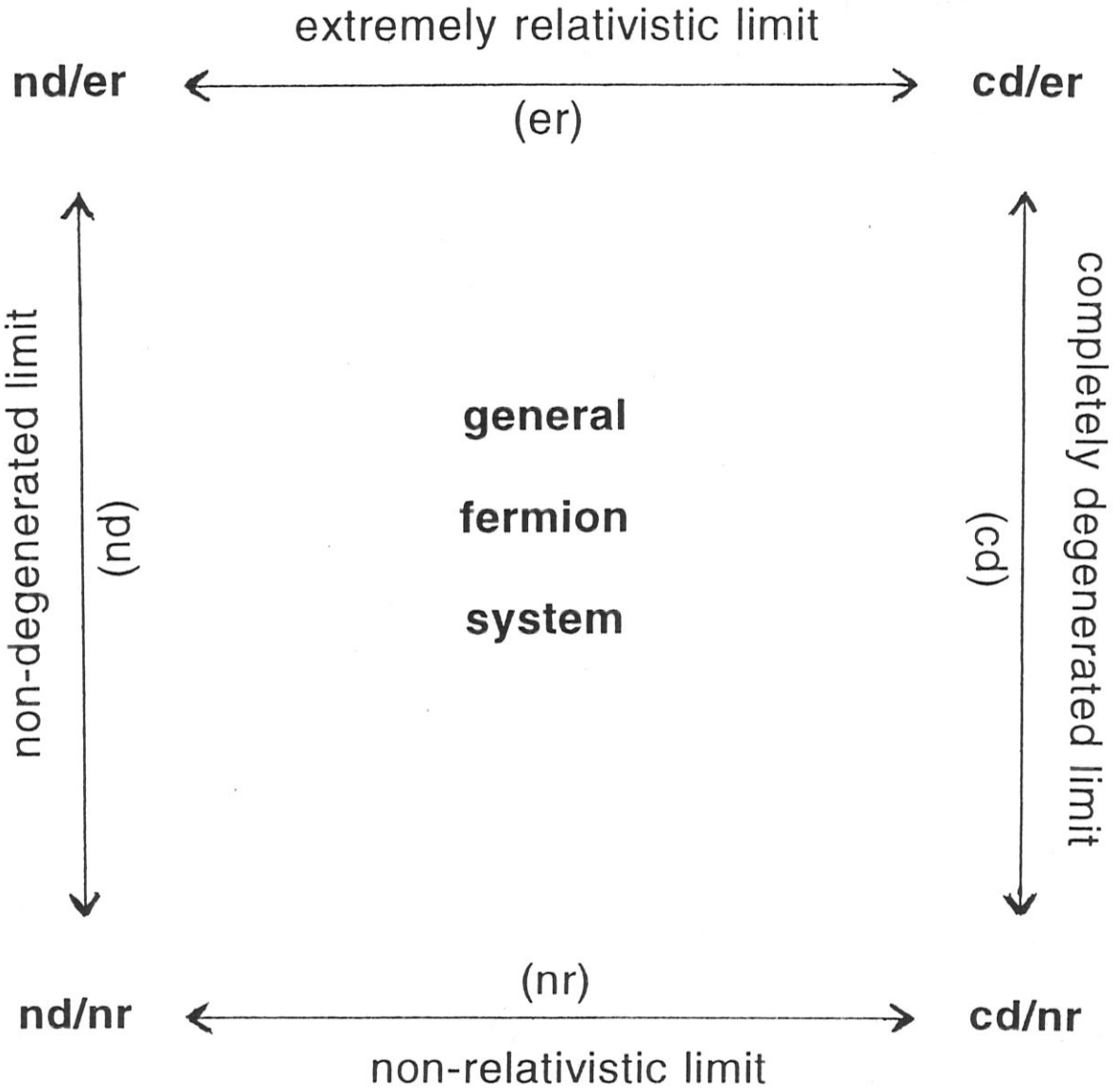

Figure 1: Schematic view of the limiting cases of statistical fermion ensembles

4. Limiting cases

There are four simple limiting ("corner") cases, see fig. 1.

1) **Non-relativistic, non-degenerated systems** (nr, nd).

Conditions: $A \gg 1$, and $\varepsilon \ll \varepsilon_R$ (most of the particles), with

$$\epsilon_R = m c^2 \quad . \qquad (7)$$

The distribution function is

$$f = f_{nd} = \frac{1}{A} \exp(-\frac{\epsilon}{kT}) \qquad (0 < \epsilon < \infty) \qquad (8)$$

(Maxwell-Boltzmann distribution function); and we have

$$\epsilon = \frac{p^2}{2m} \quad . \qquad (8a)$$

The typical particle energy is

$$\epsilon_T = kT \qquad (9a)$$

and the typical particle momentum

$$p_{0,T,nr} = (mkT)^{1/2} \quad . \qquad (9b)$$

The normalization constant becomes

$$A = \frac{g}{n h^3} (2\pi)^{3/2} p_{0,T,nr}^3 = 3\frac{\sqrt{\pi}}{4} (\frac{\epsilon_T}{\epsilon_{F,nr}})^{3/2} \qquad (10)$$

and the chemical potential

$$\mu = \epsilon_T [\frac{3}{2} \ln(\frac{\epsilon_{F,nr}}{\epsilon_T}) - \frac{1}{2} \ln\frac{9\pi}{16}] \qquad (10a)$$

(compare below equ. (14a)); and the averages are

$$p' = \sqrt{\frac{8}{\pi}}\, p_{0,T,nr}\ ; \qquad \epsilon(p') = \frac{4}{\pi}\,\epsilon_T = \frac{8}{3\pi}\,\epsilon' \qquad (11a)$$

$$p'' = \sqrt{3}\, p_{0,T,nr}\ ; \qquad \epsilon(p'') = \frac{3}{2}\,\epsilon_T = \epsilon' \qquad (11b)$$

$$p_m = \sqrt{2}\, p_{0,T,nr}\ ; \qquad \epsilon(p_m) = \epsilon_T \qquad (11c)$$

$$p^2{}_m = m\epsilon_T = p^2{}_{0,T,nr}\ ; \qquad \epsilon(p^2{}_m) = \frac{1}{2}\,\epsilon_T \qquad (11d)$$

$$\epsilon' = \frac{3}{2}\,\epsilon_T \qquad (11e)$$

$$\epsilon_m = \frac{1}{2}\,\epsilon_T \qquad (11f)$$

while the pressure becomes

$$P = \frac{2}{3}\, n\, \epsilon' \qquad (12)$$

(equation of state). Therefore, the averaged enthalpy per particle is $\chi' = (5/3)\varepsilon' = (5/2)\varepsilon_T$. And numerically, ε_T (/J) = 1.38066 $\times 10^{-23}$ T (/K); ε_T (/eV) = 8.6174 $\times 10^{-5}$ T (/K).

2) **Non-relativistic, completely degenerated systems** (nr, cd).

Conditions: A « 1, ε « ε_R (most of the particles), and

$$f = f_{cd} = 1 \quad if \quad p < p_{0,F}\ ;$$
$$f = f_{cd} = 0 \quad if \quad p > p_{0,F} \qquad (13)$$

(see (14b)), while $\varepsilon(p)$ is as in (8a).

The typical energy becomes

$$\epsilon_{F,nr} = \frac{h^2}{2m} \left(\frac{3n}{4\pi g}\right)^{2/3} \qquad (14a)$$

the typical momentum

$$p_{0,F} = h \left(\frac{3n}{4\pi g}\right)^{1/3} \qquad (14b]$$

(both defining the Fermi limit). The normalization constant is

$$A = \exp\left[- \frac{h^2}{2mkT} \left(\frac{3n}{4\pi g}\right)^{2/3}\right] = \exp\left[- \frac{\epsilon_{F,nr}}{\epsilon_T}\right] \qquad (15)$$

and

$$\mu = \epsilon_{F,nr} \qquad (15b)$$

and the averaged quantities become

$$p' = \frac{3}{4} p_{0,F} \; ; \qquad \epsilon(p') = \frac{9}{16} \epsilon_{F,nr} = \frac{15}{16} \epsilon' \qquad (16a)$$

$$p'' = \sqrt{\frac{3}{5}}\, p_{0,F} \; ; \qquad \epsilon(p'') = \frac{3}{5} \epsilon_{F,nr} = \epsilon' \qquad (16b)$$

$$p_m = p_{0,F} \; ; \qquad \epsilon(p_m) = \epsilon_{F,nr} \qquad (16c)$$

$$p^2{}_m = p_{0,F}{}^2 \; ; \qquad \epsilon(p^2{}_m) = \epsilon_{F,nr} \qquad (16d)$$

$$\epsilon' = \frac{3}{5} \epsilon_{F,nr} \qquad (16e)$$

$$\epsilon_m = \epsilon_{F,nr} \; . \qquad (16f)$$

Finally, the pressure is $P = 2n\varepsilon'/3$ as in (12), the averaged heat content $\chi' = (5/3)\varepsilon' = \varepsilon_{F,nr}$, and numerically we have $\varepsilon_{F,nr}$ (/J) = 5.842 263 $\times 10^{-38}$ (n)$^{2/3}$ (/m^{-3}), or $\varepsilon_{F,nr}$ (/eV) = 3.646 452 $\times 10^{-19}$ (n)$^{2/3}$ (/m^{-3}).

3) **Extremely relativistic, non-degenerated systems** (er, nd).

The conditions are $A \gg 1$, and $\varepsilon \gg \varepsilon_R$ (most of the particles). The distribution function is $f = f_{nd}$ as in (8), but

$$\epsilon = pc \qquad (17a)$$

The typical energy is ε_T (as in (9a)), the typical momentum

$$p_{0,T,er} = kT/c \quad . \qquad (18b)$$

The parameter of degeneracy (normalization constant) is

$$A = \frac{8\pi g}{n} \left(\frac{kT}{hc}\right)^3 = 6 \left(\frac{\epsilon_T}{\epsilon_{F,er}}\right)^3 \qquad (19)$$

(see equ. (22a)), with the chemical potential

$$\mu = \epsilon_T \left[3 \ln\left(\frac{\epsilon_{F,er}}{\epsilon_T}\right) - \ln 6\right] \qquad (19a)$$

and the averages

$$p' = 3\, p_{0,T,er}\ ; \qquad \epsilon(p') = 3\, \epsilon_T = \epsilon' \qquad (20a)$$

$$p'' = 2\sqrt{3}\, p_{0,T,er}\ ; \qquad \epsilon(p'') = 2\sqrt{3}\, \epsilon_T \qquad (20b)$$

$$p_m = 2\, p_{0,T,er}\ ; \qquad \epsilon(p_m) = 2\, \epsilon_T = \frac{2}{3}\, \epsilon' \qquad (20c)$$

$$p^2{}_m = \left(\frac{\epsilon_T}{c}\right)^2 = p^2{}_{0,T,er}\ ; \qquad \epsilon(p^2{}_m) = \epsilon_T \qquad (20d)$$

$$\epsilon' = 3\, \epsilon_T \qquad (20e)$$

$$\epsilon_m = 2\, \epsilon_T \quad . \qquad (20f)$$

The pressure of the gas is

$$P = \frac{n\,\epsilon'}{3} \qquad (21)$$

(equation of state), the averaged particle enthalpy $\chi' = (4/3)\varepsilon' = 4\varepsilon_T$, and numerically we have the same values as given at the end of part 1).

4) **Extremely relativistic, completely degenerated systems** (er, cd)

Under the conditions $A \ll 1$ and $\varepsilon \gg \varepsilon_R$ (most of the particles) the distribution function is $f = f_{cd}$ (as in (13)), with $\varepsilon(p)$ as in (17a).

The typical energy is

$$\epsilon_{F,er} = ch\,(\frac{3n}{4\pi g})^{1/3} \qquad (22a)$$

and the typical momentum $p_{0,F}$ as in (14b), (Fermi level).

The normalization constant (parameter of degeneracy) is

$$A = \exp[-\frac{ch}{kT}\,(\frac{3n}{2\pi g})^{1/3}] = \exp[-\frac{\epsilon_{F,er}}{\epsilon_T}] \qquad (23)$$

and

$$\mu = \epsilon_{F,er} \qquad (23a)$$

while the averages are

$$p' = \frac{3}{4} p_{0,F} ; \qquad \epsilon(p') = \frac{3}{4} \epsilon_{F,er} = \epsilon' \qquad (24a)$$

$$p'' = \sqrt{\frac{3}{5}} p_{0,F} ; \qquad \epsilon(p'') = \sqrt{\frac{3}{5}} \epsilon_{F,er} = \sqrt{\frac{16}{15}} \epsilon' \qquad (24b)$$

$$p_m = p_{0,F} ; \qquad \epsilon(p_m) = \epsilon_{F,er} \qquad (24c)$$

$$p^2{}_m = p_{0,F}{}^2 ; \qquad \epsilon(p^2{}_m) = \epsilon_{F,er} \qquad (24d)$$

$$\epsilon' = \frac{3}{4} \epsilon_{F,er} \qquad (24e)$$

$$\epsilon_m = \epsilon_{F,er} \quad . \qquad (24f)$$

The pressure P is nε'/3 as in (21), while the mean heat content of a particle becomes $\chi' = (4/3)\varepsilon' = \varepsilon_{F,er}$. Numerically, $\varepsilon_{F,er}$ (/J) = 9.780 721 $\times 10^{-26}$ $n^{1/3}$ (/m^{-3}), $\varepsilon_{F,er}$ (/eV) = 6.104 644 $\times 10^{-7}$ $n^{1/3}$ (/m^{-3}).

5. Marginal transitions (interconnections)

The transitions between the four special limiting cases ("corners", fig. 1) are also special solutions of equ. (4) - (6). In each of these interconnections we compile the first terms of series expansions from both sides (essentially of A and ε'), some exact values (in tables), and simple formulae representing approximately the whole transition, useful for a quick survey and approximate calculation.

While the characteristic thermal energy ε_T (= kT) is equal in both the nr and er case, the limiting Fermi energies are different and connected by

$$2\,\epsilon_{F,nr}\,\epsilon_R = \epsilon^2_{F,er} \quad . \qquad (26)$$

We use the following abbreviations of dimensionless energies which are independent of the specific properties of the Fermi particles:

$y_\alpha' = \varepsilon'/\varepsilon_\alpha$, with α = T; F; F,nr; F,er; R;
$z_\alpha = \varepsilon_T/\varepsilon_\alpha$, with α = F; F,nr; F,er; (F is omitted);
$x_\alpha = \varepsilon_\alpha/\varepsilon_R$, with α = T; F; F,nr; F,er; and:
$y_{m\alpha} = \varepsilon_m/\varepsilon_\alpha$.

Moreover, we define a general "norm energy" ε_N by the combination $\varepsilon_N = 2[\varepsilon_T^2 + (\varepsilon_F/3)^2]^{1/2}$, that is an approximate representative of ε' in the whole range of parameters with the advantage to avoid singularities of the dimensionless quantities in the limiting cases; in the dimensionless form it becomes

$y_N' = \varepsilon'/\varepsilon_N$ and $y_{mN} = \varepsilon_m/\varepsilon_N$;

while $x_N = \varepsilon_N/\varepsilon_R$.

Finally, as independent variable in intermediate regions we use the "standard energy" $\varepsilon_S = \varepsilon_T + \varepsilon_F$, and the corresponding dimensionless quantities, such as

$y_S' = \varepsilon'/\varepsilon_S$ or $x_S = \varepsilon_S/\varepsilon_R$, a.s.o.

12) **Non-relativistic systems**

(See e.g. [2], [4 - 8], [10 - 13], [29]);

a) Transition nd $\rightarrow$ cd ($z_{nr} \gg 1$):

$$A = \frac{3}{4}\sqrt{\pi} z_{nr}^{3/2} \left[1 - \frac{1}{3}\sqrt{\frac{2}{\pi}} z_{nr}^{-3/2} + \frac{2}{9\pi}\left(\left(\frac{4}{3}\right)^{3/2} - 1\right) z_{nr}^{-3} \ \ldots\right] \qquad (27a)$$

and

$$y_T' = \frac{3}{2}\left[1 + \frac{1}{3\sqrt{2\pi}} z_{nr}^{-3/2} - \frac{16}{9\pi}\left(\frac{2}{9\sqrt{3}} - \frac{1}{8}\right) z_{nr}^{-3} \ \ldots\right] \qquad (27b)$$

$$= 1.5\,[1 + 0.13298076/z_{nr}^{3/2} - 0.00186745/z_{nr}^{3} \ \ldots]$$

or, in relation to the norm energy ε_N,

$$y_N' = \frac{3}{4}\left[1 + \frac{1}{3\sqrt{2\pi}} z^{-3/2} - \frac{1}{18} z^{-2} - \frac{16}{9\pi}\left(\frac{2}{9\sqrt{3}} - \frac{1}{8}\right) z^{-3} - \right.$$

$$\left. - \frac{1}{54\sqrt{2\pi}} z^{-7/2} + \frac{1}{216} z^{-4} \ \ldots\right] \qquad (27c)$$

$$= 0.75\,[1 + 0.13298076/z^{3/2} - 0.05555556/z^{2} - 0.00186745/z^{3}$$

$$- 0.00738782/z^{7/2} + 0.00462963/z^{4} \ \ldots]$$

(with $z = z_{nr}$ in this case).

b) Transition cd $\rightarrow$ nd ($z_{nr} \ll 1$):

$$A = \exp\left[- z_{nr}^{-1}\left(1 - \frac{\pi^2}{12} z_{nr}^{2} - \frac{\pi^4}{80} z_{nr}^{4} \ \ldots\right)\right] \qquad (28a)$$

and

$$y_{F,nr}' = \frac{3}{5}\left[1 + \frac{5\pi^2}{12} z_{nr}^2 - \frac{\pi^4}{16} z_{nr}^4 \ \ldots\right] \qquad (28b)$$

$$= 0.6\,[1 + 4.11233517\, z_{nr}^2 - 6.08806819\, z_{nr}^4 \ \ldots]$$

with $y_{F,nr}' = y_T' z_{nr}$. Related to ε_N we have

$$y_N' = \frac{9}{10}\left[1 - \frac{1}{2}\left(9 - \frac{5\pi^2}{6}\right) z^2 + \frac{1}{8}\left(243 - 15\pi^2 - \frac{\pi^4}{2}\right) z^4 \ \ldots\right] \qquad (28c)$$

$$= 0.9\,[1 - 0.38766483\, z^2 + 5.78142356\, z^4 \ \ldots]$$

Interpolation: The whole range of z_{nr} is covered approximately by

$$y_T' \approx \left[\left(\frac{3}{5}\right)^{7/2} z_{nr}^{-7/2} + 2.166\, z_{nr}^{-3/2} + \left(\frac{3}{2}\right)^{7/2}\right]^{2/7} \qquad (29)$$

$$\approx \left[0.167313/z_{nr}^{7/2} + 2.166017/z_{nr}^{3/2} + 4.133514\right]^{2/7}$$

The limits of error are: - 0.8 % (with $z_{nr} \approx 0.22$), and + 0.5 % (with $z_{nr} \approx 1.16$).

