Stationary Equilibrium and Rotation of a Collisional Plasma in a Torus

HORST WOBIG

IPP 2/322 August 1994

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Abstract

This paper describes a collision dominated and isothermal plasma consisting of N particles species. Starting from the momentum balance equations of the plasma components the flux-friction relations are derived for every magnetic surface. The plasma is considered in general toroidal geometry, therefore the results apply to tokamaks and stellarators. Taking into account inertial forces leads to non-linear equations which determine the poloidal and toroidal rotation of the plasma within the magnetic surfaces.

In the first part the flux-friction relations are established in the one-fluid model. Furthermore, some integral relations of the stationary equilibrium are derived and the spin-up equations of poloidal rotation are generalised to stellarator geometry. The main part of the paper deals with the multi-fluid model and generalises the flux-friction relations to a multi-species plasma. Because of the Coriolis forces the flux-friction relations become first order differential equations instead of algebraic equations. The viscous damping is analysed for a collisional plasma showing the influence of the magnetic field topology and Pfirsch-Schlüter currents on the spin-up mechanism. The similarity between the spin-up equations and the theory of circulation in planetary atmospheres is indicated. Finally the effect of shear viscosity and gyro viscosity is considered.

Contents

Chapter 1

Introduction

In the following paper a collisional plasma in general toroidal geometry is investigated. General toroidal geometry means the absence of continuous symmetries like achsial symmetry or helical symmetry. The aim is to analyse the relation between plasma gradients, radial particle fluxes and poloidal and toroidal rotation tangential to magnetic surfaces. In particular, enhanced poloidal and toroidal rotation in the boundary region and the development of shear flow, which is of interest in conjunction with the transition to H-mode confinement, will be discussed in this paper.

The rotational instability of a toroidal plasma has been studied for the first time by Stringer[1] and the mechanism has become wellknown as Stringer spin-up. Rosenbluth and Taylor[2] included the effect of viscosity, a further analysis of the role of viscous damping or magnetic pumping in plasma rotation has been given by Hassam and Kulsrud[3]. The rotational instability of a static equilibrium has also been anlaysed by Greene, Johnson, Weimer and Windsor[4]. Recently the spin-up mechanisn has gained increasing attention in conjunction with the development of shear flow in the plasma boundary and the transition to H-mode confinement. Hassam, Antonsen, Drake and Liu have pointed out that the Stringer mechanism can be enhanced by anomalous transport and thus overcome the damping effects[5]. These authors conclude that the radial transport be poloidally asymmetric in order to provide the spin-up of poloidal rotation. However, if the plasma is turbulent and exhibits anomalous transport further anomalous effects, the turbulent Reynolds stresses, may arise and have some influence on the rotational instability. Numerical calculations by Guzdar, Drake, McCarthy, Hassam and Liu[6] confirm this idea and show that resistive ballooning instabilities in the boundary region of a tokamak can lead to the evolution of a shear flow and to a reduction of the anomalous radial transport. These authors find that both effects - turbulent Reynolds stresses and Stringer mechanism - lead to a poloidal rotation with a predominence of the Stringer spin-up in the later phase of the simulation. The role of Reynolds stresses on shear flow evolution has also been pointed out by Garcia, Carreras, Lynch and Diamond[7].

Another mechanism which often has been invoked to explain the evolution of poloidal shear

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[1]T.A. Stringer, *Phys. Rev. Lett.* **22**, 1770 (1969)

[2]M.N. Rosenbluth and J.B. Taylor, *Phys. Rev. Lett.* ,**23**, 367 (1969)

[3]A.B. Hassam, R.M. Kulsrud, *Phys. Fluids* **21** , 2271 (1978)

[4]J.M. Greene, J.L. Johnson, K.E. Weimer, N.K. Winsor, *Phys. Fluids* **14**, 1258 (1971)

[5]A.B. Hassam, T.M. Antonsen, J.F. Drake, C.S. Liu ,*Phys. Rev. Letters* **66**, No.3, 309 (1991)

[6]P.N. Guzdar, J.F. Drake, D. McCarthy, A.B. Hassam, C.S. Liu, *Phys. Fluids* **B5**, 3712 (1993)

[7]L.Garcia, B.A. Carreras, V.E. Lynch, P.H. Diamond, *Proc. 14^{th} Int Conf. on Plasma Phys. and Contr. Nuclear Fusion Research*, Würzburg, 1992, paper CN-56/D-4-6

flow is the lost orbits effect[8] and the nonlinearity of neoclassical transport coefficients on the radial electric field which may cause bifurcation and multiple solutions [9]. Lost orbits modify the ambipolar condition of radial particle fluxes and as a consequence the radial electric field changes. K.C. Shaing has extended the theory to stellarator configurations[10]. One example of an externally triggered 'lost orbit effect' is the neutral beam heating experiment in the stellarator Wendelstein VII-A[11], where a large fraction of highly energetic particles is trapped in local mirrors and rapidly transported towards the wall. Onset of plasma rotation has been measured spectroscopically. Applying a radial voltage by probes in the boundary region is another method to generate poloidal flow and a reduction of anomalous plasma transport[12]. Main interest, however, is focused on the spontaneous onset of a poloidal shear flow since in a reactor plasma there is only a little chance to apply these methods from outside.

One basic assumption in the following theory is the existence of closed magnetic surfaces which are suited to confine a plasma in equilibrium. A necessary condition is the constancy of $\oint dl/B$ on rational magnetic surfaces which is satisfied in symmetric configurations like axisymmetric tokamaks or helically invariant stellarators. This assumptions puts aside the still unsolved problem whether an ideal MHD-equilibrium exists in non-symmetric toroidal geometry. A real plasma is in stationary equilibrium, where the velocity $\mathbf{v}$, which may consist of diffusive velocity, convective velocity and rotation, has a feedback effect on the momentum balance and decouples pressure surfaces and magnetic surfaces. Plasma equilibria taking into account inertial forces have been considered by Zehrfeld and Green[13]. Hazeltine, Lee and Rosenbluth extended the work of Zehrfeld and Green and studied the formation of a weak shock in the poloidally rotating plasma[14]. Recently conditions for existence and uniqeness of stationary equilibria have been analysed by Spada and Wobig[15].

In the following paper we assume that such an equilibrium exists and that plasma losses are maintained by particle and energy sources. Furthermore, we assume that the deviation from ideal equilibrium is not too strong and that magnetic surfaces exist in lowest order. Therefore, all quantities as pressure, density and temperatures are nearly constant on magnetic surfaces. Once a stationary equilibrium is calculated self-consistently, all quantities are known and there would be no need for further analysis. However, in view of the goal to confine a high temperature plasma with small losses one is particularly interested in relations between integrated losses through magnetic surfaces, plasma gradients and poloidal and toroidal velocities. Such "flux-friction relations" have been derived by Hirshman and Sigmar[16] for tokamak plasmas, Shaing and Callen have extended the theory to non-axisymmetric stellarator plasmas including neoclassical effects[17]. Plasma momentum sources as provided by neutral beam injection have been incorporated by Coronado and Wobig[18]. In these papers the fluid equations of a multi-component plasma have been set up based on the Hamada coordinate system. In the frame of this theory the two base vectors $\mathbf{e}_p$ and $\mathbf{e}_t$ on the magnetic surface following poloidally and toroidally closed coordinate lines are utilized. In the Hamada coordinate system magnetic field

[8] S.I. Itoh, K. Itoh, *Phys. Rev. Lett.* **60**, 2276 (1988)
[9] K.C. Shaing, E.C. Crume, *Phys. Rev. Lett.* **63** 2369 (1989)
[10] K.C.Shaing, *Phys. Fluids* B5, 3841 (1993)
[11] G. Grieger et al. *Plasma Physics and Controlled Fusion* Vol. 28, 1A, 43 (1985)
[12] R.R. Weynants, G.van Oost et al. *Nuclear Fusion* **32**, 837 (1992)
[13] H.P. Zehrfeld, B.J. Green *Physical Rev. Lett* **23** No. 17 (1969) 961
[14] R.D. Hazeltine, D.P. Lee and M.N. Rosenbluth, *Phys. Fluids* **14**, 631 (1971)
[15] M. Spada, H. Wobig, *J. Phys. A: Math. Gen.* **25** (1992), 1575
[16] S.P. Hirshman, D.J. Sigmar, *Nuclear Fusion* **21**, 1079 (1981)
[17] K.C. Shaing, J.D. Callen, *Phys. Fluids* **26**, 3315 (1983)
[18] M. Coronado, H. Wobig, *Phys. Fluids* **30**, 3171 (1987)

lines and plasma current lines are straight. The Hamada coordinate system which can be defined for any toroidal geometry is well suited to evaluate radial plasma flows, poloidal and toroidal rotation and to investigate the effect of external sources on plasma losses and plasma rotation.

In the present paper the theory in these papers will be reformulated using $\mathbf{e}_p$ and $\mathbf{B}$ as base vectors on the magnetic surface. This formulation makes use of internal symmetries of the equilibrium and leads to a shorter and more compact formulation of the results. Besides these formal changes several physical effects as bootstrap currents, momentum sources and impurity diffusion are discussed in more detail. In a collision dominated plasma the bootstrap current is small and has negligible effect on the plasma equilibrium. Nevertheless it is of interest to reveal its dependence on the geometrical structure of magnetic surfaces, since this might give some guide lines how to minimize the bootstrap current in the plateau regime or in the long mean free path regime.

Besides momentum exchange by Coulomb collisions the viscosity is the dissipative mechanism of a collisional plasma. The Braginskii viscosity tensor [19] is linear in the plasma flow velocity $\mathbf{v}$ and thus momentum balance equation, equation of continuity, equation of state and energy balance equation form a closed set of equations for the plasma density, velocity and the temperature. However, the differential form of the viscosity tensor inhibits to solve the system explicitly, an ordering scheme, which neglects all dissipative terms in lowest order has to be applied. In the first part of the paper we follow the procedure which has been developed in Shaing and Callen's paper. Furthermore, we include inertial forces and turbulent effects which modify the flux-friction relations in various respect. Since the inertial forces are closely related to the Stringer spin-up it is of particular interest to study their role in the flux-friction relations. In order to clarify the basic feature of plasma rotation we consider at first an isothermal plasma and neglect the energy balance. However, as will be shown in chapter 7, the energy balance can be easily incorporated.

The applicability of a collisional model to a fusion plasma may be be questioned since in a hot plasma the mean free path is large and neoclassical theory has to be applied. However in the boundary regime the collisional model is applicable; there the plasma is in the Pfirsch-Schlüter regime or at the edge of the plateau regime. In the boundary region the transition from L-mode to H-mode occurs which is closely related to the build-up of a poloidal rotation and a radial electric field. Therefore the radial electric field and its dependence on the toroidal geometry will be extensively discussed.

A further modification of our collisional model is to take into account the full Braginskii viscosity tensor with shear viscosity and gyro viscosity. As has been pointed out by Stacey and Sigmar[20] gyro viscosity may give rise to a radial momentum transfer and therefore leads to a damping of toroidal rotation. It was suggested that gyro viscosity accounts for the enhanced momentum transport observed in tokamaks. This has been critisized by Connor et al.[21] who showed that gyro viscosity does not contribute to the balance of toroidal angular momentum in tokamaks. This theory will be generalized to non-axisymmetric configurations. Using only parallel viscosity the condition of ambipolar diffusion results in an algebraic equation for the electric field. Retaining the full viscosity tensor, however, leads to differential equations of the second order, therefore the strong coupling of the E-field to the local pressure gradients , which occurs in standard theory is removed by gyro viscosity and shear viscosity.

[19] S.I. Braginskii, *Rev. of Plasma Physics*, Vol. I, 205, Consultants Bureau, N.Y. 1965
[20] W.M. Stacey Jr., D.J. Sigmar, *Phys. Fluids* **27**, 2076 (1984) and *Phys. Fluids* **28**, 2800 (1985)
[21] J.W. Connor,S.C. Cowley, R.J. Hastie, R.L. Pan, Report CLM-P795 (1987)

Chapter 2

The One-Fluid Model

2.1 Ideal Equilibrium and Pfirsch-Schlüter Currents

Let us consider a magnetic field $\mathbf{B}$ with nested and toroidally closed magnetic surfaces $\psi =$ const. This may be either a vacuum field or a self-consistent equilibrium field. The standard equation of an ideal equilibrium is

$$\mathbf{j} \times \mathbf{B} = \nabla p \tag{2.1}$$

The problem how to find a self-consistent $\mathbf{B}$ will not be discussed here, we restrict the investigation to the task how to calculate the macroscopic plasma parameters density, pressure, velocity and plasma currents for a given field $\mathbf{B}$. In the following we call a field satisfying Eq. 2.1 an " equilibrium field ". It should be noted that $p(\psi)$ can be any scalar function of the flux function ψ, whether p is a plasma pressure or not does not play a role in defining the equilibrium field. In a realistic dissipative plasma the force balance differs from Eq. 2.1 and the magnetic field will also differ from the ideal "equilibrium field". The main task of introducing this field is to provide a coordinate system and a lowest order approximation to the real field in stationary equilibrium.

In the following we assume that the magnetic field satisfies the equation of the ideal MHD model. As introduced by Shaing and Callen [1] in their pioneering paper on neoclassical transport in non-axisymmetric equilibria, the Hamada coordinate system (s, θ, ζ) is the appropriate system to describe toroidal equilibria (s is the volume of the magnetic surface, θ and ζ are the poloidal and toroidal angles on the magnetic surface). This coordinate system exists if a solution of the ideal equilibrium condition $\mathbf{j} \times \mathbf{B} = \nabla p$ exists. With

$$\mathbf{e_p} = \nabla s \times \nabla \zeta \quad \text{and} \quad \mathbf{e_t} = -\nabla s \times \nabla \theta \tag{2.2}$$

being the poloidal and toroidal base vectors on the magnetic surface, the magnetic field is

$$\mathbf{B} = \psi'(s)\,\mathbf{e_t} + \chi'(s)\,\mathbf{e_p} \tag{2.3}$$

and the plasma current

$$\mathbf{j} = I'(s)\,\mathbf{e_t} + J'(s)\,\mathbf{e_p}. \tag{2.4}$$

$\psi(s)$ and $\chi(s)$ are the toroidal and poloidal magnetic fluxes and $I(s)$, $J(s)$ the corresponding currents. The ratio between poloidal and toroidal fluxes is the rotational transform defined by

$$\iota(s) = \frac{\chi'(s)}{\psi'(s)} \tag{2.5}$$

[1] K.C. Shaing, J.D. Callen, *Phys. Fluids* **26** (1983), 3315

Some basic properties of the base vectors $\mathbf{e_p}$ and $\mathbf{e_t}$ will be needed later. The property $\mathbf{e_t} \times \mathbf{e_p} = \nabla s$ leads to

$$\mathbf{e_p} \times \mathbf{B} = -\nabla\psi \quad ; \quad \mathbf{e_t} \times \mathbf{B} = \iota(\psi)\nabla\psi \tag{2.6}$$

Using the equilibrium condition

$$I'(s)\,\chi'(s) - J'(s)\,\psi'(s) = p'(s) \tag{2.7}$$

the plasma current can also be written in the form:

$$\psi'(s)\,\mathbf{j} = -p'(s)\,\mathbf{e_p} + I'(s)\,\mathbf{B} \tag{2.8}$$

or

$$\mathbf{j} = -p'(\psi)\,\mathbf{e_p} + I'(\psi)\,\mathbf{B} \tag{2.9}$$

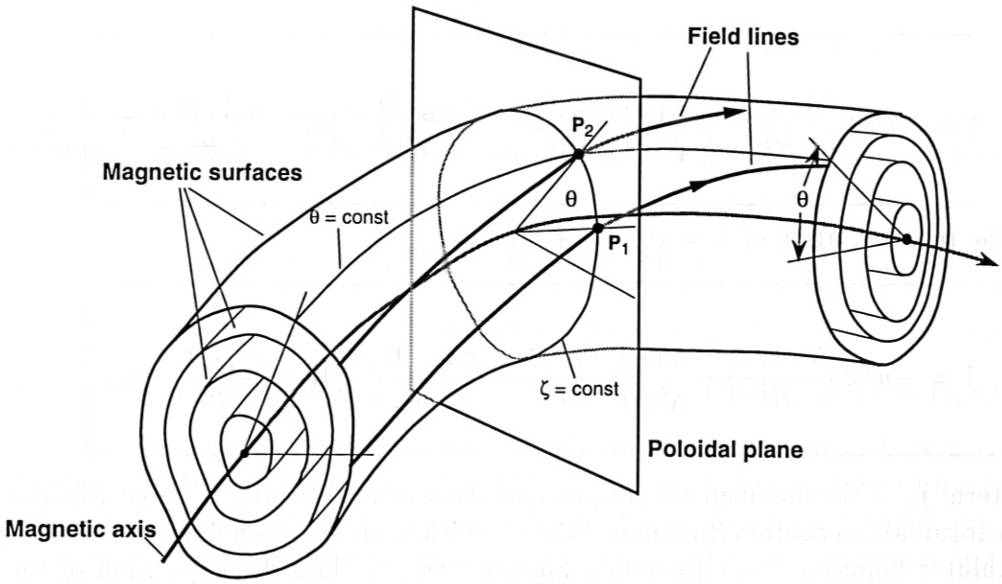

Figure 1 *Magnetic surfaces and the coordinate system s, θ, ζ. The θ=const. lines are toroidally closed and the ζ=const. lines are poloidally closed. The label s is the volume of the magnetic surface and ψ the toroidal flux .*

Since $\psi'(s) = 1/s'(\psi) \neq 0$ we may either employ the volume s or the toroidal flux ψ as radial coordinate. Averaging a scalar g over the magnetic surface is defined by

$$< g > \; = \int g \frac{df}{|\nabla\psi|} / \int \frac{df}{|\nabla\psi|} \; = \frac{1}{V'(\psi)} \int g \frac{df}{|\nabla\psi|} \tag{2.10}$$

df is the surface element on the magnetic surface and $V(\psi)$ the volume of the magnetic surface. For any vector $\mathbf{a}$ lying in the magnetic surface and $\nabla \cdot \mathbf{a} = 0$ it can be shown that

$$< \mathbf{a} \cdot \nabla g > = 0 \tag{2.11}$$

for any periodic function $g(\theta, \varphi)$. The flux of a vector $\mathbf{a}$ through a magnetic surface is

$$\Gamma = \int \mathbf{a} \cdot df = V'(\psi) < \mathbf{a} \cdot \nabla\psi > \tag{2.12}$$

The poloidal current $p'(\psi)\mathbf{e_p}$ consists of the diamagnetic current perpendicular to $\mathbf{B}$ and the Pfirsch-Schlüter current [2]

$$\mathbf{j}_{PS} = -p'(\psi)(\mathbf{e_p}\cdot\mathbf{B})\mathbf{B}/B^2 \tag{2.15}$$

however there is no need to separate these terms. It should be noted that the definition of the Pfirsch-Schlüter currents often used in the literature differs from the present one. Making use of Eq. 2.6 the current $\mathbf{j}$ can be written as

$$\mathbf{j} = -p'(\psi)\frac{\nabla s\times\mathbf{B}}{B^2} + \frac{1}{\psi'(s)}\{p'(\psi)\frac{\mathbf{e_p}\cdot\mathbf{B}}{B^2} + I'(s)\}\mathbf{B} \tag{2.16}$$

and with

$$\psi'(s) < \mathbf{j}\cdot\mathbf{B} > = p'(s) < \mathbf{e_p}\cdot\mathbf{B} > + I'(s) < B^2 > \tag{2.17}$$

$$\boxed{\mathbf{j} = -p'(\psi)\frac{\nabla s\times\mathbf{B}}{B^2} + \frac{p'(\psi)}{\psi'(s)}\{\frac{\mathbf{e_p}\cdot\mathbf{B}}{B^2} - \frac{<\mathbf{e_p}\cdot\mathbf{B}>}{<B^2>}\}\mathbf{B} + \frac{<\mathbf{j}\cdot\mathbf{B}>}{<B^2>}\mathbf{B}} \tag{2.18}$$

or using the representation of $\mathbf{B} = \psi'(s)\mathbf{e_t} + \chi'(s)\mathbf{e_p}$

$$\boxed{\mathbf{j} = -p'(\psi)\frac{\nabla s\times\mathbf{B}}{B^2} - \frac{p'(\psi)}{\chi'(s)}\{\frac{\mathbf{e_t}\cdot\mathbf{B}}{B^2} - \frac{<\mathbf{e_t}\cdot\mathbf{B}>}{<B^2>}\}\mathbf{B} + \frac{<\mathbf{j}\cdot\mathbf{B}>}{<B^2>}\mathbf{B}} \tag{2.19}$$

The first term $\mathbf{j}_\perp$ is the diamagnetic current and the second term $\mathbf{j}_{PS}$ is called Pfirsch-Schlüter current in tokamak literature (Hirshman 1978) [3] which leads also to a different definition of the Pfirsch-Schlüter diffusion flux (Hirshman, Sigmar 1981) [4]. This decomposition of the plasma current in components perpendicular and parallel to the magnetic field leads to comparatively complicated expressions of the diffusion fluxes. This decomposition is also unnatural in the sense that both components are not real currents, neither $\mathbf{j}_\perp$ nor $\mathbf{j}_{PS}$ satisfy $\nabla\cdot\mathbf{j} = 0$, only the sum does. Furthermore, neither the net toroidal current of $\mathbf{j}_\perp$ nor of $\mathbf{j}_{PS}$ is zero, nor gives the sum of both zero toroidal current. The representation used here $\psi'(s)\mathbf{j} = -p'(s)\mathbf{e_p} + I'(s)\mathbf{B}$ needs no separation in diamagnetic and Pfirsch-Schlüter currents. The implications of the various definitions of Pfirsch-Schlüter currents have been discussed in a paper by Coronado and Wobig[5].

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[2] D. Pfirsch, A. Schlüter, MPI-report PA/7 (1962). In this report a large aspect ratio expansion leads to the following ratio between parallel and perpendicular plasma currents:

$$j_\parallel \approx j_\perp \frac{2}{\iota}\cos\theta \tag{2.13}$$

The same approximation holds for the base vector ep

$$\mathbf{ep}\cdot\mathbf{b} \approx \mathbf{e}_{p,\perp}\frac{2}{\iota}\cos\theta \tag{2.14}$$

Due to these parallel current the classical diffusion is enhanced by the factor $1+2/\iota^2$, the so-called Pfirsch-Schlüter factor.

[3] S.P. Hirshman, *Phys. Fluids* 21, (1978) p. 1295

[4] S.P. Hirshman, D. Sigmar, *Nucl. Fusion*, 21, (1981), p. 1079

[5] M. Coronado, H. Wobig, *Phys. Fluids* B4, 1294 (1992)

As shown above the parallel component $\mathbf{e_p} \cdot \mathbf{B}$ is a measure of the Pfirsch-Schlüter current in a toroidal equilibrium. These are not zero locally, however it can be shown that in certain conditions the average $< \mathbf{e_p} \cdot \mathbf{B} >$ is zero. For the average values of $\mathbf{e_p} \cdot \mathbf{B}$ and $\mathbf{e_t} \cdot \mathbf{B}$ the following relations hold:

$$< \mathbf{e_p} \cdot \mathbf{B} > = -I(\psi) \quad ; \quad < \mathbf{e_t} \cdot \mathbf{B} > = J(\psi) + \oint_{s=0} \mathbf{B} \cdot \mathbf{dl} \qquad (2.20)$$

where the integration is performed along the magnetic axis.

Proof:

In the classical paper on plasma equilibria by Kruskal and Kulsrud [6] the following relations have been derived:

$$\begin{aligned} U'(\psi) &= \left(J + \oint_{s=0} \mathbf{B} \cdot \mathbf{dl}\right) - I(\psi)\,\iota(\psi) = V' < B^2 > \\ K'(\psi) &= \left(J + \oint_{s=0} \mathbf{B} \cdot \mathbf{dl}\right) I'(\psi) - I(\psi) J'(\psi) = V' < \mathbf{B} \cdot \mathbf{j} > \qquad (2.21) \end{aligned}$$

Here $U = \int B^2\, d^3x$ and $K = \int \mathbf{j} \cdot \mathbf{B}\, d^3x$ are the volume integral over the volume of a magnetic surface. Using the the representation of $\mathbf{B}$ and $\mathbf{j}$ in terms of the base vectors leads to

$$\begin{aligned} U' &= < \mathbf{e_t} \cdot \mathbf{B} > + \iota(\psi) < \mathbf{e_p} \cdot \mathbf{B} > \\ K' &= I'(\psi) < \mathbf{e_t} \cdot \mathbf{B} > + J'(\psi) < \mathbf{e_p} \cdot \mathbf{B} > \qquad (2.22) \end{aligned}$$

Inserting these equations into eqs. 2.21 yields the desired relations given in Eq. 2.20.

From these results we conclude that axisymmetric tokamak equilibria always have finite Pfirsch-Schlüter currents whereas in net-current free stellarators the surface averaged Pfirsch-Schlüter currents are zero. Locally these currents are finite also in stellarators, however, by a proper choice of the magnetic surfaces these currents can be minimized to a large extent [7].

Usually the plasma current $\mathbf{j} = -p'(\psi)\mathbf{e_p}$ decomposed in $\mathbf{j}_\perp$ and $\mathbf{j}_{||}$

$$\mathbf{j} = -p'(\psi)\,\frac{\nabla\psi \times \mathbf{B}}{B^2} + p'(\psi)\,\lambda\,\mathbf{B} \qquad (2.23)$$

The last term in this equation is the Pfirsch-Schlüter current. Since the Pfirsch-Schlüter currents give rise to the Shafranov shift of the plasma column, equilibria with low Pfirsch-Schlüter currents are to be favoured for this reason. The scalar λ must be calculated by

$$\mathbf{B} \cdot \nabla\lambda = \nabla \cdot \frac{\nabla\psi \times \mathbf{B}}{B^2} = \nabla\psi \times \mathbf{B} \cdot \nabla(\frac{1}{B^2}) \qquad (2.24)$$

This equation can be modified to

$$-\,2\mathbf{e}_p \cdot \frac{\nabla B}{B} = B^2\,\lambda\,\mathbf{B} \cdot \nabla\frac{1}{B^2} - \mathbf{B} \cdot \nabla\lambda \qquad (2.25)$$

From this relation the following conclusion can be derived: If the Pfirsch-Schlüter currents are zero, then

$$\mathbf{e}_p \cdot \frac{\nabla B}{B} = 0 \qquad (2.26)$$

[6] M. Kruskal, R. Kulsrud, *Phys. Fluids* 1 (1958)

[7] G. Grieger et al. *Physics Optimization of Stellarators*, Phys. Fluids B4,(1992) , 2081

Vice versa the result holds

$$\mathbf{e}_p \cdot \frac{\nabla B}{B} = 0 \quad \longrightarrow \quad \lambda = \frac{\text{const.}}{B^2} \tag{2.27}$$

The const. must be zero since otherwise $\lambda\,\mathbf{B}$ has a net toroidal current. In tokamaks with a net toroidal current $I(\psi)$ the current density is

$$\mathbf{j} \;=\; -p'(\psi)\,\mathbf{e}_p + I'(\psi)\mathbf{B} \tag{2.28}$$

in stellarators the total plasma current is $\mathbf{j} = -p'(\psi)\mathbf{e_p}$. The current lines are the stream lines of the base vector $\mathbf{e_p}$ which are given by the lines ζ = const. The coordinate lines are the solutions of the magnetic differential equation

$$\mathbf{B}\cdot\nabla\zeta \;=\; \psi'(s) \quad ; \quad \mathbf{B}\cdot\nabla\theta \;=\; -\iota\,\psi'(s) \tag{2.29}$$

These equations must be solved with the self consistent magnetic field, however, using the vacuum magnetic field yields an approximation to the coordinate lines. An example is given below.

