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NONEXISTENCE OF MHD EQUILIBRIA
WITH POLOIDALLY CLOSED FIELD LINES
IN THE CASE OF VIOLATED AXISYMMETRY

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Abstract

The existence of nonaxisymmetric toroidal magnetohydrodynamic (MHD) equilibria whose magnetic field lines are closed after one poloidal turn around the magnetic axis C is investigated analytically. Up-down symmetry of the configuration with respect to the equatorial plane which contains the axis is assumed. In principle, nonaxisymmetry is manifested in the form of a noncircular axis or a variation of the geometry and/or magnetic field along a circular axis. It is proved that no equilibrium with a noncircular axis exists. For a circular axis, nonexistence is proved if the ellipticity of the cross section varies along C. Nor is variation of the triangularity, etc., up to the seventh Fourier mode with respect to the poloidal angle, allowed. For variations with still higher mode numbers nonexistence is made plausible. For the magnetic field the situation is analogous. Nonaxisymmetric poloidal equilibria with equatorial mirror symmetry are thus practically ruled out. The method of investigation is an expansion in the distance from the magnetic axis, supported by the algebraic computer language REDUCE. With growing order the number of constraints on the configuration increases until the quoted results are obtained in sixth and higher orders.

I. Introduction

The existence of toroidal magnetohydrodynamic (MHD) equilibria with axisymmetry was established long ago. Their governing equation, the Grad-Shafranov equation, was derived by Hain, Lüst, Schlüter, Grad, Rubin and Shafranov [1], and many explicit solutions of it are known; see, for example, [2], [3], [4], [5], [6], [7], [8]. On the other hand, it was very soon doubted, whether the MHD equations,

$$
\mathbf{j} \times \mathbf{B} = \nabla P ,
$$
$$
\mathbf{j} = \nabla \times \mathbf{B} ,
$$
$$
\nabla \cdot \mathbf{B} = 0 ,
$$

(1)

have solutions with smooth nested flux surfaces at all in the *nonaxisymmetric* toroidal case [9]. The nature of the existence problem may be regarded in various ways.

From the Hamiltonian point of view, for example, one starts with the differential equations for magnetic field lines

$$
\frac{\mathrm{d}r^1}{\mathrm{d}r^3} = \frac{B^1}{B^3} , \qquad \frac{\mathrm{d}r^2}{\mathrm{d}r^3} = \frac{B^2}{B^3} ,
$$

(2)

where r^i, $i = 1, 2, 3$, are arbitrary coordinates and B^i are the contravariant components of the magnetic field $\mathbf{B}$. Owing to $\nabla \cdot \mathbf{B} = 0$ they may be written as a one-dimensional time-dependent Hamiltonian system [10], equivalent to a two-dimensional time-independent one. Two-dimensional Hamiltonian systems are generically nonintegrable, which for most field lines implies ergodicity and no confinement to magnetic surfaces. This, in itself, would not be a major obstacle, since integrable Hamiltonians exist as well, guaranteeing good magnetic surfaces. The problem is that with $\nabla \cdot \mathbf{j} = 0$ the current lines also constitute a Hamiltonian system which has to be integrable as well, and its surfaces must coincide with those of the field lines. In view of the close relation between $\mathbf{j}$ and $\mathbf{B}$, it is quite unclear whether these two requirements are compatible with each other [11].

From a mathematical point of view the MHD equations are non-standard in that they have both complex and real characteristics. The theory for mixed-type differential equations is not sufficiently developed to ascertain the conditions for the existence of a solution.

A deeper analysis [12], [13] shows that the existence problem for Eqs. (1) can be traced back to the "magnetic differential equation"

$$
\mathbf{B} \cdot \nabla \Phi = B \frac{\mathrm{d}\Phi}{\mathrm{d}l} = S ,
$$

(3)

where Φ is a single-valued function, l is the arclength along $\mathbf{B}$, and S is known. In poloidal and toroidal coordinates which make the field lines straight [10] a Fourier ansatz transforms the operator $\mathbf{B} \cdot \nabla$ into a factor proportional to $a_{mn} = m + nq$. Here q is the ratio of the toroidal to the poloidal magnetic fluxes, and m and n are integer Fourier mode numbers. On all surfaces with rational q, i.e. for all surfaces with closed field lines, Eq. (3) is degenerate, $a_{mn} = 0$, and an infinity of side conditions on S have to be imposed. S ultimately corresponds to a free profile function, e.g. the radial pressure profile. The side conditions are equivalent [12], [14] to the more explicit, but no less daunting requirement

$$ I \equiv \oint \frac{\mathrm{d}l}{B} = I(F), \tag{4} $$

viz. the path integral I along any closed field line must be the same for all field lines on a given surface F of constant pressure.