Another interpolating formula (in its form referring to a Padé-approximation) is

$$y_T' \approx \frac{300000\, z_{nr}^{13/2} + 40000\, z_{nr}^5 - 48}{200000\, z_{nr}^{13/2} + 320\, z_{nr}^3 - 80\, z_{nr}} \qquad (30)$$

with the limits of error: -0.1 % (if $z_{nr} \approx 1.0$) and +0.3 % (if $z_{nr} \approx 0.22$). In any case, $y_{F,nr}' = z_{nr} y_T'$. And y_N' can be approximated by

$$y_N' \approx \frac{900 + 1420\, z^2 + 747\, z^{5/2} + 978\, z^4}{1000 + 1900\, z^2 + 827\, z^{5/2} + 1304\, z^4} \qquad (30a)$$

with errors generally < 0.1 %.

Some numerical examples are given in table 1, compare fig. 2.

The most probable energy becomes

a) nd → cd:

$$y_{mT} = \frac{1}{2} + \frac{2}{3\sqrt{e\pi}}\, z_{nr}^{-3/2} + \frac{2(\sqrt{2e}-2)}{9e\pi}\, z_{nr}^{-3} \ldots \qquad (31a)$$

$$= 0.5 + 0.2281322/z_{nr}^{3/2} + 0.0086301/z_{nr}^{3} \ldots$$

($y_{mT} = \varepsilon_m/\varepsilon_T$), and

b) cd → nd:

$$y_{mT} = z_{nr}^{-1} - \ln(2 z_{nr}^{-1}) + z_{nr}[\ln(2 z_{nr}^{-1}) + 1/2 - \pi^2/12] \ldots \qquad (31b)$$

$$= 1/z_{nr} - \ln(2/z_{nr}) + z_{nr}[(\ln(2/z_{nr}) - 0.322467] + \ldots$$

where again the alternative formulation is given by $y_{mF} = y_{mT} z_F$.

13) **Non-degenerated systems**

The general dependence can be represented with the help of Whittaker functions $W_{\alpha,\beta}(z)$:

$$A = \frac{(2\pi)^{3/2} g}{n (hc)^3}\, \epsilon_R^3\, x_T^{3/2} \exp(1/x_T) \left[\frac{3}{\sqrt{2}} \sqrt{x_T}\, W_{-1/2,3/2}\left(\frac{2}{x_T}\right) + W_{0,1}\left(\frac{2}{x_T}\right) \right] \qquad (32)$$

and the reduced mean energy $\varepsilon'/\varepsilon_R = y'_R$ becomes

$$y_R' = \frac{3}{\sqrt{2}}\, x_T^{1/2}\, \frac{5 x_T^{1/2} W_{-1,2}\left(\frac{2}{x_T}\right) + \sqrt{2}\, W_{-1/2,3/2}\left(\frac{2}{x_T}\right)}{3 x_T^{1/2} W_{-1/2,3/2}\left(\frac{2}{x_T}\right) + \sqrt{2}\, W_{0,1}\left(\frac{2}{x_T}\right)} \,. \qquad (32')$$

The Whittaker functions are special forms of the confluent hypergeometric functions (see, for instance [14 - 18]), they can also be transformed into Dawson's integral and into the error integral.

Another representation is possible with modified Bessel functions $K_n(x)$(e.g. [9], see also [1], [3], [19 - 21], [30]):

$$y_R' = x_T - \frac{K_2'(1/x_T)}{K_2(1/x_T)} - 1$$

$$= \frac{K_3(1/x_T)}{K_2(1/x_T)} - 1 - x_T \qquad (32'')$$

with $K_n'(x) = dK_n(x)/dx$.

In case of

a) nr $\rightarrow$ er
(i.e. $x_T \ll 1$) the normalization constant is

$$A = \frac{3}{4}\sqrt{\pi}\, z_{nr}^{3/2} \left[1 + \frac{15}{8}x_T + \frac{105}{128}x_T^2 \ldots\right] \qquad (32a)$$

and the mean energy

$$y_T' = \frac{3}{2}\left[1 + \frac{5}{4}x_T - \frac{5}{4}x_T^2 \ldots\right] \qquad (32b)$$

b) er $\rightarrow$ nr
(i.e. $x_T \gg 1$) we have

$$A = 6\, z_{er}^{3} \left[1 + \frac{1}{x_T} + \frac{1}{4}\frac{1}{x_T^2} \ldots\right] \qquad (33a)$$

and

$$y_T' = 3\left[1 - \frac{1}{3}\frac{1}{x_T} + \frac{1}{6}\frac{1}{x_T^2} \ldots\right] . \qquad (33b)$$

Interpolation:

Approximately, we can use

$$y_T' \approx \frac{12 + 33\, x_T}{8 + 11\, x_T} \qquad (34)$$

with the maximal limits of error: + 0.79 % (if $x_T \approx 0.19$), and - 0.41 % (if $x_T \approx 2.8$).

A further and still better interpolation formula is given by

$$y_N' = y_T'/2 \approx \frac{24 + 87x_T + 81x_T^2}{32 + 76x_T + 54x_T^2} \qquad (34a)$$

Numerical examples are given in table 2. The norm energy becomes $\varepsilon_N = 2\varepsilon_T$. Accordingly, $y_R' = y_T' x_T$.

Generally, the most probable energy in the form $y_{mR} = \varepsilon_m/\varepsilon_R$ is the real solution of the equation

$$y_{mR}^3 + (3 - 2x_T)\, y_{mR}^2 + (2 - 4x_T)\, y_{mR} - x_T = 0 \quad . \qquad (35)$$

In the special case

a) nr $\rightarrow$ er:

$$y_{mT} = \frac{1}{2} + \frac{5}{8} x_T + \frac{1}{2} x_T^2 \ldots \qquad (35a)$$

($y_{mT} = \varepsilon_m/\varepsilon_T$), and

b) er $\rightarrow$ nr:

$$y_{mT} = 2 - x_T^{-1} + \frac{1}{4} x_T^{-2} \ldots \quad . \qquad (35b)$$

The pressure P becomes simply

$$P = n\epsilon_T = N\epsilon_R x_T \quad . \qquad (35c)$$

24) **Completely degenerated systems**

In this case, the normalization constant becomes simply

$$A = \exp(- z^{-1}) \qquad (36)$$

irrespective of x_F (with $z = z_F$), and the Fermi energy ε_F is calculable from the limiting (nr- and er-) expressions by

$$x_F = \sqrt{1 + 2\, x_{F,nr}} - 1 = \sqrt{1 + x_{F,er}^2} - 1 \quad . \qquad (37)$$

Moreover, in this case - as to the averaged kinetic energy ε' (or $y_R' = \varepsilon'/\varepsilon_R$) - an exact analytical formula can be derived covering the whole range of parameters x_F (e.g. [5], [22 - 26]):

$$y_R' = \frac{3\, u_F\, [2\, u_F^2 - 1]}{8\, [u_F^2 - 1]} - \frac{3}{8} \frac{\ln[\, u_F + \sqrt{u_F^2 - 1}]}{[\, u_F^2 - 1]^{3/2}} - 1 \qquad (38)$$

with

$$u_F = 1 + x_F \quad . \qquad (38a)$$

($u_F^2 = 1 + x_{F,er}^2$). Another form of the mean energy is

$$y_R' = \frac{3}{8} \frac{(p_R^2 + 2p_0^2)\,(p_R^2 + p_0^2)^{1/2}}{p_R p_0^2} -$$

$$- \frac{3}{8} \frac{p_R^3}{p_0^3} ArSin\left(\frac{p_0}{p_R}\right) - 1 \qquad (38c)$$

with $p_R = mc$, and with the Fermi momentum $p_0 = p_{0,F}$, equ. (14b).

Transitions: In case of

a) nr $\rightarrow$ er (i.e. $x_F \ll 1$, $x_{F,nr} \ll 1$)
we have

$$y_F' = \frac{3}{5}\left(1 + \frac{1}{7}x_F - \frac{5}{63}x_F^2 \ldots\right) \qquad (39a)$$

or

$$y_{F,nr}' = \frac{3}{5}\left(1 - \frac{5}{14}x_{F,nr} + \frac{5}{18}x_{F,nr}^2 \ldots\right) \qquad (39b)$$

while in the opposite case

b) er $\rightarrow$ nr (i.e $x_F \gg 1$, $x_{F,er} \gg 1$)

$$y_F' = \frac{3}{4}\left(1 - \frac{1}{3}x_F^{-1} + \frac{1}{2}x_F^{-2} \ldots\right) \qquad (40a)$$

or

$$y_{F,er}' = \frac{3}{4}\left(1 - \frac{4}{3}x_{F,er}^{-1} + x_{F,er}^{-2} \ldots\right) \quad . \qquad (40b)$$

($y_R' = y_F'(\varepsilon_F/\varepsilon_R) = y_F' x_F$).

Interpolation formulae:

Approximately, we obtain

$$y_R' \approx \left[\left(\frac{5}{3}\right)^3 x_{F,nr}^{-3} + 3.546\, x_{F,nr}^{-2} + \left(\frac{4}{3\sqrt{2}}\right)^3 x_{F,nr}^{-3/2}\right]^{-1/3} \qquad (41)$$

$$\approx \left[4.629630/x_{F,nr}^3 + 3.54617/x_{F,nr}^2 + 0.8380525/x_{F,nr}^{3/2}\right]^{-1/3}$$

with the maximum errors of + 1.9 % (if $x_F \approx 0.55$, or $x_{F,nr} \approx 0.7$, or $x_{F,er} \approx 1.2$), and - 2.8 % (if $x_F \approx 8$, or $x_{F,nr} \approx 40$, or $x_{F,er} \approx 9$).

Still better is the following four-term-formula:

$$y_R' \approx \left[\left(\frac{5}{3}\right)^{9/2} x_{F,nr}^{-7/2} \left(x_{F,nr}^{-1} + \left(\frac{45}{28}\right)\right) + \right.$$

$$\left. + \left(\frac{4}{3\sqrt{2}}\right)^{9/2} x_{F,nr}^{-9/4} \left(3\sqrt{2}\, x_{F,nr}^{-1/2} + 1\right)\right]^{-2/9} \qquad (42)$$

$$\approx \left[9.961377/x_{F,nr}^{9/2} + 16.009356/x_{F,nr}^{7/2} + \right.$$

$$\left. + 3.254941/x_{F,nr}^{11/4} + 0.767197/x_{F,nr}^{9/4}\right]^{-2/9}$$

The maximum errors are -0.47 % (if $x_F \approx 0.85$, or $x_{F,nr} \approx 1.2$, or $x_{F,er} \approx 1.5$) and + 0.69 % (if $x_F \approx 6.5$, or $x_{F,nr} \approx 28$, or $x_{F,er} \approx 7.5$). Accordingly, in both (41) and (42) the variable $x_{F,nr}$ may be replaced by $x_{F,er}$, using $x_{F,nr} = (1/2)\, x_{F,er}^2$. Moreover, $y_R' = y_F' x_F$.

A rather simple and surprisingly good interpolation is given by

$$y_F' \approx \frac{6000 + 4389\, x_F}{10000 + 5852\, x_F} \qquad (43)$$

where the error limits are not more than + 0.075 % ($x_F \approx 0.7$) and -0.035 % ($x_F \approx 13$). Another interpolation formula is

$$y_N' = 3y_F'/2 \approx \frac{2268 + 2304 x_F + 513 x_F^2}{2520 + 2200 x_F + 456 x_F^2} \qquad (43a)$$

Some more exact numerical data are given in table 3. The norm energy becomes $\varepsilon_N = (2/3)\varepsilon_F$.

The most probable energy is determined always by $\varepsilon_m = \varepsilon_F$, i.e. $y_{mF} = 1$ (compare equ. (37)).

Finally, the pressure is given by

$$P = \frac{\pi}{3}\epsilon_R\left(\frac{\epsilon_R}{hc}\right)^3\left[u_F(2u_F^2 - 5)\sqrt{(u_F^2 - 1)} + 3\,ArSin\sqrt{u_F^2 - 1}\right]$$

$$= c\frac{\pi}{h^3}\left[p_0\,(p_0^2 + p_R^2)^{1/2}\,(\frac{2}{3}p_0^2 - p_R^2) + p_R^4\,ArSin(\frac{p_0}{p_R})\right] \qquad (43b)$$

(as to p_0 see equ. (14b)), with the limiting cases

$$P \approx \frac{\pi}{3}\epsilon_R\left(\frac{mc}{h}\right)^3\frac{32\sqrt{2}}{5}x_F^{5/2}\left[1 + \frac{15}{28}x_F \ \ldots\ \right]$$

$$= \frac{2}{5}n\epsilon_{F,nr}\left(1 + \frac{15\epsilon_F}{28mc^2} \ \ldots\ \right) \qquad (43c)$$

$$P \approx \frac{\pi}{3}\epsilon_R\left(\frac{mc}{h}\right)^3 2x_F^4\left[1 + \frac{4}{x_F} \ \ldots\ \right]$$

$$= \frac{1}{4}n\epsilon_{F,er}\left(1 + \frac{4mc^2}{\epsilon_F} \ \ldots\ \right) \qquad (43d)$$

near the non-relativistic (nr) and ultra-relativistic (er) limits, respectively.

34) **Extremely relativistic systems**

(See e.g. [27], compare also [1], [3], [19 - 21], [28], [30]).
In case of the transition

a) nd $\rightarrow$ cd ($A \gg 1$, $z_{er} \gg 1$)

the parameter (normalization constant) A becomes

$$A^{-1} = \frac{1}{6}z_{er}^{-3} + \frac{1}{288}z_{er}^{-6} - \frac{5}{186624}z_{er}^{-9} \ \ldots \qquad (44a)$$

and the averaged particle energy

$$y_T' = 3\,[1 + \frac{1}{96}\,z_{er}^{-3} - \frac{47}{186624}\,z_{er}^{-6} \ldots] \qquad (44b)$$

$$= 3\,[1 + 0.01041667/z_{er}^{3} - 0.00025184/z_{er}^{6} \ldots]$$

or (related to the norm energy),

$$y_N' = \frac{3}{2}\,[1 - \frac{1}{18}\,z^{-2} + \frac{1}{96}\,z^{-3} + \frac{1}{216}\,z^{-4} -$$

$$- \frac{1}{1728}\,z^{-5} - \frac{127}{186624}\,z^{-6} \ldots\,] \qquad (44c)$$

$$= 1.5\,[1 - 0.05555556/z^2 + 0.01041667/z^3 + 0.00462963/z^4 -$$

$$- 0.00057870/z^5 - 0.00068051/z^6 \ldots]$$

(with $z = z_{er}$), while in the opposite case

b) cd $\rightarrow$ nd ($A \ll 1$, $z_{er} \ll 1$)
we have

$$A = \exp[-\,z_{er}^{-1}\,(1 - \frac{\pi^2}{3}\,z_{er}^{2} \ldots)\,] \qquad (45a)$$

and

$$y_{F,er}' = \frac{3}{4}\,(1 + \frac{2}{3}\pi^2\,z_{er}^{2} - \frac{1}{5}\pi^4\,z_{er}^{4} \ldots) \quad . \qquad (45b)$$

$$= 0.75\,(1 + 6.5797363\,z_{er}^{2} - 19.4818182\,z_{er}^{4} \ldots)$$

Moreover, in relation to the norm energy,

$$y_N' = \frac{9}{8}\,[1 + (\frac{2\pi^2}{3} - \frac{9}{2})\,z^2 - (\frac{\pi^4}{5} + 3\pi^2 - \frac{243}{8})\,z^4 \ldots\,] \qquad (45c)$$

$$= 1.125\,[1 + 2.0797363\,z^2 - 18.7156314\,z^4 \ldots]$$

Interpolation:

Approximately, we have

$$y_T' \approx [(\frac{3}{4})^5 z_{er}^{-5} + 10.085 z_{er}^{-3} + 3^5]^{1/5} \qquad (46)$$

$$\approx [0.237305/z_{er}^5 + 10.0854/z_{er}^3 + 243]^{1/5}$$

with the error limits of - 0.75 % ($z_{er} \approx 0.43$) and + 1.18 % ($z_{er} \approx 0.13$). A better interpolation is given by the four-term formula

$$y_T' \approx \left[(\frac{3}{4})^{15/2} z_{er}^{-11/2}(z_{er}^{-2} + 5\pi^2) + 3^{15/2}(\frac{5}{64} z_{er}^{-3} + 1)\right]^{2/15} \qquad (47)$$

$$\approx [0.11560044/z_{er}^{15/2} + 5.7046529/z_{er}^{11/2} +$$

$$+ 295.93712/z_{er}^3 + 3787.9951]^{2/15}$$

Error limits: + 0.31 % ($z_{er} \approx 0.32$), - 0.17 % ($z_{er} \approx 0.12$), respectively.