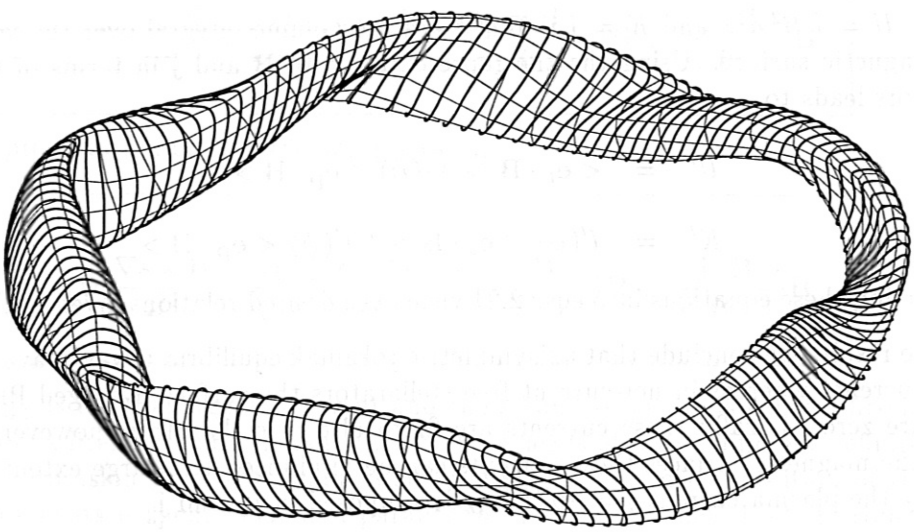

Figure 2 *Magnetic surface of a 5-period Helias configuration. The thin lines are the magnetic field lines and the orthogonal trajectories to the field lines. The thick lines are the current lines ζ = const. In some regions the current lines are nearly perpendicular to the field lines indicating very small Pfirsch-Schlüter currents in this region*

For tokamaks the Hamada coordinate system has been computed in [8].

2.2 One Fluid Model

The one-fluid model is the most simple approximation to a slowly diffusing plasma, the basic relations have already been described in the paper by Kruskal and Kulsrud. Adding Ohm's law

$$\nabla\Phi + \mathbf{v}\times\mathbf{B} \;=\; \eta\mathbf{j} \tag{2.30}$$

to the equilibrium condition describes the slowly diffusing plasma where a feedback of the velocity $\mathbf{v}$ on the momentum balance is neglected. Therefore equilibrium and diffusion are decoupled;

[8] M Coronado, J. Galindo Trejo, *Phys. Fluids* B2 , 530 (1990)

having calculated the equilibrium the macroscopic velocity is found from Ohm's law and the equation of continuity:

$$\nabla \rho \mathbf{v} = S \tag{2.31}$$

ρ is the mass density and S the source term of particles. Furthermore, and equation of state $p = p(\rho)$ links the pressure with density, the specific form of this equation is not needed here. Since the equation of ideal equilibrium remains unchanged by the diffusing plasma, pressure and density are constant on magnetic surfaces which for this reason form a set of smooth and nested tori. In the following we assume that an equilibrium field with these properties exist, the issue of existence is discussed extensively in the literature [9] .

From Ohm's law the perpendicular velocity $\mathbf{v}_\perp$ is computed which consists of a convective $\mathbf{E} \times \mathbf{B}$-drift and the perpendicular diffusion $v_D = -\eta \nabla p / B^2$. More important than the local velocity is the integrated flux $< p\mathbf{v} \cdot \nabla\psi >$ which can be obtained by averaging Ohm' law in poloidal direction. This averaging procedure yields the flux-friction relations introduced by Hirshman, Sigmar [10], which provide a relation between the integral flux Γ and the pressure gradient $p'(s)$ or $p'(\psi)$. The standard procedure to derive these relation is to apply the operator $< \mathbf{a} \cdot ... >$ to Ohm's law where $\mathbf{a}$ is a surface vector satisfying the condidion $\mathbf{a} \times \mathbf{B} = \nabla f(\psi)$. Examples are $\mathbf{a} = \mathbf{B}, \mathbf{j}, \mathbf{e_p}, \mathbf{e_t}$. The averaging procedure yields

$$< \mathbf{a} \cdot \nabla \Phi > - f'(\psi) < \mathbf{v} \cdot \nabla \psi > = < \eta \mathbf{a} \cdot \mathbf{j} > \tag{2.32}$$

$\Gamma = V'(\psi) < \mathbf{v} \cdot \nabla\psi >$ is the total flux of $\mathbf{v}$ through a magnetic surface. Since density and pressure are constant on magnetic surfaces this flux is proportional to the particle flux. Taking $\mathbf{a} = \mathbf{e_p}$ yields $f' = -1$ (see Eq. 2.6) and we obtain

$$-\Gamma = V'(\psi)\eta \left(< \mathbf{e_p} \cdot \mathbf{e_p} > p'(\psi) + I I'(\psi)\right). \tag{2.33}$$

The electric potential is single-valued in poloidal direction $< \mathbf{e_p} \cdot \nabla\Phi >= 0$ but in toroidal direction the finite loop voltage leads to $V'(\psi) < \mathbf{B} \cdot \nabla \Phi >= U_L$. $\mathbf{a} = \mathbf{B}$ yields

$$U_L = V'(\psi)\eta \left(I\, p'(\psi) + < B^2 > I'(\psi)\right) \tag{2.34}$$

These two equation can be summarized in the following form

$$\begin{pmatrix} -\Gamma \\ U_L \end{pmatrix} = V'(\psi)\eta \begin{pmatrix} < \mathbf{e}_p \cdot \mathbf{e}_p > & I(\psi) \\ I(\psi) & < B^2 > \end{pmatrix} \begin{pmatrix} p'(\psi) \\ I'(\psi) \end{pmatrix} \tag{2.35}$$

This relation is a typical example of a linear relation between thermodynamic forces and the conjugate fluxes. The fluxes are here $-\Gamma, U_L$ and the thermodynamic forces are $p'(\psi), I'(\psi)$. The transport matrix L relating these quantities is positive definit and Onsager symmetric. The matrix L is

$$L = V'(\psi)\eta \begin{pmatrix} < \mathbf{e}_p \cdot \mathbf{e}_p > & I(\psi) \\ I(\psi) & < B^2 > \end{pmatrix} \tag{2.36}$$

It can easily be shown that this matrix is positive definite by averaging Ohm' law with $\mathbf{j}$. This procedure yields

$$I'(\psi)U_L - p'(\psi)\Gamma = V'(\psi)\eta < \mathbf{j} \cdot \mathbf{j} > \tag{2.37}$$

This equation was derived by Maschke [11] , it has the typical form of an entropy production

[9] H. Grad, *Phys. Fluids* **10** (1967) 137

[10] S.P. Hirshman, D. Sigmar, *Nucl. Fusion*, **21**, (1981), p. 1079

[11] E.K. Maschke, *Plasma Physics* **13**, 905 (1971)

equation: the left hand side is the product of fluxes and thermodynamic forces while the right hand side describes the dissipated power. The dissipative power is positive.

The symmetry of the transport matrix reflects the symmetry between classical bootstrap effect and classical pinch effect. The term $II'(\psi)$ descibes an inward diffusion driven by the toroidal current I, on the other hand the term $Ip'(\psi)$ introduces the bootstrap effect in the integrated form of Ohm's law Eq. 2.34. The coefficient $< \mathbf{e_p} \cdot \mathbf{e_p} >$ characterizes the diffusive flux driven by the pressure gradient $p'(\psi)$: $-\Gamma = V'(\psi)\eta < \mathbf{e_p} \cdot \mathbf{e_p} > p'(\psi)$. In the large aspect ratio approximation close to axisymmetry the relation holds

$$< \mathbf{e_p} \cdot \mathbf{e_p} > \approx < \mathbf{e}_{p,\perp} \cdot \mathbf{e}_{p,\perp} > \left(1 + \frac{2}{\iota^2}\right) \tag{2.38}$$

This equation shows that using the base vector $\mathbf{e_p}$ in deriving the flux friction relation quite naturally incorporates the Pfirsch- Schlüter enhancement factor in $< \mathbf{e_p} \cdot \mathbf{e_p} >$.

It is unconventional to call the loop voltage U_L a thermodynamic flux driven by the force $I'(\psi)$, however, it should be noted, that there exists no strict rule in how to define fluxes and forces. One could easily reformulate the flux friction relations eqs. 2.35 by transferring the loop voltage to the right side and calling the pair $p'(\psi), U_L$ thermodynamic forces driving the "fluxes" $-\Gamma, I'(\psi)$. The transport matrix in this case is

$$D = \frac{1}{< B^2 >} \begin{pmatrix} \eta(< \mathbf{e}_p \cdot \mathbf{e}_p >< B^2 > - I^2) & I(\psi) \\ -I(\psi) & 1/\eta \end{pmatrix} \tag{2.39}$$

In this form the transport matrix D is no longer symmetric, bootstrap effect and pinch effect are antisymmetric to each other. However, since the toroidal current changes sign with $\mathbf{B} \rightarrow -\mathbf{B}$, we obtain Onsager symmetry in the following sense: $D^T[\mathbf{B}] = D[-\mathbf{B}]$. The D-matrix is also positive definit [12]. However, it is more reasonable to consider the gradients $p'(\psi)$ and $I'(s)$ as the forces which drive the fluxes Γ and U_L. The tokamak equilibrium is specified by the two functions $p'(\psi)$ and $J'(s)$ and the solution of the Grad-Shafranov equation. Thus, the toroidal current $I(s)$ is also fixed by the choice of $p(s)$ and $J(s)$. In a next step collisions are taken into account and determine the fluxes and equivalent sources, which are necessary to maintain a given pressure gradient and a given toroidal current. In the experiment, however, the sources and the loop voltage are given, the pressure gradient and the toroidal current are determined by the collision processes. Usually the loop voltage is controlled to maintain a prescribed toroidal current independently of the plasma parameters.

2.2.1 Tokamak Equilibrium

In an axisymmetric tokamak the magnetic field is represented by

$$\mathbf{B} = \nabla\varphi \times \nabla\chi(r,z) + (J(\chi) + \oint_{axis} \mathbf{B} \cdot dl)\nabla\varphi \tag{2.40}$$

Here, $r, z, \varphi \in [0,1]$ is the standard cylindrical coordinate system. The flux function χ is the solution of the Grad-Schlüter- Shafranov equation and $J(\chi)$ is the poloidal current. The Hamada coordinates ζ, θ are defined by

$$\mathbf{B} \cdot \nabla\zeta = \psi'(s) \quad ; \quad \mathbf{B} \cdot \nabla\theta = -\chi'(s) \tag{2.41}$$

[12] I should like to thank Dr. A. Boozer for valuable discussions on this point.

s is the volume of the magnetic surface. Because of axisymmetry the coordinate θ does not depend on the toroidal angle φ and the equation for θ is

$$\mathbf{B}_p \cdot \nabla\theta = -\chi'(s) \quad ; \quad \mathbf{B}_p = \nabla\varphi \times \nabla\chi \tag{2.42}$$

or

$$(\nabla s \times \nabla\varphi) \cdot \nabla\theta = 1 \tag{2.43}$$

which leads to

$$\theta = \theta_o + \int_{\varphi=\text{const.}} \frac{r\,dl}{|\nabla s|} \tag{2.44}$$

The volume of a magnetic surface is $s = \int^s \oint r\,dl/|\nabla s| ds$ and therefore

$$\oint \frac{r\,dl}{|\nabla s|} = 1 \tag{2.45}$$

The path of integration is $\varphi =$ const. on the flux surface $\chi =$ const. Thus, the period of the poloidal Hamada coordinate is 1. The base vector $\mathbf{e_t} = \nabla\theta \times \nabla s$ points in toroidal direction and because of axisymmetry the relations hold: $\mathbf{e_t} \cdot \nabla\chi = 0$ and $\mathbf{e_t} \cdot \mathbf{B}_p = 0$. This leads to

$$\mathbf{B} \cdot \mathbf{e_t} = (J(\chi) + \oint \mathbf{B} \cdot \mathbf{dl}) \tag{2.46}$$

In tokamaks the product $\mathbf{B} \cdot \mathbf{e_t}$ is a flux function and equal to the poloidal current. This leads to the following form of the plasma currents in tokamaks

$$\boxed{\mathbf{j} = -p'(\psi)\frac{\nabla s \times \mathbf{B}}{B^2} - \frac{p'(\psi)}{\chi'(s)}(J(\chi) + \oint \mathbf{B} \cdot \mathbf{dl})\left\{\frac{1}{B^2} - \frac{1}{< B^2 >}\right\}\mathbf{B} + \frac{< \mathbf{j} \cdot \mathbf{B} >}{< B^2 >}\mathbf{B}} \tag{2.47}$$

The following ansatz of the toroidal Hamada coordinate ζ

$$\zeta = \varphi + f(\theta, s) \tag{2.48}$$

leads to

$$\mathbf{B}_p \cdot \nabla f = \psi'(s) - \mathbf{B} \cdot \nabla\varphi \tag{2.49}$$

or

$$\frac{\partial f}{\partial\theta}\chi'(s) = \psi'(s) - \mathbf{B} \cdot \nabla\varphi \tag{2.50}$$

This differential equation has a periodic solution since

$$\psi'(s) = < \mathbf{B} \cdot \nabla\varphi > = \oint \mathbf{B} \cdot \nabla\varphi\, d\theta \tag{2.51}$$

Explicitly f is

$$f(\theta, s) = f_o(s) + \frac{1}{\chi'(s)}\int (\psi'(s) - \mathbf{B} \cdot \nabla\varphi)\, d\theta \tag{2.52}$$

The poloidal base vector $\mathbf{e_p}$ is

$$\mathbf{e_p} = \nabla s \times \nabla\varphi + \frac{\partial f}{\partial\theta}\nabla s \times \nabla\theta \tag{2.53}$$

or

$$\mathbf{e_p} = \nabla s \times \nabla\varphi - \frac{1}{\iota}\Big(1 - \frac{\mathbf{B}\cdot\nabla\varphi}{\psi'(s)}\Big)\,\mathbf{e_t} \tag{2.54}$$

From this form of the base vector we derive

$$< \mathbf{e_p}\cdot\mathbf{e_p} > = < |\nabla s|^2|\nabla\varphi|^2 > + \frac{1}{\iota^2} < \Big(1 - \frac{\mathbf{B}\cdot\nabla\varphi}{\psi'(s)}\Big)^2\,\mathbf{e_t}\cdot\mathbf{e_t} > \tag{2.55}$$

The second term is the Pfirsch-Schlüter enhancement arising from the Pfirsch-Schlüter currents. The first term describes the classical diffusion in a tokamak. Summarising all terms yields the plasma current in the following form

$$\mathbf{j} = -p'(\psi)\nabla s \times \nabla\varphi + \frac{p'(\psi)}{\iota}(1 - \frac{\mathbf{B}\cdot\nabla\varphi}{\psi'(s)})\,\mathbf{e_t} + I'(s)\mathbf{B} \tag{2.56}$$

In this representation of the plasma current all three components are divergence-free.

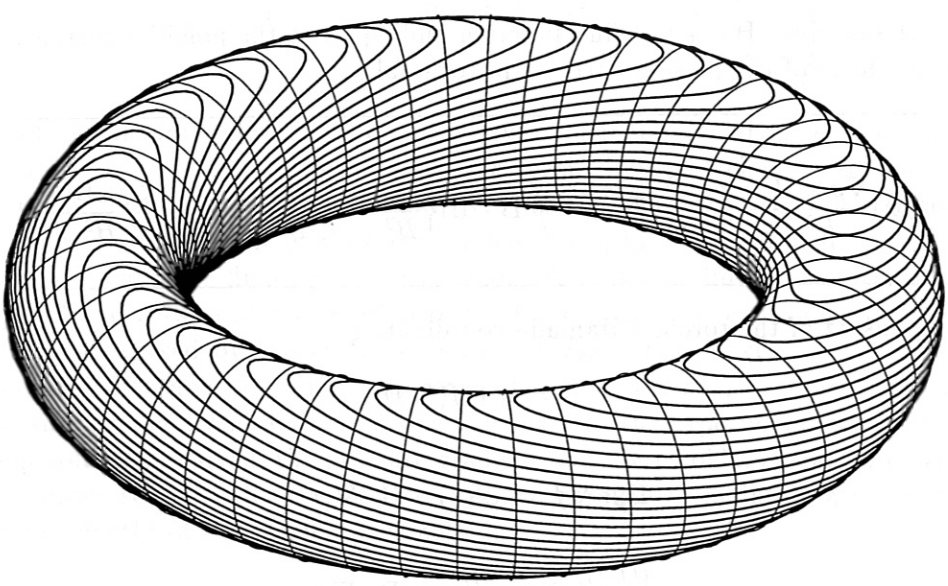

Figure 3 *Magnetic surface of a axisymmetric equilibrium. The thick lines denote the Hamada coordinate lines* ζ*=const. or the* $\mathbf{e_p}$*-lines. Aspect ratio* $A = 4$, $q = 1/\iota = 2.6$.

2.3 Stationary Equilibrium

In the preceding section we anticipated the existence of an ideal equilibrium satisfying the force balance Eq. 2.1. This model may be justified in a slowly diffusing plasma where the velocity has little influence on the the force balance. However, experiments in tokamaks and stellarators indicate the existence of a poloidal and toroidal rotation. In particular, shear flow arises in the edge region and therefore inertial and viscous forces have to be taken into account for a proper description of the stationary state. These forces, in general, are small compared with ∇p and $\mathbf{j} \times \mathbf{B}$, however in the tangential plane of the pressure surface these forces are the only ones.

In the following we consider a plasma in stationary equilibrium and include viscosity and resistivity as dissipative processes. In order to maintain a steady state, a particle source compensates the plasma losses as already outlined in the previous section. The force balance in stationary equilibrium is

$$\nabla \cdot \rho \mathbf{v} : \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} - \nabla \cdot \pi \tag{2.57}$$

ρ is the mass density and π the anisotropic pressure tensor. In order to truncate the hierarchy equations this tensor must be related to the plasma velocity $\mathbf{v}$, the detailed form of the vicous tensor valid in the collision dominated regime will be discussed in a later section. The other equations are Ohm's law

$$-\nabla \phi + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \tag{2.58}$$

the equation of continuity

$$\nabla \cdot \rho \mathbf{v} = S \tag{2.59}$$

and the equation of state

$$\rho = \rho(p, T) \tag{2.60}$$

Furthermore, a heat transport equation is necessary to compute the temperature, however, since we are interested in plasma rotation and particle transport we adopt the model of an isothermal plasma (T=constant).

An important issue in the context of stationary equilibria is the problem of existence and uniqueness of solutions. This issue will not be discussed here, for details the reader is referred to a paper by Spada and Wobig[13] . In the following we assume that solutions of these equations exist and analyse some consequences with respect to plasma rotation.

Unlike to ideal equilibria magnetic surfaces and pressure surfaces do not coincide in stationary equilibrium. Magnetic surfaces even need not exist in the rigorous sense, however, confinement of the plasma will strongly depend on the quality of magnetic surfaces and on a close coincidence of pressure surfaces and magnetic surfaces. In the following we assume that toroidally closed and nested pressure surfaces exist. The magnetic field may have a small component normal to the pressure surface. We define the average over the pressure surface by

$$< g >_p = \int_{p=const.} g \frac{df}{|\nabla p|} / \int \frac{df}{|\nabla p|} \tag{2.61}$$

We utilize the subscript p to distinguish this pressure surface average from the magnetic surface average $< .. >$ defined above. In general the plasma velocity will consist of a radial diffusive or convective velocity and a component tangential to the pressure surface $p =$ const. The latter one describes the plasma rotation which is driven by the inertial forces and damped by the viscous

[13] M. Spada, H. Wobig, *J. Phys. A: Math. Gen.* **25** (1992), 1575

forces. To demonstrate this we take the dot product of the momentum balance with the current $\mathbf{j}$ or the magnetic field $\mathbf{B}$ and average over the pressure surface. Because of $<\mathbf{j}\cdot\nabla p>_p=0$ and $<\mathbf{B}\cdot\nabla p>_p=0$ this procedure yields

$$\begin{aligned} <\mathbf{j}\cdot\nabla\cdot\rho\mathbf{v}:\mathbf{v}>_p &= -<\mathbf{j}\cdot\nabla\cdot\pi(\mathbf{v})>_p \\ <\mathbf{B}\cdot\nabla\cdot\rho\mathbf{v}:\mathbf{v}>_p &= -<\mathbf{B}\cdot\nabla\cdot\pi(\mathbf{v})>_p \end{aligned} \tag{2.62}$$

Since these equation are non-linear in velocity the inertial forces couple all three components of the velocity, toroidal and poloidal rotation are coupled to the radial diffusion and vice versa. Averaging the force balance over magnetic surfaces yields a similar result, however, since $<\mathbf{j}\cdot\nabla p>\neq 0$ in general, these equations are

$$\begin{aligned} <\mathbf{j}\cdot\nabla\cdot\rho\mathbf{v}:\mathbf{v}>+<\mathbf{j}\cdot\nabla p> &= -<\mathbf{j}\cdot\nabla\cdot\pi(\mathbf{v})> \\ <\mathbf{B}\cdot\nabla\cdot\rho\mathbf{v}:\mathbf{v}> &= -<\mathbf{B}\cdot\nabla\cdot\pi(\mathbf{v})> \end{aligned} \tag{2.63}$$

The surface average $<\mathbf{B}\cdot\nabla p>$ is zero for all p. The inertial forces may be written as

$$\nabla\cdot\rho\mathbf{v}:\mathbf{v} = \mathbf{v}S+\rho\nabla\frac{v^2}{2}-\rho\mathbf{v}\times\vec{\omega} \tag{2.64}$$

$\vec{\omega}=\nabla\times\mathbf{v}$ is the vorticity of the velocity $\mathbf{v}$. The first term arises from the equation of continuity, the second term describes the centrifugal forces and the last term in this equation represents the Coriolis force. The centrifugal forces contribute only little to the integrated force balance. Since in our isothermal model the density surface coincides with the pressure surface we find

$$<\rho\mathbf{j}\cdot\nabla\frac{v^2}{2}>_p=-\rho<\nabla\cdot\mathbf{j}\frac{v^2}{2}>_p=-\rho\frac{d}{dp}\int\frac{v^2}{2}\mathbf{j}\cdot d\mathbf{f}/\int df/|\nabla p| \tag{2.65}$$

This integral is zero if the plasma current flows in the pressure surface. The main contribution comes from the Coriolis force parallel to the plasma current or to the magnetic field and we may write the integrated momentum balance in the form

$$\boxed{\begin{aligned} <\mathbf{j}\cdot\mathbf{v}S>_p+<\rho\mathbf{v}\cdot(\mathbf{j}\times\vec{\omega})>_p &= -<\mathbf{j}\cdot\nabla\cdot\pi>_p \\ <\mathbf{B}\cdot\mathbf{v}S>_p+<\rho\mathbf{v}\cdot(\mathbf{B}\times\vec{\omega})>_p &= -<\mathbf{B}\cdot\nabla\cdot\pi>_p \end{aligned}} \tag{2.66}$$

Apart from the first term which describes the momentum loss caused by ionisation the dominating forces are the Coriolis force and the viscous force, these forces determine the poloidal and toroidal rotation of the plasma. The Coriolis forces provide a coupling of the radial diffusive flux to the tangential velocities. This is the essential feature of the Stringer spin-up mechanism, in Stringer's formulation, however, the radial derivative in the inertial forces is neglected. Such a coupling of radial, meridional and azimutal motion also exists in the atmosphere of the earth. The integrated momentum balance equations exhibit a strong similarity to the momentum balance of the zonal circulation in the earth atmosphere driven by the meridional motion in the

Hadley cell [14]. On rotating planets the vorticity consists of the rigid rotation $\vec{\Omega}$ and the relative vorticity. The angular velocity of the planet is the dominating factor in the Coriolis forces. In plasmas, however, there is only a small zeroth order vorticity: the slow diamagnetic rotation of the ions. The observed fast rotation in plasmas results from a spin-up mechanism which amplifies the the small initial rotation. This will be described in a later section.

The viscous forces provide a damping mechanism which tends to slow down the rotation of the plasma. As shown by Hassam and Kulsrud the slowing down is mainly due to magnetic pumping and it has to be expected that the magnetic field structure plays an important role in this mechanism. For this reason we analyse the viscous forces in more detail.

2.3.1 Viscous Forces

The viscous forces in the momentum balance depend on the collisionality of the plasma. In a collision dominated plasma the theory of Braginskii is applicable. In this theory the viscous stress π is locally linked to the rate of strain tensor

$$W = \frac{1}{2}(\nabla : \mathbf{v}_j + (\nabla : \mathbf{v}_j)^T) - \frac{1}{3}\nabla \cdot \mathbf{v}_j \mathbf{I} \tag{2.68}$$

In a neoclassical plasma the viscous forces have to be calculated from kinetic theory. In a strong magnetic field the distribution function may be approximated by $f(v_{||}, v_\perp)$ and the viscous stress is

$$\pi_{ik} = (p_{||} - p_\perp)(b_i b_k - \frac{1}{3}\delta_{ik}) \tag{2.69}$$

b is the unit vector along the magnetic field lines. Eq.(2.69) is the Chew-Goldberger-Low[15] form of the pressure tensor which is valid for all regimes of collisionality. The viscosity tensor of a collision dominated plasma is given by Braginskii, in this theory the leading term of the viscosity tensor yields:

$$p_{||} - p_\perp = -3\tau\, p \sum_{l,m}(b_l b_m \frac{\partial v_l}{\partial x_m} - \frac{1}{3}\frac{\partial v_m}{\partial x_m}) \tag{2.70}$$

τ is the like particle collision time. The full Braginskii viscosity tensor contains 5 terms which are of the order 1, $(\omega\tau)^{-1}$ and $(\omega\tau)^{-2}$. The term used here is of the order one and describes the bulk viscosity. In chapter 5 the effect of the other terms will be investigated.