These doubts on the very existence of nonaxisymmetric toroidal MHD equilibria have been pushed aside for years in analytical and numerical MHD studies of nonaxisymmetric devices such as stellarators. Theoretical analysis was done by, for example, expansion with respect to the distance from the axis [12], [15], [16] or some other ordering methods [17]. The convergence of such expansions remains open. Early numerical codes, e.g. [18], simply assumed, explicitly or implicitly, that island formation and ergodicity of field lines should be minor effects. Recently, the analytical study of ergodization and its elimination, both for vacuum fields [19] and in plasmas [20], has regained attention. Similarly, modern numerical MHD codes [21], down to some finite limit of resolution, are able to cope with islands and regions of ergodicity. It is found that with properly designed external coils and not too large beta values these do not affect the surfaces of constant pressure too badly.

From a theoretical point of view, in this ambiguous situation between doubts of existence and de facto arrangement with minor regions of ergodicity, proofs of either existence or nonexistence of nonaxisymmetric toroidal MHD equilibria should be enlightening even if the configurations treated are not directly related to present-day fusion devices.

Lortz [22] has given the only proof of *existence* of nonaxisymmetric toroidal MHD equilibria with volume currents, namely for toroidal configurations which are mirror-symmetric with respect to a poloidal plane and whose beta value, the ratio of plasma pressure to magnetic field energy, is not too large. Owing to the mirror symmetry all field lines are toroidally closed and hence q is infinite. Although this alleviates problems with resonances, the condition that all closed

field lines on a pressure surface have the same value of I remains nontrivial and has been coped with in the proof.

Palumbo, on the other hand, was able to prove *nonexistence* of a (somewhat special) class of nonaxisymmetric toroidal MHD configurations, namely so-called isodynamic equilibria, provided that there is a plane of symmetry in the configuration [23] or that the configuration is of the stellarator type [24]. "Isodynamic" is defined as having magnetic fields with constant amplitude B on each magnetic surface. Palumbo's result was extended by Garren and Boozer [25], who proved that no so-called quasihelical MHD stellarator equilibria exist. "Quasihelical" is synonymous with B being a function of a helical coordinate only, on each surface, while $\mathbf{B}$ may depend there on a second coordinate as well.

Not unexpectedly, it is somewhat easier to obtain (non-)existence results for nonaxisymmetric *surface current* equilibria. Recently, new explicit toroidal MHD solutions were found [26]. They are characterized by poloidally closed field lines in the current surface. In the case of zero torsion of their (noncircular) axis the shape of the cross section is arbitrary. In [27] both the existence and the nonexistence of further classes of nonaxisymmetric surface current equilibria were proved.

In the present paper we prove nonexistence of the following class of nonaxisymmetric toroidal volume current MHD equilibria. We study configurations which, so to speak, are complementary to those considered by Lortz [22]: the plasma is assumed here to be up-down symmetric with respect to an *equatorial* plane. All field lines close upon themselves after one *poloidal* turn around the axis $\mathbf{C}$ of the configuration. This axis, for symmetry reasons, is assumed to lie completely in the equatorial plane, i.e. the axis has no torsion. Its curvature $\kappa(s)$, however, is an arbitrary function of the arclength s along $\mathbf{C}$ (excluding self-intersection). Either $\kappa(s)$ is non-constant or the configuration varies in some way or other along the axis, if it is a circle.

In axisymmetry many up-down symmetric solutions with poloidally closed field lines and with finite beta are known [3], [4], [6], [8]. The simplest one has been repeatedly obtained, e.g. by Shafranov [3].

