MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

EXACT THREE-DIMENSIONAL MHD EQUILIBRIA WITHOUT CONTINUOUS SYMMETRIES

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Abstract

Exact three-dimensional solutions of the magnetohydrodynamic (MHD) equations are presented. The configurations are infinitely extended along a straight axis and have neither cylindrical nor helical symmetry. All field lines are plane and closed around the axis. The magnetic surfaces have elliptical cross-sections whose ellipticity along the axis is an arbitrary function.

In magnetohydrodynamic (MHD) theory plasma equilibria with pressure P, magnetic field $\mathbf{B}$ and current density $\mathbf{j}$ must satisfy the equations

$$
\begin{aligned}
\mathbf{j} \times \mathbf{B} &= \nabla P , \\
\mathbf{j} &= \nabla \times \mathbf{B} , \qquad (1) \\
\nabla \cdot \mathbf{B} &= 0 .
\end{aligned}
$$

Of particular interest are configurations which have smooth nested surfaces $F(\mathrm{r})=$ *const* of constant pressure. Equations (1) imply that the $\mathbf{B}$ (and $\mathbf{j}$) lines are embedded in these surfaces, which are therefore also termed magnetic or flux surfaces.

If the equilibrium is independent of at least one coordinate, Eqs. (1) can be condensed into a single quasilinear elliptic equation [1], [2], [3], of which explicit solutions with smooth nested flux surfaces are known. This refers to axisymmetric toroidal solutions (no dependence on toroidal angle) [4], cylindrical configurations (no dependence on z-coordinate) [2] and helical configurations (dependence on helical and radial coordinates only) [5]. Many more pertinent references exist.

There is a widespread suspicion, based on general considerations [6] and numerical evidence [7], that volume current configurations without these continuous symmetries are always subject to field line chaos and regions of ergodicity (though possibly confined to minor areas). No smooth nested flux surfaces would then exist. This was recently proved for for the special case of nonaxisymmetric toroidal configurations with up/down mirror symmetry and purely poloidal fields [8]. The nonexistence proof fails for straight geometry. This prompted the search for the present solution.

There is only one case without continuous symmetries in which existence of volume current MHD equilibria was proved. This is the case of mirror symmetry, but this time with respect to a poloidal plane, and small plasma beta [9]. No explicit solution, however, was given in [9], nor, to the author's knowledge, are any explicit three-dimensional solutions with pressure gradient cited elsewhere.

Here, for the first time, a simple explicit equilibrium configuration without any of the three continuous symmetries mentioned is presented. It is a "straight" configuration extending from, say, $-\infty < z < \infty$, in a Cartesian coordinate system x, y, z. The field lines are plane curves, concentric around a straight axis at $x = y = 0$. All field lines are ellipses, with half-axis ratio $\sqrt{(1+u)/(1-u)}$, where $u = u(z)$ is an arbitrary function, with $u^2 < 1$. (For $u^2 > 1$ the field lines trace out hyperbolas.)

In detail, the magnetic field is given by

$$
\begin{aligned}
B_x(x, y, z) &= \frac{-P_1}{2c} \sqrt{\frac{1+u}{1-u}}\, y \,, \\
B_y(x, y, z) &= \frac{+P_1}{2c} \sqrt{\frac{1-u}{1+u}}\, x \,, \\
B_z(x, y, z) &= 0 \,,
\end{aligned}
\tag{2}
$$

where c is a constant and $u(z)$ is arbitrary. The pressure is linear in the surface label F,

$$
P(F) = P_0 + P_1 F \,, \tag{3}
$$

and the surfaces $F(x, y, z) = const$ themselves are determined by

$$
F(x, y, z) = \frac{-P_1}{4c^2} \left[\frac{x^2}{1 + u(z)} + \frac{y^2}{1 - u(z)} \right] . \tag{4}
$$

It is elementary to check that $\mathbf{B} = (B_x, B_y, B_z)$, P and F from Eqs. (2) - (4) satisfy Eqs. (1).

In vector notation $\mathbf{B}$ can be represented as

$$
\mathbf{B} = \nabla F \times \nabla G(z) \,, \tag{5}
$$

where $G(z)$ is an arbitrary function which is related to $u(z)$ by

$$
\frac{\mathrm{d}G}{\mathrm{d}z} = c \sqrt{1 - u^2(z)} \,. \tag{6}
$$

From Eq. (5) it follows that $\mathbf{B} \cdot \nabla F = 0$, which proves that the surfaces $F = const$ are flux surfaces. In polar coordinates, with $x = r \cos\theta$, $y = r \sin\theta$, F is given by

$$
F = \frac{-P_1}{4c^2} \frac{r^2}{1 - u^2(z)} \left[1 - u(z) \cos(2\theta) \right] . \tag{7}
$$

The constant c is related to the longitudinal current density j_z as follows

$$j_z = \frac{P_1}{c\sqrt{1-u^2}}, \tag{8}$$

while the perpendicular current density is

$$j_x = \frac{P_1\, u'\, x}{2c\,(1+u)\,\sqrt{1-u^2}}, \qquad j_y = \frac{-P_1\, u'\, y}{2c\,(1-u)\,\sqrt{1-u^2}}, \tag{9}$$

with $u' = \mathrm{d}u/\mathrm{d}z$.

