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# NEW CLASSES OF THREE-DIMENSIONAL IDEAL MHD EQUILIBRIA

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# Abstract

In this paper, 6 ansatzes are investigated for their potential to allow three-dimensional (3-d) ideal magnetohydrodynamic (MHD) equilibria. The ansatzes are based on a Clebsch representation for the magnetic field, $\mathbf{B} = \nabla H \times \nabla k$, and a "generalized Clebsch representation", $\mathbf{B} = \nabla \times (\nabla K \times \nabla k)$, with $\nabla k$ being one of the coordinate directions of a cylindrical coordinate system. Three classes of equilibria, all with a straight magnetic axis, were obtained. Equilibria of the first class have a purely poloidal magnetic field of the Clebsch type with $k = z$ and include the 3-d equilibria already known. Equilibria of the two other classes have a purely toroidal (i.e. here longitudinal) magnetic field and pressure surfaces which can be chosen such that poloidal sections are closed. The second class is based on a Clebsch representation with $k = \theta$. Solutions contain a free function of $\theta$ which determines the poloidal sections of the pressure surfaces at, say, $z = 0$. The behaviour in the toroidal direction is then fixed but not periodic. For the third class the generalized Clebsch representation with $k = z$ is used. The equilibria are similar to those of the second class with two important differences. They contain no free function and field lines are not plane. Finally, 3-d vacuum fields, which exhibit 3-d magnetic surfaces, are presented. They have the same geometry as the equilibria of the third class and, in fact, can be obtained as a certain limit from these equilibria. Possible applications of the equilibria found are mentioned.

# 1. Introduction

The governing equations of ideal MHD equilibria are

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \nabla P \,, \\ \mathbf{j} &= \nabla \times \mathbf{B} \,, \end{aligned} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \,, \tag{2}$$

where $\mathbf{B}, \mathbf{j}$ and $P$ denote the magnetic field, current density and pressure, respectively. Solutions of Eqs. (1), (2) have a broad range of applications in thermonuclear fusion research, geophysics, solar and astrophysics. In most cases symmetric equilibria furnish a correct model or at least a satisfactory approximation of the true situation; in many cases, however, a 3-d model is desirable and in some cases, e.g. for stellarator-type configurations in fusion research, 3-d equilibria are absolutely necessary.

Yet, the construction of 3-d equilibria is notoriously difficult. This is because Eqs. (1), (2) represent a system of partial differential equations of mixed elliptic-hyperbolic type, for which there is no general theory on the existence and uniqueness of solutions available. It is not even clear what a well-posed initial-boundary value problem in general looks like. Only if additional assumptions are made can the problem be rendered tractable. If equilibria are required to exhibit plane, axial or helical symmetry, Eqs. (1), (2) can be reduced to a single quasilinear elliptic equation, for which an elaborate existence theory as well as many explicit solutions are known [1, 2, 3]. These symmetries, furthermore, are known to be the only continuous symmetries which allow such a reduction [4].

On the other hand, nonsymmetric magnetic fields in a bounded volume (e.g. a sphere or a torus) generically exhibit field line chaos connected with island formation and ergodic regions [5, 6, 7]. As a consequence of the equilibrium equations (1) magnetic lines lie entirely in surfaces of constant pressure. Chaotic field lines therefore do not seem to be compatible with a smooth pressure function and, consequently, 3-d fields are unlikely candidates for well-defined equilibria.

The subject has been more closely investigated for configurations of fusion interest. These have closed toroidal pressure surfaces nested around a single magnetic axis. The axisymmetric case is governed by the well-known Grad-Shafranov equation [8], which satisfactorily settles the existence problem. If axisymmetry is lost, however, only partial results are known. For example, if the magnetic axis lies in a plane, configurations with purely poloidal magnetic field (symmetric with respect to the plane) are virtually ruled out [9]. Other nonexistence results refer to quite special configurations such as the "isodynamic" [10] or the "quasihelical" [11] stellarator. In the general 3-d case with finite twist of field lines all we have is Grad's conjecture that well-defined equilibria do not exist [12]. From a practical point of view nonaxisymmetric fusion devices like stellarators obviously tolerate a certain amount of fine-scaled field line chaos. It is nevertheless of general interest whether Grad's nonexistence conjecture holds true or not.

Inspite of the negative expectations just mentioned, there are two classes of 3-d equilibria without continuous symmetries. The first is a toroidal configuration with purely toroidal magnetic field which is mirror symmetric with respect to a poloidal plane. The mirror symmetry prevents ergodization of field lines. The existence of such equilibria has been proved for low plasma pressure [13]. Note, however, that the pertinent pressure surfaces are in general not poloidally closed. Whether such equilibria having, additionally, closed toroidal pressure surfaces exist at all is not quite clear; only for configurations far from axisymmetry ("bumpy" configurations) is existence plausible [14]. The other class comprises equilibria with straight magnetic axis and field lines lying in planes orthogonal to the axis. Poloidal sections of the pressure surfaces exhibit mirror symmetry with respect to two, mutually orthogonal, planes and are either closed [15] or open [16]. In the closed case the ellipticity is an arbitrary function along the axis. In particular, a periodic function can be chosen, a case often referred to as "topological torus". Recently, the closed case was generalized so that mirror symmetry is lost [17]. To our knowledge, this is the first example of a 3-d equilibrium with the geometry of a (topological) torus without any symmetry.

As explained above, there is no general procedure available yet for constructing 3-d equilibria and, in order to make progress, one has to fall back on certain ansatzes for the magnetic field. The aim of this paper is to explore such an ansatz thoroughly and with some degree of completeness. As a result, we recovered the known straight axis equilibria and also found two new classes of 3-d equilibria. The ansatz is based on the poloidal/toroidal decomposition valid for any solenoidal vector field $\mathbf{B}$ (see, for example, [18]):

$$\mathbf{B} = \nabla H \times \nabla k + \nabla \times (\nabla K \times \nabla k). \tag{3}$$

Here, $H$ and $K$ are arbitrary scalar functions and $\nabla k$ is a particular (generalized) coordinate direction to be specified which can be completed to a nonsingular (generalized) coordinate system in the domain considered [19]. In most applications the decomposition (3) is used in Cartesian or spherical coordinates [18]; we used it in cylindrical coordinates $(r, \theta, z)$ instead. More precisely, we investigated the 6 ansatzes

$$\mathbf{B} = \nabla H \times \nabla k, \qquad k = r, \theta, z, \tag{4}$$

and

$$\mathbf{B} = \nabla \times (\nabla K \times \nabla k), \qquad k = r, \theta, z. \tag{5}$$

Note that the terms "poloidal" and "toroidal" have different meanings in the decomposition (3) (there (4) is called a toroidal field and (5) a poloidal one) and in torus (or cylinder) geometry. In order to avoid confusion, in the following (4) is called a Clebsch representation and (5) a generalized Clebsch representation.

It turned out that in 4 cases, namely for all Clebsch representations and for the generalized Clebsch representation with $k = z$, the problem can be reduced to a 2-d elliptic equation, perpendicular to the particular direction specified, and certain nonlinear compatibility conditions. In 3 out of these 4 cases nontrivial solutions of the reduced problem, i.e. 3-d equilibria, could be found. The first class is based on the Clebsch ansatz with $k = z$ and comprises the known equilibria [15, 16, 17]. In this context the extension of the open equilibria of [16] to the non-mirror-symmetric case is also discussed. The second class is based on the Clebsch ansatz with $k = \theta$. The magnetic field is purely toroidal and field lines lie on surfaces $\theta = const.$ The solution contains a free function of $\theta$ which determines the poloidal sections of the pressure surfaces at, say, $z = 0$. In particular, poloidal sections can be chosen to be closed; the behaviour in the toroidal direction is then fixed but not periodic. The third class uses the generalized Clebsch ansatz with $k = z$. The magnetic field is again purely toroidal but field lines are no longer plane. The solution does not contain a free function. The pressure surfaces again exhibit closed poloidal sections and are not periodic in the toroidal direction. Two points should be noted in connection with the third class of equilibria. First, it is an example of a 3-d equilibrium with non-planar field lines. So, plane field lines are no prerequisite of 3-d equilibria as could be assumed from the examples so far known. Second, the equilibrium has a well-defined vacuum limit; the vacuum field is also 3-d and has 3-d magnetic surfaces.

The first two classes of equilibria, based on the Clebsch ansatz, are discussed in section 2. The third class, based on the generalized Clebsch ansatz, is discussed in section 3. In section 4 some conclusions are drawn and possible applications of the new equilibria are mentioned. For the sake of completeness, the remaining ansatzes which did not lead to 3-d equilibria, are briefly dealt with in two appendices.