A rather simple interpolation is possible with

$$y_T' \approx \frac{24444 z_{er}^6 + 18 z_{er}^3 + 9}{8148 z_{er}^6 - 79 z_{er}^3 + 12 z_{er}} \qquad (48)$$

(Padé-type approximation), with the maximum errors of + 0.9 % ($z_{er} \approx 0.17$) and - 1.6 % ($z_{er} \approx 0.34$). In any case, y_T' may be replaced by $y_{F,er}'$ with $y_{F,er}' = y_T' z_{er}$.

Related to the norm energy, an approximate representation of the mean energy is possible with

$$y_N' \approx \frac{1125 + 2421\,z + 9447\,z^2 + 13923\,z^3}{1000 + 2152\,z + 6298\,z^2 + 9282\,z^3} \qquad (48a)$$

(the errors are < 0.1 %).

Further and more exact values are compiled in table 4. In the tables 1 - 4 also the particle energies $\varepsilon(p')$ and $\varepsilon(p_m)$ corresponding to the average momentum (absolute values) and to the most probable momentum, respectively, are included. As to the transitions 13, 24 and 34 compare also fig.s 3 - 5. In all these special cases (fig.s 2 - 5) the scale of the abscissa becomes linear (normalized, from 0 to 1) if we use $\zeta = 1/(1+z)$ (fig.s 2, 3), or $\xi_T = x_T/(1+x_T)$ (in fig. 4), or $\xi_F = x_F/(1+x_F)$, respectively.

The most probable energy becomes, approximately,

a) nd $\rightarrow$ cd ($z_{er} \gg 1$):

$$y_{mT} = 2\left[1 + \frac{1}{6e^2} z_{er}^{-3} - \frac{1}{2}\left(\frac{1}{9e^4} - \frac{1}{144e^2}\right) z_{er}^{-6} \ldots\right] \qquad (49a)$$

$$= 2\,[1 + 0.022556/z_{er}^3 - 0.0005476/z_{er}^6 \ldots]$$

(with $y_{mT} = \varepsilon_m/\varepsilon_T$),
and

b) cd $\rightarrow$ nd ($z_{er} \ll 1$):

$$y_{mT} = z_{er}^{-1} - \ln((2z_{er})^{-1}) + z_{er}[\ln((2z_{er})^{-1}) + (2 - \pi^2/3)] \ldots \qquad (49b)$$

$$= 1/z_{er} - \ln(0.5/z_{er}) + z_{er}[\ln(0.5/z_{er}) - 1.289868] \ldots$$

Again, in each case, we have $y_{mF} = y_{mT} z_{er}$.

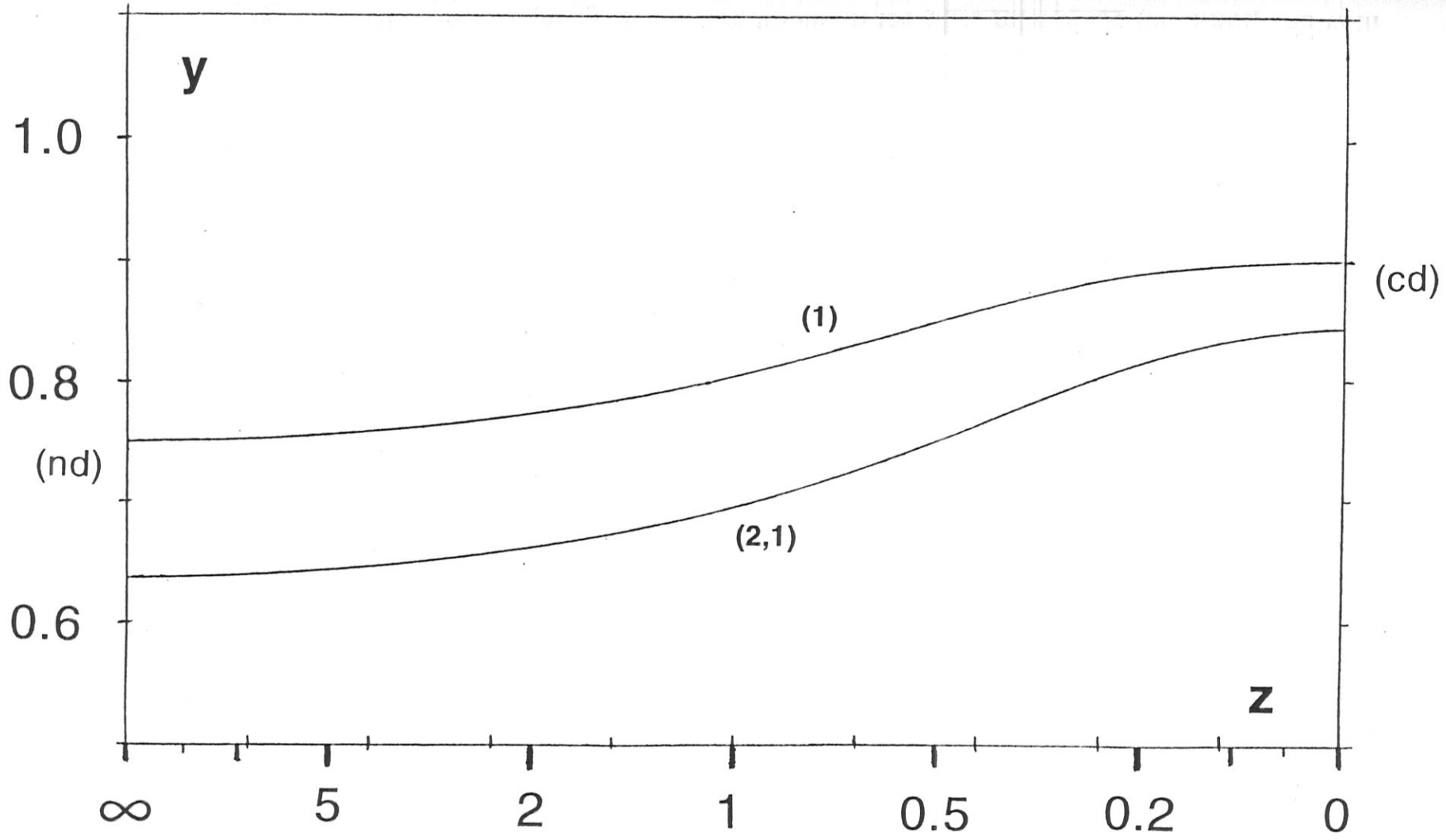

Figure 2: Mean energy ε' and energy of the mean momentum ε(p') in the dimensionless form $y_N' = \varepsilon'/\varepsilon_N$ (curve 1) and $y_N(p')$ (curve 2,1), respectively, i.e. related to the norm energy, as functions of $z = \varepsilon_T/\varepsilon_F$, in the non-relativistic (nr) limit, (left border: classical (nd) state; right border: completely degenerated (cd) state)

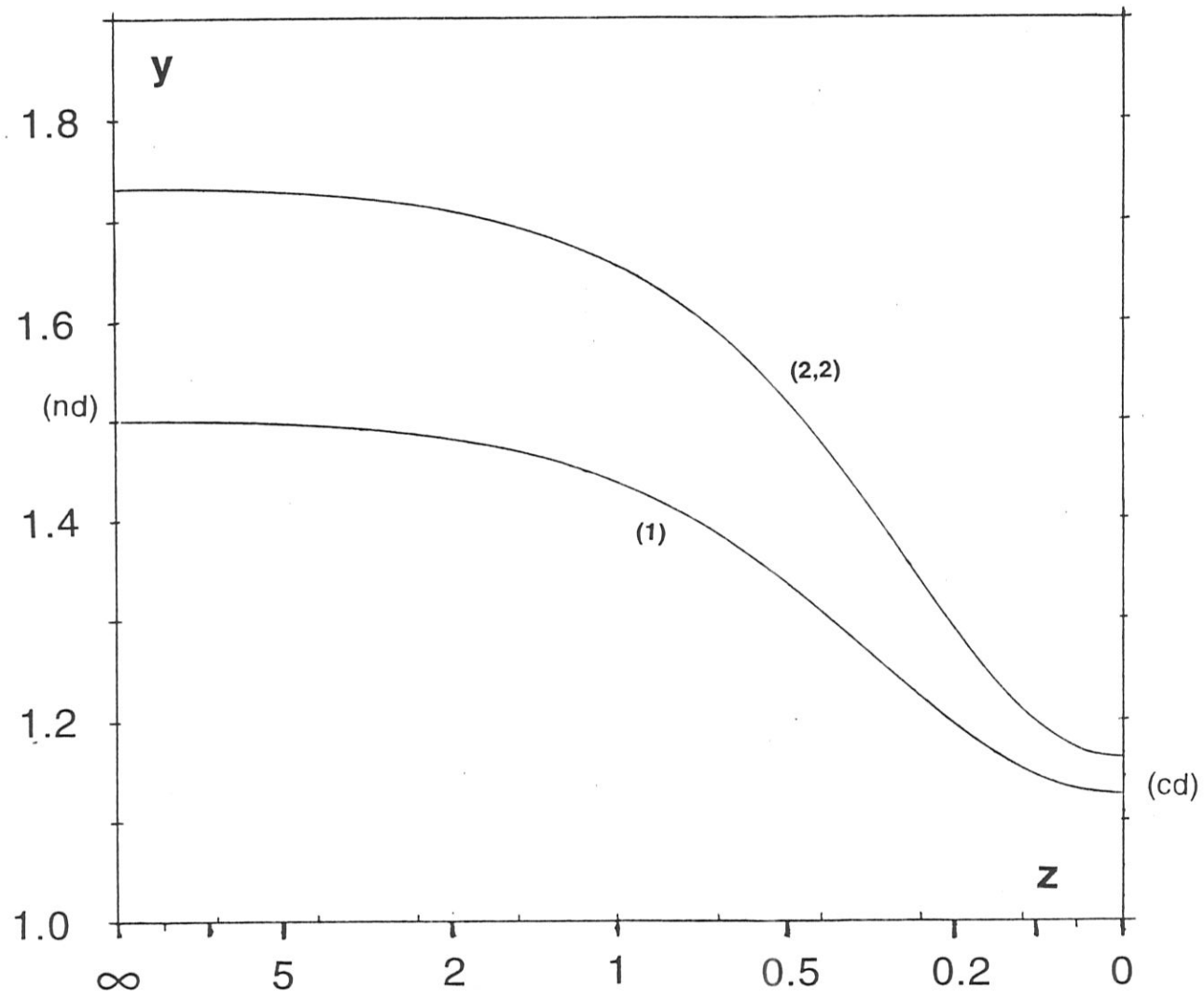

Figure 3: The same as in fig. 2, but conditions in the ultra-relativistic (er) limit, and with the energy of the mean square momentum $\varepsilon(p'')$ in the form $y_N(p'')$ (curve 2,2)

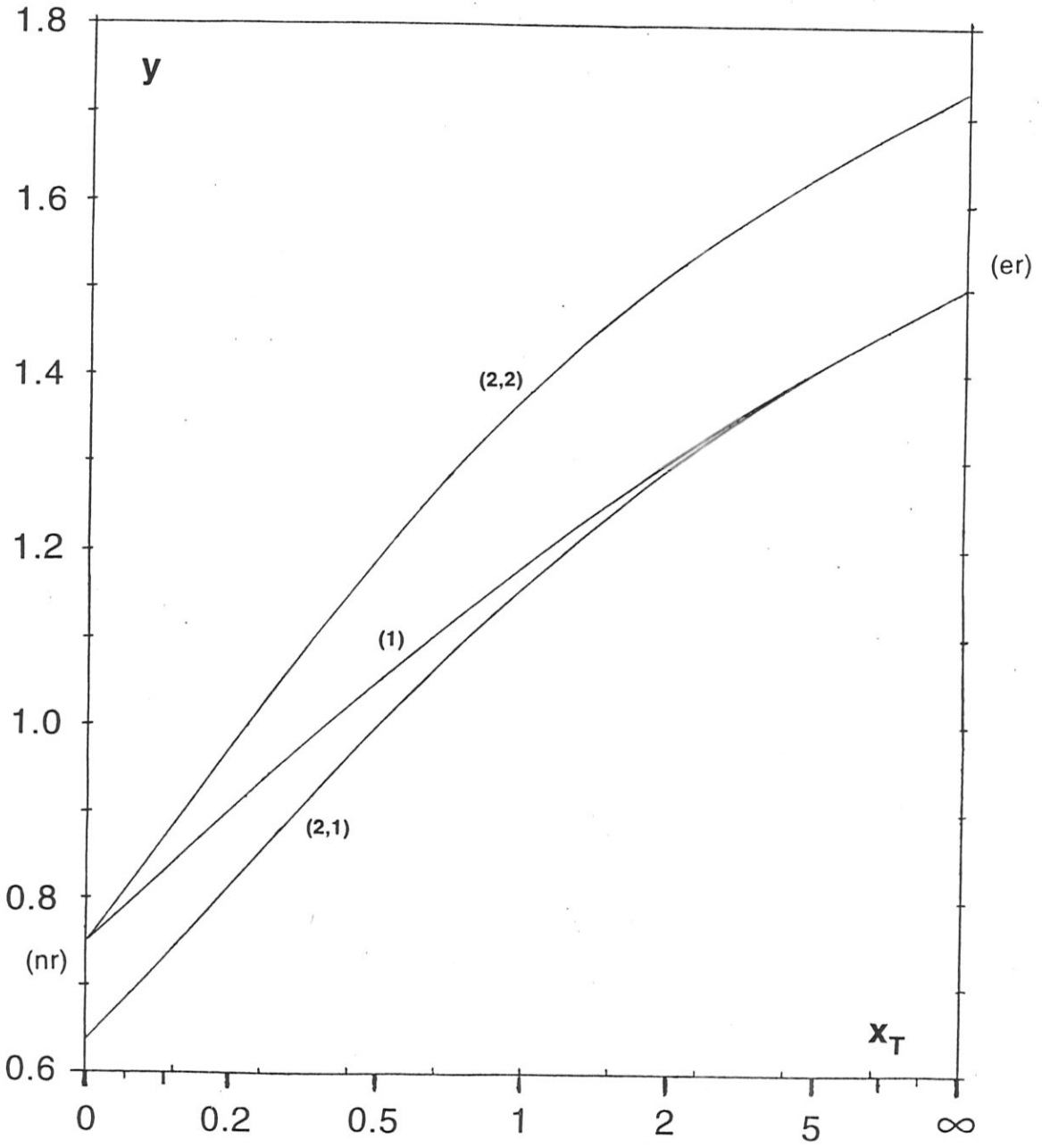

Figure 4: The same quantities as in fig.s 2 and 3 (y_N', $y_N(p')$, $y_N(p'')$) in the non-degenerated (nd) limit, as functions of $x_T = \varepsilon_T/\varepsilon_N$ (left border: non-relativistic (nr) state; right border: extremely relativistic (er) state)

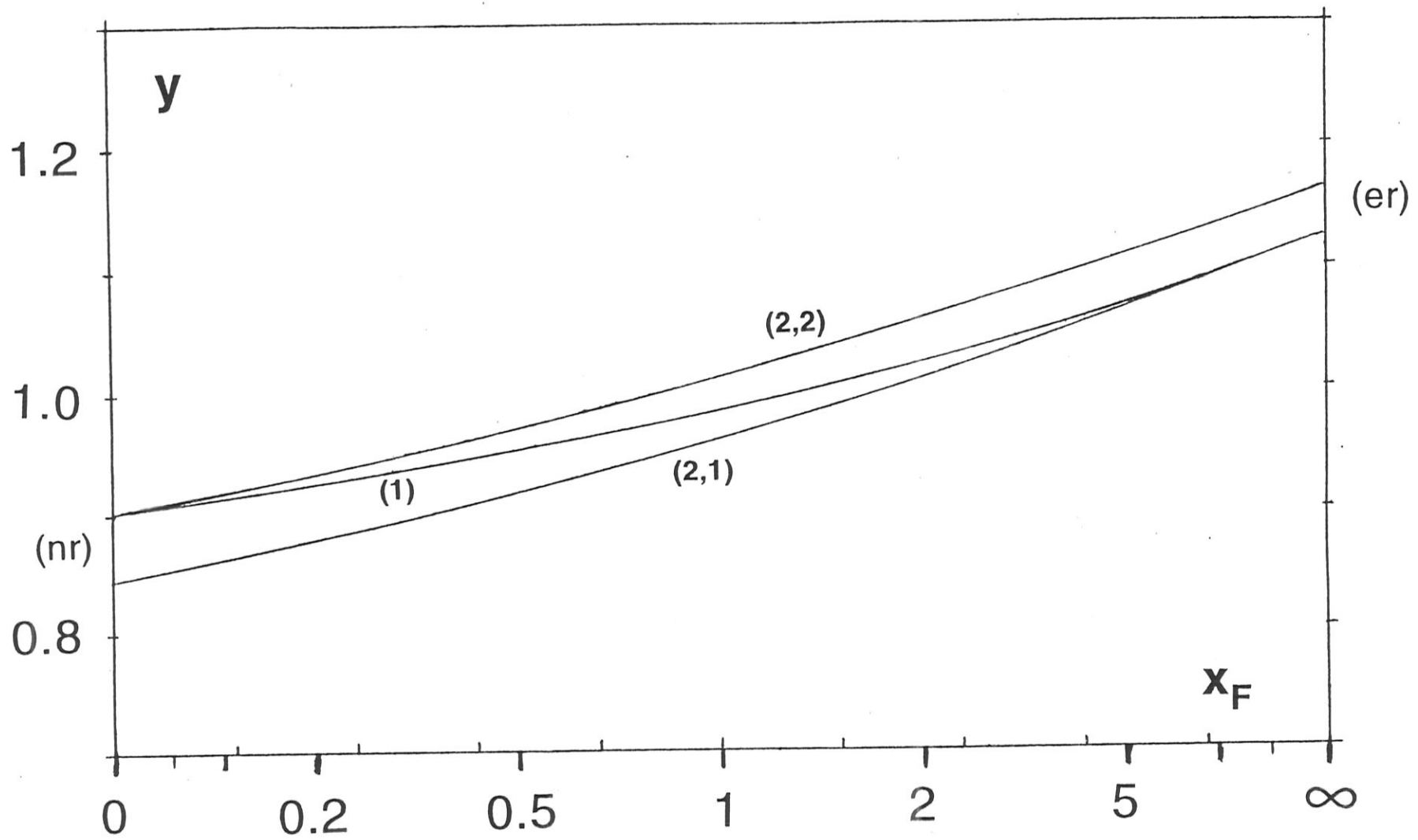

Figure 5: The same representation as in fig. 4, but in the completely degenerated (cd) limit, given as function of $x_F = \varepsilon_F/\varepsilon_R$