Another approach to the anisotropic pressure $p_{||} - p_\perp$ is based on the magnetic pumping mechanism or gyro-relaxation effect [16]. In the double adiabatic theory the variation of $p_{||}$ and $p_\perp$ is given by

$$\begin{aligned} \frac{1}{p_\perp}\frac{dp_\perp}{dt} &= -2\nabla \cdot \mathbf{v} + \mathbf{b} : \mathbf{b} \cdot \nabla : \mathbf{v} \\ \frac{1}{p_{||}}\frac{dp_{||}}{dt} &= -\nabla \cdot \mathbf{v} - 2\mathbf{b} : \mathbf{v} \cdot \nabla : \mathbf{v} \end{aligned} \tag{2.71}$$

[14] E. Palmén, C.W. Newton, *Atmospheric Circulation Systems*, Academic Press, New York, London, 1969
The meridional profile of the azimutal velocity u is determined by

$$f = \frac{u}{a}\tan\varphi - \frac{\partial u}{\partial y} + \frac{F_{xy} + F_{xz}}{\rho v} \tag{2.67}$$

v is the meriodional velocity, y the meridional coordinate (φ = geographic latitude), $f = 2\Omega \sin\varphi$ is the Coriolis parameter, F_{xy} and F_{xz} are the viscous forces. The meridional velocity v is the northward or southward component of the Hadley cell. Ω is the angular velocity of the earth, a the radius of the earth.

[15] C.F. Chew, M.L. Goldberger, F.E. Low, *Proc. Roy. Soc.* A236, 112 (1956)

[16] A. Schlüter, *Der Gyro-Relaxations Effekt* Z. Naturforschg. **12a**, 822-825 (1957)

Using the approximation $p_{\|} \approx p_{\perp} \approx p$ we obtain

$$\frac{dp_{\perp}}{dt} - \frac{dp_{\|}}{dt} = 3p(\mathbf{b}:\mathbf{b}\cdot\nabla:\mathbf{v} - \frac{1}{3}\nabla\cdot\mathbf{v}) = 3p\sum b_l b_m W_{lm} \tag{2.72}$$

$\{W_{lm}\}$ is the rate of strain tensor which has zero trace. It was suggested by A. Schlüter[17] that this equation be modified by the gyro-relaxation effect leading to the following equation:

$$\frac{dp_{\|}}{dt} - \frac{dp_{\perp}}{dt} + 3p\sum b_l b_m W_{lm} = -\nu(p_{\|} - p_{\perp}) \tag{2.73}$$

Under steady state condition this equation becomes

$$\mathbf{v}\cdot\nabla(p_{\|} - p_{\perp}) + 3p\sum b_l b_m W_{lm} = -\nu(p_{\|} - p_{\perp}) \tag{2.74}$$

If the collisions are frequent enough the convective term can be neglected and the difference in the pressure becomes

$$p_{\|} - p_{\perp} = -3p\tau\sum\{b_m b_l - \frac{1}{3}\delta_{lm}\}W_{lm} \tag{2.75}$$

This approximation leads to the same result as the bulk viscosity of Braginskii.

2.3.2 Surface Averaged Viscous Forces

Let us return to the viscous stress tensor in the CGL-approximation:

$$\pi_{ik} = (p_{\|} - p_{\perp})(b_i b_k - \frac{1}{3}\delta_{ik}) \tag{2.76}$$

which holds in strong magnetic fields without restriction to the collisionality regime. In the force balance we need the component of $\nabla\cdot\pi$ parallel to $\mathbf{j}$ and $\mathbf{B}$. Let be $\mathbf{A}$ one these vectors, then the viscous force parallel to $\mathbf{A}$ is

$$\mathbf{A}\cdot\nabla\cdot\pi = \mathbf{A}\cdot\nabla\cdot\left((p_{\|} - p_{\perp})\frac{\mathbf{B}:\mathbf{B}}{B^2}\right) - \frac{1}{3}\mathbf{A}\cdot\nabla(p_{\|} - p_{\perp}) \tag{2.77}$$

The first term on the right hand side can also be written as

$$\mathbf{B}\cdot\nabla\left(\frac{p_{\|} - p_{\perp}}{B^2}\mathbf{A}\cdot\mathbf{B}\right) - \frac{(p_{\|} - p_{\perp})}{B^2}\mathbf{B}\cdot\mathbf{B}\nabla\mathbf{A} \tag{2.78}$$

In case of $\mathbf{A} = \mathbf{B}$ we find $\mathbf{B}\cdot\mathbf{B}\nabla\mathbf{B} = \mathbf{B}\cdot\nabla B/B$. If $\mathbf{A} = \mathbf{j}$ a similar relation holds. To demonstrate this we start from $\mathbf{j}\times\mathbf{B} \approx \nabla p$ and take the curl of this equation. This procedure yields $\mathbf{j}\cdot\nabla\mathbf{B} \approx \mathbf{B}\cdot\nabla\mathbf{j}$. Therefore we may approximate the first term on the right hand side in both cases by

$$\mathbf{B}\cdot\nabla\left(\frac{p_{\|} - p_{\perp}}{B^2}\mathbf{A}\cdot\mathbf{B}\right) - \frac{(p_{\|} - p_{\perp})}{B^2}\mathbf{B}\cdot\mathbf{A}\nabla\mathbf{B} \tag{2.79}$$

In summary the viscous force parallel to $\mathbf{A}$ is approximately

$$\mathbf{B}\cdot\nabla\left(\frac{p_{\|} - p_{\perp}}{B^2}\mathbf{A}\cdot\mathbf{B}\right) - \frac{(p_{\|} - p_{\perp})}{B^2}\mathbf{A}\cdot\frac{\nabla B}{B} - \frac{1}{3}\mathbf{A}\cdot\nabla(p_{\|} - p_{\perp}) \tag{2.80}$$

[17] A. Schlüter, *Lectures on Plasma Physics*, SS 1959, University München, unpublished

The surface average over the terms with $\mathbf{B} \cdot \nabla ..$ and $\mathbf{A} \cdot \nabla ..$ is neglible since it depends on the radial components of $\mathbf{B}$ and $\mathbf{j}$. Summarizing the results yields the the surface averaged viscous forces in the following form

$$< \mathbf{A} \cdot \nabla \cdot \pi > \approx - < (p_{\|} - p_{\perp}) \mathbf{A} \cdot \frac{\nabla B}{B} > \tag{2.81}$$

In case of an ideal equilibrium this approximation is an equality.

It is obvious that this relation holds for any linear combination $\mathbf{A} = a(p)\mathbf{j} + b(p)\mathbf{B}$. Plasma equilibria with an ignorable coordinate are of special interest since there a direction exists where the magnetic pumping effect is zero. This is the toroidal direction in tokamaks and a helical direction in helically invariant stellarators. In summary the integrated momentum balance is

$$\begin{aligned} < \mathbf{j} \cdot \mathbf{v} S > + < \rho \mathbf{v} \cdot (\mathbf{j} \times \vec{\omega}) > &= < (p_{\|} - p_{\perp}) \mathbf{j} \cdot \frac{\nabla B}{B} > \\ < \mathbf{B} \cdot \mathbf{v} S > + < \rho \mathbf{v} \cdot (\mathbf{B} \times \vec{\omega}) > &= < (p_{\|} - p_{\perp}) \mathbf{B} \cdot \frac{\nabla B}{B} > \end{aligned} \tag{2.82}$$

In toroidal stellarators an invariant direction does not exist in general, therefore magnetic pumping is always present. There exists, however, a class of quasi-helically invariant stellarators [18] where a direction $\mathbf{A}$ with $\mathbf{A} \cdot \nabla B \approx 0$ exists. Magnetic pumping would not inhibit plasma rotation in this direction. Another method to reduce magnetic pumping and to facilitate poloidal plasma rotation is to reduce ∇B parallel to the plasma current. In stellarators without toroidal net current the current lines are either poloidally closed or they ergodically fill small ring-shaped domains. Therefore any minimisation of the poloidal variation of the magnetic field reduces the magnetic pumping and favours poloidal plasma rotation. Such a reduction of the poloidal component of ∇B is one of the main features of the Helias concept[19]. In the following chapters the flux-friction relations and the momentum balance equations will be extended to a multi-fluid plasma consisting of N particle species. Furthermore, it will be analysed how turbulent effects and inertial forces modify the flux-friction relations.

2.4 Time Evolution of Plasma Rotation

In the previous section the stationary equilibrium has been discussed in the one-fluid model. As analysed for the first time by T.E. Stringer a static equilibrium is unstable against a poloidal rotational perturbation[20]. This mechanism has been investigated by many authors and has gained renewed interest in context with the H-mode effect. In deriving the spin-up equations use has been made of the special geometry of tokamaks and the issue arises how the equations can be generalised to an arbitrary stellarator geometry.

For this reason we use the same approximations as in the tokamak theory and start from the momentum balance equation

$$\frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot \rho \mathbf{v} : \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} - \nabla \cdot \pi \tag{2.83}$$

and the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = S \tag{2.84}$$

[18] J. Nührenberg, R. Zille, *Phys. Letters A* Vol. 129 , No. 2 (1988) 113
[19] J. Nührenberg, R. Zille, *Phys. Letters A*, Vol 114, (1986), 129
[20] T.E. Stringer, IAEA-Conf. 1971

Ohm's law is

$$-\nabla\Phi + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j} \tag{2.85}$$

The momentum balance can also be written in the form

$$\rho(\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \mathbf{j}\times\mathbf{B} - \nabla\cdot\pi - \mathbf{v}\,S \tag{2.86}$$

In lowest order the flow velocity is tangential to the magnetic surfaces and follows from

$$-\nabla\Phi_o(\psi) + \mathbf{v}_o\times\mathbf{B} = 0 \quad ; \quad \nabla\cdot\mathbf{v}_o = 0 \tag{2.87}$$

The solution of this equation is

$$\mathbf{v}_o = -\Phi_o'(\psi)\,\mathbf{e_p} \tag{2.88}$$

$E = \Phi_o'(\psi)$ is the radial electric field. Multiplying the momentum balance with the perpendicular component of $\mathbf{e_p}$ yields the equation

$$\rho\mathbf{e}_{p,\perp}\cdot\frac{d\mathbf{v}_o}{dt} = -\mathbf{e}_{p,\perp}\cdot\nabla p + \mathbf{j}\cdot\nabla\psi - \mathbf{e}_{p,\perp}\cdot\nabla\cdot\pi - \mathbf{e}_{p,\perp}\cdot\mathbf{v}_o S \tag{2.89}$$

The parallel component of $\mathbf{e_p}$ yields

$$\rho\mathbf{e}_{p,\|}\cdot\frac{d\mathbf{v}_o}{dt} = -\mathbf{e}_{p,\|}\cdot\nabla p - \mathbf{e}_{p,\|}\cdot\nabla\cdot\pi - \mathbf{e}_{p,\|}\cdot\mathbf{v}_o S \tag{2.90}$$

Averaging Eq. 2.89 over the magnetic surface leads to

$$<\rho\mathbf{e}_{p,\perp}\cdot\frac{d\mathbf{v}_o}{dt}> = -<\mathbf{e}_{p,\perp}\cdot\nabla p> - <\mathbf{e}_{p,\perp}\cdot\nabla\cdot\pi> - <\mathbf{e}_{p,\perp}\cdot\mathbf{v}_o S> \tag{2.91}$$

The first term on the right hand side is the Stringer spin-up term. In order to get a non-zero term the pressure must be inhomogeneous on magnetic surfaces. This term can also be written in the form

$$\begin{aligned} -<\nabla p\cdot\frac{\nabla\psi\times\mathbf{B}}{B^2}> &= <p\nabla\cdot\frac{\nabla\psi\times\mathbf{B}}{B^2}> \\ &= <p\nabla\psi\times\mathbf{B}\cdot\nabla\frac{1}{B^2}> \\ &= <p\frac{|\nabla\psi|}{B}\kappa_g> \end{aligned} \tag{2.92}$$

κ_g is the geodesic curvature of the magnetic field line. The role of the geodesic curvature in the spin-up mechanism of tokamaks has also been pointed out by D.R. McCarthy et al. [21]. In tokamaks with $\mathbf{B} = \nabla\psi\times\nabla\varphi + J(\psi)\nabla\varphi$ we obtain

$$<\nabla p\cdot\frac{\nabla\psi\times\mathbf{B}}{B^2}> = J<\nabla p\cdot\frac{\nabla\psi\times\nabla\varphi}{B^2}> = <\frac{J}{B^2}\mathbf{B}_p\cdot\nabla p> \tag{2.93}$$

Using the approximation $B^2 \propto 1/R^2$ yields

$$<\mathbf{e}_{p,\perp}\cdot\nabla p> \propto <R^2\,\mathbf{B}_p\cdot\nabla p> \tag{2.94}$$

[21] D.R. McCarthy, J.F. Drake, P.N. Guzdar and A.B. Hassam, *Phys Fluids*, B5, (1993) 1188

(see also A.B. Hassam and J.F. Drake[22]). In this perpendicular momentum balance the quadratic inertial forces are neglected and inserting the explicit form of $\mathbf{v}_o$ yields

$$-<\rho \mathbf{e}_{p,\perp}\cdot \mathbf{e}_{p,\perp}> \frac{\partial E}{\partial t} = -<\mathbf{e}_{p,\perp}\cdot \nabla p> - <\mathbf{e}_{p,\perp}\cdot \nabla\cdot \pi> - <\mathbf{e}_{p,\perp}\cdot \mathbf{v}_o S> \quad (2.95)$$

In the parallel equation 2.90 the inertial force are not neglected and summing up the two equations leads to

$$-<\rho\, \mathbf{e_p}\cdot \mathbf{e_p}> \frac{\partial E}{\partial t} - <\rho \mathbf{e}_{p,\|}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}> = -<\mathbf{e_p}\cdot\nabla\cdot\pi> - <\mathbf{e_p}\cdot \mathbf{v}_o S> \quad (2.96)$$

In this formulation it is the parallel inertial force which plays the key role in the spin-up mechanism. As has been shown in the previous section the factor $<\rho \mathbf{e_p}\cdot\mathbf{e_p}>$ can be approximated by

$$<\rho\, \mathbf{e_p}\cdot \mathbf{e_p}> \approx <\rho\, \mathbf{e}_{p,\perp}\cdot \mathbf{e}_{p,\perp}> (1+\frac{2}{\iota^2}) \quad (2.97)$$

The derivation given above is not the most general one since the net toroidal flow velocity has been neglected. The general solution of eqs. 2.87 is

$$\mathbf{v}_o = -E(\psi)\,\mathbf{e_p} + \Lambda(\psi)\mathbf{B} \quad (2.98)$$

The flux function $\Lambda(\psi)$ describes the net toroidal flux of $\mathbf{v}_o$. Averaging the momentum balance with $\mathbf{e_p}\cdot ..$ and $\mathbf{B}\cdot ..$ yields the two equations

$$\begin{aligned} <\rho\, \mathbf{e_p}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> + <\rho\, \mathbf{e_p}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}> &= -<\mathbf{e_p}\cdot\nabla\cdot\pi> - <\mathbf{e_p}\cdot \mathbf{v}S> \\ <\rho \mathbf{B}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> + <\rho\mathbf{B}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}> &= -<\mathbf{B}\cdot\nabla\cdot\pi> - <\mathbf{B}\cdot \mathbf{v}S> \end{aligned} \quad (2.99)$$

Inserting the velocity $\mathbf{v}_o$ from Eq.2.98 leads to

$$\begin{aligned} <\rho\, \mathbf{e_p}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> &= -<\rho\, \mathbf{e_p}\cdot \mathbf{e_p}> \frac{\partial E}{\partial t} + <\rho \mathbf{e_p}\cdot \mathbf{B}> \frac{\partial \Lambda}{\partial t} \\ <\rho\, \mathbf{B}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> &= -<\rho\, \mathbf{B}\cdot \mathbf{e_p}> \frac{\partial E}{\partial t} + <\rho\, \mathbf{B}\cdot\mathbf{B}> \frac{\partial \Lambda}{\partial t} \end{aligned} \quad (2.100)$$

Using the approximation of constant density we may also write

$$\begin{pmatrix} <\rho\, \mathbf{e_p}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> \\ -<\rho\, \mathbf{B}\cdot \frac{\partial \mathbf{v}_o}{\partial t}> \end{pmatrix} = -\rho_o(\psi) \begin{pmatrix} <\mathbf{e_p}\cdot\mathbf{e_p}> & I \\ I & <B^2> \end{pmatrix} \begin{pmatrix} \frac{\partial E}{\partial t} \\ \frac{\partial \Lambda}{\partial t} \end{pmatrix} \quad (2.101)$$

The inertial forces have already been analysed in the previous section, they are

$$\begin{aligned} <\rho\, \mathbf{e_p}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}> &= <\rho_o \mathbf{v}\cdot(\mathbf{e_p}\times\vec{\omega})> \\ <\rho\, \mathbf{B}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}> &= <\rho_o \mathbf{v}\cdot(\mathbf{B}\times\vec{\omega})> \end{aligned} \quad (2.102)$$

[22] A.B. Hassam and J.F. Drake, *Phys. Fluids*, B5, (1993), 4022

This formulation shows that the Coriolis forces are the driving forces for poloidal or toroidal rotation. It is interesting to note that the lowest order flow $\mathbf{v}_o$ cannot excite any rotation. As has already been shown by Bineau[23] the surface averaged term $< \rho_o \mathbf{B} \cdot \nabla \cdot \mathbf{v}_o : \mathbf{v}_o >$ is zero. This also holds if we replace $\mathbf{B}$ by $\mathbf{e_p}$. As will be shown in a later chapter, this result also holds if we approximate $\rho_o \mathbf{v}$ by $\rho_o \mathbf{v}_o$ in the Coriolis force and leave the vorticity quite general. The conclusion is that a radial velocity $\mathbf{v}_1$ is needed to get a poloidal or toroidal acceleration. The vorticity of the lowest order flow is

$$\vec{\Omega} = \nabla \times \mathbf{v}_o = -E \nabla \times \mathbf{e}_p + \mathbf{e}_p \times \nabla E + \Lambda \mathbf{j} - \mathbf{B} \times \nabla \Lambda \tag{2.103}$$

and we obtain

$$\begin{aligned} -\mathbf{e}_p \times \vec{\Omega} &= E \mathbf{e}_p \times \nabla \times \mathbf{e}_p + (E_j' \mathbf{e}_p \cdot \mathbf{e}_p + \Lambda I' - \Lambda'(\mathbf{B} \cdot \mathbf{e}_p)) \nabla \psi \\ -\mathbf{B} \times \vec{\Omega} &= E \mathbf{B} \times \nabla \times \mathbf{e}_p + (E' \mathbf{B} \cdot \mathbf{e}_p + \Lambda P' - \Lambda' B^2) \nabla \psi \end{aligned} \tag{2.104}$$

In summary the surface averaged Coriolis forces are

$$\begin{aligned} - < \rho_o \mathbf{v}_1 \cdot (\mathbf{e}_p \times \vec{\Omega}) > &= R_{11} E + R_{12} \Lambda + K_{11} E' + K_{12} \Lambda' \\ < \rho_o \mathbf{v}_1 \cdot (\mathbf{B} \times \vec{\Omega}) > &= R_{21} E + R_{22} \Lambda + K_{21} E' + K_{22} \Lambda' \end{aligned} \tag{2.105}$$

The coefficients in this equation are

$$\begin{aligned} R_{11} &= < \rho_o \mathbf{v}_1 (\mathbf{e}_p \times \vec{\omega}_p) > \\ R_{12} &= < \rho_o \mathbf{v}_1 \cdot \nabla \psi > I' \\ R_{21} &= - < \rho_o \mathbf{v}_1 (\mathbf{B} \times \vec{\omega}_p) > \\ R_{22} &= - < \rho_o \mathbf{v}_1 \cdot \nabla \psi > P' \end{aligned} \tag{2.106}$$

and

$$\begin{aligned} K_{11} &= < (\rho_o \mathbf{v}_1 \cdot \nabla \psi) \, \mathbf{e}_p \cdot \mathbf{e}_p > \\ K_{12} &= < (\rho_o \mathbf{v}_1 \cdot \nabla \psi) \, \mathbf{e}_p \cdot \mathbf{B} > \\ K_{21} &= - < (\rho_o \mathbf{v}_1 \cdot \nabla \psi) \, \mathbf{e}_p \cdot \mathbf{B} > \\ K_{22} &= - < (\rho_o \mathbf{v}_1 \cdot \nabla \psi) \, B^2 > \end{aligned} \tag{2.107}$$

$\omega_p = \nabla \times \mathbf{e}_p$ is the vorticity of the base vector $\mathbf{e}_p$. Using these matrices the Coriolis forces can be summarized in the following form

$$\begin{pmatrix} - < \rho_o \mathbf{v}_1 \cdot (\mathbf{e}_p \times \vec{\Omega}) > \\ < \rho_o \mathbf{v}_1 \cdot (\mathbf{B} \times \vec{\Omega}) > \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} E \\ \Lambda \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \dfrac{\partial E}{\partial \psi} \\ \dfrac{\partial \Lambda}{\partial \psi} \end{pmatrix} \tag{2.108}$$

[23] M. Bineau, *Phys. Fluids*, **10**, (1967), 1540

The spin-up equations (without damping terms) are now

$$
-\quad \rho_o(\psi) \begin{pmatrix} <\mathbf{e_p}\cdot\mathbf{e_p}> & I \\ I & <B^2> \end{pmatrix} \begin{pmatrix} \dfrac{\partial E}{\partial t} \\ \dfrac{\partial \Lambda}{\partial t} \end{pmatrix}
$$

$$
= \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} E \\ \Lambda \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \dfrac{\partial E}{\partial \psi} \\ \dfrac{\partial \Lambda}{\partial \psi} \end{pmatrix} \tag{2.109}
$$

These coupled equations describe the evolution of the radial electric field and the toroidal net flux Λ .

These equations show that the radial derivative of E and Λ also occurs which is not the case in Stringer's original paper on the spin-up. In Stringer's paper the time derivative of E is only proportional to E and not to E'. The reason is that the parallel inertial force $\mathbf{B}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}$ is approximated by

$$
\frac{v_o}{r}\frac{\partial v_{||}}{\partial \theta} \quad ; \quad v_o = \frac{\Phi_o'(r)}{B} \tag{2.110}
$$

For this reason the spin-up equation takes the simple form

$$
(1+\frac{2}{\iota^2})\frac{\partial v_o}{\partial t} \propto v_o \tag{2.111}
$$

As shown above the velocity $\mathbf{v_1} = \mathbf{v} - \mathbf{v}_o$ is the component of $\mathbf{v}$ which leads to the spin-up of the flow velocity $\mathbf{v}_o$. This velocity $\mathbf{v_1}$ consists of the diffusion velocity $\mathbf{v}_D$ and the parallel velocity $\mathbf{v}_{||}$. These are linked by the equation of continuity

$$
\nabla\rho_o\mathbf{v}_D + \nabla\rho\mathbf{v}_{||} = S \tag{2.112}
$$

Usually the parallel velocity $\mathbf{v}_{||}$ is larger than the diffusive velocity. If we approximate $\mathbf{v_1}$ by its parallel component only, all terms with $\mathbf{v_1}\cdot\nabla\psi$ are zero and the K-matrix 2.107 is zero. In the R-matrix 2.106 only the first term R_{11} is non-zero and the spin-up equations become

$$
-\rho_o(\psi) \begin{pmatrix} <\mathbf{e_p}\cdot\mathbf{e_p}> & I \\ I & <B^2> \end{pmatrix} \begin{pmatrix} \dfrac{\partial E}{\partial t} \\ \dfrac{\partial \Lambda}{\partial t} \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ \Lambda \end{pmatrix} \tag{2.113}
$$

which leads to

$$
-\rho_o\left(<\mathbf{e_p}\cdot\mathbf{e_p}><B^2> - I^2\right)\frac{\partial E}{\partial t} = <B^2>R_{11}E \tag{2.114}
$$

and

$$
I\,E + <B^2>\Lambda = Const. \tag{2.115}
$$

Eq. 2.115 states that any poloidal rotation E leads to a toroidal net flux Λ and a toroidal rotation of the plasma. In a stellarator without toroidal current ($I = 0$), however, this linkage between E and Λ is zero and we obtain $\Lambda = const.$. There is no toroidal spin-up in a stellarator in this approximation. The radial component of $\mathbf{v}_1$, $(\mathbf{v}_1 \cdot \nabla\psi \neq 0)$, however, restores this linkage, and the coupling between poloidal and toroidal fluxes persists. In the approximation $\mathbf{v}_1 \approx \mathbf{v}_{1,\|}$ we obtain the standard spin-up equations in general toroidal geometry. In analysing the coefficient R_{11} we write the parallel velocity of $\mathbf{v}_1$ in the form $\mathbf{v}_1 \approx u\mathbf{B}$ and using $\mathbf{B} \times \mathbf{e_p} = \nabla\psi$ we find

$$\begin{aligned} R_{11} &= <\rho_o u \nabla\psi \cdot \vec{\omega}_p> \\ &= <\rho_o u \nabla \cdot (\mathbf{e_p} \times \nabla\psi)> \end{aligned} \tag{2.116}$$

where the parallel velocity is the solution of the magnetic differential equation

$$\rho_o \mathbf{B} \cdot \nabla u = -\nabla \cdot \rho_o \mathbf{v}_D + S \tag{2.117}$$

In axisymmetric tokamak geometry this approximation coincides with the equation eq. 32a

$$(1 + 2q^2) \frac{\partial E}{\partial t} = \frac{2q}{r} E \overline{\sin\theta\, u} \quad ; \quad q = \frac{1}{\iota} \tag{2.118}$$

in the paper by Hassam and Drake[24]. Because of $\mathbf{B} = \psi'(s)\,\mathbf{e_t} + \chi'(s)\,\mathbf{e_p}$ and $\nabla \cdot (\mathbf{B} \times \nabla\psi) = 0$ we have

$$\nabla \cdot (\mathbf{e_p} \times \nabla\psi) = -\frac{1}{\iota} \nabla \cdot (\mathbf{e_t} \times \nabla\psi) \tag{2.119}$$

and therefore

$$R_{11} = -\frac{1}{\iota} <\rho_o u \nabla \cdot (\mathbf{e_t} \times \nabla\psi)> \tag{2.120}$$

The surface integral $<\nabla \cdot (\mathbf{e_t} \times \nabla\psi)>$ is zero and in tokamak geometry this terms reduces to

$$\nabla \cdot (\mathbf{e_t} \times \nabla\psi) \propto \sin\theta \tag{2.121}$$

which yields the equation mentioned above.