# 2. Clebsch representation

## 2.1 Ansatz $\mathbf{B} = \boldsymbol{\nabla} H \times \boldsymbol{\nabla} z$

With the ansatz

$$\mathbf{B} = \boldsymbol{\nabla} H \times \boldsymbol{\nabla} z \tag{6}$$

for the magnetic field $\mathbf{B}$ the divergence constraint (2) is automatically satisfied and the equilibrium equations (1) reduce to

$$\begin{aligned} (\Delta_{xy} H)\, \partial_x H + \partial_x P &= 0 \,, \\ (\Delta_{xy} H)\, \partial_y H + \partial_y P &= 0 \,, \end{aligned} \tag{7}$$

$$\partial_z \Big[ (\partial_x H)^2 + (\partial_y H)^2 + 2P \Big] = 0 \,. \tag{8}$$

Here, we work for convenience with Cartesian coordinates $(x, y, z)$, i.e. $H = H(x, y, z)$, $P = P(x, y, z)$, and $\Delta_{xy}$ denotes the 2-d Laplacian in the $x$-$y$ plane,

$$\Delta_{xy} = \partial^2_{xx} + \partial^2_{yy} \,. \tag{9}$$

The integrability condition for $P$, see Eqs. (7),

$$\partial_x \partial_y P - \partial_y \partial_x P = (\partial_x H)\, \partial_y \Delta_{xy} H - (\partial_y H)\, \partial_x \Delta_{xy} H = 0 \,, \tag{10}$$

requires that $\Delta_{xy} H$ satisfy locally the relation

$$\Delta_{xy} H = f(H, z) \tag{11}$$

with arbitrary profile function $f$. Equation (8) is solved by

$$P(x, y, z) = Q(x, y) - \frac{1}{2} \Big[ (\partial_x H)^2 + (\partial_y H)^2 \Big] \,, \tag{12}$$

where $Q(x,y)$ is an arbitrary function. Equations (7) yield

$$\begin{aligned} \partial_x Q &= (\partial_y H)\, \partial^2_{xy} H - (\partial_x H)\, \partial^2_{yy} H \,, \\ \partial_y Q &= (\partial_x H)\, \partial^2_{xy} H - (\partial_y H)\, \partial^2_{xx} H \,. \end{aligned} \tag{13}$$

In order to ensure that $Q$ does not depend on $z$, $H$ has to satisfy in addition the relations following from Eqs. (13):

$$\begin{aligned} \partial_z[\,(\partial_x H)\, \partial^2_{xy} H - (\partial_y H)\, \partial^2_{xx} H\,] &= 0 \,, \\ \partial_z[\,(\partial_y H)\, \partial^2_{xy} H - (\partial_x H)\, \partial^2_{yy} H\,] &= 0 \,. \end{aligned} \tag{14}$$

So, instead of solving Eqs. (7), (8) for $H$ and $P$ direct, one can first solve the 2-d elliptic problem (11) with arbitrarily given function $f$ and then try to satisfy the compatibility conditions (14) for the $z$-dependence of $H$.

Those few MHD equilibria known so far which depend on three space coordinates independently [15], [16], [17] not only fit into this scheme, with a linear ansatz for $f$,

$$f \;=\; a_0(z) \;+\; a_1(z)\, H \,, \tag{15}$$

with $a_0(z)$ and $a_1(z)$ being arbitrary, but naturally follow from it. In the next section further 3-d solutions are obtained from a different ansatz but with the same scheme. Since Eq. (11) with $f$ given by Eq. (15) is linear, the general solution is in the form of a sum of an inhomogeneous part $H_i$ and a homogeneous part $H_h$. The homogeneous part has to satisfy

$$\partial^2_{xx} H_h \;+\; \partial^2_{yy} H_h \;=\; a_1\, H_h \,. \tag{16}$$

A product ansatz with respect to $x$ and $y$ is made, while the parametric dependence on $z$ is kept without restrictions:

$$H_h \;=\; \alpha(x,z)\, \beta(y,z) \,. \tag{17}$$

Equation (16) then yields the conditions

$$\begin{gathered} \partial^2_{xx}\alpha \;=\; -\kappa^2(z)\, \alpha \,, \qquad \partial^2_{yy}\beta \;=\; -\lambda^2(z)\, \beta \,, \\ \kappa^2 \;+\; \lambda^2 \;=\; -a_1 \,, \end{gathered} \tag{18}$$

to which we return below. The general homogeneous solution is, eventually, an arbitrary superposition of solutions of type (17).

In dealing with an inhomogeneous solution of Eqs. (11), (15) a distinction has to be made with regard to $a_1$. This leads to the consideration of two classes of solutions of the full system of Eqs. (11), (14) and (15):

**Case I**

$a_1(z) = 0$: In this case an inhomogeneous solution of Eqs. (11), (15) is

$$H_i \;=\; a(z)\,x^2 \;+\; 2b(z)\,x\,y \;+\; c(z)\,y^2 \;, \tag{19}$$

where $a(z)$, $b(z)$ and $c(z)$ are arbitrary functions. (The relation $a + c = a_0/2$, which follows from Eq. (11), does not restrict $a$ or $c$, since $a_0$ is arbitrary.) Inserting the sum of this $H_i$ and the most general solution $H_h$ from Eqs. (17) and (18) in the compatibility conditions (14) produces rather unwieldy expressions. The simplest way out is to take $H_h = 0$, i.e.

$$H \;=\; H_i \;. \tag{20}$$

In this case Eqs. (14) merely yield the condition

$$[a(z)c(z) - b^2(z)]' \;=\; 0 \;, \tag{21}$$

where the prime denotes differentiation with respect to the single argument involved. Of the three coefficients $a(z)$, $b(z)$ and $c(z)$ this leaves two of them free. If Eq. (12) for the pressure is combined with Eqs. (13), (19) and (20), one obtains

$$P \;=\; P_0 \;-\; 2\,[a(z) \;+\; c(z)]\;[a(z)\,x^2 \;+\; 2b(z)\,x\,y \;+\; c(z)\,y^2] \;, \tag{22}$$

where $P_0 = const$ is the (arbitrary) pressure on the axis $x = y = 0$. Provided that $ac - b^2 > 0$, the surfaces of constant pressure have concentric elliptical cross-section. At fixed $P$ the half-axis ratio of the ellipse and the orientation of its major axis with respect to the x axis, for example, can be chosen as arbitrary functions along the axis. In the case $ac - b^2 < 0$ the cross-section consists of a pair of hyperbolas, thus leading to open pressure surfaces.

In cylindrical coordinates $(r, \theta)$, where $x = r\cos\theta$, $y = r\sin\theta$, the pressure assumes a particularly simple form, provided that $a(z)$, $b(z)$ and $c(z)$ are written in terms of two arbitrary functions $u(z)$, $v(z)$ and an arbitrary constant $c_1$, in the form

$$
\begin{aligned}
a(z) &= \frac{-c_1}{\sqrt{1-u^2}}\left[1-u(z)\cos v(z)\right], \\
b(z) &= \frac{c_1}{\sqrt{1-u^2}}\,u(z)\sin v(z), \\
c(z) &= \frac{-c_1}{\sqrt{1-u^2}}\left[1+u(z)\cos v(z)\right],
\end{aligned}
\tag{23}
$$

namely

$$
P = P_0 - 4c_1^2\frac{r^2}{1-u^2(z)}\{1 - u(z)\cos[2\theta - v(z)]\}. \tag{24}
$$

This form shows that $u(z)$ and $v(z)$ correspond to the deviation of the cross-section from a circle ($u = 0$) and to a helical twist, respectively. For $u^2 < 1$ the cross-sections are ellipses. An example is shown in Fig. 1. The functions $u(z)$ and $v(z)$ were taken arbitrarily as $u(z) = 0.6 + 0.1\sin 2z$ and $v(z) = 0.2(z - 3.5)$. The $z$ axis extends from $-1$ to $3.5$. Two nested pressure surfaces with $\hat{P} \equiv (P - P_0)/(4c_1^2) = -1.44$ (inner surface) and $-4$ (outer surface) are presented. (In the figure captions, for the sake of simplicity, hats are omitted.) The grid lines which are closed around the axis coincide with magnetic field lines. They are plane ellipses of constant ellipticity at fixed value of $z$.

An open pressure surface, requiring $c_1^2 < 0$, is shown in Fig. 2. It has $u(z) = -6 - 0.3\sin z$, $v = 0$ and $\hat{P} = 9$. The $z$ axis in the middle extends from $-14$ to $+14$.

The MHD equilibria with magnetic field $\mathbf{B}$ and pressure $P$, given by Eqs. (6), (19), (20) and by (22) or (24), respectively, were obtained in [17], while the subcase $b(z) = 0$ (or $v(z) = 0$) was already discovered in [15]. Finally, a nontrivial harmonic polynomial for $H_h$ either does not give a physically distinct solution or does not solve Eqs. (14), as discussed in [17].

**Case II**

$a_1(z) \neq 0$: In this case an inhomogeneous solution of Eq. (11) is

$$
H_i = -\,a_0(z)/a_1(z). \tag{25}
$$

According to Eqs. (6), (7) and (8) neither the magnetic field nor the pressure is changed if an arbitrary function of $z$ is added to $H$. Nothing is lost, therefore, by putting $a_0 = H_i = 0$. Hence, from Eq. (17), it follows that

$$
H = \alpha(x, z)\,\beta(y, z), \tag{26}
$$

with $\alpha$ and $\beta$ subject to Eqs. (18). A further restriction follows from the compatibility conditions (14), namely

$$(\partial_x \alpha)^2 \;=\; -\kappa^2\,\alpha^2\;, \qquad (\partial_y \beta)^2 \;=\; -\lambda^2\,\beta^2\;. \tag{27}$$

For real-valued $H$ Eqs. (18) and (27) are only compatible provided that $\kappa^2 < 0$, $\lambda^2 < 0$ and $\alpha = \alpha_0(z)\,\exp(\pm|\kappa(z)|x)$ and an analogous $\beta$ (disregarding solutions with $\kappa = 0$ or $\lambda = 0$, which are independent of $x$ or $y$, respectively). It turns out, however, that the pressure depends on $x$ and $y$ in the combination $|\kappa|\,x \pm |\lambda|\,y$, so that the cross-sections of the surfaces of constant pressure are not closed curves. Physically more relevant solutions are obtained if the ansatz (26) is replaced by an ansatz containing the sum of two less general product terms instead of one more general product term:

$$H \;=\; a(z)g(x) \;+\; c(z)h(y)\;. \tag{28}$$

In this case one obtains from Eqs. (18) and (14)

$$g''(x) \;=\; -\kappa^2 g\;, \qquad h''(y) \;=\; -\kappa^2 h\;, \qquad [a(z)c(z)]' \;=\; 0\;, \tag{29}$$

where $\kappa^2 \;=\; -a_1$ is now an arbitrary constant. For positive as well as negative $\kappa^2$ there exist solutions of Eq. (29) with closed poloidal contours of $P$ and with one free function of $z$, say $a(z)$; these solutions correspond to the second class in Ref. [15]. For $\kappa^2 > 0$, for example, pressure surfaces which are closed around the axis at $x = y = 0$ are obtained for $c(z) = 1/a(z)$ and

$$g(x) \;=\; \frac{c_1}{\kappa}\,\cos(\kappa x)\;, \qquad h(y) \;=\; \frac{c_1}{\kappa}\,\cos(\kappa y)\;, \tag{30}$$

where $c_1$ is an arbitrary constant. In this case

$$\begin{aligned} P \;=\; P_0 \;-\; c_1^2\,\{\,a^2(z)\sin^2(\kappa x) \;+2\,[1 \;-\; \cos(\kappa x)\,\cos(\kappa y)] \\ +\; c^2(z)\sin^2(\kappa y)\,\}/2\;. \end{aligned} \tag{31}$$