6. **Tables of numerical examples to section 5:**

Table 1:

Non-relativistic case

a) mean energies (y'), related to ε_T, or ε_F, or to the norm energy ε_N:

$z = z_{nr}$	y_T'	$y_{F,nr}'$	y_N'
0	∞	0.6	0.9
0.005	120.012336549	0.6000616827	0.899991281
0.01	60.024670356	0.6002467036	0.899965162
0.02	30.049318736	0.6009863747	0.899861267
0.05	12.1229069774	0.6061453489	0.899158756
0.1	6.2428310615	0.6242831061	0.896932153
0.2	3.4575811289	0.6915162258	0.889455407
0.5	2.0432846735	1.0216423368	0.850057807
1	1.6967393538	1.6967393538	0.804834143
2	1.5701766648	3.1403533297	0.774406361
5	1.5178188825	7.5890944124	0.757228577
10	1.5063050324	15.0630503238	0.752734446
20	1.5022298051	30.044596102	0.751010603
50	1.5005641672	75.028208359	0.750265411
100	1.5001994683	150.019946834	0.750095567
200	1.5000705233	300.014104670	0.750034220
∞	1.5	∞	0.75

b) energies corresponding to the mean momentum (y(p')), related to ε_T , ε_F , and to the norm energy ε_N:

$z = z_{nr}$	$y_T(p')$	$y_{F,nr}(p')$	$y_N(p')$
0	∞	0.5625 (= 9/16)	0.84375 (= 27/32)
0.005	112.509252697	0.5625462635	0.843724482
0.01	56.268505049	0.5626850505	0.843648020
0.02	28.162007294	0.5632401459	0.843343564
0.05	11.3424633011	0.5671231651	0.841273072
0.1	5.8092982615	0.5809298262	0.834644786
0.2	3.1682359995	0.6336471999	0.815021987
0.5	1.8070780812	0.9035390406	0.751789925
1	1.4678763037	1.4678763037	0.696274866
2	1.3428880182	2.6857760364	0.662308291
5	1.2909504228	6.4547521139	0.644045586
10	1.2795084829	12.7950848286	0.639399118
20	1.2754568535	25.509137069	0.637639872
50	1.2738005781	63.690028903	0.636886136
100	1.2734379072	127.343790722	0.636715416
200	1.2733096774	254.661935475	0.636653954
∞	1.2732395447 (= 4/π)	∞	0.636619772 (= 2/π)

c) most probable energy ε_m, and energy of the most probable momentum p_m, related to ε_N:

$z = z_{nr}$	$y_{mN} = \varepsilon_m/\varepsilon_N$	$y_{pm,N} = \varepsilon(p_m)/\varepsilon_N$
0	1.5	1.5
0.01	1.4206506	1.4310127
0.02	1.3620730	1.3827046
0.05	1.2233191	1.2738914
0.1	1.0496143	1.1456841
0.2	0.8112972	0.9762648
0.5	0.4906975	0.7323230
1	0.3485114	0.6042262
2	0.2868605	0.5412698
5	0.2596597	0.5112294
10	0.2534706	0.5040942
20	0.2512409	0.5014772
50	0.2503171	0.5003802
100	0.2501127	0.5001356
∞	0.25	0.5

Table 2:

Non-degenerated (classical) case

a) mean energies y', energy of the mean momentum y(p'), and of the mean square of the momentum y(p''), related to ε_T (with $x_T = \varepsilon_T/\varepsilon_R$):

x_T	y_T' $(= \varepsilon'/\varepsilon_T)$	$y_T(p')$ $(= \varepsilon(p')/\varepsilon_T)$	$y_T(p'')$ $(= \varepsilon(p'')/\varepsilon_T)$
0	1.5	$1.2732395 = 4/\pi$	1.5
0.005	1.509328258	1.28348967	1.51309631
0.01	1.518562838	1.29369653	1.52613231
0.02	1.536758641	1.31397288	1.55201525
0.05	1.589201149	1.37358264	1.62793572
0.1	1.669889403	1.46826123	1.74775162
0.2	1.809244988	1.63787957	1.95900388
0.5	2.102348811	2.00214944	2.39397843
1	2.370441175	2.32373787	2.75650416
2	2.609695421	2.59333973	3.04670076
5	2.819292015	2.81616533	3.27820012
10	2.904939172	2.90412332	3.36768187
20	2.951245131	2.95103817	3.41500150
50	2.980199839	2.98016655	3.44424588
100	2.990049988	2.99004166	3.45413769
200	2.995012499	2.99501042	3.45911064
∞	3	3	$3.4641016 = 2\sqrt{3}$

b) the same mean values, related to the norm energy ε_N:

x_T	y_N' (= $\varepsilon'/\varepsilon_N$)	$y_N(p')$	$y_N(p'')$
0	0.75	0.6366198 = $2/\pi$	0.75
0.005	0.754664129	0.641744838	0.756548155
0.01	0.759281419	0.646847653	0.763066157
0.02	0.768379320	0.656986441	0.776007623
0.05	0.794600575	0.686791318	0.813967859
0.1	0.834944702	0.734130617	0.873875812
0.2	0.904622494	0.818939785	0.979501939
0.5	1.051174405	1.001074721	1.196989214
1	1.185220587	1.161868935	1.378252081
2	1.304847710	1.296669863	1.523350379
5	1.409646007	1.408082664	1.639100058
10	1.452469586	1.452061661	1.683840933
20	1.475622565	1.475519087	1.707500750
50	1.490099919	1.490083275	1.722122940
100	1.495024994	1.495020829	1.727068847
200	1.497506250	1.497505208	1.729555318
∞	1.5	1.5	1.7320508 = $\sqrt{3}$

c) most probable energy ε_m, energy of the most probable momentum $\varepsilon(p_m)$ and of the most proble square momentum $\varepsilon(p^2_m)$:

x_T	$y_{m,N} = \varepsilon_m/\varepsilon_N$	$y_{pm,N} = \varepsilon(p_m)/\varepsilon_N$	$y_{(p2)m,N} = \varepsilon(p^2_m)/\varepsilon_N$
0	0.25	0.5	0.25
0.01	0.253150025	0.502499938	0.250624996
0.02	0.256350178	0.504999500	0.251249969
0.05	0.266251753	0.512492197	0.253124512
0.1	0.283749113	0.524937811	0.256246099
0.2	0.322222337	0.549509757	0.262468905
0.5	0.451605963	0.618033989	0.280776406
1	0.623489802	0.707106781	0.309016994
2	0.781221355	0.809016994	0.353553391
5	0.904999876	0.909901951	0.419258240
10	0.951249998	0.952493781	0.454950976
20	0.975312500	0.975624610	0.476246891
50	0.990050000	0.990099990	0.490199920
100	0.995012500	0.995024999	0.495049995
∞	1	1	0.5

Table 3:

Completely degenerated case

a) mean energy ε_F as a function of x_F (= $\varepsilon_F/\varepsilon_R$) and of the corresponding non-relativistic and extremely relativistic Fermi energies $\varepsilon_{F,nr}$ and $\varepsilon_{F,er}$, respectively:

x_F	$x_{F,nr}$	$x_{F,er}$	y_F' (= $\varepsilon'/\varepsilon_F$)
0	0	0	0.6
0.005	0.0050125	0.10012492	0.600427384
0.01	0.01005	0.14177447	0.600852407
0.02	0.0202	0.20099751	0.601695444
0.05	0.05125	0.32015621	0.604169828
0.1	0.105	0.45825757	0.608119884
0.2	0.22	0.66332496	0.615425668
0.5	0.625	1.11803399	0.633509271
1	1.5	1.73205081	0.654956750
2	4	2.82842712	0.680705658
5	17.5	5.91607978	0.711959629
10	60	10.95445115	0.728349380
20	220	20.97617996	0.738387294
50	1300	50.99019514	0.745146854
100	5100	100.99504938	0.747537113
200	20200	200.99751242	0.748759327
∞	∞	∞	0.75

b) energy of the mean momentum p' and of the mean square momentum $(p^2)'$:

x_F	$x_{F,nr}/x_{F,er}$	$y_F(p') = \varepsilon(p')/\varepsilon_F$	$y_F(p'') = \varepsilon(p'')/\varepsilon_F$
0	0	0.5625 = 9/16	0.6
0.005	0.0500625	0.56311351	0.60059820
0.01	0.0708872	0.56372358	0.60119284
0.02	0.1004988	0.56493350	0.60237149
0.05	0.159297	0.56848317	0.60582442
0.1	0.229129	0.57414299	0.61131472
0.2	0.331662	0.58457698	0.62138773
0.5	0.559017	0.61007663	0.64575131
1	0.866025	0.63935963	0.67332005
2	1.414214	0.67260394	0.70415946
5	2.95804	0.70967027	0.73808315
10	5.47723	0.72764727	0.75440037
20	10.48809	0.73819414	0.76394103
50	25.4951	0.74511437	0.77018985
100	50.4975	0.74752888	0.77236820
200	100.4988	0.74875726	0.77347608
∞	∞	0.75	0.77459667 = $\sqrt{(3/5)}$

c) the same mean energies, related to the norm energy ε_N:

x_F	$y_N' = \varepsilon'/\varepsilon_N$	$y_N(p') = \varepsilon(p')/\varepsilon_N$	$y_N(p'') = \varepsilon(p'')/\varepsilon_N$
0	0.9	0.84375 = 27/32	0.9 = 9/10
0.005	0.900641076	0.844670262	0.900897307
0.01	0.901278610	0.845585368	0.901789254
0.02	0.902543166	0.847400252	0.903557229
0.05	0.906254742	0.852724758	0.908736629
0.1	0.912179827	0.861214487	0.916972074
0.2	0.923138502	0.876865464	0.932081594
0.5	0.950263906	0.915114941	0.968626967
1	0.982435126	0.959039447	1.009980080
2	1.021058488	1.008905910	1.056239187
5	1.067939444	1.064505405	1.107124728
10	1.092524070	1.091470902	1.131600562
20	1.107580942	1.107291208	1.145911545
50	1.117720281	1.117671556	1.155284776
100	1.121305670	1.121293316	1.158552300
200	1.123138991	1.123135883	1.160214113
∞	1.125	1.125 = 9/8	1.1618950 = $\sqrt{(27/20)}$

d) the most probable energies are given by $\varepsilon_m/\varepsilon_F$ (= $y_{m,F}$) = $\varepsilon(p_m)/\varepsilon_F$ (= $y_{mp,F}$) = $\varepsilon(p^2{}_m)/\varepsilon_F$ (= $y_{mp2,F}$) = 1, with corresponding relations to ε_N :

x_F	$y_{m,N}$	$y_{pm,N}$	$y_{(p^2)m,N}$
0 ... ∞	1.5	1.5	1.5

Table 4:

Extremely relativistic (ultra-relativistic) case

a) mean energy y', related to ε_T, to ε_F or to the norm energy ε_N :

z_{er}	y_T'	$y_{F,er}'$	y_N'
0	∞	0.75	1.125
0.005	150.024672185	0.7501233609	1.125058479
0.01	75.049333420	0.7504933342	1.125233760
0.02	37.598579438	0.7519715888	1.125932525
0.05	15.244941668	0.7622470834	1.130720776
0.1	7.9797744630	0.7979774463	1.146485660
0.2	4.6457557636	0.9291511527	1.195110811
0.5	3.2145178538	1.6072589269	1.337320263
1	3.0305298261	3.0305298261	1.437506515
2	3.0038945178	6.0077890357	1.481511650
5	3.0002499517	15.001249758	1.496802436
10	3.0000312492	30.000312492	1.499182976
20	3.0000039062	60.000078125	1.499793663
50	3.0000002500	150.000012500	1.499966793
100	3.0000000313	300.000003125	1.499991682
200	3.0000000039	600.000000781	1.499997919
∞	3	∞	1.5

b) energy corresponding to the mean square momentum $(p^2)'$, related to ε_T, to ε_F or to ε_N:

z_{er}	$y_T(p'')$	$y_{F,er}(p'')$	$y_N(p'')$
0	∞	0.7745967 = $\sqrt{(3/5)}$	1.1618950 = $(3/2)\sqrt{(3/5)}$
0.005	154.951185092	0.7747559255	1.162003170
0.01	77.523352959	0.7752335296	1.162327365
0.02	38.857074057	0.7771414811	1.163619588
0.05	15.807780624	0.7903890312	1.172466669
0.1	8.3627704397	0.8362770440	1.201512202
0.2	5.0137257110	1.0027451422	1.289770299
0.5	3.6536332307	1.8268166154	1.520003303
1	3.4906272532	3.4906272532	1.655749887
2	3.4674757953	6.9349515907	1.710148528
5	3.4643180855	17.321590428	1.728322584
10	3.4641286779	34.641286779	1.731102882
20	3.4641049980	69.282099961	1.731811986
50	3.4641018316	173.205091582	1.732012427
100	3.4641016422	346.410164220	1.732041199
200	3.4641016185	692.820323704	1.732048404
∞	3.4641016 = $2\sqrt{3}$	∞	1.7320508 = $\sqrt{3}$

c) most probable energy ε_m and energy of the most probable square momentum p^2, related to ε_N:

z_{er}	$y_{mN} = \varepsilon_m/\varepsilon_N$	$y_{(p2)m,N}$
0	1.5	1.5
0.01	1.441088	1.430647
0.02	1.402216	1.381259
0.05	1.318043	1.265287
0.1	1.220292	1.114924
0.2	1.089276	0.887333
0.5	0.960071	0.576836
1	0.969583	0.502326
2	0.989167	0.496958
5	0.997965	0.499137
10	0.999467	0.499753
20	0.999864	0.499934
50	0.999978	0.499989
100	0.999994	0.499997
∞	1	0.5

Partly, the contents of these tables are shown also in figures 2 - 5. As to table 1 (nr-case) compare also [13].