The first aim of the present paper then is to show that if the magnetic axis of such axisymmetric configurations is distorted from a circle into some other shape, the plasma is unable to find a new equilibrium. We prove this proposition. We also prove that there is no equilibrium if the circular axis is kept but the ellipticity of the cross section along it is made to vary. If the ellipticity is also fixed, equilibria with variable higher-order deformations might still exist. We prove that up to the seventh order in an expansion around the axis (corresponding essentially to

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keeping the first seven modes in a Fourier decomposition of the minor radius as a function of a poloidal angle) this is again impossible. For still higher orders nonexistence is only made plausible.

In Section II we begin with the definition of an appropriate coordinate system and a representation of the field **B** which ensures the poloidal closure of the field lines. The equilibrium equations are reduced to a system of two partial differential equations for two unknown functions, F and G. The first equation, in axisymmetry, corresponds to the Grad-Shafranov equation, while the second one describes the force balance in the toroidal direction. An expansion in the distance r from the magnetic axis is made in Section III and an outline is given in which sequence the two coupled sets of expanded equations will be solved. Both the expansion in r and the solution of the resulting equations are done with the algebraic computer language REDUCE. Section IV is devoted to the solution of a first group of four low-order equations. The ellipticity of the cross section is found as a function of the curvature of the axis, if the latter is not a circle. In the opposite case it is free. The next quadruple of equations is solved in Section V. In this section the nonexistence of up-down symmetric poloidal equilibria with noncircular axis is proved. It is also proved that poloidal equilibria with circular axis and variable ellipticity or triangularity of the cross section along it do not exist. In Section VI, finally, the nonexistence of poloidal equilibria with circular axis is extended to cases with variable quadrilaterality, etc., up to heptagonality. For variable deformations of still higher order nonexistence of an equilibrium is made plausible. Conclusions are given in Section VII.

II. Coordinate system, closed field line configurations, and force balance

We consider toroidal configurations whose magnetic axis is an arbitrary non-intersecting plane curve $\mathbf{C}(s)$, s = arclength along $\mathbf{C}$. $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the tangent, normal and binormal vectors, respectively, along $\mathbf{C}$. In the poloidal planes $s = const$, spanned by $\mathbf{n}$ and $\mathbf{b}$, polar coordinates r, θ are defined such that the position vector $\mathbf{x}$ assumes the form

$$\mathbf{x}(r, \theta, s) = \mathbf{C}(s) + r[\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta] . \tag{5}$$

With the Serret-Frenet formulas for plane curves,

$$\mathbf{t}' = \kappa\mathbf{n} , \qquad \mathbf{n}' = -\kappa\mathbf{t} , \tag{6}$$

where $\kappa(s) \geq 0$ is the curvature of $\mathbf{C}$ and primes here denote derivation with respect to s, the metric coefficients $g_{ik} = \frac{\partial\mathbf{x}}{\partial u^i} \cdot \frac{\partial\mathbf{x}}{\partial u^k}$ are found to be

$$g_{rr} = 1 , \quad g_{\theta\theta} = r^2 , \quad g_{ss} = \Delta^2 ,$$
$$g_{r\theta} = g_{rs} = g_{\theta s} = 0 , \tag{7}$$

where

$$\Delta = 1 - \kappa(s)r\cos\theta . \tag{8}$$

u^1, u^2, u^3 stands for r, θ, s, respectively. For the determinant g of the metric tensor one obtains

$$\sqrt{g} = r\Delta . \tag{9}$$

Before the equilibrium equations are expressed in this coordinate system a representation of the magnetic field $\mathbf{B}$ is given which ensures that the field lines are poloidally closed around the axis.

We start from the assumption (to be questioned) that there is an equilibrium configuration with nested toroidal surfaces of constant pressure P and label them with a scalar $F = F(r, \theta, s) = const$. F is periodic in θ and s with periods 2π and L, respectively, where L is the circumference of the axis. According to Eqs. (1) $\mathbf{B} \cdot \nabla P = 0$ and hence the magnetic field lines are embedded in the $P = const$ surfaces. We define the magnetic surfaces to coincide with the $P = const$ surfaces. This definition of magnetic surfaces, which does not restrict possible configurations, is practical and removes the ambiguity of definition inherent in cases with closed field lines.