An example is shown in Fig. 3, where $a(z) = 0.6 + 0.1\sin 2z$ and $\kappa = 1$ is chosen. $\hat{P} \equiv 2(P - P_0)/c_1^2$ is $-1.96$ (inner surface) and $-3.8025$ (outer surface). The poloidally closed grid lines again coincide with magnetic field lines. It is evident that farther away from the axis they deviate from an elliptical shape. In fact, since the pressure surfaces repeat periodically in $x$ and $y$, the field lines tend towards a rectangular shape at the separatrices $x,\ y = (2n+1)\pi/2,\ n = 0, \pm 1, \pm 2, \cdots$. The pressure $\hat{P}$ tends toward $\hat{P}_s = -(1+a^2)^2/a^2$ there.

In the case $\kappa^2 < 0$ results analogous to those of Eqs. (30) and (31) are obtained. The trigonometric functions are replaced by hyperbolic functions [15]. There again exist pressure surfaces which close around the axis.

## 2.2 Ansatz $\mathbf{B} = \nabla H \times \nabla\theta$

Here, the following ansatz is made for the magnetic field:

$$\mathbf{B} = \nabla H \times \nabla\theta \,. \tag{32}$$

It is practical to switch from polar coordinates $(r, \theta, z)$ to modified polar coordinates $(\rho, \theta, z)$ where $\rho = r^2/2$. We thus consider $H = H(\rho, \theta, z)$ and, similarly for the pressure, $P = P(\rho, \theta, z)$. The MHD equations take the form

$$\begin{aligned} \frac{1}{2\rho}(\Delta_{\rho z} H)\,\partial_\rho H \;+ \partial_\rho P &= 0\,, \\ \frac{1}{2\rho}(\Delta_{\rho z} H)\,\partial_z H \;+ \partial_z P &= 0\,, \end{aligned} \tag{33}$$

$$\partial_\theta \left[ (\partial_\rho H)^2 + \frac{1}{2\rho}(\partial_z H)^2 + 2P \right] = 0\,, \tag{34}$$

where $\Delta_{\rho z}$ denotes the 2-d Stokes operator in the $\rho$-$z$ plane,

$$\Delta_{\rho z} = 2\rho\,\partial^2_{\rho\rho} + \partial^2_{zz}\,. \tag{35}$$

The analysis of Eqs. (33) and (34) proceeds analogously to that in the previous section. The integrability condition $\partial_\rho\partial_z P - \partial_z\partial_\rho P = 0$ for $P$ yields

$$\Delta_{\rho z} H = 2\rho\, f(H, \theta) \tag{36}$$

with arbitrary profile function $f$. Equation (34) is solved by

$$P(\rho, \theta, z) = Q(\rho, z) - \frac{1}{2}\left[ (\partial_\rho H)^2 + \frac{1}{2\rho}(\partial_z H)^2 \right]\,, \tag{37}$$

where $Q(\rho, z)$ is an arbitrary function. Equations (33) yield

$$\begin{aligned} \partial_\rho Q &= \frac{1}{2\rho}\left[ (\partial_z H)\,\partial^2_{\rho z} H - (\partial_\rho H)\,\partial^2_{zz} H \;-\; \frac{1}{2\rho}(\partial_z H)^2 \right]\,, \\ \partial_z Q &= (\partial_\rho H)\,\partial^2_{\rho z} H - (\partial_z H)\,\partial^2_{\rho\rho} H\,. \end{aligned} \tag{38}$$

In order to ensure that $Q$ does not depend on $\theta$, $H$ has to satisfy the relations

$$\begin{aligned} \partial_\theta \left[ (\partial_z H)\,\partial^2_{\rho z} H - (\partial_\rho H)\,\partial^2_{zz} H \;-\; \frac{1}{2\rho}(\partial_z H)^2 \right] &= 0\,, \\ \partial_\theta[(\partial_\rho H)\,\partial^2_{\rho z} H - (\partial_z H)\,\partial^2_{\rho\rho} H] &= 0\,. \end{aligned} \tag{39}$$

By analogy with Eq. (15) the following linear ansatz for $f$ is made:

$$f \;=\; a_0(\theta) \;+\; a_1(\theta)\, H \;, \tag{40}$$

with arbitrary functions $a_0(\theta)$ and $a_1(\theta)$. Again, we seek solutions of Eqs. (36), (40) as the sum of an inhomogeneous solution $H_i$ and homogeneous solutions $H_h$. The latter have to satisfy

$$2\rho\, \partial^2_{\rho\rho} H_h \;+\; \partial^2_{zz} H_h \;=\; 2\rho\, a_1\, H_h \;. \tag{41}$$

The product ansatz

$$H_h \;=\; \alpha(\rho,\theta)\, \beta(z,\theta) \tag{42}$$

yields the conditions

$$\partial^2_{\rho\rho}\alpha \;=\; -\kappa^2(\rho,\theta)\, \alpha \;, \qquad \partial^2_{zz}\beta \;=\; -\lambda^2(\theta)\, \beta \;. \tag{43}$$

While $\lambda(\theta)$ is arbitrary, the dependence of $\kappa$ on $\rho$ is fixed and is given by

$$2\rho\, \kappa^2 \;+\; \lambda^2 \;=\; -2\rho\, a_1 \;. \tag{44}$$

Postponing the treatment of Eqs. (43), we again consider the two possible cases with respect to $a_1$.

**Case I**

$a_1(\theta) = 0$: In this case an inhomogeneous solution of Eqs. (36), (40) is

$$H_i \;=\; \frac{a_0(\theta)}{2}\, \rho^2 \;. \tag{45}$$

In contrast to the previous section, here $H_i$ does not depend on all three coordinates. It is possible, however, to bring in the missing $z$ dependence via a simple solution of the homogeneous part. For $\lambda = 0$ $\kappa$ also vanishes and Eqs. (43) are trivial to solve. The sum of $H_h$ and $H_i$ is then a polynomial of the form

$$H \;=\; \frac{a_0(\theta)}{2}\, \rho^2 \;+\; a(\theta)\rho \;+\; b(\theta) z \;+\; c(\theta) z\rho \;, \tag{46}$$

where $a$, $b$ and $c$ are arbitrary functions of $\theta$. (An irrelevant additional term which depends on $\theta$ alone was omitted.) From the nonlinear compatibility conditions (39) one obtains

$$\partial_\theta\,(\,b\,a_0 \;-\; a\,c \;-\; c^2\,z\,) \;=\; 0\;,$$
$$\partial_\theta\,(\,b^2 \;-\; c^2\rho^2\,) \;=\; 0\;. \tag{47}$$

This finally yields $b = const$, $c = const$ and

$$b\,a_0'(\theta) \;-\; c\,a'(\theta) \;=\; 0\;. \tag{48}$$

Thus, the solution in general contains one free function $a_0(\theta)$ or $a(\theta)$. In conformity with the switched roles of $z$ and $\theta$ in relation to the last section, the freedom in the present configuration refers to the angular dependence around the axis and not to the dependence along it.

From these results for $H$ and from Eqs. (37) and (38) one obtains for the pressure

$$P \;=\; P_0 \;-\; \frac{a^2(\theta)}{2} \;-\; a_0(\theta)\left[\frac{a_0(\theta)}{2}\rho^2 \;+\; a(\theta)\rho \;+\; bz \;+\; cz\rho\right]\;, \tag{49}$$

where $P_0$ is an arbitrary constant. The terms in the brackets are nothing but $H$, so that with Eq. (32) $\mathbf{B}\cdot\boldsymbol{\nabla}P = 0$ is confirmed. On switching back to the radial variable $r$ the magnetic field components $(B_r, B_\theta, B_z)$ are found to be

$$\begin{aligned}
B_r &= -\frac{b}{r} - \frac{cr}{2}\;,\\
B_\theta &= 0\;,\\
B_z &= \frac{a_0(\theta)}{2}r^2 \;+\; a(\theta) \;+\; cz\;.
\end{aligned} \tag{50}$$

In order to make the magnetic field nonsingular on the axis, it is necessary to have $b = 0$. According to Eq. (48) this implies $a(\theta) = const$. In consequence, the pressure on the axis $\rho \equiv r^2/2 = 0$ is independent of the polar angle $\theta$, as it should be. $a_0(\theta)$ is the remaining free function of the solution. The components of the current density $\mathbf{j}$ are

$$j_r \;=\; r a_0'(\theta)/2\;,\quad j_\theta \;=\; -r a_0(\theta)\;,\quad j_z \;=\; 0\;. \tag{51}$$

The surfaces of constant pressure are most easily discussed by solving Eq. (49) for $\hat{z} \equiv a + cz$. With $P_{00} = P_0 - a^2/2$ being the pressure on the $z$ axis, one has

$$\hat{z}(r,\theta) \;=\; -\frac{1}{4a_0(\theta)}\left[8\frac{P - P_{00}}{r^2} \;+\; a_0^2(\theta)r^2\right]\;. \tag{52}$$

If the pressure decreases away from the axis, i.e. $P < P_{00}$, then $\hat{z}(r)$ at fixed $\theta$ is a monotonic function of $r$, ranging from $-\infty$ to $\infty$. These limits are reached for

$r \to 0$ or $r \to \infty$, respectively (depending on the sign of $a_0$). At fixed $z = const$ the distance $r$ to the axis is univalued and for periodic $a_0(\theta)$ the curves $r(\theta)$ are closed. Thus, the surfaces $P = const$ wrap around the axis and are more and more radially compressed in one $z$ direction and blow up in the opposite direction. This feature is rather unphysical globally. With proper boundary conditions at two values of $z$, however, the solution could be physically relevant for the region in between.