7. Boundaries between limiting cases

Of course, the transitions between the four limiting existence forms of ideal Fermi systems are gradual ones, as seen from the formulas in section 5 and the numerical tables in section 6. However, in some applications the fixation of transition points are useful, in order to estimate approximately the state and the behaviour of a given system. Such boundaries cannot be given in a unique way: unavoidably, there must remain some arbitrariness. On the other hand, meaningful definitions present themselves very easily. We will compile some of such obvious boundaries.

a) Non-degenerated (classical) systems:

Limits between non-relativistic and extremely relativistic states:

a1) $\varepsilon_T = \varepsilon_R$ or $x_T = 1$:
$T = mc^2/k = T_1 = 5.929\ 861\ 8253 \times 10^9$ K

a2) $\varepsilon'_{nr} = \varepsilon_R$ or $(y_T')_{nr} x_T = 1$:
$T = (2/3)T_1 = 3.593\ 241\ 2168 \times 10^9$ K

a3) $\varepsilon'_{er} = \varepsilon_R$ or $(y_T')_{er} x_T = 1$:
$T = (1/3)T_1 = 1.976\ 620\ 6084 \times 10^9$ K

a4) $\varepsilon' = \varepsilon_R$ or $y_T' x_T = 1$:
$T = 0.479\ 206\ T_1 = 2.841\ 6254 \times 10^9$ K

b) Non-relativistic systems:

Limits between non-degenerated and completely degenerated states:

b1) $\varepsilon_T = \varepsilon_{F,nr}$ or $z_{nr} = 1$:
$T = (h^2/2mk)(3n/8\pi)^{2/3} = T_2 = 4.231\ 506\ 1753 \times 10^{-15}\ n^{2/3}$ K
or $n = 3.632\ 934\ 1961 \times 10^{21}\ T^{3/2}\ m^{-3}$

b2) $\varepsilon'_{nd} = \varepsilon'_{cd}$ or $y_T' = y_{F,nr}'$:
$T = (2/5)\ T_2 = 1.692\ 602\ 4701 \times 10^{-15}\ n^{2/3}$ K
or $n = 14.360\ 433\ 312 \times 10^{21}\ T^{3/2}\ m^{-3}$

b3) $\varepsilon'_{nd} = \varepsilon_{F,\,nr}$ or $(y_T')_{nd}\ z_{nr} = 1$:
$T = (2/3)\ T_2 = 2.821\ 004\ 117 \times 10^{-15}\ n^{2/3}$ K
or $n = 6.674\ 1263 \times 10^{21}\ T^{3/2}\ m^{-3}$

b4) $\varepsilon_T = \varepsilon'_{cd}$ or $(y_{F,nr}')/z_{nr} = 1$:
$T = (3/5)\ T_2 = 2.538\ 903\ 705 \times 10^{-15}\ n^{2/3}$ K
or $n = 7.816\ 8298 \times 10^{21}\ T^{3/2}\ m^{-3}$

c) Completely degenerated systems:

Limits between non-relativistic and extremely relativistic states:

c1) $\varepsilon_{F,nr} = \varepsilon_{F,er}$ or $z_{nr}/z_{er} = 1$:
$n = (64\pi/3)(mc/h)^3 = n_1 = 4.692\ 122\ 9717 \times 10^{36}\ m^{-3}$

c2) $\varepsilon'_{nr} = \varepsilon'_{er}$ or $y_{F,nr}'/y_{F,er}' = 1$:
$n = (125/64)\ n_1 = 9.164\ 302\ 6792 \times 10^{36}\ m^{-3}$

c3) $\varepsilon_{F,nr} = \varepsilon_R$ or $x_{F,nr} = 1$:
$n = (1/8)^{1/2}\ n_1 = 1.658\ 915\ 9857 \times 10^{36}\ m^{-3}$

c4) $\varepsilon_{F,er} = \varepsilon_R$ or $x_{F,er} = 1$:
$n = (1/8)\ n_1 = 0.586\ 515\ 371\ 46 \times 10^{36}\ m^{-3}$

c5) $\varepsilon_F = \varepsilon_R$ or $x_F = 1$:
$n = (27/64)^{1/2}\ n_1 = 3.047\ 623\ 27 \times 10^{36}\ m^{-3}$

c6) $\varepsilon'_{nr} = \varepsilon_R$ or $y_{F,nr}'x_{F,nr} = 1$:
$n = (5/6)^{3/2}\ n_1 = 3.569\ 418\ 8808 \times 10^{36}\ m^{-3}$

c7) $\varepsilon'_{er} = \varepsilon_R$ or $y_{F,er}'x_{F,er} = 1$:
$n = (2/3)^3\ n_1 = 1.390\ 258\ 6583 \times 10^{36}\ m^{-3}$

c8) $\varepsilon' = \varepsilon_R$ or $y_F'x_F = 1$:
$n = 1.489\ 3375\ n_1 = 6.988\ 1547 \times 10^{36}\ m^{-3}$

d) Extremely relativistic (ultra-relativistic) systems:

Limits between non-degenerated and completely degenerated states:

d1) $\varepsilon_T = \varepsilon_{F,er}$ or $z_{er} = 1$:
$T = (hc/2k)(3n/\pi)^{1/3} = T_3 = 7.084\ 101\ 486 \times 10^{-3}\ n^{1/3}\ K$
or $n = 2.812\ 844\ 1970 \times 10^6\ T^3\ m^{-3}$

d2) $(\varepsilon'_{er})_{nd} = (\varepsilon'_{er})_{cd}$ or $y_T'/y_{F,er}' = 1$:
$T = (1/4)T_3 = 1.771\ 025\ 3715 \times 10^{-3}\ n^{1/3}\ K$
or $n = 180.022\ 028\ 61 \times 10^6\ T^3\ m^{-3}$

d3) $(\varepsilon'_{er})_{nd} = \varepsilon_{F,er}$ or $y_T'z_{er} = 1$:
$T = (1/3)T_3 = 2.361\ 367\ 16 \times 10^{-3}\ n^{1/3}\ K$
or $n = 75.946\ 7933 \times 10^6\ T^3\ m^{-3}$

d4) $\varepsilon_T = (\varepsilon'_{er})_{cd}$ or $y_{F,er}'/z_{er} = 1$:
$T = (3/4)T_3 = 5.313\ 0761 \times 10^{-3}\ n^{1/3}\ K$
or $n = 6.667\ 482\ 54 \times 10^6\ T^3\ m^{-3}$

These different boundaries are summarized in fig. 6.

Further (similar) definitions of boundaries between nd- and cd-states may be formulated also with the condition: $\lambda = d$, i.e. mean thermal wavelength of the particles = mean distance between particles, with $\lambda = h/p$. There are many possibilities if we replace h by $\hbar = h/2\pi$ or by $h/\sqrt{(2\pi)}$, and if we use $p = p_0$ or $p = p'$ or $p = p''$ or $p = p_m$, and if we apply $d = n^{-1/3} = d_0$ or $d = d_0(3/4\pi)^{1/3} = d_1$ or $d = 2d_1$, for instance.

Approximate averages are:

Case a): $T = 3.31 \times 10^{9}$ K

Case b): $T = 2.68 \times 10^{-15} n^{2/3}$ K or $n = 7.22 \times 10^{21} T^{3/2}$ m^{-3}

Case c): $n = 2.86 \times 10^{36}$ m^{-3}

Case d): $T = 3.54 \times 10^{-3} n^{1/3}$ K or $n = 2.25 \times 10^{7} T^{3}$ m^{-3} .

From b) and d) results: $T = 4.68 \times 10^{9}$ K, and $n = 2.31 \times 10^{36}$ m^{-3}.

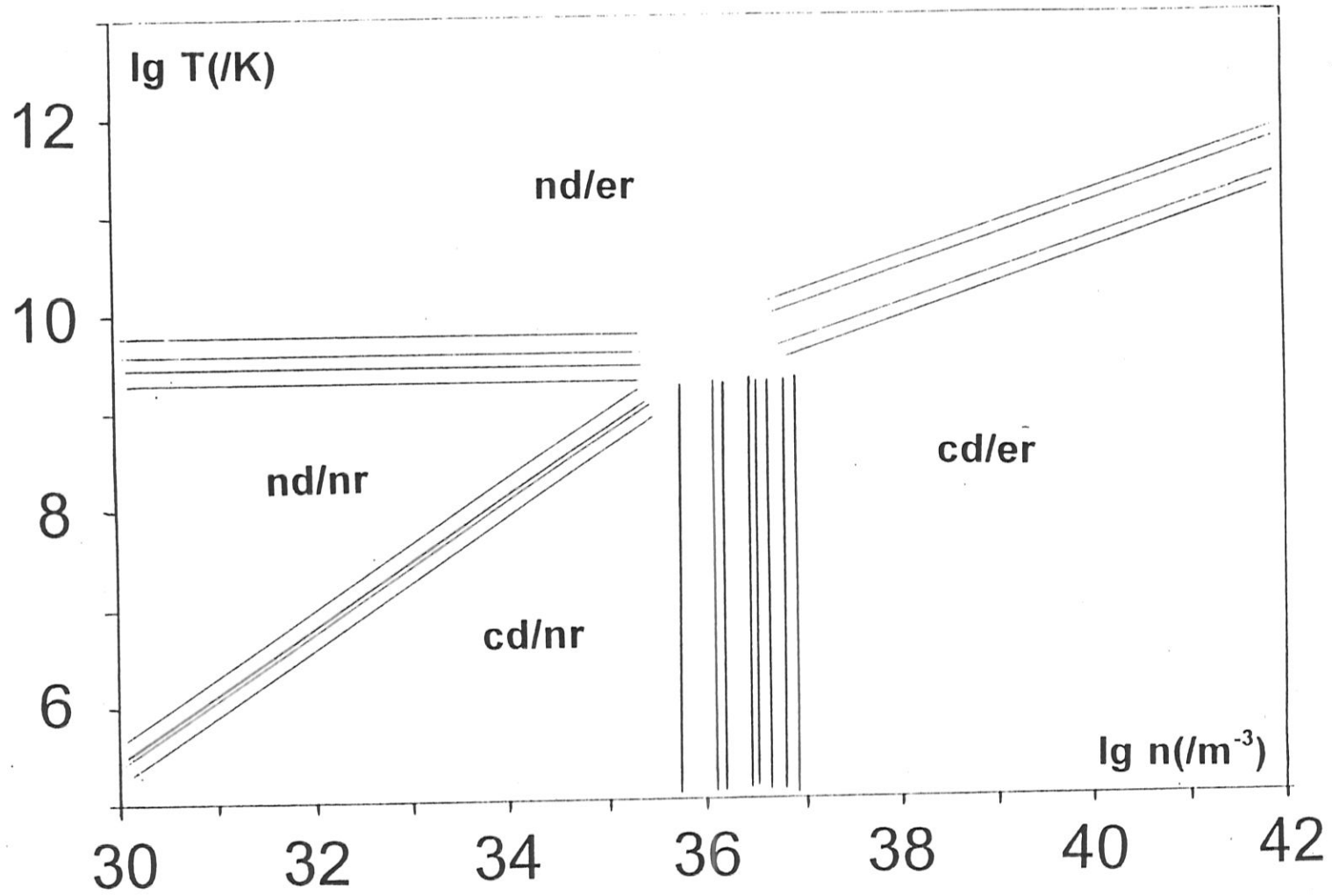

Figure 6: Different possibilities to define boundaries between the four special cases of fermion systems (see text, section 7), as functions of temperature T and density n (electron gas)

8. The general case: numerical results

1) The general case of a relativistic Fermi gas cannot be treated analytically in a closed form. However, averaged quantities can be presented by expansions e.g. with the help of infinite series of modified Bessel functions [27]. Here we give numerical examples of the mean energy, of the energy of the mean momentum, and of the energy of the mean square momentum in a dimensionless form, covering the whole range of parameters. Essentially, the basic equations are equ.s (2) - (6) in section 3.

The independent variables are:

$x = x_S = \varepsilon_S/\varepsilon_R$, with $\varepsilon_S = \varepsilon_T + \varepsilon_F$; and $z = \varepsilon_T/\varepsilon_F = x_T/x_F$.

The mean energy is given in the form $y_N' = \varepsilon'/\varepsilon_N$, where the "norm energy" $\varepsilon_N = [(2\varepsilon_T)^2 + (2\varepsilon_F/3)^2]^{1/2}$, as before. Numerical values of high precision are compiled in table 5.

Table 5

Mean energy y_N' as a function of x and z:

z → x ↓	100	50	20	10	5	2	1	.5	.2	.1	.05	.02	.01
100	1.49 4967 273	1.49 4893 143	1.49 4573 295	1.49 3721 679	1.49 0874 717	1.47 4343 628	1.42 8966 347	1.32 8503 734	1.18 9043 099	1.14 1896 073	1.12 6703 326	1.12 2142 753	1.12 1497 402
50	1.48 9993 825	1.48 9871 508	1.48 9408 178	1.48 8321 257	1.48 5019 927	1.46 7290 289	1.42 0618 951	1.31 9976 782	1.18 3198 554	1.13 7460 709	1.12 2811 798	1.11 8467 197	1.11 7872 371
20	1.47 5377 417	1.47 5116 731	1.47 4242 044	1.47 2482 404	1.46 7889 027	1.44 6807 777	1.39 6698 280	1.29 6027 193	1.16 6901 402	1.12 5016 941	1.11 1846 455	1.10 8086 175	1.10 7627 172
10	1.45 2012 174	1.45 1540 731	1.45 0041 581	1.44 7266 933	1.44 0746 467	1.41 4842 360	1.36 0320 219	1.26 0924 004	1.14 3307 029	1.10 6801 912	1.09 5670 743	1.09 2707 180	1.09 2430 483
5	1.40 8830 771	1.40 8005 523	1.40 5465 027	1.40 1018 648	1.39 1398 166	1.35 8288 301	1.29 8730 413	1.20 4793 903	1.10 6239 501	1.07 7708 566	1.06 9531 409	1.06 7691 826	1.06 7663 025
2	1.30 3362 108	1.30 1881 928	1.29 7451 024	1.29 0079 834	1.27 5411 244	1.23 3126 602	1.17 3762 688	1.10 1511 834	1.03 9994 610	1.02 4288 487	1.02 0585 107	1.02 0314 571	1.02 0590 481
1	1.18 3347 236	1.18 1503 027	1.17 6088 480	1.16 7387 835	1.15 1060 849	1.10 9919 284	1.06 3352 430	1.01 9489 936	0.98 9159 256	0.98 2118 089	0.98 1099 563	0.98 1595 506	0.98 1962 324
.5	1.04 9355 848	1.04 7593 515	1.04 2536 968	1.03 4724 029	1.02 0985 211	0.99 1267 647	0.96 6171 513	0.95 2728 299	0.94 8983 036	0.94 8138 804	0.94 8755 941	0.94 9554 019	0.94 9890 467
.2	0.90 3442 966	0.90 2339 498	0.89 9320 300	0.89 5010 381	0.88 8398 467	0.87 8980 589	0.88 0520 867	0.89 7169 872	0.91 6417 583	0.92 0233 870	0.92 1852 546	0.92 2681 949	0.92 2920 370
.1	0.83 4269 040	0.83 3672 136	0.83 2164 979	0.83 0315 690	0.82 8332 386	0.83 0419 372	0.84 4967 321	0.87 4838 999	0.90 3561 144	0.90 9142 223	0.91 1075 716	0.91 1862 245	0.91 2042 721
.05	0.79 4270 953	0.79 4019 093	0.79 3529 773	0.79 3305 918	0.79 4297 344	0.80 3428 895	0.82 5523 271	0.86 2780 586	0.89 6675 924	0.90 3187 202	0.90 5270 582	0.90 6020 973	0.90 6165 786
.02	0.76 8294 500	0.76 8285 704	0.76 8509 050	0.76 9412 345	0.77 2437 948	0.78 6270 583	0.81 3266 187	0.85 5230 324	0.89 2385 461	0.89 9471 600	0.90 1641 802	0.90 2364 980	0.90 2485 917
.01	0.75 9285 374	0.75 9364 178	0.75 9843 613	0.76 1150 053	0.76 4898 364	0.78 0381 966	0.80 9076 837	0.85 2658 291	0.89 0927 550	0.89 8208 238	0.90 0406 797	0.90 1119 903	0.90 1232 435

As to the results regarding the dimensionless mean energy see also fig.s 7 - 9. In fig. 7 the scale of the abscissa is linear (between $\zeta = 0 \ldots 1$) with $\zeta = 1/(1+z)$; similar in fig. 8 with $\xi = 1/(1+x)$.

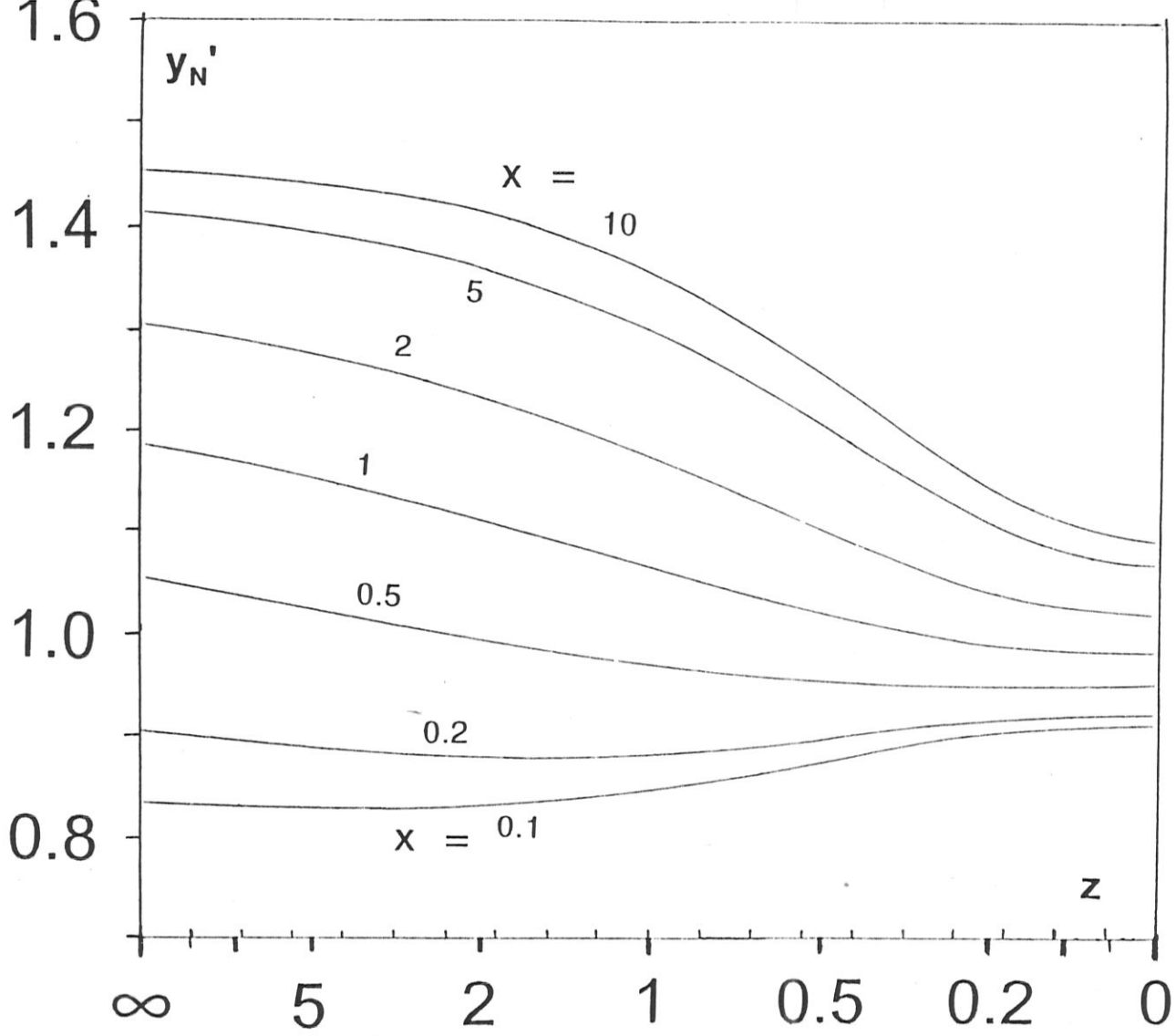

Figure 7: Mean energy ε' in the form $y_N' = \varepsilon'/\varepsilon_N$, general case of fermion systems, as function of $z = \varepsilon_T/\varepsilon_F$; parameter of the curves: $x = (\varepsilon_T + \varepsilon_F)/\varepsilon_R$