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With the assumption of flux surfaces and with the condition $\nabla \cdot \mathbf{B} = 0$ the magnetic field can always be written in the form [28]

$$\mathbf{B} = [\nabla F \times \nabla G] \,. \tag{10}$$

where the condition that $\mathbf{B}$ be a periodic function with respect to a closed poloidal or toroidal path around the torus is satisfied by the ansatz

$$G = \alpha(F)\theta + \beta(F)s + \gamma(F,\theta,s) \,. \tag{11}$$

Here α, β, and γ are arbitrary functions of F, and γ is periodic in θ and s with periods 2π and L, respectively. The magnetic field is manifestly tangential to the surfaces $F = const$, as desired. The intersection of the surfaces $F = const$ and $G = const$ defines curves tangential to $\mathbf{B}$. Thus, $G = const$ at F kept constant are the field lines. A field line is poloidally closed if it comes back to the same value of s after an increase in θ of 2π. This requires

$$\alpha(F) = 0 \,, \tag{12}$$

a condition which has a simple interpretation: $2\pi\alpha(F)$ is the toroidal flux through a poloidal surface element between F and $F + dF$ [28]. In the case of poloidally closed field lines this flux must vanish.

The labelling of the flux surfaces $F = const$ is not unique. One can always transform to another label $\tilde{F} = \tilde{F}(F)$. It is useful to exploit this arbitrariness by the following definition of $\tilde{F}$:

$$\beta\left(F(\tilde{F})\right) = 1 \,. \tag{13}$$

Altogether, omitting the tilde, we are left with the representation

$$G(F,\theta,s) = s + \gamma(F,\theta,s) \,. \tag{14}$$

If γ is independent of θ, the field lines $G = const$ at $F = const$ are given by $s = const$, i.e. they are plane, otherwise not.

The contravariant components of the magnetic field are obtained from $[\mathbf{a} \times \mathbf{b}]^i = \epsilon^{ikl} a_k b_l / \sqrt{g}$, $i, k, l = r, \theta, s$, where ϵ^{ikl} is the antisymmetric Ricci tensor with values 0, ± 1, summation over repeated indices is implied, and a_k, b_l are the covariant components of the vectors $\mathbf{a}$, $\mathbf{b}$. From Eqs. (10) and (14) it follows that

$$B^r = \frac{1}{\sqrt{g}}\frac{\partial G}{\partial s}\frac{\partial F}{\partial \theta} - \frac{1}{\sqrt{g}}\frac{\partial G}{\partial \theta}\frac{\partial F}{\partial s} \,,$$

$$B^\theta = -\frac{1}{\sqrt{g}}\frac{\partial G}{\partial s}\frac{\partial F}{\partial r} \,, \tag{15}$$

$$B^s = \frac{1}{\sqrt{g}} \frac{\partial G}{\partial \theta} \frac{\partial F}{\partial r} ,$$

where the partial derivatives of G are taken at fixed F. The covariant components of $\mathbf{B}$, $B_i = g_{ik} B^k$, are required for the current density vector $\mathbf{j}$, whose components obey the relations

$$\sqrt{g}\, j^i = \epsilon^{ikl} \frac{\partial B_k}{\partial x^l} , \qquad i, k, l = r, \theta, s . \tag{16}$$

The covariant components of the the MHD force balance equation, finally, are

$$\sqrt{g}\, j^\theta B^s - \sqrt{g}\, j^s B^\theta - \frac{\partial P}{\partial r} = 0 , \tag{17}$$

$$\sqrt{g}\, j^s B^r - \sqrt{g}\, j^r B^s - \frac{\partial P}{\partial \theta} = 0 , \tag{18}$$

$$\sqrt{g}\, j^r B^\theta - \sqrt{g}\, j^\theta B^r - \frac{\partial P}{\partial s} = 0 . \tag{19}$$

Equations (15)-(19) can be combined into a system of two coupled partial differential equations for F and G. The first one is a generalization of the Grad-Shafranov equation [1] to the special nonsymmetric case considered here, while the second describes the toroidal force balance. We shall discuss them presently.