An example of this configuration is given in Fig. 4. The pressure $\hat{P} \equiv 8(P - P_{00})$ on the two surfaces shown, nested around the $z$ axis, has the values $-0.4$ (inner surface) and $-0.9$ (outer surface). The free function $a_0(\theta)$ is chosen as $a_0(\theta) = -1 + 0.1\cos\theta + 0.1\cos 2\theta$. $\hat{z}$ extends from $-2.6$ to $0.6$. The longitudinal grid lines correspond to constant values of the poloidal angle $\theta$ and thus coincide with magnetic field lines.

If the pressure increases away from the axis, $\hat{z}(r)$ is no longer monotonic. Starting from small values of $r$, $|\hat{z}(r)|$ first decreases and then increases again as $r$ goes to $\infty$. In other words, the pressure surfaces start close to the axis, then increase their distance from it until at some point $z = z_0$ they turn around and, still increasing in distance from the axis, continue back in the direction from which they started. This gives a kind of arcade-like surface with a specific "head" or turn-around region. Figure 5 shows an example of such a configuration. It has $\hat{P} = 0.8$ and $a_0(\theta) = 1 + 0.2\cos\theta + 0.2\cos 2\theta$. The distance to the axis extends from 0.38 to 2.1. Again, the longitudinal grid lines correspond to magnetic field lines.

It remains to discuss the case $\lambda(\theta) \neq 0$. Here, the regular solution of Eq. (43) is

$$\begin{aligned} \alpha(\rho,\theta) &= \sqrt{\rho}\, I_1\left(\lambda(\theta)\sqrt{2\rho}\right) \ , \\ \beta(z,\theta) &= c_1(\theta)\sin\left(\lambda(\theta)\, z\right) \ + \ c_2(\theta)\cos\left(\lambda(\theta)\, z\right) \ , \end{aligned} \tag{53}$$

for $\lambda^2 > 0$. The coefficients $c_1(\theta)$, $c_2(\theta)$ are arbitrary functions, and $I_1$ is a modified Bessel function. For $\lambda^2 < 0$, $\lambda$ and $I_1$ of Eqs. (53) are replaced by $|\lambda|$ and $J_1$, respectively. Also, the trigonometric functions are replaced by hyperbolic ones. $H(\rho,\theta,z)$ is the sum of $H_h$, Eq. (42), and $H_i$, Eq. (45). It turns out, however, that the compatibility conditions (39) can only be satisfied provided that $a_0(\theta)$, $\lambda(\theta)$, $c_1(\theta)$ and $c_2(\theta)$ are constant. This implies an axisymmetric and hence, in the present context, irrelevant solution.

**Case II**

$a_1(z) \neq 0$: In this case an inhomogeneous solution of Eq. (36) is $H_i = -a_0(\theta)/a_1(\theta)$. Again, as in the last section, nothing is lost if $H_i = H_i(\theta)$ is omitted by putting $a_0(\theta) = 0$. Thus, $H = H_h$ is given by the ansatz (42) or, as will again prove useful, as the sum of two such product terms. A case for which Eqs. (39) can readily be analyzed is $\lambda = 0$. From Eqs. (43), (44) it follows that

$$\partial^2_{\rho\rho}\alpha \;=\; a_1(\theta)\,\alpha\;, \tag{54}$$

and $\beta(\theta, z) = b(\theta) + c(\theta)z$, where $b$ and $c$ are arbitrary functions of $\theta$. The equations (39), however, can only be satisfied provided that $H$ – and, consequently, the total configuration – is independent of $\theta$ or $z$. A three-dimensional solution, however, can in fact be found from the two-term ansatz

$$H \;=\; \beta_1(\theta, z)\,\sin(\kappa\rho) \;+\; \beta_2(\theta, z)\,\cos(\kappa\rho)\;, \tag{55}$$

where $\kappa = \sqrt{-a_1}$, for $a_1 < 0$, and

$$H \;=\; \beta_1(\theta, z)\,\sinh(|\kappa|\rho) \;+\; \beta_2(\theta, z)\,\cosh(|\kappa|\rho)\;, \tag{56}$$

for $a_1 > 0$. Here

$$\beta_i(\theta, z) \;=\; b_i(\theta) \;+\; c_i z\;, \quad i \;=\; 1,\,2\;. \tag{57}$$

The functions $b_1$, $b_2$, $c_1$, $c_2$ are arbitrary. From Eqs. (39), (55) one obtains

$$\begin{aligned}
&\partial_\theta\{\,[\,c_1^2 - c_2^2 + 4c_1c_2\kappa\rho\,]\cos(2\kappa\rho) \;+\; 2[\,(c_1^2 - c_2^2)\kappa\rho - 2c_1c_2\,]\sin(2\kappa\rho)\\
&\qquad -\;(c_1^2 + c_2^2)\,\} \;=\; 0\;,\\
&\partial_\theta\{\,a_1\,[\,b_1c_1 + b_2c_2 + (c_1^2 + c_2^2)\,z\,]\,\} \;=\; 0\;.
\end{aligned} \tag{58}$$

In consequence, $\kappa$, $a_1$ and $c_1$, $c_2$ have to be constant, and $b_1(\theta)$, $b_2(\theta)$ are related by

$$c_1\,b_1'(\theta) \;+\; c_2\,b_2'(\theta) \;=\; 0\;. \tag{59}$$

The pressure is found from Eqs. (37), (38) to be

$$P \;=\; P_0 \;+\; \frac{\kappa^2}{2}\left\{\,[\,\beta_1\sin(\kappa\rho) + \beta_2\cos(\kappa\rho)\,]^2 - b_1^2 - b_2^2\right\}\;, \tag{60}$$

where $P_0$ is an arbitrary constant. Since $P - P_0$ is proportional to $H^2 - b_1^2(\theta) - b_2^2(\theta)$, the condition $\mathbf{B}\cdot\boldsymbol{\nabla}P = 0$, using Eq. (32), is again confirmed. The magnetic field components are

$$\begin{aligned} B_r &= -\frac{1}{r}\left[\, c_1 \sin\Phi \;+\; c_2 \cos\Phi \,\right] \,, \\ B_\theta &= 0 \,, \\ B_z &= \kappa \left[\, \beta_1 \cos\Phi \;-\; \beta_2 \sin\Phi \,\right] \,, \end{aligned} \tag{61}$$

where $\Phi(r) = \kappa r^2/2$. The current density components are

$$\begin{aligned} j_r &= \frac{\kappa}{r}\left[\, b_1' \cos\Phi \;-\; b_2' \sin\Phi \,\right] \,, \\ j_\theta &= \kappa^2 r \left[\, \beta_1 \sin\Phi \;+\; \beta_2 \cos\Phi \,\right] \,, \\ j_z &= 0 \,. \end{aligned} \tag{62}$$

The field is nonsingular on the axis for $c_2 = 0$. In this case Eq. (59) implies $b_1'(\theta) = 0$. The remaining free function is $b_2(\theta)$. The pressure on axis is then $P_{00} = P_0 - \kappa^2 b_1^2/2$.

For $a_1 > 0$ the plus sign in Eq. (59) changes into a minus sign and the pressure assumes the form

$$P \;=\; P_0 \;-\; \frac{\kappa^2}{2}\left\{ \left[\, \beta_1 \sinh(|\kappa|\rho) + \beta_2 \cosh(|\kappa|\rho) \,\right]^2 - b_1^2 + b_2^2 \right\} \,. \tag{63}$$

The magnetic field is analogous to Eq. (61) with the minus sign in $B_z$ changed into a plus sign, and $\kappa$ replaced by $|\kappa|$.

The surfaces of constant pressure $z = z(r,\theta)$ are easily discussed in terms of $Z \equiv (b_1 + c_1 z)/b_2(\theta)$. From Eq. (60) and $c_2 = 0$ one obtains

$$Z \;=\; \frac{U - \cos\Phi}{\sin\Phi} \,, \tag{64}$$

where

$$U(\theta) \;=\; \sqrt{1 + 2\,\frac{P - P_{00}}{\kappa^2 b_2^2(\theta)}} \,. \tag{65}$$

Similarly to the situation in Eq. (31), the periodicity of the trigonometric functions brings about an infinity of solution branches. Considering the branch which comes close to the axis, $\Phi \to 0$, it is evident from $\cos^2\Phi \le 1$ that, for fixed $\theta$, the solution extends from $z = -\infty$ to $+\infty$ provided that $0 < |U| < 1$. These limits are reached for $r \to 0$ and $r \to \sqrt{2\pi/\kappa}$, respectively (depending on the sign of $U-1$). At fixed $z = const$ the distance $r$ to the axis is univalued within the branch and for periodic $b_2(\theta)$ the curves $r(\theta)$ are closed around the $z$ axis. Thus, the surfaces $P = const$ wrap around the axis. They are more and more radially compressed in one $z$ direction but reach a constant extent in the opposite direction.