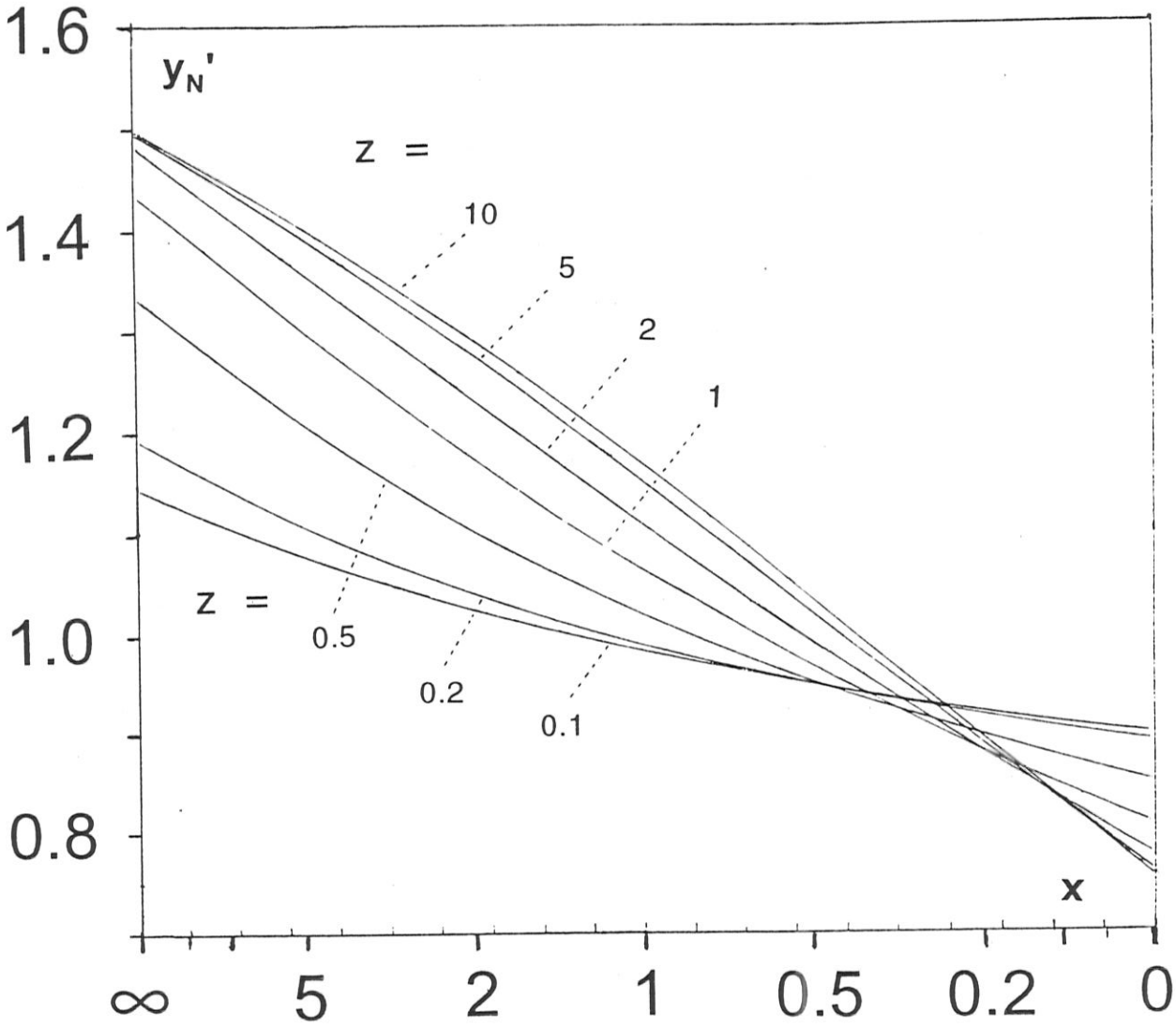

Figure 8: Mean energy ε' as in fig. 7, but as function of x; parameter of the curves: z

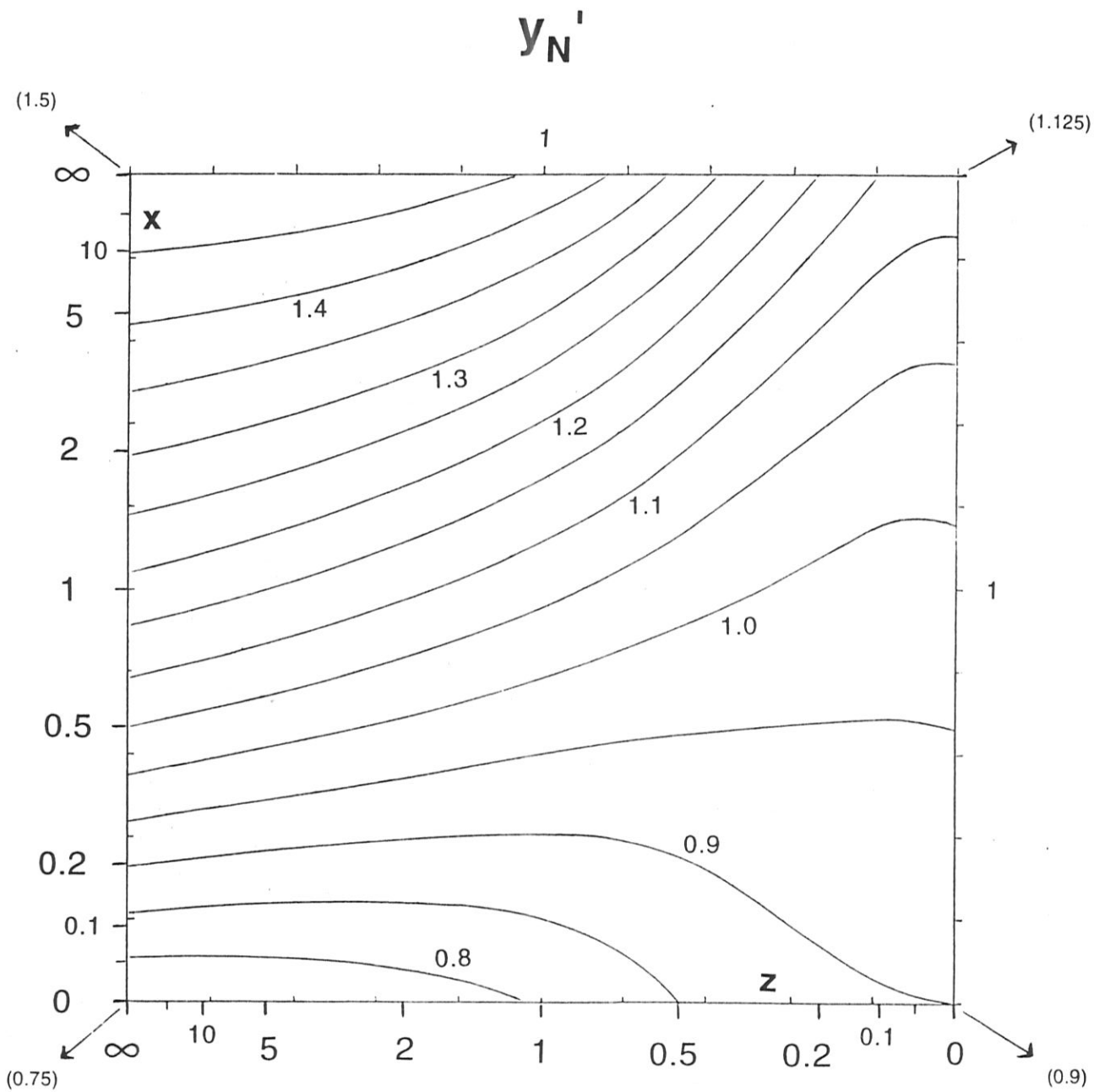

Figure 9: Isolines of the mean energy $y_N' = \varepsilon'/\varepsilon_N$ (parameter of the curves) as function of x and z, covering the whole range of these variables (general case of fermion systems)

These numerical data have to be supplemented by the boundary values at $x \to 0$, $x \to \infty$, and at $z \to 0$, $z \to \infty$, respectively, as discussed in the former sections. Especially, the "corner" values (see section 4) of the mean energy in this dimensionless form are:

$y_N' = 0.75$ $(x = 0, z = \infty)$,

$y_N' = 0.9$ $(x = 0, z = 0)$,

$y_N' = 1.125$ $(x = \infty, z = 0)$,

$y_N' = 1.5$ $(x = \infty, z = \infty)$.

2) Interpolation:

Similar to the interpolating formulae in the limiting cases simple expressions can be derived which are useful for approximate calculations. Starting from combinations of corresponding rational formulae from the four limiting cases (especially equ. (30a), (34a), (43a), (48a)), partly modified, supplemented by an adaption to the steepness of the dependence on the two independent variables x and z, and partly optimized regarding the deviations from the exact figures (table 5), we get - for instance - this result:

$$y_N' \approx R_1/R_2 \qquad (50)$$

$$\begin{aligned} R_1 = {} & [0.9 + 1.42z^2 + 0.746z^{5/2} + 0.978z^4] \\ & + [0.914 + 1.447z + 3.78z^2 + 1.578z^3 + \\ & \quad + 3.545z^4]x \\ & + [0.2034 + 0.44z + 2.262z^2 + 2.239z^3 + \\ & \quad + 3.3z^4]x^2 \qquad (50a) \end{aligned}$$

$$\begin{aligned} R_2 = {} & [1.0 + 1.9z^2 + 0.826z^{5/2} + 1.304z^4] \\ & + [0.873 + 1.64z + 3.02z^2 + 2.78z^3 + \\ & \quad + 3.097z^4]x \\ & + [0.1808 + 0.391z + 1.61z^2 + 1.493z^3 + \\ & \quad + 2.2z^4]x^2 \qquad (50b) \end{aligned}$$

In the whole range of parameters, the maximum deviations between the results of equ. (50) and the exact data (as compiled in table 5) are between about - 0.3 % and about + 0.5 %. Partly, equ. (50) has been used for further investigations. (More precisely: extreme differences are +0.49% ($z \approx 0.21$, $x \approx 5.4$), +0.57% ($z \approx 3.1$, $x \approx 2.8$), -0.21% ($z \approx 6.5$, $x \approx 0.17$), -0.31% ($z \approx 0.6$, $x \approx 1.2$).)

As a much simpler approach to the mean energy in the general case the following formula may be used:

$$y_N' \approx \frac{9 + 5.4x + 11.7z^2 + 27xz^2}{10 + 4.8x + 15.6z^2 + 18xz^2} \qquad (51)$$

Equ. (51) is built in such a way that the four "corner" values are given exactly as in (50), and that the most essential dependences on x and z can be represented. However, this expression is less flexible than equ. (50) and cannot sufficiently be adapted to the exact function $y_N'(x,z)$. Therefore, in spite of largely optimized coefficients, equ. (51) is much less correct than (50), and not only in the central region of the x-z-plane, but also at the boundaries (i.e.: $x,z \rightarrow 0$, $x,z \rightarrow \infty$, respectively). The maximum deviations from corresponding exact values (table 5) are - 2.9 % and + 3.2 %, respectively.

The most simple average of the mean energy in the general case is $y_N' = c_0 = \text{const}$, since the variation of y_N' is not large (table 5), with $c_0 \approx 1.05$. This crude approximation is worst near the four "corners" of the x-z-field (or n-T-field).

3) In addition, numerical values of the energy of the mean momentum $\varepsilon(p')$ and of the mean square momentum $\varepsilon(p'')$, in the dimensionless form $y_N(p') = \varepsilon(p')/\varepsilon_N$, and $y_N(p'') = \varepsilon(p'')/\varepsilon_N$, respectively, are compiled in table 6, covering the main part of the general x-z-plane.

Table 6: Dimensionless energy of the averaged momentum $y_N(p')$ (upper numbers) and of the averaged square momentum $y_N(p'')$ (lower numbers), related to the norm energy ε_N, as function of the dimensionless parameters x and z

x↓ z→	10	5	2	1	.5	.2	.1
10	1.446775 490 1.678358 763	1.440165 237 1.671261 999	1.413963 111 1.641766 698	1.358919 975 1.575162 936	1.258849 352 1.437227 108	1.141433 230 1.234432 621	1.105392 494 1.162101 850
5	1.399147 841 1.629809 330	1.389201 476 1.619189 454	1.355044 676 1.581029 127	1.293789 429 1.506747 856	1.197970 335 1.371375 691	1.100346 574 1.191883 863	1.073227 479 1.131220 608
2	1.280521 021 1.506088 756	1.264455 594 1.488534 355	1.218103 505 1.435562 241	1.153361 922 1.353940 975	1.077188 071 1.236679 649	1.020531 759 1.108674 277	1.009166 672 1.069439 221
1	1.141021 033 1.354995 337	1.121844 435 1.333067 695	1.073418 903 1.274762 304	1.019401 370 1.201107 103	0.973148 062 1.116500 773	0.953824 024 1.038407 513	0.954111 631 1.015829 827
.5	0.980618 167 1.172773 697	0.963428 494 1.151736 114	0.926252 386 1.102899 199	0.895933 435 1.053883 346	0.885021 549 1.011883 738	0.898512 665 0.979523 249	0.907369 511 0.969783 900
.2	0.806773 608 0.963832 201	0.798236 487 0.952015 906	0.785687 451 0.930657 315	0.787306 811 0.919650 082	0.811786 030 0.923311 459	0.852817 535 0.930405 884	0.867919 388 0.930461 762
.1	0.728307 497 0.865778 338	0.725534 960 0.860908 831	0.726919 216 0.856634 353	0.743678 395 0.864787 274	0.782992 010 0.888202 553	0.834755 306 0.910871 840	0.852094 976 0.914558 933

In the whole range of the parameters x and z holds the relation $y_N(p') \leq y_N' \leq y_N(p'')$ (as can be seen also in the limiting cases, compare tables 1 - 4). The behaviour of $y_N(p')$ and of $y_N(p'')$ is shown also in fig.s 10 and 11, respectively. The scales in fig.s 9, 10 and 11 are linear from 0 to 1 in the variables $\xi = x/(1+x)$ (ordinate), and $\zeta = 1/(1+z)$ (abscissa), respectively.

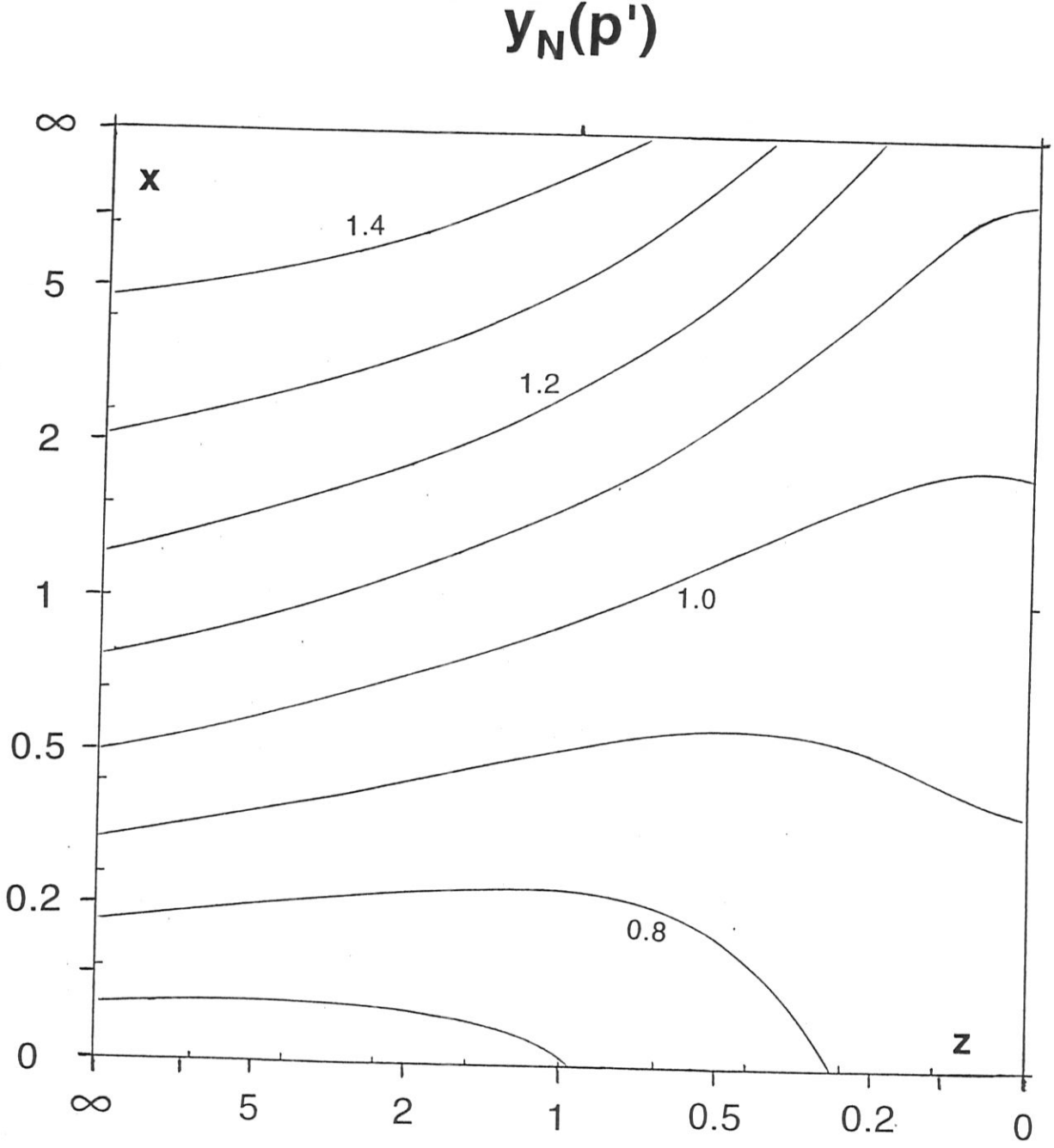

Figure 10: Isolines of the energy of the mean momentum $y_N(p')$ (parameter) in the whole range of variables, as in fig. 9