With $\partial P / \partial r = (dP/dF) \partial F / \partial r$ and Eqs. (15)-(16) a factor $\partial F / \partial r$ cancels from Eq. (17) for the radial force balance (provided $\partial F / \partial r \neq 0$), leaving

$$\sigma \frac{\partial}{\partial r} \left(\frac{g_{\theta\theta}}{\sqrt{g}} \sigma \frac{\partial F}{\partial r} \right) + \tau \frac{\partial}{\partial r} \left(\frac{g_{ss}}{\sqrt{g}} \tau \frac{\partial F}{\partial r} \right)$$
$$+ \left(\sigma \frac{\partial}{\partial \theta} - \tau \frac{\partial}{\partial s} \right) \left[\frac{g_{rr}}{\sqrt{g}} \left(\sigma \frac{\partial}{\partial \theta} - \tau \frac{\partial}{\partial s} \right) F \right] + \sqrt{g} \frac{dP(F)}{dF} = 0 , \tag{20}$$

where

$$\sigma \equiv \left. \frac{\partial G}{\partial s} \right|_F , \qquad \tau \equiv \left. \frac{\partial G}{\partial \theta} \right|_F . \tag{21}$$

If $\tau = 0$ on a field line, this line is plane, as mentioned above.

If G is considered given, Eq. (20) is a quasilinear parabolic p.d.e. for $F(r, \theta, s)$: with three independent variables there are only two second-order derivatives, $\partial^2 / \partial r^2$ and the combination $(\sigma \partial / \partial \theta - \tau \partial / \partial s)^2$. The operator

$$\mathcal{L} = \sigma \partial / \partial \theta - \tau \partial / \partial s , \tag{22}$$

when applied to arbitrary functions $A(G)$ which only depend on the field line label G, at a fixed magnetic surface $F = const$, gives zero: $\mathcal{L} A(G) = A'(G) \mathcal{L} G = A' (\sigma \partial G / \partial \theta - \tau \partial G / \partial s) = 0$, by virtue of Eqs. (21). The characteristic surfaces

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[29] S of Eqs. (20) are the poloidal surfaces obtained by the union of all field lines with arbitrary F and fixed $G = G_0 = const$. There are no derivatives in Eq. (20) pointing out of S, in conformity with the poloidally closed field lines having no mutual toroidal connection. Inside S Eq. (20) is elliptic.

In the axisymmetric case the field lines are plane, i.e. $\tau = 0$, and by a suitable transformation from F to $\hat{F}(F)$ $\sigma = 1$ can be achieved. In this case Eq. (20) trnsforms to the Grad-Shafranov equation [1] with zero toroidal field.

The force balance in the θ direction, Eq. (18), also leads to Eq. (20), provided $\partial F / \partial \theta \neq 0$.

Finally, the toroidal force balance, Eq. (19), becomes

$$
\sigma \frac{\partial F}{\partial r} \left\{ \frac{\partial}{\partial s} \left(\frac{g_{\theta\theta}}{\sqrt{g}} \sigma \frac{\partial F}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{g_{ss}}{\sqrt{g}} \tau \frac{\partial F}{\partial r} \right) \right\}
$$
$$
+ \left(\sigma \frac{\partial F}{\partial \theta} - \tau \frac{\partial F}{\partial s} \right) \left\{ \frac{\partial}{\partial s} \left[\frac{g_{rr}}{\sqrt{g}} \left(\sigma \frac{\partial F}{\partial \theta} - \tau \frac{\partial F}{\partial s} \right) \right] - \frac{\partial}{\partial r} \left(\frac{g_{ss}}{\sqrt{g}} \tau \frac{\partial F}{\partial r} \right) \right\}
$$
$$
+ \sqrt{g} \frac{dP}{dF} \frac{\partial F}{\partial s} = 0 . \tag{23}
$$

We do not have any a priori insight regarding possible solutions of this equation. With respect to the existence of nonaxisymmetric equilibria, however, later experience shows it to be more critical than Eq. (20).

In passing, we note that for *plane* field lines, but not necessarily axisymmetry, Eq. (23) can be integrated with respect to s because it then assumes the form

$$
\frac{\partial}{\partial s} \left[g_{\theta\theta} \left(\frac{\sigma}{\sqrt{g}} \frac{\partial F}{\partial r} \right)^2 + g_{rr} \left(\frac{\sigma}{\sqrt{g}} \frac{\partial F}{\partial \theta} \right)^2 + 2P(F) \right] = 0 . \tag{24}
$$

Of course, the sum of the first two terms in the square brackets constitutes B^2, and Eq. (24) expresses the conservation of the total energy density along s. It would be interesting to find out analytically whether in this rather restricted set of geometries the pair of equations (20) (with $\tau = 0$) and (24) can have a solution. Indeed, if even further restrictions on the magnetic field or geometry are introduced, nonexistence can be proved [30].