An example of this configuration is given in Fig. 6. The normalized pressure $\hat{P} \equiv 2(P - P_{00})/(\kappa b_1)^2$ on the two surfaces shown has the values $-0.1$ and $-0.5$ on the inner and the outer surface, respectively. The normalized free function $\hat{b}_2(\theta) \equiv b_2(\theta)/b_1$ is chosen as $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.1\cos 2\theta$ and $\kappa = \sqrt{2}$ is assumed. $\hat{z} = 1 + c_1 z/b_1$, on the axis, extends from $-2.6$ to $1.3$. Again, the longitudinal grid lines represent magnetic field lines.

For $|U| > 1$ there is no sign change of $Z$ in the $r$ interval quoted. $|Z|$, as a function of $r$, first decreases and then increases again as $r$ goes from small values to its final value. This gives an arcade-like surface as also encountered in case I solutions. Figure 7 shows an example of this configuration. It corresponds to the choices $\hat{P} = 0.2$, $\kappa = \sqrt{2}$ and $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.1\sin 2\theta$. The region $r = 0.3$ to $1.6$ is shown.

For $a_1 > 0$ the trigonometric functions in Eq. (64) are replaced by hyperbolic ones. The pressure surfaces, instead of approaching the axis, blow up towards $r \to \infty$ at a finite value of $z$. An example is shown in Fig. 8. It corresponds to $\hat{P} = 0.2$, $\kappa = \sqrt{2}$ and $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.05\cos 3\theta$. The region shown extends from $r = 0.3$ to $2.6$.

It remains to consider the postponed case $\lambda(\theta) \neq 0$ as well. $H(\rho, \theta, z)$ assumes the form

$$H \;=\; \alpha_1(\rho,\theta)\,\sin\left(\lambda(\theta)\,z\right) \;+\; \alpha_2(\rho,\theta)\,\cos\left(\lambda(\theta)\,z\right)\;, \tag{66}$$

where $\alpha_{1,2}(\rho, \theta)$, from Eqs. (43), (44), are confluent hypergeometric functions of $\rho$, with $\theta$ as a parameter. The analysis of Eqs. (39), focussed in particular on terms with secular $z$ dependence, shows that only axisymmetric solutions, i.e. $\partial_\theta(\alpha_1,\ \alpha_2,\ \lambda) = 0$, survive the compatibility conditions, just as with the Bessel function analogue of case I.

# 3. Generalized Clebsch representation

In this section the ansatz

$$\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} K \times \boldsymbol{\nabla} z) \tag{67}$$

is investigated. We are again working in Cartesian coordinates $(x, y, z)$, i.e. $K = K(x, y, z)$, $P = P(x, y, z)$, and $\Delta$ and $\Delta_{xy}$ denote the 3-d and the 2-d Laplacian in the $x$-$y$ plane, respectively. The Cartesian components of the magnetic field $\mathbf{B}$ and current density $\mathbf{j}$ read

$$\mathbf{B} = (\partial^2_{xz} K,\ \partial^2_{yz} K,\ -\Delta_{xy} K)^\top \tag{68}$$

and

$$\mathbf{j} = (-\Delta \partial_y K,\ \Delta \partial_x K,\ 0)^\top. \tag{69}$$

The divergence constraint (2) is, of course, satisfied by the ansatz and the equilibrium equations (1) reduce to

$$\begin{aligned} (\Delta_{xy} K)\partial_x(\Delta_{xy} K + \partial^2_{zz} K) + \partial_x P = 0, \\ (\Delta_{xy} K)\partial_y(\Delta_{xy} K + \partial^2_{zz} K) + \partial_y P = 0, \end{aligned} \tag{70}$$

$$(\partial^2_{xz} K)\partial_x(\Delta_{xy} K + \partial^2_{zz} K) + (\partial^2_{yz} K)\partial_y(\Delta_{xy} K + \partial^2_{zz} K) + \partial_z P = 0. \tag{71}$$

The integrability condition $\partial_x \partial_y P - \partial_y \partial_x P = 0$ for the pressure $P$ together with Eqs. (70) furnish, similarly to the Clebsch cases in the last section, a relation between $\Delta_{xy} K$ and $\partial^2_{zz} K$ with an arbitrary profile function $f$:

$$\Delta_{xy} K = f(\partial^2_{zz} K, z). \tag{72}$$

Here, $\partial^2_{zz} K \not\equiv 0$ is assumed, the case $\partial^2_{zz} K \equiv 0$ being dealt with later on. Using Eq. (72), one can integrate Eqs. (70) and obtain

$$f^2(\partial^2_{zz} K, z) + 2\, F(\partial^2_{zz} K, z) + 2\, P = C(z) \tag{73}$$

with $C(z)$ being an arbitrary function of $z$, and $F$ an integral function of $f$,

$$F(u, z) := \int_0^u f(\tilde{u}, z)\, d\tilde{u}. \tag{74}$$

Equation (71) can be put in the form

$$\left(1 + \partial_u f|_{u=\partial^2_{zz} K}\right) \partial_z \left[(\partial^2_{xz} K)^2 + (\partial^2_{yz} K)^2\right] + 2\, \partial_z P = 0. \tag{75}$$

Differentiating Eq. (73) with respect to $z$, one can eliminate $\partial_z P$ from Eqs. (73), (75):

$$\left(1 + \partial_u f|_{u=\partial^2_{zz}K}\right) \partial_z\left[(\partial^2_{xz}K)^2 + (\partial^2_{yz}K)^2\right] - \partial_z\left(f^2 + 2\,F\right) = -C'. \qquad (76)$$

Analogously to the last section, Eqs. (70), (71) are replaced by Eqs. (72), (76), which determine the functions $C$ and $K$ once the profile function $f$ is given. The pressure $P$ is then given by Eq. (73).

In order to proceed, a linear profile function

$$f = a_0(z) + a_1(z)\,\partial^2_{zz}K \qquad (77)$$

is once more chosen. In the case $a_1 = 0$, Eq. (72) is again a 2-d elliptic equation and Eq. (76) can be integrated once with respect to $z$:

$$(\partial^2_{xz}K)^2 + (\partial^2_{yz}K)^2 - 2\,a_0(z)\,\partial^2_{zz}K = \bar{C}(z) + Q(x,y). \qquad (78)$$

Here, $\bar{C}(z) := a_0^2 - C$ and $Q(x,y)$ are arbitrary functions. The pressure function is now given by

$$P = -a_0(z)\,\partial^2_{zz}K - \frac{1}{2}\bar{C}(z). \qquad (79)$$

The general solution of Eq. (72) has the form

$$K = a(z)\,x^2 + c(z)\,y^2 + K_{harm}(x,y;z), \qquad (80)$$

with $2(a+c) = a_0$ and $K_{harm}$ being a harmonic function in $x$ and $y$ that, additionally, may depend on $z$. For $K_{harm}$, second and some higher-order harmonic polynomials in $x$ and $y$ with $z$ dependent coefficients have been tried. However, no 3-d solutions satisfying Eq. (78) with poloidally closed pressure surfaces have been found.

In the case $a_1 \neq 0$ the coefficient $a_0$ can again be made zero; furthermore, $a_1$ is assumed to be independent of $z$ and $\neq -1$. In that case Eq. (76) can again be integrated,

$$(\partial^2_{xz}K)^2 + (\partial^2_{yz}K)^2 - a_1\,(\partial^2_{zz}K)^2 = \tilde{C}(z) + Q(x,y), \qquad (81)$$

and the pressure function $P$ takes the form

$$P = -\frac{1+a_1}{2}\left(a_1\,(\partial^2_{zz}K)^2 + \tilde{C}(z)\right), \qquad (82)$$

where $\tilde{C}(z) = -C(z)/(1+a_1)$.

The ansatz for $K$,

$$K = a(z)\,g(x) + c(z)\,h(y), \tag{83}$$

is now found to be successful. Equation (72) furnishes the equations

$$\begin{aligned} \frac{g''(x)}{g} &= a_1 \frac{a''(z)}{a} = -\kappa^2, \\ \frac{h''(y)}{h} &= a_1 \frac{c''(z)}{c} = -\lambda^2, \end{aligned} \tag{84}$$

which for $a_1 < 0$, for example, are solved by

$$\begin{aligned} g(x) &= g_0 \cos \kappa x, \quad a(z) = \exp \bar{\kappa} z, \quad a_1 \bar{\kappa}^2 = -\kappa^2, \\ h(y) &= h_0 \cos \lambda y, \quad c(z) = \exp(-\bar{\lambda} z), \quad a_1 \bar{\lambda}^2 = -\lambda^2. \end{aligned} \tag{85}$$

Condition (81) is satisfied if $\bar{\kappa} = \bar{\lambda}$ (and hence $\kappa = \pm\lambda$) and if one takes

$$\begin{aligned} \tilde{C}(z) &:= \kappa^2 \bar{\kappa}^2 \left( g_0^2 \exp 2\bar{\kappa} z + h_0^2 \exp(-2\bar{\kappa} z) \right) + \tilde{C}_0, \\ Q(x,y) &:= 2\,\kappa^2 \bar{\kappa}^2\, g_0 h_0 \cos \kappa x \cos \kappa y - \tilde{C}_0, \end{aligned} \tag{86}$$

where $\tilde{C}_0$ is an arbitrary constant. Inserting (85) in (83), one obtains

$$K = g_0 \exp \bar{\kappa} z \cos \kappa x + h_0 \exp(-\bar{\kappa} z) \cos \kappa y, \tag{87}$$

and the pressure takes the form

$$\begin{aligned} P &= \frac{1 + a_1}{2} \left( \kappa^2 \bar{\kappa}^2\, K^2 - \tilde{C} \right) \\ &= P_0 - \tfrac{1}{2} \kappa^2 (\bar{\kappa}^2 - \kappa^2) [g_0^2 \exp 2\bar{\kappa} z \sin^2 \kappa x + h_0^2 \exp(-2\bar{\kappa} z) \sin^2 \kappa y \\ &\quad + 2 g_0 h_0 (1 - \cos \kappa x \cos \kappa y)], \end{aligned} \tag{88}$$

where $P_0 = (\bar{\kappa}^2 - \kappa^2)(g_0 h_0 \kappa^2 - \tilde{C}_0/2\bar{\kappa}^2$ is the pressure on axis.