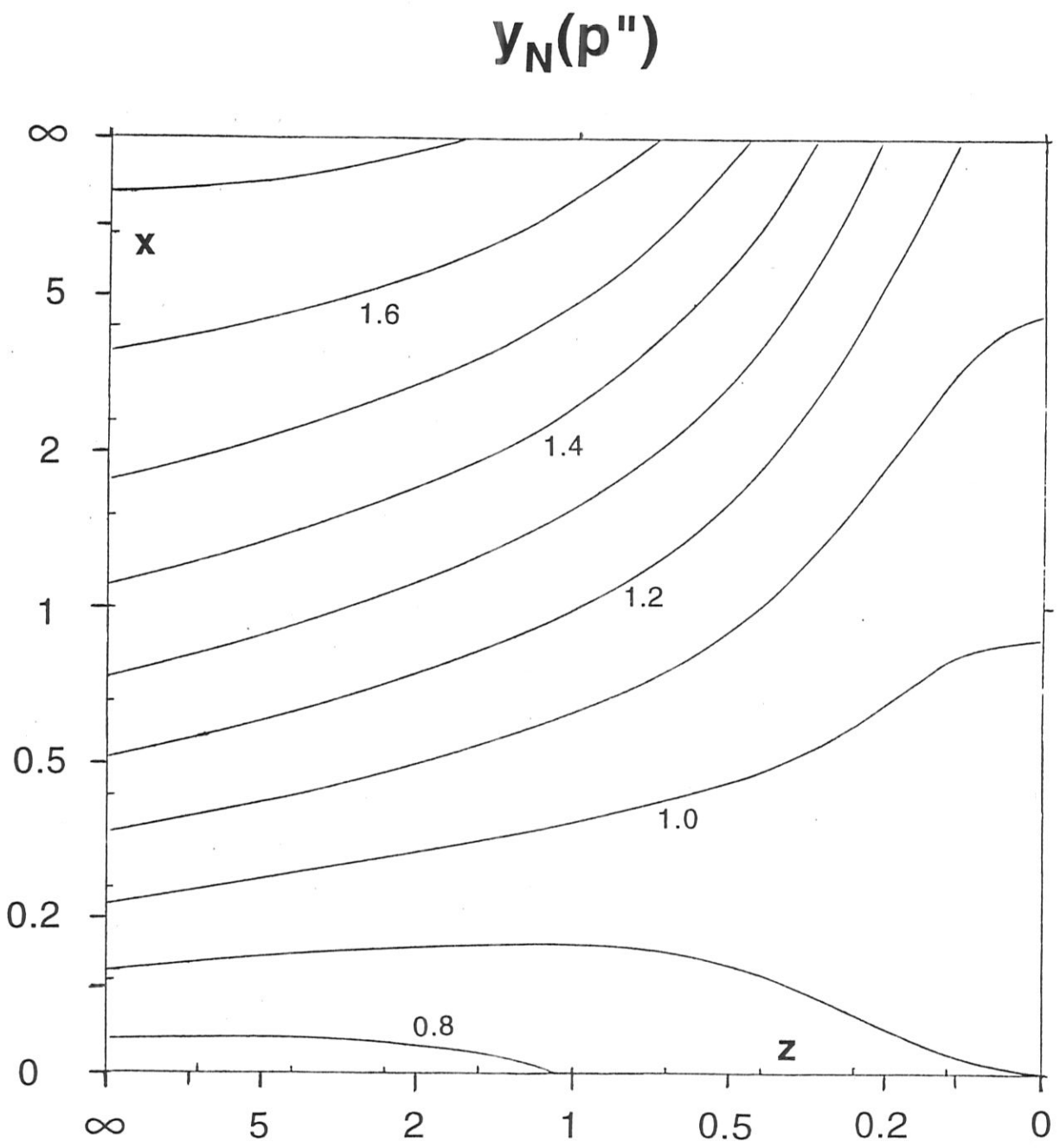

Figure 11: Isolines of the energy of the mean square momentum $y_N(p")$ (parameter) in the whole x-z-plane, as in fig. 9

In addition, some numerical data to the particle energy of the most probable momentum p_m (maximum of the distribution function f(p)) in the general case, related to the norm energy ε_N, i.e. $y_N(p_m) = \varepsilon(p_m)/\varepsilon_N$, are compiled in table 6a).

Table 6a: Dimensionless energy $y_N(p_m) = y_{pm,N}$, that is: energy ε of the most probable momentum p_m, normalized to ε_N, in the central part of the x-z-area ($x = (\varepsilon_T + \varepsilon_F)/\varepsilon_R$; $z = \varepsilon_T/\varepsilon_F$))

x↓, z→	10	5	2	1	0.5	0.2	0.1
10	0.9476179	0.9420802	0.9234647	0.8966903	0.8987165	1.0620292	1.2065516
5	0.9017325	0.8933139	0.8687589	0.8411091	0.8579833	1.0441336	1.1968915
2	0.7960113	0.7841985	0.7571525	0.7432179	0.7971153	1.0160454	1.1800278
1	0.6945513	0.6847491	0.6693192	0.6796028	0.7636051	0.9984476	1.1675959
0.5	0.6105363	0.6064396	0.6082766	0.6410601	0.7456535	0.9873605	1.1582609
0.2	0.5482730	0.5503204	0.5679199	0.6180032	0.7364448	0.9804364	1.1511519
0.1	0.5262856	0.5307987	0.5544550	0.6108530	0.7340875	0.9782609	1.1484907

As to the supplementing data at the boundaries of the x-z-area see tables 1c), 2c), 3d) (where $y_{mp,N} = \text{const}$), 4c) (with $y_{mp,N} = y_{mN}$). Everywhere in the x-z-field of parameters we have $0.5 \leq y_N(p_m) \leq 1.5$.

4) The chemical potential:

As already mentioned above (see equ. (2)) the chemical potential μ of the particles can be calculated from

$$\mu = \epsilon_T \ln\frac{1}{A} \qquad (52)$$

Near the four "corners" of the x-z-space of parameters (fig. 1) approximate formulae are given by combination of the equations (27a), (28a), (32a), (33a), (36), (44a) and (45a); these are (essentially up to the first order only):

a) ($x \ll 1$, $z \gg 1$):

$$\mu \approx \epsilon_T \left[-\frac{3}{2}\ln z_{nr} - \ln\frac{3\sqrt{\pi}}{4} + \sqrt{\frac{2}{9\pi}}\frac{1}{z_{nr}^{3/2}} - \frac{15}{8}x_T \ldots \right] \qquad (52a)$$

$$\approx \epsilon_T \left[-1.5\ln z - 0.2846829 + 0.2659815\frac{1}{z^{3/2}} - \right.$$

$$\left. - 0.375x\left(5 - \frac{7}{z} + \ldots\right) \ldots \right]$$

b) ($x \gg 1$, $z \gg 1$):

$$\mu \approx \epsilon_T \left[-3\ln z_{er} - \ln 6 + \frac{1}{48}\frac{1}{z_{er}^3} - \frac{1}{x_T} \ldots \right] \qquad (52b)$$

$$\approx \epsilon_T \left[-3\ln z - 1.7917595 + 0.0208333\frac{1}{z^3} + \right.$$

$$\left. + \frac{z}{x}\left(3 + \frac{2}{z} - \ldots\right) \ldots \right]$$

c) (x « 1, z « 1):

$$\mu \approx \epsilon_T \left(\frac{1}{z_{nr}}\right) \left[1 - \frac{\pi^2}{12} z_{nr}^2 \ldots\right] \qquad (52c)$$

$$\approx \epsilon_F \left[1 - 0.8224670\, z^2 (1 - x + \ldots) \ldots\right]$$

d) (x » 1, z « 1):

$$2\mu \approx \epsilon_T \left(\frac{1}{z_{er}}\right) \left[1 - \frac{\pi^2}{3} z_{er}^2 \ldots\right] \qquad (52d)$$

$$\approx \epsilon_F \left[1 - 3.2898681\, z^2 \left(1 - \frac{2}{x} - \ldots\right) \ldots\right]$$

Here we have used the connections $z_{nr} = z/(1 + x_F/2)$; $z_{er} = z/(1 + 2/x_F)^{1/2}$; $x_T = xz/(z + 1)$; $x_F = x/(z + 1)$; (as easily derivable from the definition of these dimensionless quantities). Moreover, we have $x_N = \varepsilon_N/\varepsilon_R = 2x\,(z^2 + 1/9)^{1/2}/(z + 1)$; $y_T = \varepsilon/\varepsilon_T = 2y_N(z^2 + 1/9)^{1/2}/z$; and $y_F = \varepsilon/\varepsilon_F = zy_T$.

In table 7 some examples of the chemical potential in the central part of the x-z-area are given.

Table 7

Chemical potential μ in the form $\alpha = \mu/\varepsilon_T = \ln(1/A)$ as function of x and z

x↓ z→	10	5	2	1	.5	.2	.1
10	-7.061 669	-5.553 280	-3.305 451	-1.444 262	+0.608 284	+4.410 705	+9.702 177
5	-6.377 841	-5.009 373	-2.958 714	-1.226 626	+0.728 852	+4.459 327	+9.725 739
2	-5.452 088	-4.216 900	-2.401 044	-0.856 460	+0.944 972	+4.552 528	+9.772 151
1	-4.848 575	-3.680 512	-2.001 143	-0.579 149	+1.116 228	+4.632 213	+9.813 149
.5	-4.395 369	-3.273 487	-1.691 471	-0.358 772	+1.258 233	+4.702 366	+9.850 281
.2	-4.031 833	-2.946 399	-1.440 427	-0.176 530	+1.379 763	+4.765 396	+9.884 494
.1	-3.888 107	-2.817 125	-1.340 718	-0.103 158	+1.429 777	+4.792 138	+9.899 258

9. Connection with temperature and density

The mean energy per particle and the other characteristic parameters of the ensemble of fermions are formulated as functions of the dimensionless quantities x and z. In real cases the temperature T and the density n are given. For the application of formulae, tables and figures compiled in the former sections 3 - 8 the connection of the physical parameters with the dimensionless quantities must be known. In order to simplify this transformation (e.g. with equ.s (7), (9a), (15a), (23a)) some further explicit formulae and numerical tables are added, assuming that the Fermions considered are electrons (or positrons), i.e. we use $g = 2$ and $m = 9.10939*10^{-31}$ kg.

We have

$$x_T = 1.686\,38*10^{-10}\ T_{/K} \qquad (53a)$$

and

$$x_F = [1 + 1.427\,185*10^{-24}\ n^{2/3}{}_{/m^{-3}}]^{1/2} - 1 \qquad (53b)$$

since

$$x_{F,nr} = 7.135\,9252*10^{-25}\ n^{2/3}{}_{/m^{-3}} \qquad (53c)$$

$$x_{F,er} = 1.194\,6485*10^{-12}\ n^{1/3}{}_{/m^{-3}} \qquad (53d)$$

From equations (53a,b) we get $x\ (= x_S) = x_T + x_F$, and $z = x_T/x_F$. In table 8 these relations are interpreted in numerical examples. Vice-versa, we have $\lg n = 35.768 + (3/2)\lg(u(u+2))$, and $\lg T = 9.773 + \lg(uz)$, with $u = x/(z+1)$.

Table 8

Connection between {x,z} and {T, n}.
a) x (upper numbers) and z (lower numbers) as function of temperature (in K) and density (in m^{-3}); A+B = $A*10^{+B}$

log n→ log T↓	30	31	32	33	34	35
14	1.686+4 2.363+8	1.686+4 5.092+7	1.686+4 1.098+7	1.686+4 2.372+6	1.686+4 5.174+5	1.686+4 1.176+5
13	1.686+3 2.363+7	1.686+3 5.092+6	1.686+3 1.098+6	1.686+3 2.372+5	1.686+3 5.174+4	1.687+3 1.176+4
12	1.686+2 2.363+6	1.686+2 5.092+5	1.686+2 1.098+5	1.686+2 2.372+4	1.687+2 5.174+3	1.688+2 1.176+3
11	1.686+1 2.363+5	1.686+1 5.092+4	1.687+1 1.098+4	1.687+1 2.372+3	1.690+1 5.174+2	1.701+1 1.176+2
10	1.686 2.363+4	1.687 5.092+3	1.688 1.098+3	1.693 2.372+2	1.719 5.174+1	1.830 1.176+1
9	1.687-1 2.363+3	1.690-1 5.092+2	1.702-1 1.098+2	1.757-1 2.372+1	2.012-1 5.174	3.121-1 1.176
8	1.694-2 2.363+2	1.719-2 5.092+1	1.840-2 1.098+1	2.397-2 2.372	4.945-2 5.174-1	1.603-1 1.176-1
7	1.758-3 2.363+1	2.018-3 5.092	3.223-3 1.098	8.797-3 2.372-1	3.428-2 5.174-2	1.451-1 1.176-2
6	2.400-4 2.363	4.998-4 5.092-1	1.705-3 1.098-1	7.279-3 2.372-2	3.276-2 5.174-3	1.436-1 1.176-3

log n→ log T↓	36	37	38	39	40	41	42
14	1.686+4 3.022+4	1.687+4 9.575+3	1.687+4 3.639+3	1.687+4 1.535+3	1.689+4 6.812+2	1.692+4 3.097+2	1.698+4 1.423+2
13	1.687+3 3.022+3	1.688+3 9.575+2	1.691+3 3.639+2	1.697+3 1.535+2	1.711+3 6.812+1	1.741+3 3.097+1	1.805+3 1.423+1
12	1.692+2 3.022+2	1.704+2 9.575+1	1.733+2 3.639+1	1.796+2 1.535+1	1.934+2 6.812	2.231+2 3.097	2.871+2 1.423
11	1.742+1 3.022+1	1.863+1 9.575	2.150+1 3.639	2.785+1 1.535	4.162+1 6.812-1	7.132+1 3.097-1	1.353+2 1.423-1
10	2.244 3.022	3.448 9.575-1	6.321 3.639-1	1.267+1 1.535-1	2.644+1 6.812-2	5.615+1 3.097-2	1.202+2 1.423-2
9	7.266-1 3.022-1	1.930 9.575-2	4.803 3.639-2	1.116+1 1.535-2	2.493+1 6.812-3	5.463+1 3.097-3	1.186+2 1.423-3
8	5.748-1 3.022-2	1.778 9.575-3	4.651 3.639-3	1.101+1 1.535-3	2.477+1 6.812-4	5.448+1 3.097-4	1.185+2 1.423-4
7	5.596-1 3.022-3	1.763 9.575-4	4.636 3.639-4	1.099+1 1.535-4	2.476+1 6.812-5	5.446+1 3.097-5	1.185+2 1.423-5
6	5.581-1 3.022-4	1.761 9.575-5	4.635 3.639-5	1.099+1 1.535-5	2.476+1 6.812-6	5.446+1 3.097-6	1.185+2 1.423-6

log n→ log T↓	43	44	45	46	47	48
14	1.712+4 6.578+1	1.742+4 3.047+1	1.806+4 1.413+1	1.944+4 6.555	2.241+4 3.042	2.881+4 1.412
13	1.943+3 6.578	2.240+3 3.047	2.880+3 1.413	4.259+3 6.555-1	7.230+3 3.042-1	1.363+4 1.412-1
12	4.250+2 6.578-1	7.221+2 3.047-1	1.362+3 1.413-1	2.741+3 6.555-2	5.713+3 3.042-2	1.211+4 1.412-2
11	2.732+2 6.578-2	5.704+2 3.047-2	1.211+3 1.413-2	2.590+3 6.555-3	5.561+3 3.042-3	1.196+4 1.412-3
10	2.581+2 6.578-3	5.552+2 3.047-3	1.195+3 1.413-3	2.574+3 6.555-4	5.546+3 3.042-4	1.195+4 1.412-4
9	2.565+2 6.578-4	5.537+2 3.047-4	1.194+3 1.413-4	2.573+3 6.555-5	5.544+3 3.042-5	1.195+4 1.412-5
8	2.564+2 6.578-5	5.535+2 3.047-5	1.194+3 1.413-5	2.573+3 6.555-6	5.544+3 3.042-6	1.195+4 1.412-6
7	2.564+2 6.578-6	5.535+2 3.047-6	1.194+3 1.413-6	2.573+3 6.555-7	5.544+3 3.042-7	1.195+4 1.412-7
6	2.564+2 6.578-7	5.535+2 3.047-7	1.194+3 1.413-7	2.573+3 6.555-8	5.544+3 3.042-8	1.195+4 1.412-8

b) x (upper data) and z (lower data) in the "central region" of the n-T plane (compare fig. 6), i.e. within $5 \cdot 10^{35}\ m^{-3} \leq n \leq 1 \cdot 10^{37}\ m^{-3}$, and $2 \cdot 10^{9}\ K \leq T \leq 1 \cdot 10^{10}\ K$