A necessary condition for the existence of equilibria is that $I = \oint \mathrm{d}l/B$ be the same for all closed field lines on a given flux surface, $I = I(F)$, ($B = |\mathbf{B}|$, $\mathrm{d}l =$ length element along $\mathbf{B}$) [10]. Evaluation of I with Eqs. (2) - (4) yields $I = 4\pi c/P_1$, which is indeed a constant, not only on $F = const$, but also absolutely, for all F.

An MHD equilibrium of finite radial extent can of course be obtained by bounding the plasma with a conducting wall at some $F = F_0$.

Some examples of flux surfaces are presented in Figures (1) - (4). The z-axis is in the vertical direction. The half-axes a_x and a_y of the ellipses are $k\sqrt{1+u}$ and $k\sqrt{1-u}$, respectively, where $k = 2c\sqrt{-F/P_1}$. It follows that $0 \le a_x,\ a_y \le k\sqrt{2}$.

Figures 1 and 2 show two periodic solutions, namely $u = 0.4\sin z$, with $z \in [0,\, 6\pi]$, and $u = 0.4 + 0.2\sin z$, $z \in [\pi/2,\, 6\pi + \pi/4]$, respectively, both with $k = 3$. A solution which deviates from axial symmetry only locally is shown in Figure 3, with $u(z) = 0.4 - 0.8\exp(-0.3z^2)$, $k = 3$ and $z \in [-8,\, 8]$. A quasiperiodic flux surface constructed with $u = 0.2(\sin z + \sin\sqrt{2}z)$ is shown in Figure 4, with $k = 5$ and $z \in [0,\, 6\pi]$.

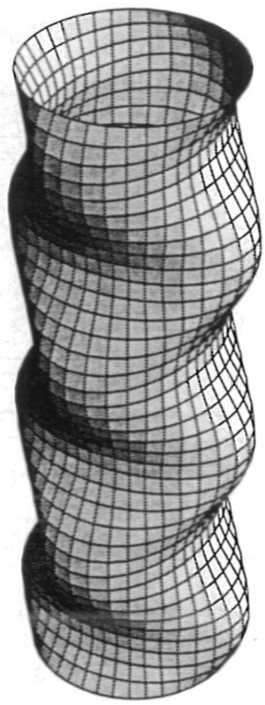

Fig. 1: Magnetic surface $F = const$. Deviation from circular cross section: $u(z) = 0.4 \sin z$. Vertical interval: $z \in [0,\, 6\pi]$.

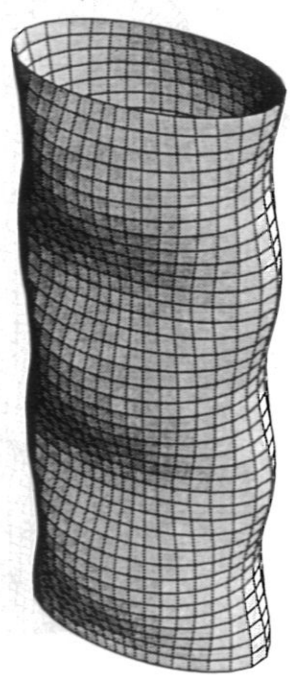

Fig. 2: Same as Fig. 1, $u(z) = 0.4 + 0.2 \sin z$, $z \in [\pi/2,\, 6\pi + \pi/4]$.

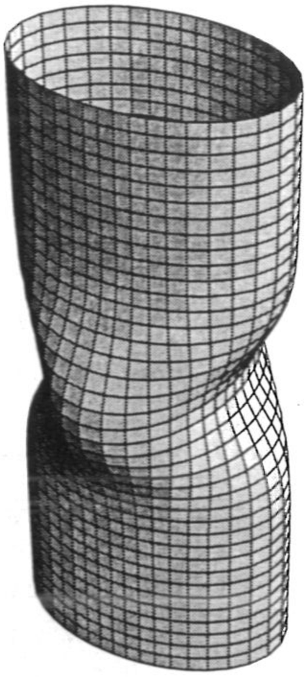

Fig. 3: Same as Fig. 1, $u(z) = 0.4 - 0.8\exp(-0.3z^2)$, $z \in [-8, 8]$.

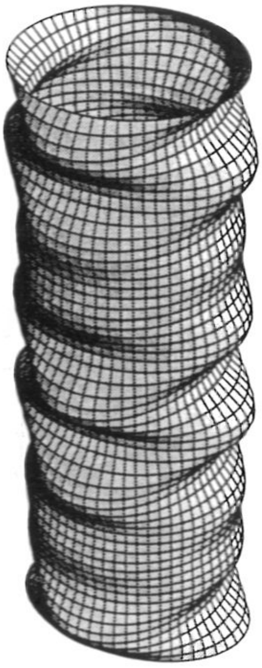

Fig. 4: Same as Fig. 1, $u(z) = 0.2(\sin z + \sin\sqrt{2}z)$, $z \in [0, 6\pi]$.

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