Another form of $\nabla \cdot (\mathbf{e_p} \times \nabla\psi)$ is obtained by inserting $\nabla\psi = \mathbf{B} \times \mathbf{e_p}$ which yields

$$\mathbf{e_p} \times \nabla\psi = \mathbf{B}\,\mathbf{e_p}^2 - \mathbf{e_p}\,(\mathbf{e_p} \cdot \mathbf{B}) \tag{2.122}$$

and

$$\nabla \cdot (\mathbf{e_p} \times \nabla\psi) = \mathbf{B} \cdot \nabla \mathbf{e_p}^2 - \mathbf{e_p} \cdot \nabla(\mathbf{B} \cdot \mathbf{e_p}) \tag{2.123}$$

This formulation explicitely shows the role of the parallel component of $\mathbf{e_p} : \mathbf{e_p} \cdot \mathbf{B}$ which is related to the Pfirsch-Schlüter currents (see Eq. 2.15).

In the discussion above we have neglected the damping effect by magnetic pumping. The viscous forces have been analysed in the previous section (see eq. 2.81) and to complete the simplified spin-up equations by the damping forces we write

$$-\rho_o(\psi) \begin{pmatrix} <\mathbf{e_p} \cdot \mathbf{e_p}> & I \\ I & <B^2> \end{pmatrix} \begin{pmatrix} \frac{\partial E}{\partial t} \\ \frac{\partial \Lambda}{\partial t} \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ \Lambda \end{pmatrix} + \begin{pmatrix} <(p_\| - p_\perp)\,\mathbf{e_p} \cdot \frac{\nabla B}{B}> \\ -<(p_\| - p_\perp)\,\mathbf{B} \cdot \frac{\nabla B}{B}> \end{pmatrix} \tag{2.124}$$

[24] A.B. Hassam, J.F. Drake, Phys. Fluids **B5**, (1993), 4024

The equation for the poloidal velocity E becomes

$$- \rho_o \left(< \mathbf{e_p} \cdot \mathbf{e_p} > < B^2 > -I^2\right) \frac{\partial E}{\partial t} = < B^2 > R_{11} E + < (p_\| - p_\perp) \mathbf{h} \cdot \frac{\nabla B}{B} > \tag{2.125}$$

where the vector $\mathbf{h}$ is $\mathbf{h} =< B^2 > \mathbf{e_p} + I\mathbf{B}$. In a stellarator without net current ($I = 0$) the damping term is simplified to $< p_\| - p_\perp < B^2 > \mathbf{e_p} \cdot \nabla B/B >$ and the spin-up equation is

$$-\rho_o < \mathbf{e_p} \cdot \mathbf{e_p} > \frac{\partial E}{\partial t} = R_{11} E + < (p_\| - p_\perp) \mathbf{e_p} \cdot \frac{\nabla B}{B} > \tag{2.126}$$

In conclusion we find that in stellarator configurations with small $\mathbf{e_p} \cdot \nabla B$ magnetic pumping is reduced and therefore a poloidal spin-up is easier to achieve than in standard stellarators. As will be shown in later chapters the term $< (p_\| - p_\perp) \mathbf{e_p} \cdot \frac{\nabla B}{B} >$ is proportional to the neoclassical particle flux through a magnetic surface. Thus, Helias configurations with reduced neoclassical transport see a small damping against poloidal spin-up. Since neoclassical transport in tokamaks is smaller than in stellarators this implies also a smaller damping against poloidal spin-up, which may be one of the reasons why H-mode operation in tokamaks can be achieved more easily than in stellarators. In a collision dominated plasma the viscous damping is described by the Braginskii viscosity which is linear in the velocity $\mathbf{v}_o$, however, in the long mean free path regime, where the term $p_\| - p_\perp$ must be calculated from a kinetic equation, the radial electric field modifies the particle orbits and the viscous damping depends non-linearily from the radial electric field E. This non-linearity is the reason for multiple solutions of the saturated state the bifurcation phenomenon. The effect of lost orbits is also included in the term $p_\| - p_\perp$.

Non-linearities also arise in a collision dominated plasma if the inertial forces are retaind to a higher order. If we abandon the approximation $\rho = \rho_o(\psi)$ in the centripetal forces, these can be written as

$$- < \frac{v^2}{2} \mathbf{e_p} \cdot \nabla \rho > \quad ; \quad - < \frac{v^2}{2} \mathbf{B} \cdot \nabla \rho > \tag{2.127}$$

The variation of ρ within the magnetic surface can be found approximately from

$$\begin{aligned} \rho_o \mathbf{e_p} \cdot (\mathbf{v}_o \cdot \nabla) \mathbf{v}_o &= -c_s^2 \mathbf{e_p} \cdot \nabla \rho \\ \rho_o \mathbf{B} \cdot (\mathbf{v}_o \cdot \nabla) \mathbf{v}_o &= -c_s^2 \mathbf{B} \cdot \nabla \rho \end{aligned} \tag{2.128}$$

The non-linear saturated state of the poloidal rotation has been analysed by many authors utilysing the axisymmetric tokamak geometry. The zeroth order density distribution has been computed by Zehrfeld and Green[25] and by Hazeltine, Lee and Rosenbluth[26]. Numerical calculations of the spin-up by Green et al.[27] showed that the rotation saturation in the vicinity of the poloidal sound speed. The results of Zehrfeld and Green have been generalised to stellarator geometry by Kovrizhnikh and Shchepetov[28], however, in this paper the specific features of non-axisymmetry are eliminated by averaging the equations in toroidal direction.

[25] H.P. Zehrfeld, B.J. Green, *Phys Rev. Lett.* **25** (1969), 961
[26] R.D. Hazeltine, E.P. Lee, M.N. Rosenbluth, *Phys. Fluids* **14** (1971) 361
[27] J.M. Green, J.L. Johnson, K. E. Weimer, N.K. Winsor, *Phys. Fluids* **14** (19710, 1258
[28] L.M. Kovrizhnikh, S.V. Shchepetov, *Nuclear Fusion* **29** (1989), 667

Chapter 3

The Multi-Fluid Model

3.1 Basic Equations

Let us consider a magnetic field $\mathbf{B}$ with nested and toroidally closed magnetic surfaces $\psi =$ const. This may be either a vacuum field or a self-consistent equilibrium field satisfying the condition

$$\mathbf{j} = \nabla \times \mathbf{B} \quad , \quad \nabla \cdot \mathbf{B} = 0 \tag{3.1}$$

In this magnetic field we consider a turbulent plasma consisting of N particle species. The macroscopic quantities of this plasma are written as $f + \delta f(t)$ where f denotes the time averaged quantities and δf the fluctuating part. The plasma is assumed to be isothermal with a temperature T, the modifications due to temperature gradients will be discussed in a later chapter. The momentum balance of each particle species is

$$\boxed{m_j \frac{\partial n_j \mathbf{v}_j}{\partial t} + m_j \nabla \cdot n_j \mathbf{v}_j : \mathbf{v}_j = -\nabla p_j + q_j n_j \mathbf{E} + q_j n_j \mathbf{v}_j \times \mathbf{B} - \mathbf{F}_{j,1} - \nabla \cdot \pi_j} \tag{3.2}$$

with the following notations: $p_j = n_j kT =$ pressure of each particle species, $n_j =$ density, $q_j =$ charge, $\mathbf{v}_j =$ macroscopic velocity. $\pi_j =$ anisotropic part of the pressure tensor. The friction force [1] is

$$\mathbf{F}_{j,1} = \sum_k l_{11}^{jk} \mathbf{v}_k - \frac{2}{5} l_{12}^{jk} \frac{\mathbf{q}_k}{p_k} \tag{3.3}$$

$\mathbf{q}_j$ is the heat flux vector and the coefficients l^{jk} are the friction coefficients defined by

$$l_{11}^{jk} = \int m_j \mathbf{v} \cdot C_{jk}^l \frac{2\mathbf{v}}{v_k^2} d^3\mathbf{v} \quad ; \quad l_{12}^{jk} = \int m_j \mathbf{v} \left(\frac{v^2}{v_j^2} - \frac{2}{5} \right) \cdot C_{jk}^l \frac{2\mathbf{v}}{v_k^2} d^3\mathbf{v} \tag{3.4}$$

C_{jk}^l is the linearized Coulomb collision operator which links the particle species with index j and k. $v_j^2 = 2kT_j/m_j$ is the thermal velocity of the particle species j. In the following we will neglect the heat flux in the friction force, the effect of this term will be discussed in a later section. The friction force (without $\mathbf{q}_j$) can also be written as

$$\mathbf{F}_{j,1} = \sum_k \alpha_{jk} (\mathbf{v}_j - \mathbf{v}_k) \tag{3.5}$$

[1] S.P. Hirshman, D. Sigmar, *Nuclear Fusion* **21**, 1079 (1981)

Since Coulomb collisions conserve momentum the matrix α_{jk} is symmetric and the friction coefficients α_{jk} are given by

$$\alpha_{ik} = n_i\, n_k\, m_{ik}\, \frac{4}{3}\sqrt{2\pi}\, ln\, \Lambda \frac{q_i^2\, q_k^2}{\sqrt{m_{ik}}\, kT^{3/2}} \tag{3.6}$$

where $m_{ik} = m_i\, m_k/(m_i+m_k)$ is the reduced mass and $ln\Lambda$ the Coulomb logarithm. The inertial forces on the left hand side can be modified using the equation of continuity

$$\boxed{\frac{\partial n_j}{\partial t} + \nabla \cdot n_j \mathbf{v}_j = S_j} \tag{3.7}$$

where S_j is a source term describing all ionisation processes. We consider her the source term as a given term independent of time. The inertial forces are also written in the form

$$m_j n_j \frac{\partial \mathbf{v}_j}{\partial t} + m_j n_j \mathbf{v}_j \cdot \nabla \mathbf{v}_j + m_j S_j \mathbf{v}_j \tag{3.8}$$

Usually the source term is neglected, however it may be important in the boundary region where neutral gas provides a source for ions and electrons. Since the source S_j is positive (exept for recombination) this term tends to reduce the velocity of the fluid.

These equations describe the evolution of the plasma on all time scales. We consider a toroidal plasma in a stationary turbulent state. On a fast time scale the plasma instability has grown to a finite level where nonlinear effects stabilize a further increase. All plasma parameters consist of a time averaged term and a fluctuating term.

$$n_j \longrightarrow n_j + \delta n_j \quad , \quad p_j \longrightarrow p_j + \delta p_j \tag{3.9}$$

$$\mathbf{v}_j \longrightarrow \mathbf{v}_j + \delta \mathbf{v}_j \quad , \quad \mathbf{E} \longrightarrow \mathbf{E} + \delta \mathbf{E} \tag{3.10}$$

$$\mathbf{B} \longrightarrow \mathbf{B} + \delta \mathbf{B} \tag{3.11}$$

For simplicity we assume that the friction coefficients α_{ik} are time independent and constant on magnetic surfaces. Taking the average over the fast time scale eliminates all terms which are linear in the fluctuations and the result is the evolution equation on the slow time scale:

$$\xi_j = -\nabla p_j + q_j n_j \mathbf{E} + q_j n_j \mathbf{v}_j \times \mathbf{B} - \sum_k \alpha_{jk} (\mathbf{v}_j - \mathbf{v}_k) - \nabla \cdot \pi_j \tag{3.12}$$

The term ξ includes also the quadratic terms arising from the turbulence

$$\xi_j = m_j n_j \frac{\partial \mathbf{v}_j}{\partial t} + \overline{m_j (n_j + \delta n_j)(\mathbf{v}_j + \delta \mathbf{v}_j) \cdot \nabla (\mathbf{v}_j + \delta \mathbf{v}_j)} - q_j\, \overline{\delta n_j \delta \mathbf{E}} - \overline{\delta \mathbf{j}_j \times \delta \mathbf{B}} + m_j S_j \mathbf{v}_j \tag{3.13}$$

$\delta \mathbf{j} = \delta q_j n_j \mathbf{v}_j$ is the fluctuating part of the current of the particle species j. Explicitly the inertial term is

$$n_j \mathbf{v}_j \cdot \nabla \mathbf{v}_j + \overline{n_j \delta \mathbf{v}_j \cdot \nabla \delta \mathbf{v}_j} + \overline{\delta n_j \delta \mathbf{v}_j} \cdot \nabla \mathbf{v}_j + \overline{\delta n_j \mathbf{v}_j \cdot \nabla \delta \mathbf{v}_j} + O(\delta^3) \tag{3.14}$$

Turbulence gives rise to three extra forces: the second term on the right hand side in Eq.(3.13) includes the turbulent Reynolds stresses, the third one arises from the fluctuating electric fields and the last one is the turbulent force by fluctuating magnetic fields. Furthermore, we need the equation of continuity on the slow time scale

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \overline{n_j \mathbf{v}_j} = S_j \quad ; \quad \overline{n_j \mathbf{v}_j} = n_j \mathbf{v}_j + \overline{\delta n_j \delta \mathbf{v}_j} \tag{3.15}$$

These equations describe a stationary state and a slow instability of this state. This could be the rotational instability of a diffusing plasma which spins up according to the Stringer mechanism or driven by the poloidal Reynolds stresses arising from the turbulence. We will not consider here any reaction of the stationary solution on the turbulence level which is a characteristic feature of the L-H-transition in tokamaks.

In the frame of these equations several scenarios can be analysed

- A quiescent equilibrium under the influence of inertial forces and viscous forces.
- A stationary turbulent equilibrium under the influence of additional turbulent forces.
- Rotational instability of a stationary equilibrium.

The momentum balance of the whole plasma is obtained by summing up the momentum balance of all particle species

$$\sum_j \xi_j = -\nabla p + \mathbf{j} \times \mathbf{B} - \sum_j \nabla \cdot \pi_j \tag{3.16}$$

Because of quasineutrality $\sum q_j n_j = 0$ and the momentum conservation of Coulomb interaction the electric field and the friction forces do not occur. Neglecting the inertial forces and the viscous forces yields the familiar equation describing the ideal equilibrium. This equation shows that pressure surfaces, magnetic surfaces and current surfaces in general do not coincide. In stationary equilibrium the magnetic surface may exhibit islands and stochasticity while pressure surfaces are smooth and nested.

In the following we assume that a stationary equilibrium exists and that pressure surfaces p=const. are nested toroidal surfaces. The ∇p-force is mainly balanced by the $\mathbf{j} \times \mathbf{B}$ force, in radial direction the inertial and viscous forces are small. The tangential force balance, however, is governed by these forces. To demonstrate this we average Eq. 3.16 over the pressure surface. Multiplying this equation with $\mathbf{B}$ or $\mathbf{j}$ and averaging yields the equations

$$\sum_j < \mathbf{B} \cdot \xi_j >_p = - \sum_j < \mathbf{B} \cdot \nabla \cdot \pi_j >_p \tag{3.17}$$

and

$$\sum_j < \mathbf{j} \cdot \xi_j >_p = - \sum_j < \mathbf{j} \cdot \nabla \cdot \pi_j >_p \tag{3.18}$$

The subscript p indicates that averaging takes place over the pressure and not over the magnetic surface (which may not exist everywhere). The average values of $< \mathbf{j} \cdot \nabla p >$ and $< \mathbf{B} \cdot \nabla p >$ are zero because of $\nabla \cdot \mathbf{j} = 0$ and $\nabla \cdot \mathbf{B} = 0$. These equations 3.17 and 3.18 also hold if we take a linear combination $\mathbf{a} = f(p)\mathbf{B} + g(p)\mathbf{j}$ instead of $\mathbf{B}$ or $\mathbf{j}$.

In a two-component plasma consisting of hydrogen and electrons this balance is mainly the balance of ions, because of small mass electrons play a minor role and the inertial and viscous forces are those of the hydrogen. As will be shown later the viscous forces tend to slow down any rotation of the plasma. If a finite poloidal or toroidal rotation of the plasma exists, it must be driven by the inertial or turbulent components in ξ_j. The motion of the plasma tangential to the isobars has a strong similarity to zonal circulation in the atmospheres of planets which is driven by the combined effect of Coriolis forces and cellular convection. In plasma such a rotation is of particular interest in conjunction with the development of shear flow in boundary regions and the effect on radial plasma transport.

3.2 Expansion Technique

Since we are mainly interested in the momentum balance of a toroidal plasma we adopt the model of an isothermal plasma with a given temperature. The equation of state links pressure and density

$$p_j = n_j kT. \tag{3.19}$$

In general, an energy equation would be needed to compute the temperature separately. The source terms S_j describe all ionisation and recombination processes. Since the charge is conserved by these processes, we get:

$$\sum_j q_j S_j = 0 \tag{3.20}$$

in every volume element. In steady state the conservation of charge leads to the ambipolarity condition

$$\sum_j q_j \Gamma_j = 0 \tag{3.21}$$

with Γ_j being the particle flux through a magnetic surface

$$\Gamma_j = \int n_j \mathbf{v}_j \cdot d\mathbf{f} \tag{3.22}$$

$d\mathbf{f}$ is the surface element on a magnetic surface. The plasma current density is

$$\mathbf{j} = \sum n_j q_j \mathbf{v}_j \quad \text{with} \quad \nabla \cdot \mathbf{j} = 0 \tag{3.23}$$

In a given magnetic field these equations determine the density, the flow velocity of each particle species and the electric field. However, in practice solutions can only be found approximately by an expansion technique. The main difficulty arises from the first order and second order derivatives of $\mathbf{v}$ in Eq. (3.2) which makes this equation a nonlinear differential equation rather than an algebraic equation. The expansion technique proposed by Shaing and Callen neglects these nonlinear and dissipative terms in lowest order. Furthermore all source terms are omitted in lowest order. Mathematically, this procedure may be critizised, since higher order derivatives of the flow velocity are neglected in lowest order. Boundary layer effects, which may arise around rational magnetic surfaces, cannot be treated by this method. Following the method applied by Shaing and Callen we expand all unknown quantities into two terms

$$\begin{aligned} p_j &\longrightarrow P_j + p_j \\ \mathbf{v}_j &\longrightarrow \mathbf{V}_j + \mathbf{v}_j \\ \Phi &\longrightarrow \Phi + \phi \\ n_j &\longrightarrow N_j + n_j \end{aligned}$$

Here $P_j, \mathbf{V}_j, \Phi, N_j$ are the lowest order quantities and $p_j, \mathbf{v}_j, \phi, n_j$ are the perturbations. Φ and ϕ are the potential of the electric field.

Lowest Order Equations

In lowest order the momentum balance and the equation of continuity are:

$$0 = -\nabla P_j + q_j N_j(-\nabla\Phi + \mathbf{V}_j \times \mathbf{B}) \tag{3.24}$$

and

$$\nabla \cdot N_j \mathbf{V}_j = 0 \tag{3.25}$$

with $P_j = N_j kT$. In lowest order the density N_j and the electric potential Φ are functions of the magnetic surface ψ = const and the lowest order flow is incompressible

$$\nabla \cdot \mathbf{V}_j = 0 \tag{3.26}$$

Defining the functions

$$U_j(\psi) = \frac{kT}{q_j}\, ln\, N_j(\psi) + \Phi(\psi) \tag{3.27}$$

and

$$E_j(\psi) := U_j'(\psi) = \frac{kT}{q_j}\frac{N_j'}{N_j} + \Phi'(\psi) \tag{3.28}$$

the momentum balance in lowest order can be written

$$0 = -\nabla\, U_j(\psi) + \mathbf{V}_j \times \mathbf{B} \tag{3.29}$$

The parallel component of this equation yields $\mathbf{B}\cdot\nabla U = 0$, which implies that in zeroth order the flow of each particle species stays in the magnetic surface and is determined by the flow potential function $U_j(\psi)$. At this point it may be noted that the existence of closed magnetic surfaces is required by the specific expansion technique which solves Eq.(3.29) instead of Eq.(3.2). A rigorous treatment of Eq.(3.2) would not anticipate the existence of magnetic surfaces, however, the solution would strongly depend on the topology of $\mathbf{B}$. In the following calculations the two vectors $\mathbf{B}$ and $\mathbf{e}_p$ will be used as base vectors on the magnetic surface instead of the two vectors $\mathbf{e}_p$ and $\mathbf{e}_t$. Using the two incompressible base vectors $\mathbf{e}_p$ and $\mathbf{B}$ the zeroth order flow velocity can be written

$$\boxed{\mathbf{V}_j = -E_j(\psi)\,\mathbf{e}_p + \Lambda_j(\psi)\,\mathbf{B}} \tag{3.30}$$

In lowest order these functions $E_j(\psi)$ and $\Lambda_j(\psi)$ are undetermined, they have to be calculated from the first order equations. $\Lambda_j(\psi)$ is the toroidal net flow of the particle species and the toroidal plasma current between two adjacent magnetic surfaces is

$$d\,I = \sum q_j N_j\, \Lambda_j(\psi)\, d\psi \tag{3.31}$$

From Eq.(3.30) we obtain the plasma current in the following form

$$\mathbf{j} = -P'(\psi)\,\mathbf{e}_p + I'(\psi)\mathbf{B} \tag{3.32}$$

$P = \sum P_j$ is the total pressure of the plasma. This formulation of the plasma current is particularly convenient to identify the role of the net toroidal current $I(\psi)$.

First Order Equations

In order to find conditions for $\Lambda_j(\psi)$ and $E_j(\psi)$ the first order equations have to be taken into account:

$$
\boxed{\begin{aligned} q_j N_j \mathbf{E}_o - q_j N_j \nabla\phi + q_j N_j \mathbf{v}_j \times \mathbf{B} &= \sum_j \alpha_{jk}(\mathbf{V}_j - \mathbf{V}_k) - \nabla\cdot\pi_j \\ &+ \xi_j + \nabla p_j + \frac{n_j}{N_j}\nabla P_j \end{aligned}} \tag{3.33}
$$

The induced electric field $\mathbf{E}_o$ is also of first order since it is needed to compensate the collisional dissipation. The matrix elements α_{ik} depend on the lowest order densities and are constant on the magnetic surface. The structure of these first order equations is the same as Ohm's law in chapter 2, therefore the same procedure can be applied to derive the flux-friction relations in a multi-species plasma. The equation of continuity is in first order

$$
\nabla\cdot(N_j\mathbf{v}_j + n_j\mathbf{V}_j) = S_j \tag{3.34}
$$

Chapter 4

Flux-Friction Relations

4.1 Surface Averaged Momentum Balance

In lowest order the functions $E_j(\psi)$ and $\Lambda_j(\psi)$ remain undetermined, they must be calculated from the first order equations. The required periodicity of the first order quantities $n_j, \mathbf{v}_j, p_j, \phi$ leads to conditions of integrability and to the desired relations between $\Lambda_j(\psi)$ and $E_j(\psi)$. For this purpose the first order momentum balance in the magnetic surface is considered and by averaging the momentum balance over the magnetic surface the surface functions $E_j(\psi)$ and $\Lambda_j(\psi)$ are correlated to the radial particle flux. This averaging process described in chapter II will be applied to the first order equations Eqs. 3.33 with $\mathbf{a} = \mathbf{B}$ and $\mathbf{a} = \mathbf{e}_p$ as surface vectors. The result of the averaging process is

$$\begin{aligned} q_j N_j < \mathbf{a} \cdot \mathbf{E_o} > - f' \, q_j < N_j \mathbf{v}_j \cdot \nabla \psi > &= \sum \alpha_{jk} < \mathbf{a} \cdot (\mathbf{V}_j - \mathbf{V}_k) > + < \mathbf{a} \cdot \nabla \cdot \pi > \\ &+ < \mathbf{a} \cdot \xi_j > \end{aligned} \quad (4.1)$$

Furthermore, we obtain $< \mathbf{e}_p \cdot \nabla \phi >= 0$ and $< \mathbf{e}_p \cdot \mathbf{E}_o >= 0$ since there is no poloidal loop voltage. Similar relations are also valid for the toroidal base vector $\mathbf{e_t}$: $< \mathbf{e_t} \cdot \nabla \phi >= 0$, $< \mathbf{e_t} \cdot \nabla p >= 0$ but $< \mathbf{e_t} \cdot \mathbf{E}_o > \neq 0$ because of the toroidal loop voltage. The radial particle flux is also

$$q_j N_j < \mathbf{e_t} \cdot (\mathbf{v}_j \times \mathbf{B}) > = \iota(\psi) < q_j N_j \mathbf{v}_j \cdot \nabla \psi > = \frac{\iota}{V'(\psi)} q_j \Gamma_j \quad (4.2)$$

or

$$q_j N_j < \mathbf{e}_p \cdot (\mathbf{v}_j \times \mathbf{B}) > = - < q_j N_j \mathbf{v}_j \cdot \nabla \psi > = - \frac{1}{V'(\psi)} q_j \Gamma_j \quad (4.3)$$

Parallel Momentum Balance

By multiplying Eq.(3.33) with $\mathbf{B}$ and taking the surface average we obtain the parallel momentum balance of each particle species, which relates the integral toroidal flow $\Lambda_j(\psi)$ to the driving terms.

$$\sum_k \alpha_{jk} < \mathbf{B} \cdot (\mathbf{V}_j - \mathbf{V}_k) > - < \mathbf{B} \cdot \nabla \cdot \pi_j > = - < \mathbf{B} \cdot \xi_j > + < q_j N_j \mathbf{B} \cdot \mathbf{E}_o > \quad (4.4)$$

If the velocity $\mathbf{V}_j$ and $\mathbf{V}_k$ are replaced by Eq.(3.30) this equation is modified to

$$\begin{aligned}\sum_k \alpha_{jk}\{< B^2 > (\Lambda_j(\psi) - \Lambda_k(\psi)) \quad &+ \quad < \mathbf{B}\cdot\mathbf{e}_p > (E_j(\psi) - E_k(\psi))\} \\ + < \mathbf{B}\cdot\nabla\cdot\pi_j > \quad &= \quad - < \mathbf{B}\cdot\xi_j > + < q_j N_j \mathbf{B}\cdot\mathbf{E}_o >\end{aligned} \tag{4.5}$$

In a steady state stellarator this driving term $q_j N_j < \mathbf{B}\cdot\mathbf{E}_o >$ does not exist. Summing up this equation over all particles makes the first terms in Eq.(4.6) vanish due to the momentum conservation of Coulomb interaction and the result is

$$\sum_j < \mathbf{B}\cdot\nabla\cdot\pi_j > \quad = \quad -\sum_j < \mathbf{B}\cdot\xi_j > \tag{4.6}$$

Also the toroidal electric field has dropped out because of charge neutrality.