$n/10^{36}\ m^{-3} \rightarrow$ $T/10^{9}\ K \downarrow$	0.5	0.7	1	1.5	2
10	2.064 4.461	2.144 3.684	2.244 3.023	2.381 2.429	2.493 2.090
7	1.559 3.122	1.638 2.579	1.738 2.116	1.875 1.701	1.988 1.463
5	1.221 2.230	1.301 1.842	1.401 1.511	1.537 1.215	1.650 1.045
4	1.053 1.784	1.132 1.473	1.232 1.209	1.369 0.972	1.482 0.836
3	0.884 1.338	0.964 1.105	1.064 0.907	1.200 0.729	1.313 0.627
2	0.715 0.892	0.795 0.737	0.895 0.604	1.031 0.486	1.144 0.418

$n/10^{36}\ m^{-3} \rightarrow$ $T/10^{9}\ K \downarrow$	3	4	5	7	10
10	2.679 1.700	2.830 1.474	2.961 1.323	3.181 1.128	3.448 0.957
7	2.173 1.190	2.324 1.032	2.455 0.926	2.675 0.790	2.942 0.670
5	1.835 0.850	1.987 0.737	2.118 0.662	2.338 0.564	2.604 0.479
4	1.667 0.680	1.818 0.590	1.949 0.529	2.169 0.451	2.436 0.383
3	1.498 0.510	1.650 0.442	1.780 0.397	2.000 0.339	2.267 0.287
2	1.329 0.340	1.481 0.295	1.612 0.265	1.832 0.226	2.099 0.191

c) Temperature T (in K, upper numbers) and density n (in m^{-3}, lower numbers) as function of x and z, (with A+B = A*10^B)

x↓ z→	10^3	10^2	10^1	1	10^{-1}	10^{-2}	10^{-3}
10^3	5.924+12 3.042+36	5.871+12 7.502+38	5.391+12 4.553+41	2.965+12 7.375+43	5.391+11 4.421+44	5.871+10 5.710+44	5.924+9 5.865+44
10^2	5.924+11 5.635+34	5.871+11 2.988+36	5.391+11 5.938+38	2.965+11 7.776+40	5.391+10 4.553+41	5.871+9 5.866+41	5.924+8 6.024+41
10^1	5.924+10 1.669+33	5.871+10 5.557+34	5.391+10 2.522+36	2.965+10 1.214+38	5.391+9 5.938+38	5.871+8 7.502+38	5.924+7 7.689+38
1	5.924+9 5.242+31	5.871+9 1.646+33	5.391+9 4.861+34	2.965+9 8.197+35	5.391+8 2.522+36	5.871+7 2.988+36	5.924+6 3.042+36
10^{-1}	5.924+8 1.657+30	5.871+8 5.172+31	5.391+8 1.448+33	2.965+8 1.925+34	5.391+7 4.861+34	5.871+6 5.557+34	5.924+5 5.635+34
10^{-2}	5.924+7 5.238+28	5.871+7 1.634+30	5.391+7 4.550+31	2.965+7 5.887+32	5.391+6 1.448+33	5.871+5 1.646+33	5.924+4 1.669+33
10^{-3}	5.924+6 1.656+27	5.871+6 5.168+28	5.391+6 1.438+30	2.965+6 1.855+31	5.391+5 4.550+31	5.871+4 5.172+31	5.924+3 5.242+31

Moreover, $\varepsilon'/eV = 5.110*10^5\ y_R'$, a.s.o. In a similar way (using equ. (53a,b)) the dimensionless norm energy x_N can be calculated from T and n.

The deviations of the mean energy ε' of an electron gas (in the dimensionless form y_N') from the four limiting values (fig. 1) are demonstrated in fig. 12, with isolines of constant errors (compare also fig. 6). In case of the nd-cd transitions the error limits are valid with the used dimensionless quantities only, while in case of nr-er transitions the limits are generally applicable (e.g. also to $\varepsilon'(T)$ or $\varepsilon'(n)$, respectively). Obviously, the same kind of arbitrariness (i.e. the dependence on the choice of variables) is relevant to fig.s 2 and 3, and also to fig.s 7 - 11 regarding the z-dependence.

The situation as shown in fig. 12 turns out to be very similar if instead of the mean energy (ε') the energy corresponding to the mean momentum ($\varepsilon(p')$) or to the rms momentum ($\varepsilon(p'')$) is considered.

In case of neutrons the numerical constant of x_T, equ. (53a), has to be replaced by $9.17167 *10^{-14}$, the numerical coefficient in x_F, equ. (53b), by $4.221495 *10^{-31}$.

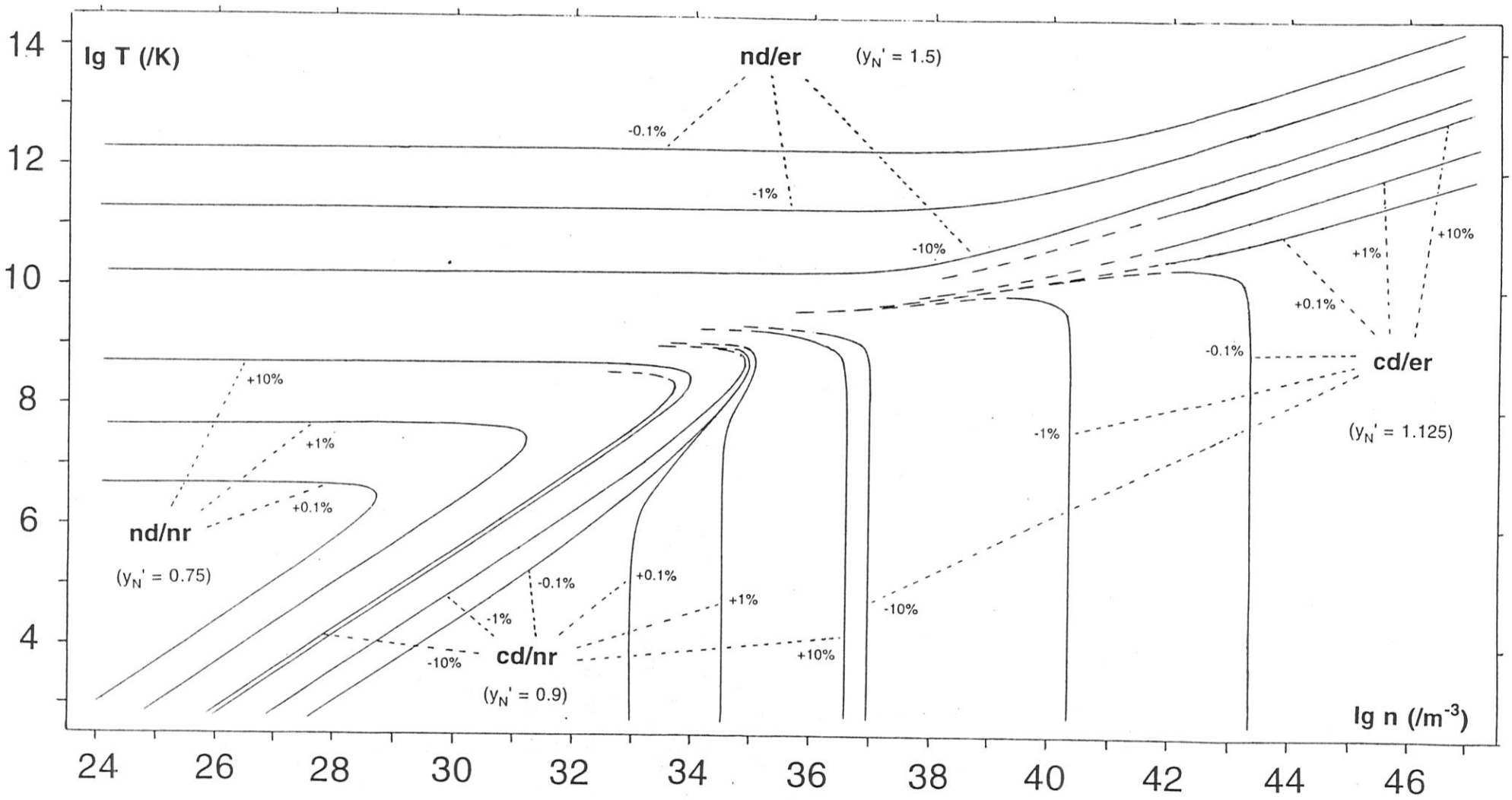

Figure 12: Margins of error (in %), if the normalized mean energy $y_N' = \varepsilon'/\varepsilon_N$ of the special limiting cases of fermion systems are generally applied (isolines of constant deviations), as functions of n and T (electron gas)

10. Summary and conclusion

In principle, the properties of ideal statistical ensembles of equal and interaction-free fermions in thermal equilibrium are well known. They depend only on temperature (T) and density (n) of the ensemble, if the properties of the particles are fixed (spin, mass). They are given by the distribution function in the phase space.

Because of the somewhat complicated integrals that have to be solved it is convenient to have simple formulae at hand for a quick and easy evaluation. Such formulae, tables and diagrams are compiled, especially for the determination of the averaged (kinetic) energy per particle. Moreover, some further data on the mean momentum and mean square momentum (or the particle energy corresponding to these averages) are given, partly also on the most probable values of these parameters and on the chemical potential and the equation of state.

The behaviour of the system is not uniform, it differs very much, depending on both T and n. There are two essential contrasts determining this behaviour: 1) between non-relativistic and extremely relativistic (ultra-relativistic) conditions, and 2) between classical (non-degenerated) and completely degenerated (quantumphysical) conditions. The second one is the more fundamental one. Correspondingly, there exist four limiting forms of the state of such systems: 1. the non-degenerated (classical) state, 2. the completely degenerated quantum state, 3. the non-relativistic state, 4. the extremely (ultra-)relativistic state. They are the "boundaries" of the general T-n-area or of the area in corresponding dimensionless parameters of the system. The properties of the ensemble in these special cases are extremely different. Several special (or limiting) cases have been treated in detail already many years ago (e.g. [19 - 28]).

Starting from combinations of these limiting situations (the "corners" of the T-n-plane) approximate or interpolating formulae can be derived covering more or less of the whole n-T-space. By definition of suited dimensionless quantities - such as $x_T = \varepsilon_T/mc^2$ (with $\varepsilon_T = kT$), and $x_F = \varepsilon_F/mc^2$ (where $\varepsilon_F(n)$ = Fermi energy), or the combinations $z = x_T/x_F$, $x = x_T + x_F$, and the dimensionless "norm energy" x_N by $(x_N/2)^2 = (x_T)^2 + (x_F/3)^2$ - the results can be presented universally in a form which is free of singularities and independent of the special particle properties. Especially, the averaged particle energy ε' is represented very approximately by x_N only: the deviations in the whole n-T-area are limited to $3/2 > \varepsilon'/(x_N\varepsilon_R) > 3/4$ (with $\varepsilon_R = mc^2$). Measured in such units, the variations in the non-degenerated state are generally greater than in the completely degenerated state, e.g. concerning the mean energy by factor 2 and factor $5/4 = 1.25$, respectively, or in the energy of the mean momentum by factor $3\pi/4 \approx 2.3562$ and $4/3 \approx 1.333$, respectively, and also in the energy of the mean square momentum: factors of the variations $4/\sqrt{3} \approx 2.3094$ (non-degenerated) and $\sqrt{(5/3)} \approx 1.291$ (completely degenerated), respectively.

As seen from the "central point" of any fermion system within a double logarithmic n-T-scale, which cannot be defined very sharply (compare fig.s 6 and 12, i.e. approximately from $n \approx 2.5 \bullet 10^{36}\ m^{-3}$, $T \approx 4 \bullet 10^{9}$ K in case of electrons), a sector of only 9.36 % of the whole plane can be associated with common nd/nr states, 15.64 % belong to cd/nr states, 30.12 % to cd/er states, and not less than 44.88 % are occupied by nd/er states.

In any case, as the "natural" basic physical quantities of energy we use ε_T (thermal energy) and ε_F (Fermi energy), though the meaning and the origin of both parameters are quite different. (Of course, other alternative basic quantities may be chosen also, but necessarily they must be simple functions of ε_T and ε_F).

Essentially, we have confined ourselves to the calculation of the mean energy and related quantities, such as the energy of the lowest powers of the mean momentum. We consider structureless particles only, i.e. particles without internal degrees of freedom, and without interactions (or potential energies due to mutual forces). The further statistical and thermodynamic functions of such perfect Fermi gases (partition function, entropy, free energy, enthalpy, thermodynamic potential, heat capacity, and others) may be treated in similar ways, partly they are easily derivable from the given results.

Thus, a good survey over the properties of ideal fermion systems is achievable, with a collection of simple formulae, as a basis of investigations on such general statistical ensembles including interactions between the particles (their mutual potential energy), bound states, particle sources and sinks, additional degrees of freedom, or deviations from thermal equilibrium, as they exist in a great diversity of real many-fermion systems. Moreover, these data may be used as a basis to the investigation of many further applications (as they are mentioned already in the introduction, for instance), where ideal fermion systems are a simple and sufficient approximation.

11. Appendix: Quick survey of the results

As a quick reference for practical use, in the following compilation most of the equations, tables and figures of this paper are listet, where special properties of ideal fermion systems can be found. Moreover, the definition of used symbols related to physical quantities or to dimensionless units is repeated.

Meaning of abbreviations:

EE = exact equation,
AS = approximate series (expansion),
AI = approximate interpolating formula
T = table (numerical values)
F = figure
S = section (of this paper)

Quantity	basic equations	limiting cases	non-relativistic case	non-degenerated (classical) case	completely degenerated case	extremely (ultra-)relativistic case	general case
parameter A, chemical potential μ	EE 2, 4 (S 3)	EE 10, 10a, 15, 15b, 19, 19a, 23, 23a (S 4)	AS 27a, 28a (S 5)	EE 32 AS 32a, 33a (S 5)	EE 36 (S 5)	AS 44a, 45a (S 5)	(EE 2, 4) (S 3) EE 51 AS 52a,b,c,d T 7 (S 8)
energy: typical		EE 9a, 14a, 22a, (S 4)		(EE 9a) (S 4) EE 53a (S 9)	EE 37 (S 5) EE 35b,c,d (S 9)		
mean	EE 6 (S 3)	EE 11e, 16e, 20e, 24e (S 4)	AS 27b,c, 28b,c AI 29, 30, 30a (S 5) T 1a (S 6) F 2	EE 32', 32" AS 32b, 33b AI 34, 34a (S 5) T 2a,b (S 6) F 4	EE 38, 38c AS 39a,b, 40a,b AI 41, 42, 43, 43a T 3a,c (S 6) F 5	AS 44b,c, 45b,c AI 46, 47, 48, 48a (S 5) T 4a (S 6) F 3	T 5 (S 8) AI 50,a,b, 51 (S 8) F 7,8,9, F 12
most probable		EE 11f, 16f, 20f, 24f (S 4)	AS 31a,b (S 5) T 1c (S 6)	EE 35 AS 35a,b (S 5) T 2c (S 6)	(EE 37) (S 5) (T 3d) (S 6)	AS 49a,b (S 5) T 4c (S 6)	

Quantity	basic equations	limiting cases	non-relativistic case	non-degenerated (classical) case	completely degenerated case	extremely (ultra-) relativistic case	general case
momentum: mean	EE 5 (S 3)	EE 11a, 16a, 20a, 24a (S 4)	T 1b (S 6) F 2	T 2a,b (S 6) F 4	T 3b,c (S 6) F 5	(T 4a) (S 6) (F 3)	T 6 (S 8) F 10
most probable		EE 11c, 16c, 20c, 24c (S 4)	T 1c (S 6)	T 2c (S 6)	(T 3d) (S 6)	(T 4c) (S 6)	T 6a (S 8)
square momentum: mean		EE 11b, 16b, 20b, 24b (S 4)	(T 1a) (S 6) (F 2)	T 2a,b (S 6) F 4	T 3b,c (S 6) F 5	T 4b (S 6) F 3	T 6 (S 8) F 11
most probable		EE 11d, 16d, 20d, 24d (S 4)	(T 1c) (S 6)	T 2c (S 6)	(T 3d) (S 6)	T 4c (S 6)	
boundaries (limits)			S 7b)	S 7a)	S 7c)	S 7d)	F 6
pressure		EE 12, 21 (S 4)		EE 35c (S 5)	EE 43b,c,d (S 5)		
units, symbols	S 2, S 5, S 8 EE 53a-d, T 8a-c (S 9), S 11						

Summary of some important definitions of used symbols related to single particles:

ε_T (classical) thermal energy (= kT)[(9a)]
ε_F Fermi energy [(14a), (22a), (37)]
ε_N "norm energy" as a typical measure of energy in the whole field of parameters (= $2[\varepsilon_T^2 + (\varepsilon_F/3)^2]^{1/2}$)
ε_R relativistic energy (= mc^2) [(7)]

ε' mean energy

ε_m most probable energy

p particle momentum

μ chemical potential, = - kT ln A

$x_T = \varepsilon_T/\varepsilon_R$; $x_F = \varepsilon_F/\varepsilon_R$; $x = x_S = x_T + x_F = (\varepsilon_T + \varepsilon_F)/\varepsilon_R$;
$z = z_T = \varepsilon_T/\varepsilon_F$;
$y_T' = \varepsilon'/\varepsilon_T$; $y_F' = \varepsilon'/\varepsilon_F$; $y_N' = \varepsilon'/\varepsilon_N$.

Averaged quantities $u = u' (= \langle u \rangle) = \int uf(v)dv / \int f(v)dv$,
and $u'' = \sqrt{(\langle u^2 \rangle)}$; (f(v) distribution function).

As to relations between dimensionless units see also section 5 (p. 15), section 8, 4) (p. 60).

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