For $|\kappa x|, |\kappa y| << 1$ the cross-sections $z = const$ of the pressure surfaces are elliptical, provided that $g_0 h_0 > 0$, otherwise hyperbolical. For $z \to \pm\infty$ they shrink to a vertical line on one side and to a horizontal line on the other.

The magnetic field components are

$$\begin{aligned} B_x &= -\kappa \bar{\kappa} g_0 \exp \bar{\kappa} z \sin \kappa x, \\ B_y &= \kappa \bar{\kappa} h_0 \exp(-\bar{\kappa} z) \sin \kappa y, \\ B_z &= \kappa^2 \left( g_0 \exp \bar{\kappa} z \cos \kappa x + h_0 \exp(-\bar{\kappa} z) \cos \kappa y \right), \end{aligned} \tag{89}$$

while the components of the current density are

$$j_x = \kappa(\bar{\kappa}^2 - \kappa^2) h_0 \exp(-\bar{\kappa} z) \sin \kappa y,$$
$$j_y = -\kappa(\bar{\kappa}^2 - \kappa^2) g_0 \exp \bar{\kappa} z \sin \kappa x, \tag{90}$$
$$j_z = 0.$$

Similarly to the configuration in (61), (62) the current is purely poloidal. The magnetic field lines, however, are not plane any more. This follows from, for example, the integration of the field line equations $dx/B_x = dy/B_y = dz/B_z$, which, for $|\bar{\kappa} z| >> 1$, can be done analytically. For $\bar{\kappa} z \to \infty$ and $\bar{\kappa} z \to -\infty$ the field lines approach $x \to 0$, $y \to const$ and $y \to 0$, $x \to const$, respectively. The connection between these asymptotic values requires a field line twist.

Figure 9 shows two nested pressure surfaces. Also shown as thicker lines are three field lines, two of them at the symmetry positions $x = 0$ and $y = 0$ and one in between. Note that the "longitudinal" grid lines in *this* figure are generated by some auxiliary construction and do not correspond to constant polar angles in the $x$-$y$ plane or to field lines. Figure 9 is drawn with $g_0 = h_0 = 1$, $\kappa = 1$, $\bar{\kappa} = 0.5$, $z \in [-2.5, 2.5]$. The normalized pressure $\hat{P} = 2(P - P_0)/(\kappa^2(\bar{\kappa}^2 - \kappa^2))$ has the values $\hat{P} = -0.2$ and $-0.7$.

The case $a_1 = -1$ has been excluded so far. In fact, Eq. (87) with $\kappa = \bar{\kappa}$ describes a 3-d vacuum field, which exhibits, moreover, 3-d magnetic surfaces $S(x, y, z) = const$, i.e.

$$\mathbf{B} \cdot \nabla S = 0, \tag{91}$$

where $S$ is given by

$$\begin{aligned} S &= \kappa^4 \left( K^2 - g_0^2 \exp 2\kappa z - h_0^2 \exp(-2\kappa z) \right) \\ &= g_0^2 \exp 2\bar{\kappa} z \sin^2 \kappa x + h_0^2 \exp(-2\bar{\kappa} z) \sin^2 \kappa y \\ &\quad + 2 g_0 h_0 (1 - \cos \kappa x \cos \kappa y). \end{aligned} \tag{92}$$

Analogously to the equilibrium problem 3-d vacuum fields generically show field line chaos, which prevents the formation of magnetic surfaces. 3-d vacuum fields with magnetic surfaces are, therefore, exceptions as rare as 3-d equilibria. The vacuum field can, furthermore, be obtained as a continuous limit ($a_1 \to -1$) of a family of equilibria, a limit which is otherwise often singular [20].

It remains to consider the case $\partial^2_{zz} K \equiv 0$. In that case Eqs. (70) can immediately be integrated to yield

$$(\Delta_{xy} K)^2 + 2\, P = C(z). \tag{93}$$

Differentiating Eq. (93) with respect to $z$ leads together with Eq. (71) to

$$(\partial^2_{xz}K)\,\Delta_{xy}\partial_x K + (\partial^2_{yz}K)\,\Delta_{xy}\partial_y K - (\Delta_{xy}K)\,\Delta_{xy}\partial_z K = -C'. \tag{94}$$

The general solution of Eq. (94) under the constraint $\partial^2_{zz}K \equiv 0$ has the form

$$\begin{aligned} K &= K_0(x,y) + K_1(x,y)\,z, \\ C' &= C_0 + C_1\,z, \end{aligned} \tag{95}$$

which inserted in Eq. (94) leads to

$$(\partial_x K_1)\partial_x\Delta_{xy}K_0 + (\partial_y K_1)\partial_y\Delta_{xy}K_0 - (\Delta_{xy}K_1)\Delta_{xy}K_0 = -C_0, \tag{96}$$

$$(\partial_x K_1)\partial_x\Delta_{xy}K_1 + (\partial_y K_1)\partial_y\Delta_{xy}K_1 - (\Delta_{xy}K_1)^2 = -C_1. \tag{97}$$

Equation (97) is a 2-d nonlinear equation for $K_1$. Once $K_1$ is determined and inserted in Eq. (96), this is a 2-d linear equation for $\Delta_{xy}K_0$. On switching to cylindrical coordinates Eqs. (96), (97) read

$$\begin{aligned} (\partial_r K_1)\partial_r\Delta_{r\theta}K_0 + \frac{1}{r^2}(\partial_\theta K_1)\partial_\theta\Delta_{r\theta}K_0 - (\Delta_{r\theta}K_1)\Delta_{r\theta}K_0 &= -C_0, \\ (\partial_r K_1)\partial_r\Delta_{r\theta}K_1 + \frac{1}{r^2}(\partial_\theta K_1)\partial_\theta\Delta_{r\theta}K_1 - (\Delta_{r\theta}K_1)^2 &= -C_1, \end{aligned} \tag{98}$$

with $\Delta_{r\theta}$ being the 2-d Laplacian in the $r$-$\theta$ plane,

$$\Delta_{r\theta} = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial^2_{\theta\theta}. \tag{99}$$

An implicit solution of Eqs. (98) is given by

$$\begin{aligned} K_1(r) &= -\int^r \left(c_1 \sin\frac{k\tilde{r}^2}{2} + c_2\cos\frac{k\tilde{r}^2}{2}\right)\frac{d\tilde{r}}{\tilde{r}}, \\ \Delta_{r\theta}K_0(r,\theta) &= -k\left(b_1(\theta)\cos\frac{kr^2}{2} - b_2(\theta)\sin\frac{kr^2}{2}\right), \end{aligned} \tag{100}$$

where $c_1$, $c_2$, $b_1(\theta)$ and $b_2(\theta)$ obey the relation

$$c_1 b_1' + c_2 b_2' = 0. \tag{101}$$

Comparing the magnetic field (68) of the solution (100) with Eqs. (61), one realizes that the solution of section 2.2, case II, has been regained in a different representation. This shows that the "simplicity" of an equilibrium heavily depends on the representation chosen. Solutions of Eqs. (96), (97) (or Eqs. (98)) representing new equilibria have not been found so far.

# 4. Outlook

Our primary interest in this study of 3-d MHD equilibria is not so much in applications but is theoretical. It relates to the supposed nonexistence of 3-d equilibria in toroidal geometry with sheared field lines, a problem which arises in fusion-oriented MHD. Simplified equilibria such as those considered here may help to distinguish the essential features from the irrelevant ones. Basically, it is learned from the examples given in this paper that 3-d equilibria are not ruled out per se. Closed field line equilibria such as those of the first class or equilibria with infinitely extended field lines such as those of the second and third classes do indeed exist. Neither is a discrete symmetry like mirror symmetry necessary nor have field lines to be plane. All three classes include representatives with poloidally closed pressure surfaces. The closing in the toroidal direction, however, seems to cause trouble. For the purely poloidal equilibria, for example, pressure surfaces can indeed be chosen to be periodic in the toroidal direction; but actual closing requires distorting the straight axis, and this is ruled out [9]. In the purely toroidal case not even periodic solutions have been found so far. To construct such equilibria (whether of Clebsch or generalized Clebsch type or not) would be of considerable interest. The main shortcoming of all known 3-d equilibria is, of course, the missing field line twist. This property combined with double periodicity carries the potential of field line chaos, which is at the heart of the problem. So, the construction of a twisted 3-d equilibrium (periodic or not) or proof of its impossibility would be an important step ahead. Note in this context that any 3-d equilibrium based on a Clebsch representation with $k = r$ (see Appendix A) would be at least locally twisted. Yet, no such equilibrium has been found so far.

Nevertheless, we want to mention some possible applications of the new equilibria. The open equilibria in section 2.1 (see Fig. 2), for example, might be of some interest in the reconnection context. They generalize the well-known 2-d geometry for reconnection via X-points in that they allow variation of the magnetic field in the third direction. In particular, rotation of the X-point and variation of the angle of the intersecting field lines along the third direction are allowed. Whether these new features do affect the reconnection process remains to be explored.

Other possible applications concern the modelling of magnetic arcade structures in the solar corona [21] or the geomagnetic tail [22]. The former application is supported by Fig. 5 and Fig. 7, which can be interpreted as 3-d generalizations of an ordinary 2-d arcade configuration. Up to the second application 3-d equilibria are useful for modelling magnetotail configurations under quiescent conditions. In Ref. [23], for example, approximate 3-d equilibria were constructed on the basis of a Clebsch representation for the magnetic field and a Taylor series expansion in one variable and an asymptotic expansion in another variable.