Poloidal Momentum Balance

By taking the vector $\mathbf{e}_p$ instead of $\mathbf{B}$ the averaging process yields the momentum balance in poloidal direction.

$$\frac{1}{V'(\psi)} q_j \Gamma_j \; = \; \sum_k \alpha_{jk} < \mathbf{e}_p\cdot(\mathbf{V}_j - \mathbf{V}_k) > + < \mathbf{e}_p\cdot\nabla\cdot\pi_j > + < \mathbf{e}_p\cdot\xi_j > \tag{4.7}$$

or

$$\begin{aligned}-\frac{1}{V'(\psi)} q_j \Gamma_j \quad &= \quad \sum_k \alpha_{jk} < \mathbf{e}_p\cdot\mathbf{e}_p > (E_j(\psi) - E_k(\psi)) \\ &- \quad \sum_k \alpha_{jk} < \mathbf{B}\cdot\mathbf{e}_p > (\Lambda_j(\psi) - \Lambda_k(\psi)) \\ &- \quad < \mathbf{e}_p\cdot\nabla\cdot\pi_j > - < \mathbf{e}_p\cdot\xi_j >\end{aligned} \tag{4.8}$$

The first term in Eq.(4.8) describe the classical and the Pfirsch-Schlüter diffusion. In conventional derivations of the collisional diffusion the lowest order flow is divided in a perpendicular flow and a parallel flow which leads to the classical diffusion flux and the Pfirsch-Schlüter diffusion flux. However, if we make use of the natural decomposition of the lowest order flow in components parallel to $\mathbf{e}_p$ and $\mathbf{B}$, both diffusion fluxes are treated together and described by one geometrical coefficient $< \mathbf{e}_p\cdot\mathbf{e}_p >$. In standard stellarators with large aspect ratio this coefficient is proportional to $1 + 2/\iota^2$. Any reduction of this geometrical coefficient by proper choice of the magnetic field reduces the collisional diffusion of all particles species. Therefore also the collisional inward diffusion of impurity ions is reduced by this effect. The second term is the classical pinch effect. This particle flux is proportional to the toroidal fluxes $\Lambda_k(\psi)$ and the the toroidal current $I(\psi)$. In stellarators with zero toroidal current this term does not arise.

The neoclassical diffusive flux is described by the third term.

$$\frac{1}{V'(\psi)} q_j \Gamma_j = < \mathbf{e}_p\cdot\nabla\cdot\pi_j > . \tag{4.9}$$

In this fluid description the neoclassical flux is proportional to the surface averaged poloidal viscous force, whereas the classical and Pfirsch-Schlüter diffusion is determined by the poloidal frictional force. The formulation of the neoclassical diffusion in terms of the viscous tensor holds for all regimes of collisionality, however in every regime another approximation to the viscous tensor must be found. All turbulent effects and the diffusion caused by external momentum sources are summarised in the last term. Eq.(4.8) shows that the poloidal inertial force, the poloidal electric field and the poloidal $\delta\mathbf{j} \times \delta\mathbf{B}$-forces may lead to anomalous diffusion fluxes. The anomalous diffusion flux is

$$< \mathbf{e}_p \cdot \xi_j > = < \mathbf{e}_p \cdot \nabla \cdot \overline{m_j n_j \mathbf{v}_j : \mathbf{v}_j} > - < q_j \overline{\delta n_j \mathbf{e}_p \cdot \delta\mathbf{E}} > - < \mathbf{e}_p \cdot \overline{\delta\mathbf{j}_j \times \delta\mathbf{B}} > \qquad (4.10)$$

The first term in this relation describes the anomalous flux driven by the turbulent Reynolds stresses, the second term is the anomalous flux due to poloidal electric field fluctuations and the last term arises from magnetic field fluctuations Summing up the fluxes yields the total poloidal momentum balance

$$\sum_j < \mathbf{e}_p \cdot \nabla \cdot \pi_j > = -\sum_j < \mathbf{e}_p \cdot \xi_j > \qquad (4.11)$$

To arrive at this relation the ambipolarity condition and the symmetry of the matrix α_{jk} has been used. It should be noted that the fluctuating electric field does not occur on the right hand side of this equation. Because of the quasineutrality the term $\sum \delta q_j n_j \delta\mathbf{E}$ is zero. The fluctuating electric field may be the dominant term in the anomalous radial flux, however it does not affect the poloidal force balance and the poloidal rotation of the total plasma.

The viscous forces may be simplified by using the Chew-Goldberger-Low representation of the pressure tensor. In this approximation the parallel and poloidal viscous forces are [1]

$$< \mathbf{B} \cdot \nabla \cdot \pi_j > \;=\; - < (p_{\|} - p_{\perp})_j \mathbf{B} \cdot \frac{\nabla B}{B} > \qquad (4.12)$$

and

$$< \mathbf{e}_p \cdot \nabla \cdot \pi_j > \;=\; - < (p_{\|} - p_{\perp})_j \mathbf{e}_p \cdot \frac{\nabla B}{B} > \qquad (4.13)$$

In axisymmetricc tokamaks these viscous forces are not independent of each other. Because of the symmetry the relation holds

$$\mathbf{B} \cdot \nabla B = B^{\theta} \mathbf{e}_p \cdot \nabla B \qquad (4.14)$$

and therefore

$$< \mathbf{B} \cdot \nabla \cdot \pi_j > = B^{\theta} < \mathbf{e}_p \cdot \nabla \cdot \pi_j > \qquad (4.15)$$

In a stellarator the toroidal current I(ψ) and the loop voltage are zero and therefore these equations may be simplified to

$$\boxed{\sum_k \alpha_{jk} < B^2 > (\Lambda_j(\psi) - \Lambda_k(\psi)) - < (p_{\|} - p_{\perp})_j \mathbf{B} \cdot \frac{\nabla B}{B} > = - < \mathbf{B} \cdot \xi_j >} \qquad (4.16)$$

and

$$\boxed{\begin{aligned} -\frac{1}{V'(\psi)} q_j \Gamma_j \;&=\; \sum_k \alpha_{jk} < \mathbf{e}_p \cdot \mathbf{e}_p > (E_j(\psi) - E_k(\psi)) \\ &+ \; < (p_{\|} - p_{\perp})_j \mathbf{e}_p \cdot \frac{\nabla B}{B} > - < \mathbf{e}_p \cdot \xi_j > \end{aligned}} \qquad (4.17)$$

[1] M Coronado, H. Wobig, *Phys. Fluids* **29**(2) (1986), 527

Summation over all particle species yields

$$\sum_j < (p_{\|} - p_{\perp})_j \mathbf{e}_p \cdot \frac{\nabla B}{B} > \; = \; \sum_j < \mathbf{e}_p \cdot \xi_j > \tag{4.18}$$

and

$$\sum_j < (p_{\|} - p_{\perp})_j \mathbf{B} \cdot \frac{\nabla B}{B} > \; = \; \sum_j < \mathbf{B} \cdot \xi_j > \tag{4.19}$$

The flux friction relations determine the unknown functions $E_j(\psi)$, $\Lambda_j(\psi)$ and the electric potential $\Phi(\psi)$ if we consider the radial fluxes and the toroidal loop voltage as given. These are 2N+1 equations, where N is the number of particle species. It should be noted that these equations are valid in all regimes of collisionality. The pressure anisotropy $p_{\|} - p_{\perp}$ and the forces ξ_j are the unknown quantities in these relations, they must be calculated separately in every collisionality regime.

4.2 Vector Notation

In order to facilitate further analysis it is convenient to introduce a shorter notation which avoids the summation over particle species. For this purpose we introduce the N-dimensional vector space R^N and the vectors

$$\vec{\Gamma} = \frac{1}{V'(\psi)} \begin{pmatrix} \vdots \\ q_j \Gamma_j \\ \vdots \end{pmatrix} \quad ; \quad \vec{E} = \begin{pmatrix} \vdots \\ E_j \\ \vdots \end{pmatrix} \quad ; \quad \vec{\Lambda} = \begin{pmatrix} \vdots \\ \Lambda_j \\ \vdots \end{pmatrix} \tag{4.20}$$

and the transport matrix

$$D_\alpha \; = \; \{\sum_l \alpha_{lj} \delta_{jk} - \alpha_{jk}\} \tag{4.21}$$

The viscous forces are summarized in the vectors

$$\vec{\Pi}_b = \begin{pmatrix} \vdots \\ < \mathbf{B} \cdot \nabla \cdot \pi_j > \\ \vdots \end{pmatrix} \quad ; \quad \vec{\Pi}_p = \begin{pmatrix} \vdots \\ < \mathbf{e}_p \cdot \nabla \cdot \pi_j > \\ \vdots \end{pmatrix} \tag{4.22}$$

and the driving forces

$$\vec{K}_b = \begin{pmatrix} \vdots \\ < \mathbf{B} \cdot \xi_j > \\ \vdots \end{pmatrix} \quad ; \quad \vec{K}_p = \begin{pmatrix} \vdots \\ < \mathbf{e}_p \cdot \xi_j > \\ \vdots \end{pmatrix} \tag{4.23}$$

Let us introduce three more vectors

$$\vec{e} = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \quad ; \quad \vec{q} = \begin{pmatrix} \vdots \\ q_j \\ \vdots \end{pmatrix} \quad ; \quad \vec{N}_q = \begin{pmatrix} \vdots \\ q_j N_j \\ \vdots \end{pmatrix} \tag{4.24}$$

and summarize the toroidal electric field in the vector $\vec{U}_L$

$$\vec{U}_L = \begin{pmatrix} \vdots \\ < q_j N_j \mathbf{B} \cdot \mathbf{E}_o > \\ \vdots \end{pmatrix} = \vec{N}_q < \mathbf{B} \cdot \mathbf{E}_o > \tag{4.25}$$

The scalar product in the vector space R^N is

$$\vec{a} \cdot \vec{b} = \sum_j^N a_j b_j \tag{4.26}$$

and in this notation the ambipolar condition is

$$\sum_j^N q_j \Gamma_j = V'(\psi) \vec{\Gamma} \cdot \vec{e} = 0 \Longrightarrow \vec{e} \cdot \vec{\Pi}_p = \vec{e} \cdot \vec{K}_p \tag{4.27}$$

and the parallel momentum balance

$$\vec{e} \cdot \vec{\Pi}_b = \vec{e} \cdot \vec{K}_b \tag{4.28}$$

The flux-friction relations are in this notation

$$\begin{aligned} -\vec{\Gamma} &= < \mathbf{e}_p \cdot \mathbf{e}_p > D_\alpha \vec{E} - < \mathbf{B} \cdot \mathbf{e}_p > D_\alpha \vec{\Lambda} - \vec{\Pi}_p - \vec{K}_p \\ \vec{U}_L &= < \mathbf{B} \cdot \mathbf{e}_p > D_\alpha \vec{E} + < B^2 > D_\alpha \vec{\Lambda} + \vec{\Pi}_b + \vec{K}_b \end{aligned} \tag{4.29}$$

or

$$\begin{pmatrix} -\vec{\Gamma} \\ \vec{U}_L \end{pmatrix} = D_\alpha \begin{pmatrix} < \mathbf{e}_p \cdot \mathbf{e}_p > & I(\psi) \\ I(\psi) & < B^2 > \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} -\vec{\Pi}_p \\ \vec{\Pi}_b \end{pmatrix} + \begin{pmatrix} -\vec{K}_p \\ \vec{K}_b \end{pmatrix} \tag{4.30}$$

The structure of the flux-friction relations Eqs. 4.30 is the same as the coresponding equations in the one-fluid model (Eq. 2.35). However, two more terms arise, these are the viscous terms Π_p, Π_b and the turbulent terms K_p, K_b. The resistivity is replaced by the matrix D_{alpha}. D_α is the classical transport matrix, its properties and the relation to the collision operator have been extensively discussed in the review paper of Hirshman and Sigmar[2]. The classical transport matrix is singular (Det D_α=0), this follows from the conservation of Coulomb collisions. The equation $D_\alpha \vec{x} = 0$ has the non-trivial solution $\vec{x} = \vec{e}$. For this reason there exists a unique solution of the inhomogeneous equation $D_\alpha \vec{x} = \vec{y}$ if the vector $\vec{y}$ is orthogonal to $\vec{e}$. This implies that the matrix D_α^{-1} exists in the subspace of R^N which is orthogonal to $\vec{e}$. One example is the vector $\vec{N}_q$ which is perpendicular to $\vec{e}$. $\vec{e} \cdot \vec{N}_q = 0$ is the immediate result of the quasineutrality of the plasma. Furthermore we obtain $\vec{e} \cdot \vec{U}_L = 0$ and therefore the vector $D_\alpha^{-1} \vec{U}_L$ exists.

[2]S.P. Hirshman, D.J. Sigmar *Nuclear Fusion* **21**, (1981), 1079. D_α is equivalent to the matrix l_{11}^{ab} in this paper

4.2.1 Generalised Ohm's Law

In that case we can eliminate the parallel fluxes and derive a generalised Ohm's law. The vector of parallel fluxes is

$$\vec{\Lambda} = \frac{1}{< B^2 >}\left\{D_\alpha^{-1}\vec{U}_L + < \mathbf{B}\cdot\mathbf{e}_p > \vec{E} - D_\alpha^{-1}(\vec{\Pi}_b + \vec{K}_b)\right\} \tag{4.31}$$

The toroidal plasma current is the sum of all particle currents

$$I'(\psi) = \sum_j q_j N_j \Lambda_j = \vec{N}_q \cdot \vec{\Lambda} \tag{4.32}$$

and we get Ohm's law in the following form

$$\boxed{< B^2 > I' = \vec{N}_q \cdot D_\alpha^{-1}\vec{U}_L + < \mathbf{B}\cdot\mathbf{e}_p > \vec{N}_q \cdot \vec{E} - \vec{N}_q \cdot D_\alpha^{-1}(\vec{\Pi}_b + \vec{K}_b)} \tag{4.33}$$

Because of the quasineutrality the electric potential drops out from $\vec{N}_q \cdot \vec{E}$ and we obtain

$$\vec{N}_q \cdot \vec{E} = \sum kT\, N_j' = P'(\psi) \tag{4.34}$$

The generalised Ohm's law correlates the toroidal plasma current to 4 driving forces. The first term on the right hand side of Eq.(4.33) is proportional to the toroidal loop voltage. The second term is proportional to the density gradients and the toroidal current $I(\psi) = - < \mathbf{e}_p \cdot \mathbf{B} >$, this is the classical bootstrap effect. The third term is the neoclassical bootstrap current drive represented by the parallel viscous forces. These forces, however, depend on the toroidal fluxes, this effect will lead to the neoclassical modification of the resistivity. The last term describes the driving forces arising from external momentum input, inertial forces and turbulent forces. An example of turbulent forces driving a current is the so-called α-effect of the solar dynamo. The classical bootstrap effect is proportional to the current I, therefore the bootstrap effect needs a seed current $I(0) \neq 0$ as initial condition, otherwise the equation $I' \propto I(\psi)$ leads to $I = 0$. For the first time this classical bootstrap effect has been described by Kruskal and Kulsrud[3].

4.3 The First Order Equations

In the previous chapter the first order equations have been used to derive integrability conditions which eliminate the ambiguity of the zeroth order solutions. When these conditions are satisfied the first order equations can be used to calculate the functions $p_j, n_j, \mathbf{v}_j$ and ϕ. These first order equations are

$$\begin{aligned}\nabla p_j + q_j N_j \nabla\phi &= \frac{n_j}{N_j}\nabla P_j(\psi) + q_j N_j \mathbf{v}_j \times \mathbf{B}) \\ &- \sum_j \alpha_{jk}(\mathbf{V}_j - \mathbf{V}_k) - \nabla\cdot\pi_j + q_j N_j \mathbf{E}_o - \xi_j\end{aligned} \tag{4.35}$$

and the equation of continuity

$$\nabla\cdot(N_j \mathbf{v}_j + n_j \mathbf{V}_j) = S_j \tag{4.36}$$

[3]M. Kruskal, R. Kulsrud, *Phys. Fluids* 1 (1958)

From the parallel component of the momentum balance we obtain a magnetic differential equation for the pressure p_j and the electric potential ϕ.

$$\mathbf{B}\cdot\nabla h_j = -\sum_j \alpha_{jk}\,\mathbf{B}\cdot(\mathbf{V}_j - \mathbf{V}_k) - \mathbf{B}\cdot\nabla\cdot\pi_j - \mathbf{B}\cdot\xi_j \tag{4.37}$$

where we have introduced $h_j = n_j\,kT + q_j N_j \phi$. When a solution h_j of this magnetic differential equation is given the density n_j is eliminated by the condition of quasineutrality:

$$\sum q_j n_j = 0 \Longrightarrow \sum q_j h_j = \phi \sum q_j^2 N_j \tag{4.38}$$

which determines the electric potential.

The momentum balance contains only the perpendicular component of the first order velocity $\mathbf{v}_j$. The parallel component of the first order flow can be calculated from the equation of continuity. With $\mathbf{v}_j = \mathbf{v}_{j,\perp} + \lambda_j\,\mathbf{B}$ the magnetic differential equation for λ_j is

$$N_j\,\mathbf{B}\cdot\nabla\lambda_j + \mathbf{V}_j\cdot\nabla n_1 + \nabla\cdot N_j\,\mathbf{v}_{j,\perp} = S_j \tag{4.39}$$

This magnetic differential equation contains two inhomogeneous terms S_j and ξ_j, the latter term is hidden in $\mathbf{v}_{j,\perp}$. Usually these inhomogeneous terms are localized in poloidal and toroidal direction. This is especially the case with neutral beam injection as the heating method. The beam is a localized source for particles and momentum to the plasma, and in Fourier analysing the source terms we obtain a whole spectrum of poloidal and toroidal modes. The general form of the magnetic differential equations is

$$\mathbf{B}\cdot\nabla f = g \tag{4.40}$$

where f stands for either p_j or λ_j and g for the remainder of Eqs.(4.37). In magnetic coordinates the operator $\mathbf{B}\cdot\nabla$ is

$$\mathbf{B}\cdot\nabla = C_o\,B^2\,(\iota\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}) \tag{4.41}$$

and the magnetic differential equations becomes

$$C_o\,(\iota\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi})f = \frac{g}{B^2} \tag{4.42}$$

Because the Fourier series of the right hand side contains a whole spectrum of poloidal and toroidal harmonics resonances occur on all rational magnetic surfaces with $l\iota - m = 0$ (l is the poloidal mode number and m the toroidal mode number). On these resonant magnetic surfaces the parallel flow velocity goes to infinity if the right hand side of the magnetic differential contains resonant coefficients. Because of the poloidal and the toroidal localisation of the source terms, however, these resonant terms always exist. This singular behaviour around resonant surfaces demonstrates the collapse of the ordering scheme. We have considered the viscous terms as small corrections and have treated them as being of first order. The viscous terms only contain the lowest order flow velocity V_j. This approximation has to be abandoned around resonant magnetic surfaces and thus the momentum balance (4.35) yields a differential equation for the velocity $\mathbf{v}_j$. The viscous dissipation introduces a damping term which prevents the parallel flow velocity from going to infinity. However, the parallel flow may be large and so is the parallel plasma current

$$\mathbf{j} = \sum_j q_j N_j \mathbf{v}_j\cdot\mathbf{b} \tag{4.43}$$

Such a large parallel current in the vicinity of rational magnetic surfaces could lead to a perpendicular magnetic field which perturbes the topology of the magnetic surfaces and creates islands and stochasticity. In low shear systems it can be expected that this resonance phenomenon is restricted to low order rational surfaces which are separeted from each other. With larger shear these resonant surfaces can overlap and enhanced losses may arise over a large part of the plasma radius. In the following, however, we assume the first order corrections to be small compared with the zeroth order quantities and calculate the plasma losses from the surface averaged first order equations.

Chapter 5

The Collision Dominated Plasma

5.1 Flux-Friction Relations in a Collisional Plasma

The flux-friction relations derived above do not represent a closed set which can be solved for the unknown functions $E_j(\psi)$ and $\Lambda_j(\psi)$. For this purpose the anisotropy of the pressure has to be calculated from the kinetic equations and its correlation to the macroscopic flow velocity $\mathbf{V}_j$ has to be found. Only in a collision dominated plasma the general solution of this problem has been established and is summarised in Braginskii's paper. In a neoclassical plasma details of the particle orbits are essential in determining the particle distribution function and thus the pressure anisotropy.

Surface Averaged Viscous Forces

In the following study we concentrate on the collision dominated regime and start from Braginskii's formulation of the viscosity

$$p_{\parallel} - p_{\perp} = -3\tau\, p \sum_{l,m} (b_l b_m \frac{\partial v_l}{\partial x_m} - \frac{1}{3}\frac{\partial v_m}{\partial x_m}) \tag{5.1}$$

and the lowest order flow velocity Eq.(3.30). Inserting 3.30 in Eq.(5.1) yields

$$(p_{\parallel} - p_{\perp})_j = -3\tau_j\, P_j \frac{\mathbf{B}}{B^2} \cdot \mathbf{B}\nabla\mathbf{V}_j \tag{5.2}$$

Because of $\nabla \times (\mathbf{V}_j \times \mathbf{B}) = 0$ we find

$$(\mathbf{B}\nabla)\mathbf{V}_j = (\mathbf{V}_j\nabla)\mathbf{B} \tag{5.3}$$

and therefore the anisotropy of the pressure is

$$\begin{aligned}(p_{\parallel} - p_{\perp})_j = & - 3\tau_j\, P_j \mathbf{V}_j \cdot \frac{\nabla B}{B} \\ & - 3\tau_j\, P_j \left(-E_j(\psi)\, \mathbf{e}_p \cdot \frac{\nabla B}{B} + \Lambda_j(\psi)\, \mathbf{B} \cdot \frac{\nabla B}{B}\right)\end{aligned} \tag{5.4}$$

This linear relation between $p_{\parallel} - p_{\perp}$ and the unknown functions E_j and $\Lambda_j(\psi)$ closes the momentum equation. For further use we introduce the following abbreviations

$$C_p = < (\mathbf{e}_p \cdot \frac{\nabla B}{B})^2 > \quad , \quad C_t = < (\mathbf{B} \cdot \frac{\nabla B}{B})^2 > \tag{5.5}$$

and

$$C_b = - < (\mathbf{e}_p \cdot \frac{\nabla B}{B})(\mathbf{B} \cdot \frac{\nabla B}{B}) > \tag{5.6}$$

Thus in a collision dominated plasma the poloidal and parallel viscous forces are

$$< \mathbf{e}_p \cdot \nabla \cdot \pi_j > = -3\tau_j P_j (C_p E_j + C_b \Lambda_j)$$

$$< \mathbf{B} \cdot \nabla \cdot \pi_j > = 3\tau_j P_j (C_b E_j + C_t \Lambda_j) \tag{5.7}$$

To abbreviate the notation we define the diagonal matrix

$$L = \{3\tau_j P_j \delta_{jk}\} \tag{5.8}$$

and write the viscous forces in the following form

$$\begin{pmatrix} -\vec{\Pi}_p \\ \vec{\Pi}_b \end{pmatrix} = L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} \tag{5.9}$$

Combining this equation with Eq.(4.30) yields the flux-friction relations of a collision dominated plasma:

$$\begin{pmatrix} -\vec{\Gamma} \\ \vec{U}_L \end{pmatrix} = \left[D_\alpha \begin{pmatrix} < \mathbf{e_p} \cdot \mathbf{e_p} > & I \\ I & < B^2 > \end{pmatrix} + L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \right] \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} -\vec{K}_p \\ \vec{K}_b \end{pmatrix} \tag{5.10}$$

These flux-friction relations represent a linear relation between the "forces" $\vec{E}, \vec{\Lambda}$ and the "fluxes" $-\vec{\Gamma}, \vec{U}_L$. The transport matrix which correlates these quantities has several symmetry properties. D_α is the classical transport matrix and symmetric, the matrix L is also symmetric. Furthermore, there exists a symmetry between bootstrap effect and pinch effect: the matrix $D_\alpha I(\psi) + C_b L$ describes both effects, bootstrap effect and pinch effect.

Inverting the system (5.10) yields the forces E_j, Λ_j in terms of the fluxes $\vec{\Gamma}$ and $\vec{U}_L$ and the external forces $\vec{K}_p, \vec{K}_b$. The formal solution of Eq. (5.10) is

$$\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} = \left[D_\alpha \begin{pmatrix} < \mathbf{e_p} \cdot \mathbf{e_p} > & I \\ I & < B^2 > \end{pmatrix} + L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \right]^{-1} \begin{pmatrix} -\vec{\Gamma} + \vec{K}_p \\ \vec{U}_L - \vec{K}_b \end{pmatrix} \tag{5.11}$$

In general the transport matrix T

$$T = D_\alpha \begin{pmatrix} < \mathbf{e}_p \cdot \mathbf{e}_p > & I(\psi) \\ I(\psi) & < B^2 > \end{pmatrix} + L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \tag{5.12}$$

is non-singular and therefore the system (5.10) can be inverted since T^{-1} exists.

Entropy Production

The entropy production rate of the one-fluid model has been derived in Eq. 2.37:

$$I'(\psi)U_L - p'(\psi)\Gamma = V'(\psi)\eta < \mathbf{j}\cdot\mathbf{j} > \tag{5.13}$$

The equivalent equation of the multi-fluid model is obtained from the flux-friction relations by introducing again the vector $\mathbf{V}_j$

$$\vec{U}_L \cdot \vec{\Lambda} - \vec{\Gamma} \cdot \vec{E} = \frac{1}{2}\sum_{jk} \alpha_{jk} < (\mathbf{V}_j - \mathbf{V}_k)^2 > + \sum_j 3\tau_j P_j < \left(\mathbf{V}_j \cdot \frac{\nabla B}{B}\right)^2 > . \tag{5.14}$$

The entropy production due to the forces $\vec{K}$ has been omitted here. The viscosity always increases the entropy production. However, there are cases where the viscous entropy production is zero. In case of achsial symmetry any uniform toroidal velocity $\mathbf{V}_j = \mathbf{V}_k$ has zero viscous entropy production, a toroidal rotation of all particle species is not damped.

Achsial Symmetry

However in case of achsial symmetry the coefficients C_p, C_b, C_t are proportional to each other. In Hamada coordinates the magnetic field is

$$\mathbf{B} = B^\theta \mathbf{e}_p + B^\zeta \mathbf{e}_\mathrm{t} \tag{5.15}$$

Because of the achsial symmetry we find $\mathbf{e}_\mathrm{t} \cdot \nabla B = 0$ and therefore

$$C_b = -B^\theta C_p \quad ; \quad C_t = -B^\theta C_b \tag{5.16}$$

In axisymmetric tokamaks the transport matrix

$$T = D_\alpha \begin{pmatrix} < \mathbf{e}_\mathrm{p} \cdot \mathbf{e}_\mathrm{p} > & I \\ I & < B^2 > \end{pmatrix} + LC_p \begin{pmatrix} 1 & -B^\theta \\ -B^\theta & (B^\theta)^2 \end{pmatrix} \tag{5.17}$$

is singular and the the homogeneous equation $T\vec{X} = 0$ of the system (5.10) has the solution $\vec{X} = (\vec{E}, \vec{\Lambda}) = (B^\theta \Lambda_0(\psi)\vec{e}, \Lambda_0(\psi)\vec{e})$; Λ_0 is an arbitrary flux function which describes a toroidal rotation of all particle species. The inhomogeneous term in Eq.(5.10) must be orthogonal to this homogeneous solution which yields

$$B^\theta\, \vec{e} \cdot \vec{K}_p - \vec{e} \cdot \vec{K}_b = 0 \tag{5.18}$$

Because of $\vec{e} \cdot \vec{\Gamma} = 0$ and $\vec{e} \cdot \vec{U}_L = 0$ the orthogonality condition only applies to external forces. The poloidal rotation in this case is zero, the poloidal velocity is

$$\begin{aligned} \mathbf{V}_p &= -E_j \mathbf{e}_p + \Lambda_j B^\theta \mathbf{e}_p \\ &= -\Lambda_0 B^\theta \mathbf{e}_p + \Lambda_0 B^\theta \mathbf{e}_p = 0 \end{aligned} \tag{5.19}$$

The magnetic pumping effect which is described by the coefficient C_p slows down any poloidal rotation if there is no driving term. If the magnetic field has a helical or quasi-helical symmetry $B = B(\psi, \theta - M\zeta)$ there also exists an invariant direction $\mathbf{e}_b = \mathbf{e}_\mathrm{t} + M\mathbf{e}_p$ with $\mathbf{e}_b \cdot \nabla B = 0$. Because of the helical invariance the relations hold

$$\mathbf{e}_\mathrm{t} \cdot \nabla B = -M\mathbf{e}_p \cdot \nabla B \tag{5.20}$$

and

$$\mathbf{B} \cdot \nabla B = (B^\theta - MB^\zeta)\,\mathbf{e}_p \cdot \nabla B \tag{5.21}$$

In this case B^θ in Eq. 5.17 is replaced by $B^\theta - MB^\zeta$ and the transport is singular again.