Equations (20) and (23) are two coupled equations for the two unknown functions $F(r, \theta, s)$ and $G(F, \theta, s)$. They constitute the basis of our investigations in the following sections.

III. Expansion around the magnetic axis

In order to find out whether Eqs. (20) and (23) are compatible, an expansion in the distance r from the magnetic axis is made.

For this purpose boundary conditions on the axis are required. The restriction to purely poloidal fields (no toroidal flux) implies that the toroidal magnetic field component on axis must vanish. Furthermore, we consider only configurations whose current density is non-singular also on axis. With $|\nabla\theta| = O(1/r)$ this leads to

$$B^r(r = 0) = 0 , \tag{25}$$

$$B^\theta(r = 0) = O(1) , \tag{26}$$

$$B^s(r = 0) = 0 . \tag{27}$$

Equation (27) is in contrast to previous expansions of the MHD equations around a magnetic axis [12, 15, 16]. There, $B^s \neq 0$ on axis is explicitly assumed and the analysis would break down in case of violation of this assumption.

The poloidal flux through a ribbon of size $dS = L\,dr$ along the axis (length L) is $d\Psi_p = dS\,\mathbf{n}\cdot\mathbf{B} = L\,dr\,B^\theta/|\nabla\theta| = O(r)dr$. On the other hand, one obtains from Eqs. (10)-(13) for this same poloidal flux [28] $d\Psi_p = -L\,dF$. Hence, $dF/dr = O(r)$, so that F is a parabolic function of r in the neighbourhood of the axis. On the axis itself F has to be constant. If we take this arbitrary constant to be zero, the following expansion results:

$$F(r, \theta, s) = f_2(s, \theta)r^2 + f_3(s, \theta)r^3 + f_4(s, \theta)r^4 + \dots . \tag{28}$$

For $G(F, \theta, s)$ we make the straightforward ansatz

$$G(F, \theta, s) = G_0(s, \theta) + G_1(s, \theta)F + G_2(s, \theta)F^2 + \dots , \tag{29}$$

where capitals are used to indicate an expansion with respect to F. For F, in a second step, the series (28) has to be inserted. The ansatz (29) is only compatible with Eqs. (15) and (28) provided that

$$\frac{\partial G_0}{\partial \theta} = 0 . \tag{30}$$

Equation (14) implies that $\gamma(F, \theta, s)$ has a similar expansion to G. Indeed, $G_0 = s + \gamma_0$, $G_i = \gamma_i$, $i = 1, 2, \dots$. Periodicity of γ implies that the functions γ_i, $i = 0, 1, 2 \dots$, and G_i, $i = 1, 2 \dots$, are periodic in θ and s. $G_0(\theta, s)$ is periodic in θ but only $\partial G_0/\partial s$ is also periodic in s.

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The pressure $P(F)$ is likewise expanded in the form

$$P(F) = P_0 + P_1 F + P_2 F^2 + \dots . \tag{31}$$

From Eqs. (28)-(31) one obtains a series expansion in powers of r of the basic Eqs. (20) and (23). Instead of quoting these equations by equation number the more vivid names – but to be taken with a grain of salt – "Grad-Shafranov" (GS) equation and "force balance" (FB) equation (without "toroidal" preceding it) will also be used.