Finally, it should be mentioned that for astrophysical purposes Low investigated 3-d equilibria in a series of papers [24] (see also [25]). The basic difference to our investigation is the inclusion of a gravitational field. This additional freedom is crucial for his construction of 3-d equilibria. In fact, for vanishing gravitational field these equilibria have no well-defined limit.

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# Appendix A

This appendix investigates the ansatz $\mathbf{B} = \nabla H \times \nabla r$. The problem is again reduced to a 2-d elliptic equation on the cylinder $r = const$ and two compatibility conditions for the $r$ dependence; 3-d solutions, however, have not been found.

Equations (1) here take the form

$$\partial_r(\frac{1}{r}\partial_\theta H)^2 + \frac{1}{r^2}\partial_r(r\partial_z H)^2 + 2\,\partial_r P = 0, \tag{A.1}$$

$$\begin{aligned}(\Delta_{\theta z} H)\partial_\theta H + \partial_\theta P = 0,\\ (\Delta_{\theta z} H)\partial_z H + \partial_z P = 0,\end{aligned} \tag{A.2}$$

where $\Delta_{\theta z}$ denotes the 2-d Laplacian on the $\theta$,$z$ cylinder,

$$\Delta_{\theta z} = \frac{1}{r^2}\partial^2_{\theta\theta} + \partial^2_{zz}. \tag{A.3}$$

From the integrability condition $\partial_z\partial_\theta P - \partial_\theta\partial_z P = 0$ for $P$ and Eqs. (A.2) one deduces in the usual way a 2-d elliptic equation for $H$ with an arbitrary profile function $f(H, r)$:

$$\Delta_{\theta z} H = f(H, r). \tag{A.4}$$

Note that, contrary to the previously considered cases, (A.1) is no longer a total $r$ derivative. The compatibility conditions are, therefore, derived from the remaining integrability conditions for $P$, $\partial_r\partial_\theta P - \partial_\theta\partial_r P = 0$ and $\partial_r\partial_z P - \partial_z\partial_r P = 0$. One obtains

$$\begin{aligned}
\partial_r\Big(\left(\partial_\theta H\right)\partial^2_{zz}H - \left(\partial_z H\right)\partial^2_{\theta z}H\Big) - \frac{2}{r}\left(\partial_z H\right)\partial^2_{\theta z}H = 0,\\
\partial_r\left(\frac{1}{r^2}\left[\left(\partial_z H\right)\partial^2_{\theta\theta}H - \left(\partial_\theta H\right)\partial^2_{\theta z}H\right]\right) - \frac{2}{r}\left(\partial_z H\right)\partial^2_{zz}H = 0.
\end{aligned} \tag{A.5}$$

Assuming, as usual, a linear profile function,

$$f = a_0(r) + a_1(r)\, H, \tag{A.6}$$

one can again be distinguish the cases $a_1 = 0$ and $a_1 \neq 0$. In the latter case

$$H_i = -\frac{a_0(r)}{a_1(r)} \tag{A.7}$$

is an inhomogeneous solution and the separation ansatz

$$H_h = \alpha(r,\theta)\,\beta(r,z) \tag{A.8}$$

for the homogeneous part leads to

$$\begin{aligned}
\partial^2_{\theta\theta}\alpha &= -m^2\,\alpha,\\
\partial^2_{zz}\beta &= -\lambda^2(r)\,\beta,
\end{aligned} \tag{A.9}$$

with $m^2/r^2 + \lambda^2 + a_1 = 0$. This suggests the general ansatz for $H_h$,

$$\begin{aligned}
H_h = {} & u_1(r)\sin(m\theta + \lambda z) + u_2(r)\sin(m\theta - \lambda z)\\
& + u_3(r)\cos(m\theta + \lambda z) + u_4(r)\cos(m\theta - \lambda z).
\end{aligned} \tag{A.10}$$

Nontrivial (i.e. 3-d) solutions of type (A.10), which, additionally, satisfy the compatibility conditions (A.5) have, however, not been found.

In the case $a_1 = 0$ an ansatz of the form

$$\begin{aligned}
H_i &= \lambda(r) + \mu(r)\,z + \nu(r)\,z^2,\\
H_h &= \left(u_1(r)\sinh\Phi + u_2(r)\cosh\Phi\right)\,\sin m\theta\\
&\quad + \left(u_3(r)\sinh\Phi + u_4(r)\cosh\Phi\right)\,\cos m\theta,
\end{aligned} \tag{A.11}$$

with $2\nu = a_0$ and $\Phi := mz/r$ was tried. However, nontrivial solutions satisfying Eqs. (A.5) could not be found either. Whether more general ansatzes are successful or 3-d equilibria of this Clebsch type do not exist at all remains an open problem.

# Appendix B

This appendix shows that the remaining two generalized Clebsch cases cannot by treated with the method which has been successful so far. More precisely, none of the integrability conditions for the pressure can be resolved.

For the ansatz $\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} K \times \boldsymbol{\nabla}\theta)$ Eqs. (1) furnish

$$\begin{aligned}
&\frac{1}{r^2}(\Delta_{rz}K)\left(\partial_r(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_r K\right) - \frac{2}{r^5}(\partial^2_{\theta z}K)^2 + \partial_r P = 0,\\
&\frac{1}{r^2}(\Delta_{rz}K)\left(\partial_z(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_z K\right) + \frac{2}{r^5}(\partial^2_{\theta z}K)\,\partial^2_{r\theta}K + \partial_z P = 0,
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&\frac{1}{r^2}(\partial^2_{r\theta}K)\left(\partial_r(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_r K\right) +\\
&\frac{1}{r^2}(\partial^2_{\theta z}K)\left(\partial_z(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_z K\right) + \partial_\theta P = 0,
\end{aligned} \tag{B.2}$$

with $\Delta_{rz}$ being the 2-d Stokes operator in the $r$-$z$ plane,

$$\Delta_{rz} = r\,\partial_r(\frac{1}{r}\,\partial_r) + \partial^2_{zz}. \tag{B.3}$$

For the ansatz $\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} K \times \boldsymbol{\nabla} r)$ we obtain

$$\begin{aligned}
&(\Delta_{\theta z}K)\left(\partial_\theta(\Delta_{\theta z}K) + \partial_r(r\partial_r(\frac{1}{r}\partial_\theta K))\right)\\
&+\frac{2}{r^5}(\partial^2_{\theta z}K)\,\partial_r(r\partial_z K) + \partial_\theta P = 0,\\
&(\Delta_{\theta z}K)\left(\partial_z(\Delta_{\theta z}K) + \partial_r(\frac{1}{r}\partial_r(r\partial_z K))\right)\\
&-\frac{2}{r^5}(\partial^2_{\theta z}K)\,\partial_r(\frac{1}{r}\partial_\theta K) + \partial_z P = 0,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
&\frac{1}{r}(\partial_r(\frac{1}{r}\partial_\theta K))\left(\partial_\theta(\Delta_{\theta z}K) + \partial_r(r\partial_r(\frac{1}{r}\partial_\theta K))\right) +\\
&\frac{1}{r}(\partial_r(r\partial_z K))\left(\partial_z(\Delta_{\theta z}K) + \partial_r(\frac{1}{r}\partial_r(r\partial_z K))\right) + \partial_r P = 0,
\end{aligned} \tag{B.5}$$

where $\Delta_{\theta z}$ is the 2-d Laplacian on the $\theta$,$z$ cylinder given in Eq. (A.3). If one tries to proceed as in section 3, one finds that, unfortunately, the integrability condition $\partial_r\partial_z P - \partial_z\partial_r P = 0$ applied to Eqs. (B.1) does not lead to a simple relation between $\Delta_{rz}K$ and $\partial^2_{\theta\theta}K$, and the condition $\partial_\theta\partial_z P - \partial_z\partial_\theta P = 0$ applied to Eqs. (B.4) does not work either.

For these generalized Clebsch ansatzes one has, therefore, to solve Eqs. (B.1), (B.2) or Eqs. (B.4), (B.5) direct. This is a formidable task, which is not followed up here.

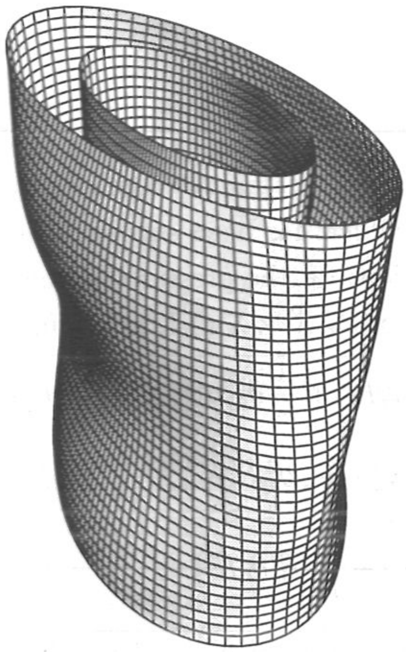

Fig. 1: Nested pressure surfaces $P(r,\theta,z) = -r^2\{1 - u(z)\cos[2\theta - v(z)]\}/[1 - u^2(z)]$. Closed grid lines are magnetic field lines. With $u(z) = 0.6 + 0.1\sin(2z)$, $v(z) = 0.2(z - 3.5)$, $P = -1.44$ (inner surface) and $P = -4$ (outer surface).