Stellarators with Reduced Pfirsch-Schlüter Currents

Another case of special interest are stellarator configurations with zero or reduced Pfirsch-Schlüter currents. As shown in chapter 2.4 zero Pfirsch-Schlüter currents imply $\mathbf{e}_p \cdot \nabla B = 0$. In such a configuration the coefficients C_p, C_b are zero and therefore the transport matrix is singular. The solution of the homogeneous equation is a arbitrary poloidal rotation $E(\psi)\vec{e}, \vec{\Lambda} = 0)$. These configurations are equivalent to a mirror configuration with poloidal invariance, in poloidal direction the magnetic pumping effect is zero or negligibly small. As will be show in a later chapter these configurations are of special interest in conjunction with a spin-up of poloidal and toroidal rotation by Coriolis forces which is related to the L-H-transition in the plasma boundary. This degeneracy of the viscous matrix is absent in stellarators without continuous symmetries. In these configurations the viscous damping exists in all directions.

5.1.1 Effect of Atomic Processes

The singularity of the transport matrix is also eliminated by atomic processes, which diminuish the momentum of charged particles. These processes are important in the boundary region where ionisation and charge exchange effects lead to a loss of momentum. Let us assume that the neutral background is at rest. The friction force arising from these atomic processes is proportional to the velocity of the constituents and instead of Eq.3.2 we get

$$\xi_j = -\nabla p_j + q_j n_j \mathbf{E} + \mathbf{j}_j \times \mathbf{B} - \sum_k \alpha_{jk} (\mathbf{v}_j - \mathbf{v}_k) - \beta_j \mathbf{v}_j - \nabla \cdot \pi_j \tag{5.22}$$

The friction coefficients β_j are proportional to the density of neutrals and we consider these coefficients as constant on magnetic surfaces. The coefficients are positive and as a consequence the matrix $D_\beta = \{\beta_j \delta_{jk}\}$ is also positive definit and non-singular. The derivation of the flux-friction relations is also valid in this case except that everywhere the matrix D_α has to be replaced by $D_\alpha + D_\beta$. Since momentum of charged particles is not conserved by these atomic processes we find $D_\beta \vec{e} \neq 0$ and the transport matrix T

$$T = (D_\alpha + D_\beta) \begin{pmatrix} <\mathbf{e}_p \cdot \mathbf{e}_p> & I(\psi) \\ I(\psi) & <B^2> \end{pmatrix} + L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \tag{5.23}$$

is positive and non-singular in all cases.

This transport matrix summarizes all mechanisms which contribute to the radial and toroidal fluxes. The system Eq. (5.10) is now

$$\boxed{\begin{pmatrix} -\vec{\Gamma} \\ \vec{U}_L \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} -\vec{K}_p \\ \vec{K}_b \end{pmatrix}} \tag{5.24}$$

and the T-matrices are

$$\begin{aligned} T_{11} &= (D_\alpha + D_\beta) <\mathbf{e}_p \cdot \mathbf{e}_p> + LC_p \\ T_{12} &= (D_\alpha + D_\beta)\, I(\psi) + LC_b \\ T_{21} &= (D_\alpha + D_\beta)\, I(\psi) + LC_b \\ T_{22} &= (D_\alpha + D_\beta) <B^2> + LC_t \end{aligned} \tag{5.25}$$

T_{11} describes the radial diffusion flux, T_{12} and T_{21} the bootstrap and pinch effect and T_{22} the resistivity of the plasma. The matrices T_{11} and T_{22} are positive definit and the inverse of these matrices exists. The ambipolar condition ($\vec{e} \cdot \vec{\Gamma} = 0$) and the quasineutrality ($\vec{e} \cdot \vec{N}_q = 0 \longrightarrow \vec{e} \cdot \vec{U}_L = 0$) are taken into account when the flux-friction relations (5.10) are reduced to two equations by summing up over particle species. The scalar product with $(\vec{e}, \vec{e})$ yields

$$\begin{pmatrix} -\vec{e} \cdot \vec{K}_p \\ \vec{e} \cdot \vec{K}_b \end{pmatrix} = \begin{pmatrix} \vec{e} \cdot T_{11} & \vec{e} \cdot T_{12} \\ \vec{e} \cdot T_{21} & \vec{e} \cdot T_{22} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} \tag{5.26}$$

Because of the momentum conservation of Coulomb collisions all terms with D_α are zero and these two conditions are in explicit form

$$\begin{pmatrix} -\vec{e} \cdot \vec{K}_p \\ \vec{e} \cdot \vec{K}_b \end{pmatrix} = \begin{pmatrix} < \mathbf{e_p} \cdot \mathbf{e_p} > & I(\psi) \\ I(\psi) & < B^2 > \end{pmatrix} \begin{pmatrix} \vec{e} \cdot D_\beta \cdot \vec{E} \\ \vec{e} \cdot D_\beta \cdot \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \begin{pmatrix} \vec{e} \cdot L \cdot \vec{E} \\ \vec{e} \cdot L \cdot \vec{\Lambda} \end{pmatrix} \tag{5.27}$$

5.1.2 Alternative Form of the Flux-Friction Relations

The derivation of the flux friction relations is asymmetric with respect to the toroidal and poloidal direction. We have used the base vectors $\mathbf{e}_p$ and $\mathbf{B}$ instead of the vectors $\mathbf{e}_p$ and $\mathbf{e_t}$. In this formulation the toroidal current $I(\psi)$ explicitly occurs and this form of the flux friction relations is particularly suited to identify the role of the toroidal current and to analyse the special case of a stellarator without net toroidal current. In a currentless stellarator the classical transport matrix is diagonal; a pinch effect and a classical bootstrap effect does not exist. However, viscosity and — as will be shown later —inertial forces provide a coupling between toroidal and poloidal directions and in such a case it is more appropriate to use the base vectors $\mathbf{e}_p$ and $\mathbf{e_t}$. The plasma velocity is

$$\begin{aligned} \mathbf{V}_j &= (-E_j + B^\theta \Lambda_j(\psi))\, \mathbf{e}_p + \Lambda_j(\psi) B^\zeta \mathbf{e_t} \\ &= V_p^j\, \mathbf{e}_p + V_t^j\, \mathbf{e_t} \end{aligned} \tag{5.28}$$

The relation between the components E_j, Λ_j and V_p^j, V_t^j is in matrix form

$$\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} = \frac{1}{B^\zeta} \begin{pmatrix} -B^\zeta & B^\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{V}_p \\ \vec{V}_t \end{pmatrix} \tag{5.29}$$

Averaging the momentum balance equations with $\mathbf{e}_p$ and $\mathbf{e_t}$ yields the fluxes

$$\begin{pmatrix} -\vec{\Gamma} \\ \frac{1}{B^\zeta} \vec{U}_L - \iota \vec{\Gamma} \end{pmatrix} \tag{5.30}$$

instead of $-\vec{\Gamma}$, $\vec{U}_L$. Using these definitions the flux friction relations are

$$\begin{pmatrix} \vec{\Gamma} \\ \frac{1}{B^\zeta} \vec{U}_L - \iota(\psi) \vec{\Gamma} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \vec{V}_p \\ \vec{V}_t \end{pmatrix} + \begin{pmatrix} \vec{K}_p \\ \vec{K}_b \end{pmatrix} \tag{5.31}$$

The transport matrix T is now

$$
\begin{aligned}
T_{11} &= (D_\alpha + D_\beta) < \mathbf{e}_p \cdot \mathbf{e}_p > + LC_p \\
T_{12} &= (D_\alpha + D_\beta) < \mathbf{e}_t \cdot \mathbf{e}_p > + LC_b \\
T_{21} &= (D_\alpha + D_\beta) < \mathbf{e}_t \cdot \mathbf{e}_p > + LC_b \\
T_{22} &= (D_\alpha + D_\beta) < \mathbf{e}_t \cdot \mathbf{e}_t > + LC_t
\end{aligned}
\tag{5.32}
$$

The matrix of viscous coefficients is here

$$
L \begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} = L \begin{pmatrix} < (\mathbf{e}_p \cdot \frac{\nabla B}{B})^2 > & < (\mathbf{e}_p \cdot \frac{\nabla B}{B}) \cdot (\mathbf{e}_t \cdot \frac{\nabla B}{B}) > \\ < (\mathbf{e}_t \cdot \frac{\nabla B}{B}) \cdot (\mathbf{e}_p \cdot \frac{\nabla B}{B}) > & < (\mathbf{e}_t \cdot \frac{\nabla B}{B})^2 > \end{pmatrix} \tag{5.33}
$$

The coefficients $< \mathbf{e_t} \cdot \mathbf{e_p} >$ and $< \mathbf{e_p} \cdot \mathbf{e_p} >$ are correlated by

$$
< \mathbf{e_p} \cdot \mathbf{e_t} > = -I - B^\theta \; < \mathbf{e_p} \cdot \mathbf{e_p} > \tag{5.34}
$$

The various representations of the velocity $\mathbf{V}_j$ are equivalent. In stellarators without net toroidal current the formulation with E_j, Λ_j is more appropriate than in terms of V_p^j, V_t^j. Because of $< \mathbf{e}_p \cdot \mathbf{B} >= 0$ the transport matrix T (Eq. 6.3) is nearly diagonal, parallel and poloidal directions are only coupled by D_β and C_b. In axisymmetric tokamaks the representation in terms of V_p^j, V_t^j is more appropriate since the viscous coefficients $C_t = C_b = 0$ lead to a simple form of the viscous part of the transport matrix. This form clearly shows the singularity of the viscous transport which reflects the wellknown fact that bulk viscosity does damp not a toroidal rotation of a tokamak plasma.

5.2 Generalised Ohm's Law in a Collisional Plasma

In Eq. (4.33) Ohm's law for each magnetic surface has been derived. This equation does not allow to compute the toroidal plasma current explicitly since the viscous forces depend on the toroidal fluxes $\Lambda_k(\psi)$. The second equation in (5.10) provides a method to eliminate the toroidal fluxes and to express these in terms of the thermodynamic forces E_j. This equation is

$$
\vec{U}_L = T_{21}\vec{E} + T_{22}\vec{\Lambda} + \vec{K}_b \tag{5.35}
$$

leading to

$$
\vec{\Lambda} = T_{22}^{-1}\vec{U}_L - T_{22}^{-1}T_{21}\vec{E} - T_{22}^{-1}\vec{K}_b \tag{5.36}
$$

The matrix T_{22} is the generalisation of the plasma resistivity. Since the matrix C_tL is positive the resistivity of the plasma is increased by viscous effects. The toroidal plasma current I' is given by $I' = \vec{N}_q \cdot \vec{\Lambda}$

$$
I'(\psi) = \vec{N}_q \cdot T_{22}^{-1}\vec{U}_L - \vec{N}_q \cdot T_{22}^{-1}T_{21}\vec{E} - \vec{N}_q \cdot T_{22}^{-1}\vec{K}_b \tag{5.37}
$$

The vector $\vec{U}_L$ is proportional to the loop voltage U_L

$$
\vec{U}_L = \frac{1}{V'}\vec{N}_q U_L \tag{5.38}
$$

Introducing the volume $V(\psi)$ of the magnetic surface instead of the flux as radial variable we may write Ohm's law in the form

$$\begin{aligned} I'(V) &= (\psi'(V))^2 \vec{N}_q \cdot T_{22}^{-1} \cdot \vec{N}_q U_L - \psi'(V)\vec{N}_q \cdot T_{22}^{-1} T_{21} \vec{E} \\ &- \psi'(V)\vec{N}_q \cdot T_{22}^{-1} \vec{K}_b \end{aligned} \tag{5.39}$$

The first term on the right hand side is the current driven by a toroidal loop voltage U_L. The conductivity of a magnetic surface is

$$\sigma(V) = (\psi'(V))^2 \vec{N}_q \cdot T_{22}^{-1} \cdot \vec{N}_q \tag{5.40}$$

since the matrix $C_t L$ is positive definit matrix this viscosity always enhances the resistivity of the plasma and reduces the conductivity. The second term on the right hand side describes the classical and the viscous bootstrap effects. We introduce the coefficient

$$D_{bcl} = \psi'(V)\vec{N}_q \cdot T_{22}^{-1}(D_\alpha + D_\beta)\vec{E} = \psi'(V)/ < B^2 > P'(V) \tag{5.41}$$

and write Ohm's law in shorter form

$$\boxed{I'(V) = \sigma(V)\, U_L - D_{bcl}\, I(V) - \psi'(V)\vec{N}_q \cdot T_{22}^{-1} C_b\, L\vec{E} - \psi'(V)\vec{N}_q \cdot T_{22}^{-1} \vec{K}_b} \tag{5.42}$$

The matrix $C_b L$ is the relevant matrix of viscous bootstrap effects. Since the coefficient C_b can be positive or negative the viscosity driven bootstrap current may be positive or negative. The viscous contribution to the bootstrap current $C_b L\vec{E}$ provides an effect by the radial electric field. Explicitly the vector $L\vec{E}$ is

$$L\,\vec{E} = \begin{pmatrix} \vdots \\ 3\tau_j P_j \dfrac{kT}{q_j} \dfrac{N_j'}{N_j} \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ 3\tau_j P_j \\ \vdots \end{pmatrix} \Phi'(\psi) \tag{5.43}$$

Thus, the viscous bootstrap current drive is proportional to the density gradients and the radial electric field. The last term in Eq. (5.42) is the current driven by external forces, the inertial forces or the anomalous effects arising from plasma fluctuations. These terms will be discussed in a later chapter.

5.3 The Radial Electric Field

The radial electric field does not explicitly occur in the flux- friction relations Eqs.(4.30), it is one of components in E_j.

$$E_j(\psi) = \frac{kT}{q_j} \frac{N_j'}{N_j} + \Phi'(\psi) \tag{5.44}$$

To compute the electric field we proceed in the following way: First we compute the vectors $\vec{E}$ and $\vec{\Lambda}$ form Eq. 5.24. Secondly, the electric field is written in terms of the density gradients and in a third step the density gradients will be eliminated. Let us first consider a stellarator without toroidal current, external forces and friction forces (D_β is neglected). The system Eq. 5.26 is reduced to

$$\begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.45}$$

with

$$X = \vec{e} \cdot L\vec{E} = \sum_j 3\tau_j P_j (\frac{kT}{q_j}\frac{N_j'}{N_j} + \Phi') \tag{5.46}$$

and

$$Y = \vec{e} \cdot L\vec{\Lambda} = \sum_j 3\tau_j P_j \Lambda_j(\psi) \tag{5.47}$$

Since the matrix in stellarators is not singular Det=$C_t C_p - C_b^2 \neq 0$) the homogeneous system has only the trivial solution $X = \sum_j 3\tau_j P_j E_j = 0$ and $Y = \sum_j 3\tau_j P_j \Lambda_j = 0$. This leads to

$$\Phi' = -\frac{1}{\sum \tau_j P_j} kT \sum_j \tau_j \frac{P_j}{q_j}\frac{N_j'}{N_j} \tag{5.48}$$

In the simple case of a two-component plasma with $\tau_e << \tau_i, P_e = P_i$ this yields

$$\Phi' = -kT\frac{N_i'}{q_i N_i} \tag{5.49}$$

The electric field is mainly determined by the ion pressure gradient, the macroscopic velocity of the ions is very small. we obtain

$$-\frac{E_i}{E_e} = \frac{\tau_e}{\tau_i} \ll 1 \quad ; \quad -\frac{\Lambda_i}{\Lambda_e} = \frac{\tau_e}{\tau_i} \ll 1 \tag{5.50}$$

Therefore the velocity of the ions is small but not zero. Taking into account external forces yields the system

$$\begin{pmatrix} C_p & C_b \\ C_b & C_t \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = -\begin{pmatrix} \vec{e} \cdot \vec{K}_p \\ \vec{e} \cdot \vec{K}_b \end{pmatrix} \tag{5.51}$$

Since the matrix of this system is not singular (Det $= C_b^2 - C_t C_p \neq 0$) the solution of this system is given by

$$X = -\frac{C_b \vec{e} \cdot \vec{K}_b - C_t \vec{e} \cdot \vec{K}_p}{\text{Det}} \tag{5.52}$$

$$Y = \frac{C_p \vec{e} \cdot \vec{K}_b - C_b \vec{e} \cdot \vec{K}_p}{\text{Det}} \tag{5.53}$$

The electric field in terms of the density gradients is then

$$\Phi' = \frac{1}{\sum \tau_j P_j}\{X - kT \sum_j \tau_j \frac{P_j}{q_j}\frac{N_j'}{N_j}\} \tag{5.54}$$

The hydrogen ions provide the main contribution to the radial electric field, $\tau_j P_j$ is the weight function and if impurities represent only a small fraction of the ions the hydrogen ions are the dominating ones. In stellarators viscous damping reduces the poloidal diamagnetic flow of ions to a small value so that ion pressure is basically balanced by the radial electric field. The picture may be changed when external forces are relevant.

The electric field is expressed in terms of the density gradients. These, however, are determined by the transport coefficients and the fluxes $\vec{\Gamma}$ and $\vec{U_L}$. The radial fluxes are fixed by the particle sources and $\vec{U_L}$ is proportional to the applied loop voltage. $\vec{\Gamma}$ and $\vec{U_L}$ are the quantities controlled externally and the plasma gradients result from the diffusion proccess. When the thermodynamic forces E_j are given by the solution of Eq. (5.10) the density gradients are found from

$$E_j = \frac{kT}{q_j}\frac{N_j'}{N_j} + \frac{1}{\sum \tau_j P_j}\{X - kT\sum_j \tau_j \frac{P_j}{q_j}\frac{N_j'}{N_j}\} \tag{5.55}$$

This is an equation for the density gradients and can be solved if the thermodynamic forces E_j and the external forces $\vec{K_p}$ and $\vec{K_b}$ in X are given. Inserting the solution in Eq.(5.54) yields the electric field in terms of the externally controlled quantities. However, it should be noted that this is only a formal solution. The inertial terms and the anomalous terms in K_p, K_b may depend non-linearily on the lowest order velocity and therefore this equation must be solved self-consistently. This will be analysed in the next chapter.

In axisymmetric tokamaks this procedure to compute the radial electric field fails since the determinant Det=$C_b^2 - C_t C_t$=0 and the solutions X and Y diverge to infinity. In this case interactions with neutrals ($D_\beta \neq 0$) or a small ripple viscosity must be retained to keep the rotation velocity finite.

Chapter 6

Plasma Rotation

6.1 Slowly Diffusing Plasma

The standard model of a slowly diffusing plasma only retains the collisional forces and neglects inertial forces and turbulent effects. The relation between fluxes and thermodynamic forces $\vec{E}, \vec{\Lambda}$ is given in Eq. 5.24 without the terms $\vec{K}_b$ and $\vec{K}_p$. In lowest order the flow of the plasma in the magnetic surfaces is given by Eq. (3.30). $E_j \mathbf{e}_p$ is the poloidal velocity and $\Lambda_j(\psi)\mathbf{B}$ the parallel velocity. The total plasma velocity is $\mathbf{V}_j + \mathbf{v}_j$ where $\mathbf{v}_j$ describes the radial diffusion velocity. If we neglect all "external" forces $\vec{K}_p$ and $\vec{K}_b$ the vectors $\vec{E}$ and $\vec{\Lambda}$ are given by the solution of Eq.(5.24):

$$\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} = T^{-1} \begin{pmatrix} -\vec{\Gamma} \\ \vec{U_L} \end{pmatrix} \tag{6.1}$$

or in explicit form

$$\begin{aligned} \vec{E} &= -(T_{11} - T_{12}T_{22}^{-1}T_{21})^{-1} \left(\vec{\Gamma} + T_{12}T_{22}^{-1}\,\vec{U}_L\right) \\ \vec{\Lambda} &= T_{22}^{-1}\left(\vec{U}_L - T_{21}\,\vec{E}\right) \end{aligned} \tag{6.2}$$

The particle fluxes are given by the source term. The matrix of trnsport coefficients is

$$\begin{aligned} T_{11} &= (D_\alpha + D_\beta) < \mathbf{e}_p \cdot \mathbf{e}_p > + LC_p \\ T_{12} &= (D_\alpha + D_\beta)\, I(\psi) + LC_b \\ T_{21} &= (D_\alpha + D_\beta)\, I(\psi) + LC_b \\ T_{22} &= (D_\alpha + D_\beta) < B^2 > + LC_t \end{aligned} \tag{6.3}$$

Since in this approximation there are no non-linearities the solution of the the flux-friction relations is unique. In stellarators (zero loop voltage $\vec{U}_L = 0$) the solution is

$$\begin{aligned} \vec{E} &= (T_{11} - T_{12}T_{22}^{-1}T_{21})^{-1}\,\vec{\Gamma} \\ \vec{\Lambda} &= -T_{22}^{-1}T_{21}\vec{E} \end{aligned} \tag{6.4}$$

The viscosity gives rise to a small bootstrap current if the coefficient C_b is non-zero. Ohm's law is

$$I'(V) = -D_{bcl}\, I(V) - \psi'(s)\vec{N}_q T_{22}^{-1} C_b L \vec{E} \tag{6.5}$$

This differential equation determines the toroidal current in stellarators, in tokamaks the collisional bootstrap effect is negligible in comparison with the current driven by the loop voltage. The seed current on the magnetic axis is zero $I(0) = 0$, therefore the solution would be identical to zero in stellarators with $C_b = 0$. This implies $T_{12} = T_{21} = 0$. In a stellarator with C_b there is no collisional bootstrap effect and no pinch effect. In this case Eq. 6.2 yields $\Lambda_j = 0$ for all particle species and the poloidal velocity is given by

$$\vec{E} = T_{11}^{-1}\vec{\Gamma} \tag{6.6}$$

The geometrical coefficients in the T_{11}-matrix are $< \mathbf{e}_p \cdot \mathbf{e}_p >$ and C_p. This implies that in Helias configurations where the Pfirsch-Schlüter transport coefficient is small and the poloidal variation of B is reduced a higher polidal rotation may be expected than in standard stellarators. In case of a two-component plasma without plasma neutral interaction the matrix T_{11} is

$$T_{11} = < \mathbf{e_p} \cdot \mathbf{e_p} > \begin{pmatrix} \alpha_{ei} & -\alpha_{ei} \\ -\alpha_{ei} & \alpha_{ei} \end{pmatrix} + C_p \begin{pmatrix} 3\tau_e P_e & 0 \\ 0 & 3\tau_i P_i \end{pmatrix} \tag{6.7}$$

which yields

$$E_e = \Gamma \frac{1}{C_p 3\tau_e P_e + < \mathbf{e_p} \cdot \mathbf{e_p} > (1 + \tau_e/\tau_i)} \tag{6.8}$$

and

$$E_i = -\Gamma \frac{\tau_e/\tau_i}{C_p 3\tau_e P_e + < \mathbf{e_p} \cdot \mathbf{e_p} > (1 + \tau_e/\tau_i)} \tag{6.9}$$

$\Gamma_e = -\Gamma_i = \Gamma$ is the ambipolar particle loss. This approximation explicitely shows the effect of the geometrical coefficients $< \mathbf{e}_p \cdot \mathbf{e}_p >$ and C_p.

6.2 The Effect of Inertial Forces

In the model of a slowly diffusing plasma the solutions $\vec{E}$ and $\vec{\Lambda}$ are unique proportional to the source terms $\vec{\Gamma}$ and $\vec{U}_L$. M Next we consider the effect of the non-linear inertial forces in ξ_j and neglect the turbulent terms. The inertial forces provide a feedback of the diffusion velocity on the momentum balance and since the inertial forces are non-linear, the solution is no more unique; multiple solutions may arise. The implications of these non-linearities and conditions for uniqueness have been studied in [1]. Besides a slowly diffusing solution also solutions with strong poloidal and toroidal rotation may exist, these are of particular interest in conjunction with the H-mode phenomenon in the boundary of tokamaks and stellarators.

$$\begin{aligned} \vec{K}_p &= \{< \mathbf{e}_p \cdot \nabla m_j n_j \mathbf{v}_j : \mathbf{v}_j >\} \\ \vec{K}_b &= \{< \mathbf{B} \cdot \nabla m_j n_j \mathbf{v}_j : \mathbf{v}_j >\} \end{aligned} \tag{6.10}$$

Taking into account the equation of continuity ($\nabla \cdot n_j \mathbf{v}_j = S_j$) leads to the following formulation of the inertial forces

$$\nabla \cdot n_j \mathbf{v}_j : \mathbf{v}_j = \mathbf{v}_j S_j + n_j \nabla \frac{\mathbf{v}_j^2}{2} - n_j \mathbf{v}_j \times \vec{\omega} \tag{6.11}$$

[1] M. Spada, H. Wobig, *J. Phys. A: Math. Gen* **25** (1992), 1575

$\vec{\omega} = \nabla \times \mathbf{v}_j$ is the vorticity of the velocity $\mathbf{v}_j$. The second term in this equation is the centrifugal force and the last one the Coriolis force. In lowest order the inertial forces are

$$\nabla \cdot N_j \mathbf{V}_j : \mathbf{V}_j \; = \; N_j \nabla \frac{\mathbf{V}_j^2}{2} - N_j \mathbf{V}_j \times \vec{\Omega}_j \tag{6.12}$$

$\vec{\Omega}$ is the vorticity of $\mathbf{V}_j$. In this order the surface average forces $< \mathbf{e}_p \cdot ... >$ and $\mathbf{B} \cdot ... >$ are zero [2]. The averaged centrifugal forces are zero ($< \mathbf{e}_p \cdot \nabla \mathbf{V}_j^2 >= 0$ and $< \mathbf{B} \cdot \nabla \mathbf{V}_j^2 >= 0$) since the density N_j is constant on magnetic surfaces. This also holds for the averaged Coriolis forces. This average is

$$< \mathbf{e}_p \cdot (\mathbf{V}_j \times \vec{\Omega}_j) > = < (\mathbf{e}_p \times \mathbf{V}_j) \cdot \vec{\Omega}_j > \tag{6.13}$$

Because of $\nabla \times (\mathbf{e}_p \times \mathbf{V}_j) = 0$ and $\mathbf{e}_p \times \mathbf{V}_j \propto \nabla \psi$ the averaged Coriolis force is zero. The same holds for the Coriolis force averaged with $\mathbf{B}$. This results leads to the conclusion that the radial diffusive or convective velocity has to be retained to provide a finite effect by the inertial forces. In general the surface averaged inertial forces are

$$\begin{aligned} < \mathbf{e}_p \cdot \nabla n_j \mathbf{v}_j : \mathbf{v}_j > &= < \mathbf{v}_j \cdot \mathbf{e}_p \, S_j > - < \frac{\mathbf{v}_j^2}{2} \mathbf{e}_p \cdot \nabla n_j > + < n_j \mathbf{v}_j \cdot (\mathbf{e}_p \times \vec{\omega}) > \\ < \mathbf{B} \cdot \nabla n_j \mathbf{v}_j : \mathbf{v}_j > &= < \mathbf{v}_j \cdot \mathbf{B} \, S_j > - < \frac{\mathbf{v}_j^2}{2} \mathbf{B} \cdot \nabla n_j > + < n_j \mathbf{v}_j \cdot (\mathbf{B} \times \vec{\omega}) > \end{aligned} \tag{6.14}$$

The first term on the right hand side is only important in the boundary region where ionisation takes place. This term is a damping term which slows down the poloidal and parallel rotation of the plasma. The second term describes the parallel and poloidal centrifugal forces; these are only non-zero if the density n_j is inhomogeneous on magnetic surfaces. n_j is the first order density variation on the magnetic surface which must be found from the first order equations Eqs.(3.33). The last terms are the poloidal and parallel Coriolis forces which mainly are caused by the radial component of the flux $n_j \mathbf{v}_j$. This term is the origin of the spin-up mechanism described for the first time by Stringer [3]. This mechanism leads to an amplification of a small initial vorticity $\vec{\omega}$ by the radial diffusive or convective flux $n_j \mathbf{v}_j \cdot \nabla \psi$ in the Coriolis force.