It is advantageous to know in advance which quantities are determined by which of the two equations and in which order of expansion. Letting REDUCE do the work, one can look for f_m, G_n and P_k with the highest indices m, n, k which may contribute to the order r^N. The results for the GS equation are collected in Table I:

Table I: GS equation

N	1	2	3	4	5	6	7
F	f_2	f_3	f_4	f_5	f_6	f_7	f_8
G	G_0	G_0	G_1	G_1	G_2	G_2	G_3
P	P_1	P_1	P_2	P_2	P_3	P_3	P_4
M	1		2		3		4

and those for the FB equation in Table II:

Table II: FB equation

N	1	2	3	4	5	6	7	8
F			f_2	f_3	f_4	f_5	f_6	f_7
G			G_1	G_1	G_2	G_2	G_3	G_3
P			P_1	P_1	P_2	P_2	P_3	P_3
M			1		2		3	

Looking at G_n and P_k shows that the equations can be grouped in pairs – indicated in the tables by vertical bars. The pairs have been labeled with an index, M, which shows which pairs in the two equations belong together by virtue of having the same pair of f_m. A natural procedure, consisting of four steps for each M, to solve the equations seems to be as follows. Consider $M = 1$: f_2 is determined – partly or fully – from the GS equation with $N = 1$, for G_0 given. The same applies to f_3, with $N = 2$. Next G_1 is determined from the FB equation with

$N = 3$. The FB equation with $N = 4$, finally, either serves to determine G_0 and/or to determine f_2, f_3, and G_1 more completely and/or gives a necessary condition to be satisfied by the configuration. Such necessary conditions are what will ultimately prevent the existence of solutions.

It is instructive to see the explicit analytical expressions for the highest-order terms f_m and G_n quoted schematically in Tables I and II. After some tedious but straightforward work one obtains the two equations

$$\sigma_0^2 \left(\frac{\partial^2 f_m}{\partial \theta^2} + m^2 f_m \right) r^{m-1} = \left\{ - \sigma_0 f_2^{n-1} \left[4(n+2) f_2'^2 + 2 \ddot{f}_2 f_2 + n \dot{f}_2^2 \right] G_n' \right.$$

$$\left. + \left[A_1(\theta, s) + A_2(\theta, s) \frac{\partial}{\partial \theta} + A_3(\theta, s) \frac{\partial}{\partial s} \right] \dot{G}_n \right\} r^{2n+1}$$

$$+ \text{ terms with lower order } \{f_{\tilde{m}}\}, \{G_{\tilde{n}}\}, \qquad m \geq 2, \ n \geq 1, \tag{32}$$

from the GS equation, and

$$4 \sigma_0 f_2^{n+1} \frac{\partial}{\partial \theta} \left(\dot{G}_n f_2 \right) r^{2n+1} = - \frac{\partial}{\partial s} \left[2m \sigma_0^2 f_2 f_m + \sigma_0^2 \dot{f}_2 \dot{f}_m + P_1 f_m \right] r^{m+1}$$

$$+ \text{ terms with lower order } \{f_{\tilde{m}}\}, \{G_{\tilde{n}}\}, \qquad m \geq 3, \ n \geq 1, \tag{33}$$

from the FB equation. (The details of the coefficients A_1, A_2, and A_3 in Eq. (32) are not relevant, since these terms will not play a role; see below).

In Eqs. (32) and (33) the terms with factor r^{2n+1} are valid for odd orders, $N = 2n + 1$, only. In even orders, $N = 2n + 2$, the terms are a lot more involved and need not be discussed here, except for the fact that they also depend on (derivatives of) G_n. In Eq. (32) and (33) one has $m = 2n + 2$ and $m = 2n$, respectively. For values of m and n less than those indicated the respective terms either do not exist or do not conform to the general expressions given.

Regarding the notation for derivatives, we use throughout the paper, for compactness, both an explicit notation and the abbreviations $\dot{\ }$ and $'$ for $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial s}$, respectively. The partial derivatives of course turn into total ones if there is only one argument.

Relations (32) and (33) show that the sequence of steps envisaged above is reasonable: the functions f_m are determined by the GS equations, in the form of an ordinary harmonic differential equation in the poloidal angle θ whose solution is straightforward once the right-hand sides are expressed as functions of θ. The

functions G_n are solutions of the FB equations of odd order and can be solved by quadrature. Periodicity conditions with respect to θ have to be satisfied here. Both solution processes are also done with the help of REDUCE. It is evident, however, that the dependence on s, the arclength along the axis, has to be found in a more indirect way than the dependence on the poloidal angle θ.

In the following sections the quadruples of equations for consecutive Ms will be treated one by one.