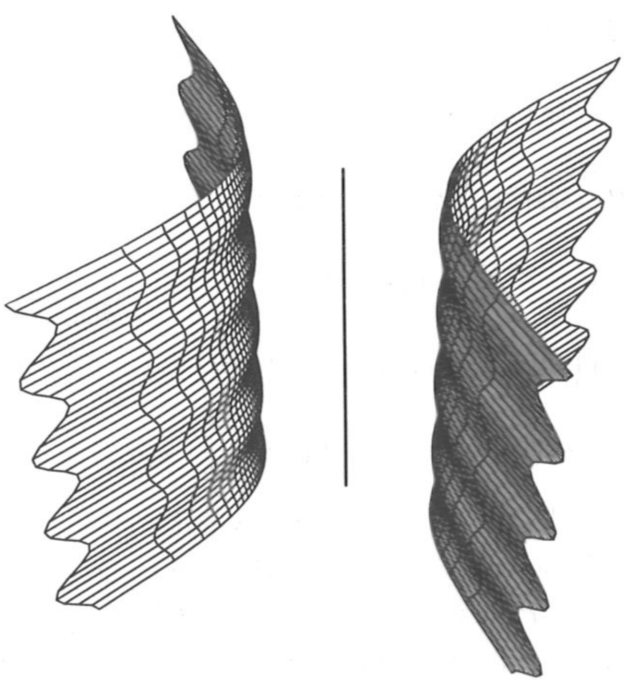

Fig. 2: Two branches of open pressure surface $P = 9$, with axis in between. With $u(z) = -6 - 0.3\sin z$, $v(z) = 0$. $P$ defined as in Fig. 1.

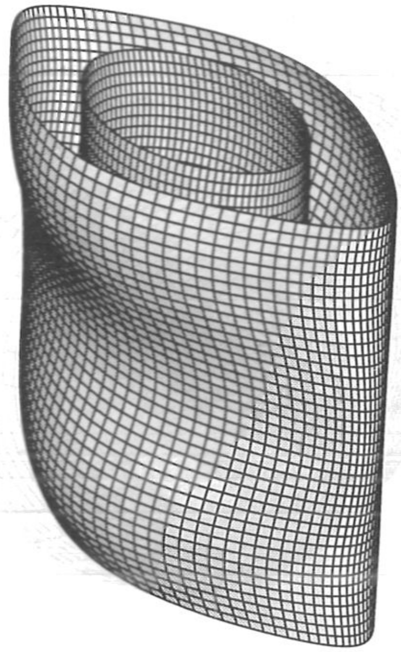

Fig. 3: Nested pressure surfaces $P(x, y, z) = -[a^2(z)\sin^2 z + 2(1 - \cos x \, \cos y) + a^{-2}(z)\sin^2 y]$. Closed grid lines are magnetic field lines. With $a(z) = 0.6 + 0.1\sin(2z)$, $P = -1.96$ (inner surface) and $P = -3.8025$ (outer surface).

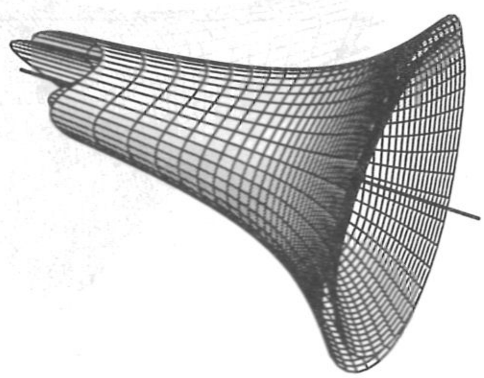

Fig. 4: Nested pressure surfaces $P(r, \theta, z) = -r^2 a_0(\theta)[r^2 a_0(\theta) + 4z]$, with $z$ axis. Longitudinal grid lines are magnetic field lines. With $a_0(\theta) = -1 + 0.1\cos\theta + 0.1\cos(2\theta)$, $P = -0.4$ (inner surface) and $P = -0.9$ (outer surface).

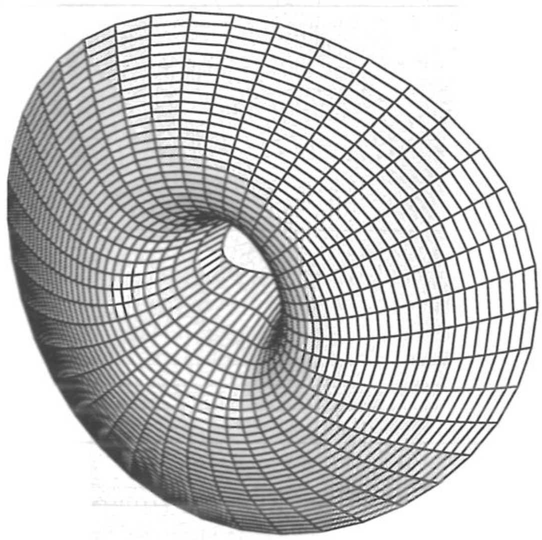

Fig. 5: Pressure surface $P = 0.8$ with arcade geometry. With $a_0(\theta) = 1 + 0.2\cos\theta + 0.2\cos(2\theta)$. $P$ defined as in Fig. 4.

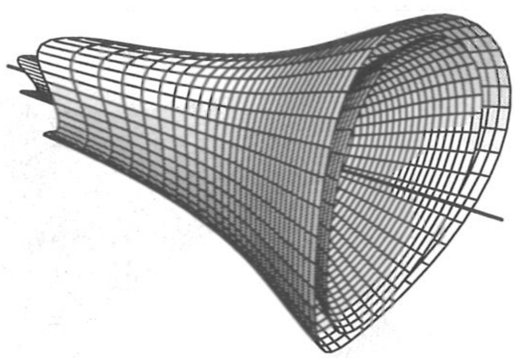

Fig. 6: Nested pressure surfaces $P(r,\theta,z) = [z\sin(r^2) + b_2(\theta)\cos(r^2)]^2 - b_2^2(\theta)$, with $z$ axis. Longitudinal grid lines are magnetic field lines. With $b_2(\theta) = 1 + 0.15\cos\theta + 0.1\cos(2\theta)$, $P = -0.1$ (inner surface) and $P = -0.5$ (outer surface).

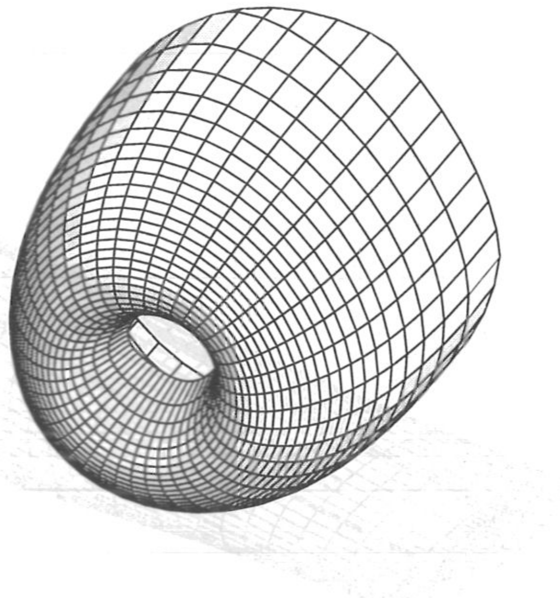

Fig. 7: Pressure surface $P = 0.2$ with arcade geometry. With $b_2(\theta) = 1 + 0.15\cos\theta + 0.1\sin(2\theta)$. $P$ defined as in Fig. 6.

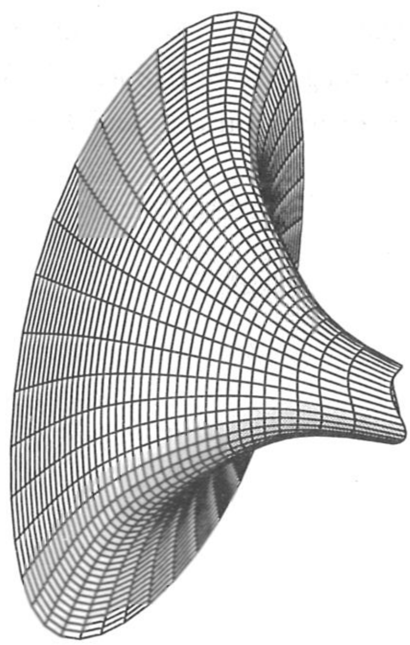

Fig. 8: Pressure surface $P(r,\theta,z) = [z\sinh(r^2) + b_2(\theta)\cosh(r^2)]^2 - b_2^2(\theta)$. With $b_2(\theta) = 1 + 0.15\cos\theta + 0.05\cos(3\theta)$ and $P = 0.2$. Field lines tend towards $r = \infty$ at finite $z$.

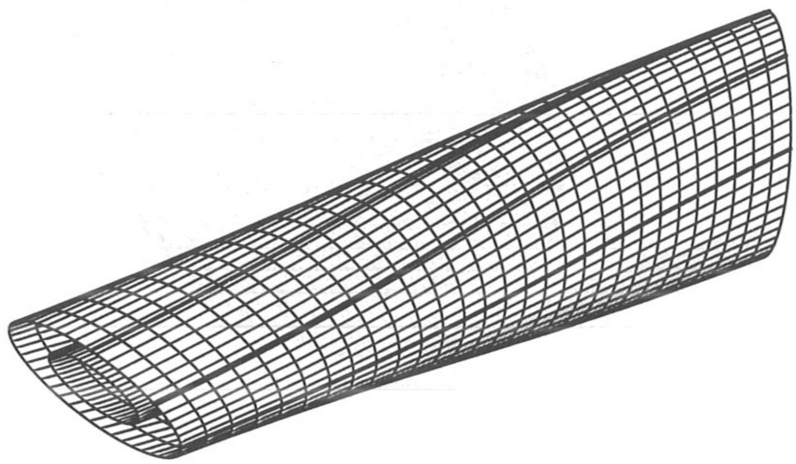

Fig. 9: Nested pressure surfaces $P(x, y, z) = -[\exp z\ \sin^2 x + 2\,(1 - \cos x \cos y) + \exp(-z) \sin^2 y]$ with three magnetic field lines (thick). The middle one has finite torsion. With $P = -0.2$ (inner surface) and $P = -0.7$ (outer surface).

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