Proof:

> In order to demonstrate that a radial flux is needed we write the averaged Coriolis force in the form $< (\mathbf{e}_p \times n_j \mathbf{v}_j) \cdot \vec{\omega}_j >$. If $n_j \mathbf{v}_j$ is replaced by $N_j \mathbf{V}_j$ this term is zero for every vorticity $\vec{\omega}$. Because of
>
> $$\mathbf{e}_p \times N_j \mathbf{V}_j = N_j \Lambda_j(\psi) \nabla \psi = \nabla f(\psi) \tag{6.15}$$
>
> we obtain $\nabla f(\psi) \cdot \vec{\omega}_j = \nabla \cdot (\nabla f \times \mathbf{v}_j)$. Since the vector $\nabla f \times \mathbf{v}_j$ is tangential to the magnetic surface $\psi =$ const. the surface average of $\nabla \cdot (\nabla f \times \mathbf{v}_j)$ is zero for every $\mathbf{v}_j$. Thus the surface averaged Coriolis force is non-zero only if the particle flux $n_j \mathbf{v}_j$ is not equal to $N_j \mathbf{V}_j$ In first order the poloidal Coriolis force is $< n_j \mathbf{v}_j \cdot (\mathbf{e}_p \times \vec{\Omega}) >$. This is also valid if we replace $\mathbf{e}_p$ by $\mathbf{B}$. $n_j \mathbf{v}_j$ are here the first order fluxes.

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[2] M. Bineau, *Phys. Fluids* **10**, (1967), 1540

[3] T.E. Stringer, *Proc. of the .. IAEA-Conf. of Plasma Phys. and Controlled Thermonuclear Fusion* Vol. II, 383, Vienna 1971

6.2.1 Coriolis Forces

In order to calculate the first order Coriolis forces we start from the vorticity of

$$\mathbf{V}_j = -E_j\,\mathbf{e_p} + \Lambda_j(\psi)\,\mathbf{B} \tag{6.16}$$

which is given by

$$\vec{\Omega} = -E_j\nabla\times\mathbf{e}_p + \mathbf{e}_p\times\nabla E_j + \Lambda_j(\psi)\mathbf{j} - \mathbf{B}\times\nabla\Lambda_j(\psi) \tag{6.17}$$

and we obtain

$$\begin{aligned} -\mathbf{e}_p\times\vec{\Omega} &= E_j\mathbf{e}_p\times\nabla\times\mathbf{e}_p + (E_j'\mathbf{e}_p\cdot\mathbf{e}_p + \Lambda_j I' - \Lambda_j'(\mathbf{B}\cdot\mathbf{e}_p))\nabla\psi \\ -\mathbf{B}\times\vec{\Omega} &= E_j\mathbf{B}\times\nabla\times\mathbf{e}_p + (E_j'\mathbf{B}\cdot\mathbf{e}_p + \Lambda_j P' - \Lambda_j' B^2)\nabla\psi \end{aligned} \tag{6.18}$$

It is interesting to note that the averaged Coriolis forces are proportional to the velocity components E_j, Λ_j and also to the radial derivatives of E_j and Λ_j. These derivatives represent the velocity shear and they play an important role in the physics of the H-mode.

In summary the surface averaged Coriolis forces are

$$\begin{aligned} -< m_j n_j \mathbf{v}_j\cdot(\mathbf{e}_p\times\vec{\Omega}) > &= R^j_{11}\,E_j + R^j_{12}\,\Lambda_j + K^j_{11}\,E_j' + K^j_{12}\,\Lambda_j' \\ < m_j n_j \mathbf{v}_j\cdot(\mathbf{B}\times\vec{\Omega}) > &= R^j_{21}\,E_j + R^j_{22}\,\Lambda_j + K^j_{21}\,E_j' + K^j_{22}\,\Lambda_j' \end{aligned} \tag{6.19}$$

The coefficients in this equation are

$$\begin{aligned} R^j_{11} &= < m_j n_j \mathbf{v}_j(\mathbf{e}_p\times\vec{\omega}_p) > \\ R^j_{12} &= < m_j n_j \mathbf{v}_j\cdot\nabla\psi > I' \\ R^j_{21} &= -< m_j n_j \mathbf{v}_j(\mathbf{B}\times\vec{\omega}_p) > \\ R^j_{22} &= -< m_j n_j \mathbf{v}_j\cdot\nabla\psi > \; P' \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} K^j_{11} &= < (m_j n_j \mathbf{v}_j\cdot\nabla\psi)\mathbf{e}_p\cdot\mathbf{e}_p > \\ K^j_{12} &= < (m_j n_j \mathbf{v}_j\cdot\nabla\psi)\mathbf{e}_p\cdot\mathbf{B} > \\ K^j_{21} &= -< (m_j n_j \mathbf{v}_j\cdot\nabla\psi)\mathbf{e}_p\cdot\mathbf{B} > \\ K^j_{22} &= -< (m_j n_j \mathbf{v}_j\cdot\nabla\psi)B^2 > \end{aligned} \tag{6.21}$$

$\omega_p = \nabla\times\mathbf{e}_p$ is the vorticity of the base vector $\mathbf{e}_p$. Similar to the matrices T we introduce the diagonal matrices R and K by $R = \{R^j\delta_{jk}\}$ and $K = \{K^j\delta_{jk}\}$. The K-matrices are antisymmetric in the following sense $K_{12} = -K_{21}$, however, this does not hold for the R-matrices: $R_{12} \neq R_{21}$. Using these matrices the Coriolis forces can be summarized in the following form

$$\begin{pmatrix} -\vec{K}_p \\ \vec{K}_b \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} \dfrac{\partial\vec{E}}{\partial\psi} \\ \dfrac{\partial\vec{\Lambda}}{\partial\psi} \end{pmatrix} \tag{6.22}$$

All forces arising due to ionisation can be summarized in the equation

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} \tag{6.23}$$

where the components of the S-matrices are

$$\begin{aligned} S^j_{11} &= -< S_j\, \mathbf{e}_p \cdot \mathbf{e}_p > \\ S^j_{12} &= -< S_j\, \mathbf{e}_p \cdot \mathbf{B} > \\ S^j_{21} &= < S_j\, \mathbf{e}_p \cdot \mathbf{B} > \\ S^j_{22} &= < S_j\, B^2 > \end{aligned} \tag{6.24}$$

Since $S_j \geq 0$ is a positive source term the matrices S^j_{ik} are positive, all eigenvalues are positive. This means that the ionisation slows down toroidal and poloidal rotation of the plasma.

Representing the plasma velocity in terms of poloidal and toroidal components V^j_p, V^j_t yields an alternative formulation of the surface averaged Coriolis forces. The vorticity of $\mathbf{V}_j$ is

$$\vec{\Omega} = V^j_p \nabla \times \mathbf{e}_p - \mathbf{e}_p \times \nabla V^j_p + V^j_t \nabla \times \mathbf{e}_t - \mathbf{e}_t \times \nabla V^j_t \tag{6.25}$$

and the two matrices R and K are in this version

$$\begin{aligned} R^j_{11} &= < m_j n_j \mathbf{v}_j (\mathbf{e}_p \times \vec{\omega}_p) > \\ R^j_{12} &= < m_j n_j \mathbf{v}_j (\mathbf{e}_p \times \vec{\omega}_t) > \\ R^j_{21} &= < m_j n_j \mathbf{v}_j (\mathbf{e}_t \times \vec{\omega}_p) > \\ R^j_{22} &= < m_j n_j \mathbf{v}_j (\mathbf{e}_t \times \vec{\omega}_t) > \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} K^j_{11} &= < (m_j n_j \mathbf{v}_j \cdot \nabla\psi)\, \mathbf{e}_p \cdot \mathbf{e}_p > \\ K^j_{12} &= < (m_j n_j \mathbf{v}_j \cdot \nabla\psi)\, \mathbf{e}_p \cdot \mathbf{e}_t > \\ K^j_{21} &= < (m_j n_j \mathbf{v}_j \cdot \nabla\psi)\, \mathbf{e}_t \cdot \mathbf{e}_p > \\ K^j_{22} &= < (m_j n_j \mathbf{v}_j \cdot \nabla\psi)\, \mathbf{e}_t \cdot \mathbf{e}_t > \end{aligned} \tag{6.27}$$

$\vec{\omega}_p = \nabla \times \mathbf{e}_p$ and $\vec{\omega}_t = \nabla \times \mathbf{e}_t$ are the vorticities of the base vectors. This leads to Coriolis forces in the form

$$\boxed{\begin{pmatrix} \vec{K}_p \\ \vec{K}_t \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \vec{V}_p \\ \vec{V}_t \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{V}_p}{\partial \psi} \\ \frac{\partial \vec{V}_t}{\partial \psi} \end{pmatrix}} \tag{6.28}$$

It should be noted that the non-diagonal terms is these matrices are the result of the toroidal geometry. In a straight and cylindrical plasma the base vectors $\mathbf{e}_p$ and $\mathbf{e}_t$ are perpendicular to each other and the non-diagonal components in the K-matrix are zero. Furthermore, since $\omega_t = 0$ and $\mathbf{e}_t \times \omega_p = 0$ the non- diagonal terms of the R-matrices are zero, too. The coupling between toroidal and poloidal flow is the combined effect of Coriolis forces and toroidal curvature.

6.2.2 Centripetal Forces

The remaining term is the centripetal force which is quadratic in the velocity. In lowest order the density is constant on the magnetic surface and the surface averaged centripetal forces are zero. The first non-trivial approximation to the centrifugal forces is

$$-<\frac{\mathbf{V}_j^2}{2}\mathbf{e}_p\cdot\nabla n_j> \quad ; \quad -<\frac{\mathbf{V}_j^2}{2}\mathbf{B}\cdot\nabla n_j> \tag{6.29}$$

where the first order density variation in the magnetic surface must be calculated from the first order equations 4.37

$$\mathbf{B}\cdot\nabla h_j = -\sum_j \alpha_{jk}\,\mathbf{B}\cdot(\mathbf{V}_j - \mathbf{V}_k) - \mathbf{B}\cdot\nabla\cdot\pi_j + q_j N_j \mathbf{B}\cdot\mathbf{E}_o \tag{6.30}$$

with $h_j = n_j\,kT + q_j N_j \phi$. Neglecting the viscous forces and inserting the explicit form of the velocity $\mathbf{V}_j$ yields the magnetic differential equation

$$\mathbf{B}\cdot\nabla h_j = \sum_j \alpha_{jk}\mathbf{B}\cdot\mathbf{e}_p\,(E_j - E_k) - \sum_j \alpha_{jk}B^2\,(\Lambda_j - \Lambda_k) + q_j N_j\mathbf{B}\cdot\mathbf{E}_o \tag{6.31}$$

Since $\mathbf{e}_p\cdot\mathbf{B}$ and B^2 are the only terms which vary on the magnetic surface, the solution of the magnetic differential equation can be written in the form

$$h_j = f_p\sum_j \alpha_{jk}\,(E_j - E_k) - f_b\sum_j \alpha_{jk}\,(\Lambda_j - \Lambda_k) \tag{6.32}$$

The two functions f_p, f_b are solutions of the magnetic differential equations

$$\begin{aligned}\mathbf{B}\cdot\nabla f_p &= \mathbf{B}\cdot\mathbf{e}_p - <\mathbf{B}\cdot\mathbf{e}_p> \\ \mathbf{B}\cdot\nabla f_b &= B^2 - <B^2>\end{aligned} \tag{6.33}$$

In vector notation ($\vec{h} = \{h_j\}$) these solutions are written as

$$\vec{h} = -\,f_p D_\alpha \vec{E} - f_b D_\alpha \vec{\Lambda}. \tag{6.34}$$

The electric potential ϕ is given by

$$\phi = \frac{\vec{q}\cdot\vec{h}}{\vec{q}\cdot\vec{N}_q} \tag{6.35}$$

and the vector of first order densities

$$\vec{n} = \frac{1}{kT}\left(\vec{h} - \vec{N}_q\frac{\vec{q}\cdot\vec{h}}{\vec{q}\cdot\vec{N}_q}\right) \tag{6.36}$$

The tangential density gradients which are needed in the centrifugal forces are linear in the thermodynamic forces $\vec{E}, \vec{\Lambda}$, the tangential derivatives act only on f_p and f_b. Evaluation of these terms yields

$$\begin{aligned}\{\mathbf{B}\cdot\nabla n_j\} &= +\ \frac{\mathbf{e}_p\cdot\mathbf{B} - <\mathbf{e}_p\cdot\mathbf{B}>}{kT}\left(D_\alpha\vec{E} - \frac{\vec{N}_q}{\vec{q}\cdot\vec{N}_q}\vec{q}\cdot D_\alpha\vec{E}\right) \\ &\quad -\ \frac{B^2 - <B^2>}{kT}\left(D_\alpha\vec{\Lambda} - \frac{\vec{N}_q}{\vec{q}\cdot\vec{N}_q}\vec{q}\cdot D_\alpha\vec{\Lambda}\right)\end{aligned} \tag{6.37}$$

and

$$\{\mathbf{e}_p \cdot \nabla n_j\} = + \frac{\mathbf{e}_p \cdot \nabla f_p}{kT}\left(D_\alpha \vec{E} - \frac{\vec{N}_q}{\vec{q}\cdot\vec{N}_q}\vec{q}\cdot D_\alpha \vec{E}\right) - \frac{\mathbf{e}_p \cdot \nabla f_b}{kT}\left(D_\alpha \vec{\Lambda} - \frac{\vec{N}_q}{\vec{q}\cdot\vec{N}_q}\vec{q}\cdot D_\alpha \vec{\Lambda}\right). \tag{6.38}$$

In a two-component plasma consisting of electrons and single charged ions these terms are zero. Using the relations $N_e = N_i$, $q_e = -q_i$ and $(D_\alpha \vec{E})_e = (D_\alpha \vec{E})_i$ it can easily be shown that n_e and n_i are constant on magnetic surfaces. However, since $h_e = -h_i$ the electric field is not constant and there is a finite convective flow through the magnetic surface. In this order the centrifugal forces do not provide a finite tangential force but the Coriolis forces are finite. Therefore the contribution of the centripetal forces will be neglected in the following

6.2.3 Flux-Friction Relations with Coriolis Forces

M Inserting these results of the inertial forces into the flux-friction relations in $\vec{E}, \vec{\Lambda}$ representation yields

$$\begin{pmatrix} -\vec{\Gamma} \\ \vec{U}_L \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} \\ \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} \frac{\partial \vec{E}}{\partial \psi} \\ \frac{\partial \vec{\Lambda}}{\partial \psi} \end{pmatrix} \tag{6.39}$$

In the $\vec{V}_p, \vec{V}_t$ representation these equation are

$$\begin{pmatrix} \vec{\Gamma} \\ \frac{1}{B^\zeta}\vec{U}_L - \iota(\psi)\vec{\Gamma} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} \vec{V}_p \\ \vec{V}_t \end{pmatrix} \\ \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} \vec{V}_p \\ \vec{V}_t \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} \frac{\partial \vec{V}_p}{\partial \psi} \\ \frac{\partial \vec{V}_t}{\partial \psi} \end{pmatrix} \tag{6.40}$$

It should be noted that the matrices in these equations are defined differently according to the representation of the tangential velocity V_j. The Coriolis forces modify the flux-friction relations in various ways. Firstly, a radial coupling of the flow on different magnetic surfaces is provided by the radial derivatives $\partial/\partial\psi$: the flux-friction relation are differential equations instead of algebraic equations. Secondly, the Coriolis forces lead to a coupling of the radial diffusive flux to the tangential velocities. This is the essential feature of the Stringer spin-up mechanism, in Stringer's formulation, however, the radial derivative in the inertial forces is neglected. Such a coupling of radial, meridional and azimutal motion also exists in the wind system of the earth as has been pointed out already in chapter II. The convective radial diffusive flux is calculated from the perpendicular component of

$$\nabla h_j = q_j N_j \mathbf{v}_j \times \mathbf{B} - \sum_j \alpha_{jk} (\mathbf{V}_j - \mathbf{V}_k) \tag{6.41}$$

and the parallel velocity from

$$\nabla \cdot N_j \mathbf{v}_j = S_j \tag{6.42}$$

The solution is the classical diffusion velocity plus the Pfirsch-Schlüter convective velocity. Inserting these solutions into the Coriolis forces yields R and K-matrices which are linear in the velocities $\vec{E}$ and $\vec{\Lambda}$. As a consequence the flux friction relation are quadratic in the thermodynamic forces $\vec{E}, \vec{\Lambda}$.

6.3 Effect of Plasma Turbulence

In a next approximation we analyse the effect of turbulent terms on the flux friction relations. As already mentioned before, the turbulent fluctuations $\delta n_j, \delta \mathbf{v}_j, \delta \mathbf{E}$, etc. are considered as given quantities. The issue here is how these turbulent forces modify the poloidal and toroidal force balance. The surface averaged forces are

$$\begin{aligned} \vec{K}_p &= \{< \mathbf{e}_p \cdot \nabla \overline{m_j n_j \mathbf{v}_j : \mathbf{v}_j} > + < \mathbf{e}_p \cdot \xi_{j,an} >\} \\ \vec{K}_b &= \{< \mathbf{B} \cdot \nabla \overline{m_j n_j \mathbf{v}_j : \mathbf{v}_j} + < \mathbf{B} \cdot \xi_{j,an} >\} \end{aligned} \tag{6.43}$$

where $\xi_{j,an}$ denotes all terms originating from electric and magnetic fluctuations. The inertial stress tensor can be decomposed in the following form

$$\overline{n\mathbf{v} : \mathbf{v}} = \overline{n\mathbf{v}} : \mathbf{v} + \overline{\delta(n\mathbf{v}) : \delta\mathbf{v}} \tag{6.44}$$

$\overline{n\mathbf{v}}$ is the total time averaged particle flux. The last term is the turbulent Reynolds stress. In fluid dynamics and gas dynamics this terms represents the eddy viscosity leading to enhanced perpendicular momentum transfer and thus enhances the effect of viscosity. The Coriolis forces are computed in the same procedure as above except for the time averaging which leads to $\overline{n_j \mathbf{v}_j} = n_j \mathbf{v}_j + \overline{\delta n_j \, \delta \mathbf{v}_j}$ instead of $n_j \mathbf{v}_j$ in the Coriolis forces. The matrices R and K remain unchanged, only the Pfirsch-Schlüter diffusion flux has to to be replaced by the anomalous particle fluxes. These matrices are now

$$\begin{aligned} R^j_{11} &= < \overline{n_j \mathbf{v}_j} \cdot (\mathbf{e}_p \times \vec{\omega}_p) > \\ R^j_{12} &= < \overline{n_j \mathbf{v}_j} \cdot \nabla \psi > I' \\ R^j_{21} &= - < \overline{n_j \mathbf{v}_j} \cdot (\mathbf{B} \times \vec{\omega}_p) > \\ R^j_{22} &= - < \overline{n_j \mathbf{v}_j} \cdot \nabla \psi > P' \end{aligned} \tag{6.45}$$

and

$$\begin{aligned} K^j_{11} &= < (\overline{n_j \mathbf{v}_j} \cdot \nabla \psi) \mathbf{e}_p \cdot \mathbf{e}_p > \\ K^j_{12} &= < (\overline{n_j \mathbf{v}_j} \cdot \nabla \psi) \mathbf{e}_p \cdot \mathbf{B} > \\ K^j_{21} &= - < (\overline{n_j \mathbf{v}_j} \cdot \nabla \psi) \mathbf{e}_p \cdot \mathbf{B} > \\ K^j_{22} &= - < (\overline{n_j \mathbf{v}_j} \cdot \nabla \psi) B^2 > \end{aligned} \tag{6.46}$$

The other anomalous terms $< \mathbf{e_p} \cdot \xi_{j,an} >$ and $< \mathbf{B} \cdot \xi_{j,an} >$ cannot be linked to the lowest order velocity without going into the details of the instabilities leading to this turbulence. Here, we consider these terms as given and focus our interest on the the effect of these anomalous terms on the force balance. Summarizing all components discussed in the previous chapters yields the flux- friction relations in the following form

$$\boxed{\begin{pmatrix} -\vec{\Gamma} + \vec{K}_{p,an} \\ \\ \vec{U}_L - \vec{K}_{b,an} \end{pmatrix} = T \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix} + (R+S) \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix} + K \begin{pmatrix} \dfrac{\partial \vec{E}}{\partial \psi} \\ \dfrac{\partial \vec{\Lambda}}{\partial \psi} \end{pmatrix}} \qquad (6.47)$$

T, R, S, K are the 2x2 matrices decribed above. $\vec{K}_{p,an} = \{< \mathbf{e_p} \cdot \xi_{j,an} >\}$ and $\vec{K}_{b,an} = \{< \mathbf{B} \cdot \xi_{j,an} >\}$ are the anomalous fluxes caused by the $\delta\mathbf{E}$ and $\delta\mathbf{B}$ fluctuations in the plasma. In V_p, V_t representation the structure of the flux friction relations is the same, the matrices T, R, S, K must be defined properly using the base vector $\mathbf{e_t}$ instead of $\mathbf{B}$.

6.4 Stationary Solution

In chapter 6.1 we considered a slowly diffusing plasma neglecting the anomalous losses and the inertial forces. In this approximation the velocity of ions is small since viscous forces or magnetic pumping slow down ion flow. The situation may be changed completely when inertial forces and anomalous radial losses are taken into account. Without inertial forces the plasma velocity is determined by

$$\begin{pmatrix} -\vec{\Gamma} + \vec{K}_{p,an} \\ \\ \vec{U}_L - \vec{K}_{b,an} \end{pmatrix} = T \begin{pmatrix} \vec{E}_0 \\ \\ \vec{\Lambda}_0 \end{pmatrix} \qquad (6.48)$$

where $\vec{\Gamma}, \vec{U}_L, \vec{K}_{p,an}, \vec{K}_{b,an}$ are given terms. Since T is positive definit there is always a unique solution $\vec{E}_0, \vec{\Lambda}_o$. The anomalous effects reduce the inhomogeneous terms on the left hand side of this equation and consequently the plasma velocity $\vec{E}, \vec{\Lambda}$ is reduced. On the other hand the anomalos radial loss $\overline{n_j \mathbf{v}_j \cdot \nabla\psi}$ enhances the Coriolis forces and may lead to a strong modification of the lowest order solution. We look for the solution of the whole system in the form $\vec{E}_0 + \vec{E}_1, \vec{\Lambda}_0 + \vec{\Lambda}_1$ where the terms with index 1 are solutions of

$$\begin{aligned} 0 = T \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} &+ (R+S) \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} + K \begin{pmatrix} \dfrac{\partial \vec{E}_1}{\partial \psi} \\ \dfrac{\partial \vec{\Lambda}_1}{\partial \psi} \end{pmatrix} \\ &+ (R+S) \begin{pmatrix} \vec{E}_0 \\ \\ \vec{\Lambda}_0 \end{pmatrix} + K \begin{pmatrix} \dfrac{\partial \vec{E}_0}{\partial \psi} \\ \dfrac{\partial \vec{\Lambda}_0}{\partial \psi} \end{pmatrix} \end{aligned} \qquad (6.49)$$

In V_p, V_t representation this system would exhibit the same structure . In a quiescent plasma the matrices R, K are computed with the radial fluxes $n_j \mathbf{v}_j \cdot \nabla\psi$, which describe classical and Pfirsch-Schlüter diffusion, in a turbulent plasma these radial fluxes are replaced by the time averaged fluxes $\overline{n_j \mathbf{v}_j \cdot \nabla\psi}$ which certainly enhances the influence of the Coriolis forces[4].

The important issue is how much the the solution $\vec{E}_1, \vec{\Lambda}_1$ differ from the lowest order terms $\vec{E}_0, \vec{\Lambda}_0$. As pointed out previously for the case of a two-component plasma the ions are slowly rotating as compared to the electrons. Does Eq. 6.49 provide solutions with a rapid rotation of the ions?.

As already mentioned these equations equations exhibit a strong similarity to the equations describing the zonal circulation in the atmospheres of the planets. An example with strong circulation is the superrotation found in the atmosphere of Venus [5] where the velocity of solar driven Hadley cells is transferred to a fast zonal circulation. In our case the role of the Hadley cells is played by the radial diffusive or anomalous fluxes and the role of the planets rotation is taken over by the lowest order rotation described by $\vec{E}_0, \vec{\Lambda}_0$. In the zonal circulation on earth the viscous damping and the rotation of the earth are the dominating factors, the effect of the Coriolis force $\mathbf{v}_H \times \nabla \times \mathbf{v}_{rel}$ is negligble. $\mathbf{v}_H$ is the meridional motion of the Hadley cells and $\mathbf{v}_{rel}$ is the azimutal zonal circulation. However, the spin-up in a toroidal plasma resembles more the situation on Venus [6] where the slow rotation of the planet acts as a seed effect which is strongly enhanced by the non-linear Coriolis forces. In Eq. 2.109 these enhancenment terms are the second and the third ones in the first line. Anomalous transport increases the effect, therefore rotational spin-up is more likely in a turbulent plasma than in a quiescent plasma[7]. Anomalous radial transport plays the role of Hadley cells; this transport may depend on the toroidal and

[4] The role of anomalous radial transport on the formation of poloidal rotation has been pointed out by A.B. Hassam, T.M. Antonsen, J.F. Drake and C.S. Liu, *Phys. Rev. Letters*, Vol. 66, No. 3, (1991), 309

[5] H.G. Mayr, I. Harris, K.H. Schatten, D.R. Stevens-Rayburn and K.L. Shan, *Earth, Moon and Planets* **41**, (1988), 45. These authors discuss the energy balance, meridional and zonal velocity in the atmosphere of Venus and write the zonal momentum balance in the form

$$V(\Omega_v + U/r) - U\, K_r/H^2 \tag{6.50}$$

Here U is the zonal velocity, V the poleward velocity of the Hadley cell, Ω_v the angular velocity of Venus, K_r the eddy viscosity representing the slowing-down mechanism and H a length scale. r is the radius of the rotating zone. The other equations describing the meridional motion are omitted here. This equation shows how the rotation of the planet acts as the driving term. The solution for U is

$$U = \Omega_p r/(K_r r/VH^2 - 1) \tag{6.51}$$

If the eddy viscosity is small enough or the meridional velocity V of the Hadley cells large enogh a singularity arises and the zonal velocity U may become rather large in the vicinity of this singularity. The spin-up of the atmosphere in azimutal direction is caused by the Coriolis force which links the circulation U to the meridional velocity V. This leads to the strong wind relativ to velocity of the surface $U = \Omega_p r$.

[6] The anology between shear flow in plasmas and the superrotation of the Venus atmosphere has been mentioned in a paper by J.F. Drake et al. (J.F. Drake, J.M. Finn, P. Guzdar, V. Shapiro, V. Shevchenko, F. Waelbroeck, A.B. Hassam, C.S. Liu, R. Sagdeev , *Phys. Fluids* **B4**, (1992), 488.) In this paper the development of a shear flow is explained by the "peeling" instability of convective cells. Numerically the two-dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla\omega - \mu\Delta\omega = 0 \tag{6.52}$$

is solved, however, this theory does not describe the Hadley mechanism and the development of zonal circulation.

[7] This has been pointed out in a paper by Hassam et al. (A.B. Hassam, T.M. Antonsen, J.F. Drake, C.S. Liu, *Phys. Rev. Lett.* **66** No. 3 (1991), 309). These authors conclude that the radial plasma loss be poloidally asymmetric to provide an effect on rotational spin-up. This result is not supported by the present theory. As long as the matrices R, K are finite poloidal and toroidal rotation is possible

poloidal plasma velocity and the matrices R, K are non-linear functions of $\vec{E}, \vec{\Lambda}$. For the very moment we consider the anomalous tansport as a free parameter and analyse its implications on the solutions of Eq. 2.109. If the anomalous transport is large enough, a singularity in Eq. 2.109 arises if the homogeneous equations has a non-trivial solution

$$0 = T \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} + (R+S) \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} + K \begin{pmatrix} \dfrac{\partial \vec{E}_1}{\partial \psi} \\ \dfrac{\partial \vec{\Lambda}_1}{\partial \psi} \end{pmatrix} \tag{6.53}$$

Close to this singularity the solution of the inhomogeneous system may become very large. It is this singularity which is invoked by Mayr et al. to explain the strong enhancement of the circulation on Venus. This singularity also indicates the limits of the present theory, close to this singularity the feedback of the plasma velocity on anomalous transport has to be retained. Theory and numerical calculations indicate a reduction of anomalous transport by poloidal rotation and shear flow in the plasma[8]. where the slow rotation of the planet acts as a seed effect which is strongly enhanced by the non-linear Coriolis forces. In Eq. 2.109 these enhancenment terms are the second and the third ones in the first line. Anomalous transport increases the effect, therefore rotational spin-up is more likely in a turbulent plasma than in a quiescent plasma[9]. Anomalous radial transport plays the role of Hadley cells; this transport may depend on the toroidal and poloidal plasma velocity and the matrices R, K are non-linear functions of $\vec{E}, \vec{\Lambda}$. For the very moment we consider the anomalous tansport as a free parameter and analyse its implications on the solutions of Eq. 2.109. If the anomalous transport is large enough, a singularity in Eq. 2.109 arises if the homogeneous equations has a non-trivial solution

$$0 = T \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} + (R+S) \begin{pmatrix} \vec{E}_1 \\ \\ \vec{\Lambda}_1 \end{pmatrix} + K \begin{pmatrix} \dfrac{\partial \vec{E}_1}{\partial \psi} \\ \dfrac{\partial \vec{\Lambda}_1}{\partial \psi} \end{pmatrix} \tag{6.54}$$

Close to this singularity the solution of the inhomogeneous system may become very large. It is this singularity which is invoked by Mayr et al. to explain the strong enhancement of the circulation on Venus. This singularity also indicates the limits of the present theory, close to this singularity the feedback of the plasma velocity on anomalous transport has to be retained. Theory and numerical calculations indicate a reduction of anomalous transport by poloidal rotation and shear flow in the plasma[10].

[8] The theory of Guzdar et al. (P.N. Guzdar, J.F. Drake, D. McCarthy, A.B. Hassam and C.S. Liu, *Phys. Fluids* **B5**, (1993), 3712,) shows the the suppression of resistive ballooning mode fluctuations by shear flow in the plasma boundary region.

[9] This has been pointed out in a paper by Hassam et al. (A.B. Hassam, T.M. Antonsen, J.F. Drake, C.S. Liu, *Phys. Rev. Lett.* **66** No. 3 (1991), 309). These authors conclude that the radial plasma loss be poloidally asymmetric to provide an effect on rotational spin-up. This result is not supported by the present theory. As long as the matrices R, K are finite poloidal and toroidal rotation is possible

[10] The theory of Guzdar et al. (P.N. Guzdar, J.F. Drake, D. McCarthy, A.B. Hassam and C.S. Liu, *Phys. Fluids* **B5**, (1993), 3712,) shows the the suppression of resistive ballooning mode fluctuations by shear flow in the plasma boundary region.

6.5 The Effect of Shear Viscosity and Gyro Viscosity

In the preceding chapters only the leading term of the Braginskii viscosity tensor have been retained. This term describes the magnetic pumping effect and the resulting damping term is linear in the velocity $\mathbf{v}_j$. The inertial terms introduce first order derivatives of the velocity and the flux friction relations turn out to be first order differential equations. In general the viscous forces include second order derivatives of the velocity which leads to a momentum transfer across the magnetic surface. This effects are described by the complete Braginskii viscosity tensor which incorporates shear viscosity and gyroviscosity. In the following the total Braginskii viscosity tensor will be taken into account and the balance equation for a general nonaxisymmetric configuration will be formulated.

The full viscosity tensor in a collisional plasma consists of 5 terms which are of the order 1, $(\omega\tau)^{-1}$ and $(\omega\tau)^{-2}$, (ω is the gyro frequency and τ the collision time).

$$\pi_{ik} = -\sum_{0}^{4} \eta_l \, W_{l,ik} \tag{6.55}$$

η_l are the viscosity coefficients and $W_{l,ik}$ is related to the rate of strain tensor

$$W_{\mu\nu} = \frac{\partial V_\mu}{\partial x_\nu} + \frac{\partial V_\nu}{\partial x_\mu} - \frac{2}{3} \nabla \cdot \mathbf{V} \, \delta_{\mu\nu} \tag{6.56}$$

by

$$W_{l,ik} = A^{\mu\nu}_{l,ik}(\mathbf{x}) \, W_{\mu\nu} \tag{6.57}$$

The viscous coefficient are assumed to be constant on magnetic surfaces. In the following we adopt the Einstein convention with summation over equal indices. $A(\mathbf{x})$ is a tensor which depends only on the magnetic field. In the following this tensor only appears in the combination

$$C^{\mu\nu}_{ik} = \sum_{0}^{4} \eta_l \, A^{\mu\nu}_{l,ik} \tag{6.58}$$

thus the pressure tensor has the simple form

$$\pi_{ik} = -C^{\mu\nu}_{ik}(\mathbf{x}) \, W_{\mu\nu} \tag{6.59}$$

The rate of strain tensor $W_{\mu\nu}$ is a linear first order differential operator on $\mathbf{V} = -E\mathbf{e_p} + \Lambda\mathbf{B}$ and since the lowest order plasma flow is incompressible we obtain

$$W_{\mu\nu}[\mathbf{V}] = -E(\psi) \, W_{\mu\nu} [\mathbf{e}_p] + \Lambda(\psi) \, W_{\mu\nu} [\mathbf{B}] - E'(\psi) \, W^{\star}_{\mu\nu} [\mathbf{e}_p] + \Lambda'(\psi) \, W^{\star}_{\mu\nu} [\mathbf{B}] \tag{6.60}$$

Here we introduce the function $E(\psi)$ instead of U'. In explicit form the tensors $W_{\mu\nu} [\mathbf{B}]$ and $W^{\star}_{\mu\nu} [\mathbf{B}]$ are

$$W_{\mu\nu} [\mathbf{B}] = \frac{\partial B_\mu}{\partial x_\nu} + \frac{\partial B_\nu}{\partial x_\mu} \tag{6.61}$$

and

$$W^{\star}_{\mu\nu} [\mathbf{B}] = B_\mu \frac{\partial \psi}{\partial x_\nu} + B_\nu \frac{\partial \psi}{\partial x_\mu} \tag{6.62}$$

In $W_{\mu\nu} [\mathbf{e}_p]$ and $W^{\star}_{\mu\nu} [\mathbf{e}_p]$ the vector $\mathbf{B}$ has to be replaced by $\mathbf{e}_p$. With these notations the general form of the viscosity tensor is

$$\begin{aligned} \pi_{ik} = & + \; E(\psi) \, C^{\mu\nu}_{ik} \, W_{\mu\nu} [\mathbf{e}_p] + E'(\psi) \, C^{\mu\nu}_{ik} \, W^{\star}_{\mu\nu} [\mathbf{e}_p] \\ & - \; \Lambda(\psi) \, C^{\mu\nu}_{ik} \, W_{\mu\nu} [\mathbf{B}] - \Lambda'(\psi) \, C^{\mu\nu}_{ik} W^{\star}_{\mu\nu} [\mathbf{B}] \end{aligned} \tag{6.63}$$

The index j of the particle species has been omitted for simplicity. All the terms $C_{ik}^{\mu\nu} W_{\mu\nu}$ depend only on the geometry of the magnetic field and the coefficients of viscosity. Combining them in matrices Π_{ik}

$$\Pi_{ik}[\mathbf{e}_p] = C_{ik}^{\mu\nu} W_{\mu\nu} [\mathbf{e}_p] \tag{6.64}$$

$$\Pi_{ik}[\mathbf{B}] = C_{ik}^{\mu\nu} W_{\mu\nu} [\mathbf{B}] \tag{6.65}$$

$$\Pi_{ik}^{\star}[\mathbf{e}_p] = C_{ik}^{\mu\nu} W_{\mu\nu}^{\star} [\mathbf{e}_p] \tag{6.66}$$

$$\Pi_{ik}^{\star}[\mathbf{B}] = C_{ik}^{\mu\nu} W_{\mu\nu}^{\star} [\mathbf{B}] \tag{6.67}$$

the pressure tensor takes the simpler form

$$\begin{aligned} \pi_{ik} = & + E(\psi)\, \Pi_{ik}[\mathbf{e}_p] - \Lambda(\psi)\, \Pi_{ik}[\mathbf{B}] \\ & + E'(\psi) \Pi_{ik}^{\star}[\mathbf{e}_p] - \Lambda'(\psi)\, \Pi_{ik}^{\star}[\mathbf{B}] \end{aligned} \tag{6.68}$$

or in V_p, V_t-notation

$$\begin{aligned} \pi_{ik} = & - V_p(\psi)\, \Pi_{ik}[\mathbf{e}_p] - V_t(\psi)\, \Pi_{ik}[\mathbf{B}] \\ & - V_p'(\psi) \Pi_{ik}^{\star}[\mathbf{e}_p] - V_t'(\psi)\, \Pi_{ik}^{\star}[\mathbf{B}] \end{aligned} \tag{6.69}$$

From this equation the toroidal and poloidal averages $< \mathbf{B} \cdot \nabla \cdot \pi >$ and $< \mathbf{e}_p \cdot \nabla \cdot \pi >$ have to be calculated. The viscous forces are

$$\begin{aligned} \nabla \cdot \pi \ = \ & E(\psi) \nabla \cdot \Pi[\mathbf{e_p}] - \Lambda(\psi)\, \nabla \cdot \Pi[\mathbf{B}] \\ + \ & E'(\psi) \nabla \cdot \Pi^{\star}[\mathbf{e_p}] - \Lambda'(\psi)\, \nabla \cdot \Pi^{\star}[\mathbf{B}] \\ + \ & E'(\psi) \Pi[\mathbf{e_p}] \cdot \nabla\psi - \Lambda'(\psi)\, \Pi[\mathbf{B}] \cdot \nabla\psi \\ + \ & E''(\psi) \Pi^{\star}[\mathbf{e_p}] \cdot \nabla\psi - \Lambda''(\psi)\, \Pi^{\star}[\mathbf{B}] \cdot \nabla\psi \end{aligned} \tag{6.70}$$

This equation shows that the first order and second order derivatives of $E(\psi)$ and $\Lambda(\psi)$ occur if the total viscous tensor is retained. These terms are essential in determining the velocity shear. The surface averaged forces are

$$\begin{aligned} < \mathbf{e_p} \cdot \nabla \cdot \pi > \ = \ & E(\psi) < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{e_p}] > - \Lambda(\psi) < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{B}] > \\ + \ & E'(\psi) (< \mathbf{e_p} \cdot \nabla \cdot \Pi^{\star}[\mathbf{e_p}] > + < \mathbf{e_p} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) \\ - \ & \Lambda'(\psi) (< \mathbf{e_p} \cdot \nabla \cdot \Pi^{\star}[\mathbf{B}] > + < \mathbf{e_p} \cdot \Pi[\mathbf{B}] \cdot \nabla\psi >) \\ + \ & E''(\psi) < \mathbf{e_p} \cdot \Pi^{\star}[\mathbf{e_p}] \cdot \nabla\psi > - \Lambda''(\psi) < \mathbf{e_p} \cdot \Pi^{\star}[\mathbf{B}] \cdot \nabla\psi > \end{aligned} \tag{6.71}$$

and the parallel viscous force

$$\begin{aligned} < \mathbf{B} \cdot \nabla \cdot \pi > \ = \ & E(\psi) < \mathbf{B} \cdot \nabla \cdot \Pi[\mathbf{e_p}] > - \Lambda(\psi) < \mathbf{B} \cdot \nabla \cdot \Pi[\mathbf{B}] > \\ + \ & E'(\psi) (< \mathbf{B} \cdot \nabla \cdot \Pi^{\star}[\mathbf{e_p}] > + < \mathbf{B} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) \\ - \ & \Lambda'(\psi) (< \mathbf{B} \cdot \nabla \cdot \Pi^{\star}[\mathbf{B}] > + < \mathbf{B} \cdot \Pi[\mathbf{B}] \cdot \nabla\psi >) \\ + \ & E''(\psi) < \mathbf{B} \cdot \Pi^{\star}[\mathbf{e_p}] \cdot \nabla\psi > - \Lambda''(\psi) < \mathbf{B} \cdot \Pi^{\star}[\mathbf{B}] \cdot \nabla\psi > \end{aligned} \tag{6.72}$$

or in V_p, V_t-notation

$$\begin{aligned} < \mathbf{e_p} \cdot \nabla \cdot \pi > = \ & - V_p(\psi) < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{e_p}] > - V_t(\psi) < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{e_t}] > \\ - \ & V_p'(\psi) (< \mathbf{e_p} \cdot \nabla \cdot \Pi^{\star}[\mathbf{e_p}] > + < \mathbf{e_p} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) \\ - \ & V_t'(\psi) (< \mathbf{e_p} \cdot \nabla \cdot \Pi^{\star}[\mathbf{e_t}] > + < \mathbf{e_p} \cdot \Pi[\mathbf{e_t}] \cdot \nabla\psi >) \\ - \ & V_p''(\psi) < \mathbf{e_p} \cdot \Pi^{\star}[\mathbf{e_p}] \cdot \nabla\psi > - V_t''(\psi) < \mathbf{e_p} \cdot \Pi^{\star}[\mathbf{e_t}] \cdot \nabla\psi > \end{aligned} \tag{6.73}$$

and the parallel viscous force

$$
\begin{aligned}
< \mathbf{e_t} \cdot \nabla \cdot \pi > = & - V_p(\psi) < \mathbf{e_t} \cdot \nabla \cdot \Pi[\mathbf{e_p}] > - V_t(\psi) < \mathbf{e_t} \cdot \nabla \cdot \Pi[\mathbf{e_t}] > \\
& - V_p'(\psi) (< \mathbf{e_t} \cdot \nabla \cdot \Pi^*[\mathbf{e_p}] > + < \mathbf{e_t} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) \\
& - V_t'(\psi) (< \mathbf{e_t} \cdot \nabla \cdot \Pi^*[\mathbf{e_t}] > + < \mathbf{e_t} \cdot \Pi[\mathbf{B}] \cdot \nabla\psi >) \\
& - V_p''(\psi) < \mathbf{e_t} \cdot \Pi^*[\mathbf{e_p}] \cdot \nabla\psi > - V_t''(\psi) < \mathbf{e_t} \cdot \Pi^*[\mathbf{e_t}] \cdot \nabla\psi >
\end{aligned} \quad (6.74)
$$

The main result of the present analysis is the linear dependence of the surface averaged viscous forces on the functions $E_j(\psi)$, $\Lambda_j(\psi)$ and the first order and second order derivatives with respect to the radial coordinate ψ. Inserting these terms into the flux-friction relations 4.8 and 4.6 yields a system of second order differential equations instead of the algebraic system considered so far. This implies that also the electric field results from the solution of a second order equation rather than from an algebraic equation. The ambipolarity condition and the general form of the poloidal viscous forces lead to a second order differential equation for the electric field $\Phi'(\psi)$. The radial derivatives of $E_j(\psi)$ and $\Lambda_j(\psi)$ are introduced by the shear viscosity and gyro viscosity. In chapter 5 the viscosity has been evaluated retaining only the bulk viscosity, there no derivatives of E and Λ occured. Therefore first order and second order derivatives are the result of gyro viscosity and shear viscosity. In the next step we return to the vector notation of the viscous forces and write these in the following form

$$
\begin{pmatrix} -\vec{\Pi}_p \\ \vec{\Pi}_b \end{pmatrix} = L_0 \begin{pmatrix} \vec{E} \\ \vec{\Lambda} \end{pmatrix} + L_1 \begin{pmatrix} \vec{E}' \\ \vec{\Lambda}' \end{pmatrix} + L_2 \begin{pmatrix} \vec{E}'' \\ \vec{\Lambda}'' \end{pmatrix} \quad (6.75)
$$

The three matrices L_0, L_1 and L_2 are

$$
L_0 = \begin{pmatrix} - < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{e_p}] > & < \mathbf{e_p} \cdot \nabla \cdot \Pi[\mathbf{B}] > \\ < \mathbf{B} \cdot \nabla \Pi[\mathbf{e_p}] > & - < \mathbf{B} \cdot \nabla \cdot \Pi[\mathbf{B}] > \end{pmatrix} \quad (6.76)
$$

andL_1

$$
\begin{pmatrix} -(< \mathbf{e_p} \cdot \nabla \cdot \Pi^*[\mathbf{e_p}] > + < \mathbf{e_p} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) & (< \mathbf{e_p} \cdot \nabla \cdot \Pi^*[\mathbf{B}] > - < \mathbf{e_p} \cdot \Pi[\mathbf{B}] \cdot \nabla\psi >) \\ (< \mathbf{B} \cdot \nabla \cdot \Pi^*[\mathbf{e_p}] > + < \mathbf{B} \cdot \Pi[\mathbf{e_p}] \cdot \nabla\psi >) & -(< \mathbf{B} \cdot \nabla \cdot \Pi^*[\mathbf{B}] > - < \mathbf{B} \cdot \Pi[\mathbf{B}] \cdot \nabla\psi >) \end{pmatrix} \quad (6.77)
$$

and also

$$
L_2 = \begin{pmatrix} - < \mathbf{e_p} \cdot \Pi^*[\mathbf{e_p}] \cdot \nabla\psi > & < \mathbf{e_p} \cdot \Pi^*[\mathbf{B}] \cdot \nabla\psi > \\ < \mathbf{B} \cdot \Pi^*[\mathbf{e_p}] \cdot \nabla\psi > & - < \mathbf{B} \cdot \Pi^*[\mathbf{B}] \cdot \nabla\psi > \end{pmatrix} \quad (6.78)
$$

In the similar manner the matrices can be written in the V_p, V_t- notation.

6.6 Summary and Conclusions

Equation 6.75 is the most general form of the viscous forces in the collisional regime. The dominant term is the matrix L_o, it represent the magnetic pumping effect. Shear viscosity and gyro-viscosity are the reason of the radial derivatives of E_j and $\Lambda_j(\psi)$. We are now in the position to sum up all terms in the flux friction relations. For this purpose we start from Eq. 4.30 and the Coriolis forces in Eq. 6.22. Together with the ionisation term Eq. 6.23 and the viscous forces in Eq. 6.75 the flux friction relations are

$$\begin{pmatrix} -\vec{\Gamma} \\ \\ \vec{U}_L \end{pmatrix} = (D_\alpha + D_\beta) \begin{pmatrix} <\mathbf{e}_p \cdot \mathbf{e}_p> & I(\psi) \\ \\ I(\psi) & <B^2> \end{pmatrix} \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix}$$
$$+ \begin{pmatrix} R_{11} & R_{12} \\ \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{E}}{\partial \psi} \\ \\ \frac{\partial \vec{\Lambda}}{\partial \psi} \end{pmatrix}$$
$$+ L_0 \begin{pmatrix} \vec{E} \\ \\ \vec{\Lambda} \end{pmatrix} + L_1 \begin{pmatrix} \frac{\partial \vec{E}}{\partial \psi} \\ \\ \frac{\partial \vec{\Lambda}}{\partial \psi} \end{pmatrix} + L_2 \begin{pmatrix} \frac{\partial^2 \vec{E}}{\partial \psi^2} \\ \\ \frac{\partial^2 \vec{\Lambda}}{\partial \psi^2} \end{pmatrix} \qquad (6.79)$$

The matrix D_α describes the effect of Coulomb collisions and D_β the interaction with neutrals. Retaining only these terms generalises the flux-friction relation derived in chapter 2 (eq. 2.35). The S-matrix describes the momentum losses by ionisation and the R and K-matrices are the result of Coriolis forces. These matrices are linear in the diffusion fluxes, either classical Pfirsch-Schlüter diffusion or anomalous diffusion. Retaining only the parallel component of the diffusion flux yields $R_{ij} = K_{ij} = 0$, except for $R_{11} \neq 0$. In this approximation the Stringer spin-up mechanism is found again. The L-matrices describe the effects of the Braginskii viscosity, L_o represents the magnetic pumping effect, the other two matrices are the result of shear viscosity and gyro viscosity. An approximation to the L_o-matrix is given in eq. 5.33. The terms arising from shear viscosity and gyro viscosity usually are rather small and may be neglected, however, these terms introduce second order derivatives in the radial coordinate ψ and therefore they are important in boundary layers or singularities. On the left hand side the fluxes $\vec{\Gamma}$ and $\vec{U}_L$ are consider as given quantities. This is obvious in case of the loop voltage, the fluxes $\vec{\Gamma}$ are linked to the source terms by the equation of continuity. If we also take into account the anomalous particle fluxes the left hand side of the flux-friction relation must be replaced by

$$\begin{pmatrix} -\vec{\Gamma} + \vec{K}_{p,an} \\ \\ \vec{U}_L - \vec{K}_{b,an} \end{pmatrix} \tag{6.80}$$

Without giving more information about the physics of the anomalous transport this only a formal extension of the flux-friction relation. It depends on the details of the plasma turbulence and the underlying instability mechanism how these anomalous terms depend on the thermodynamic forces $\vec{E}$ and $\vec{\Lambda}$. However, the modified flux-friction relations show, that anomalous or turbulent losses occur on two places, on one hand they modify the lowest order solution, which follows from Eq. 6.48 and on the other hand they enhance the R and K-matrices, thus modifying the spin-up mechanism.

As pointed out in the previous sections these equations exhibit a strong similarity to the the equations which govern the zonal circulation in planetary atmospheres. There temperature gradient driven Hadley cells couple into azimutal rotation. The coupling is provided by the Coriolis force which may amplify the rotation of the planet leading to a strong rotation of the planetary atmosphere as observed on Venus. In the toroidal plasma the Hadley cells are replaced by the radial diffusive flux and the associated parallel mass flow. The slow rotation which can be enhanced by the Coriolis forces are the solutions $\vec{E}_0, \vec{\Lambda}_0$ as described in Eq. 6.48.

Formally the flux-friction equations are linear in the forces $\vec{E}$ and $\vec{\Lambda}$ and therefore multiple solutions and bifurcations cannot occur. However, the diffusion fluxes, either classical or anomalous, depend on these forces, and this way non-linearities are introduced. To obtain quantitative results it needs to analyse the anomalous tranport mechanism in more detail, the present analysis is restricted to the general role of anomalous effects in the flux-friction relations.

These relations Eq. 6.79 hold for any toroidal equilibrium, there is no limitation to a specific configuration or a special coordinate system. The influence of the specific geometry is represented by the two vetors $\mathbf{B}$ and $\mathbf{e_p}$, where the parallel component of $\mathbf{e_p}$ is proportional to the Pfirsch-Schlüter currents. Reducing the Pfirsch-Schlüter currents by a proper choice of the equilibrium field reduces also the magnetic pumping effect and thus facilitates the poloidal spin-up and the increase of the radial electric field. In deriving the flux-friction relations we made use of the properties of the magnetic field which is the field of an ideal MHD-equilibrium. This is a slight inconsistency, since the force balance of the rotating plasma is modified by the inertial forces and the viscous forces and the self-consistent magnetic field may differ from that of an ideal equilibrium. However, since these forces are small compared with ∇p and $\mathbf{j} \times \mathbf{B}$ the modification of the ideal equilibrium is expected to be small, too. If islands occur, this conclusion is no longer true, in this this case the self-consistent treatment of the stationary equilibrium is needed, however, this is beyond the scope of the present paper. A further deficiency of the present model is the assumption of constant temperature, the effect of temperature gradients will be considered in a subsequent paper.