IV. Equations in group $M = 1$

A. "Grad-Shafranov" equation, order $N = 1$

The GS equation (20) to order $N = 1$ gives the differential equation

$$\frac{\partial^2 f_2}{\partial \theta^2} + 4 f_2 = \frac{-P_1}{\sigma_0^2} , \tag{34}$$

where, by virtue of Eqs. (21), (29), and (30),

$$\sigma_0 = \sigma_0(s) \equiv \frac{dG_0(s)}{ds} . \tag{35}$$

The function $\sigma_0(s)$ is related to the current density j_0^s on axis via $j_0^s = P_1/\sigma_0$. The solution of Eq. (34) is

$$f_2(\theta, s) = f_{20}(s) + f_{22}(s) \cos(2\theta) , \tag{36}$$

where

$$f_{20}(s) = \frac{-P_1}{4\sigma_0^2} . \tag{37}$$

$\sigma_0(s)$ and $f_{22}(s)$ are as yet undetermined. A term containing $\sin(2\theta)$, also with undetermined coefficient, has been omitted in Eq. (36), i.e. this coefficient has been made zero. Antisymmetric terms with respect to θ, in $F(r, \theta, s)$, would violate the assumed up-down symmetry of the configuration.

It is advantageous to introduce a quantity $u(s)$,

$$u(s) = -\frac{f_{22}}{f_{20}} , \tag{38}$$

such that

$$f_2(\theta, s) = f_{20}(s)[1 - u(s) \cos(2\theta)] . \tag{39}$$

To the present lowest order the cross sections of the magnetic surfaces, for $|u| < 1$, are ellipses with s-dependent ellipticity. With Cartesian coordinates x, y in the planes $s = const$, $x = r \cos \theta$, $y = r \sin \theta$, one obtains from Eq. (28) for the cross sections

$$F = f_{20}(s) \left[(1 - u)x^2 + (1 + u)y^2 \right] = const , \tag{40}$$

so that the half-axis ratio ϵ of the ellipses is $\epsilon(s) = \sqrt{(1 + u)/(1 - u)}$.

The field lines of $\mathbf{B}$, determined so far by $G = G_0 = const$, are embedded in the $s = const$ planes, as evidenced by Eq. (30).

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B. "Grad-Shafranov" equation, order $N = 2$

To order $N = 2$ the GS equation (20) gives

$$\frac{\partial^2 f_3}{\partial \theta^2} + 9 f_3 = \kappa \left(\frac{\partial^2 f_2}{\partial \theta^2} \cos \theta + \frac{\partial f_2}{\partial \theta} \sin \theta + 2 f_2 \cos \theta + \frac{3 P_1}{\sigma_0^2} \cos \theta \right) . \tag{41}$$

If f_2 from Eq. (39) is inserted, this becomes

$$\frac{\partial^2 f_3}{\partial \theta^2} + 9 f_3 = \frac{\kappa P_1}{2 \sigma_0^2} (5 - u) \cos \theta . \tag{42}$$

A cancellation of Fourier modes 3θ on the r.h.s. of Eq. (41) has taken place. This avoids the occurrence of secular terms from resonances with the left-hand side. The solution of Eq. (42) is

$$f_3(\theta, s) = f_{31}(s) \cos \theta + f_{33}(s) \cos(3\theta) . \tag{43}$$

where

$$f_{31}(s) = \frac{\kappa P_1}{16 \sigma_0^2} (5 - u) , \tag{44}$$

and $f_{33}(s)$ is as yet undetermined. It adds triangularity to the shape of the cross sections. f_{31}, which is proportional to the curvature $\kappa(s)$ of the magnetic axis, corresponds to the Shafranov shift. The field lines are still plane.

C. "Force balance" equation, order $N = 3$

From the FB equation (23) one obtains to lowest order, $N = 3$, a differential equation for G_1:

$$4 \sigma_0 f_2^2 \frac{\partial}{\partial \theta} \left(\dot{G}_1 f_2 \right) = -\frac{\partial f_2'}{\partial \theta} \frac{\partial f_2}{\partial \theta} \sigma_0^2 - 4 f_2' f_2 \sigma_0^2 - f_2' P_1$$

$$- \left(\frac{\partial f_2}{\partial \theta} \right)^2 \sigma_0' \sigma_0 - 4 f_2^2 \sigma_0' \sigma_0 . \tag{45}$$

With Eqs. (37) and (39) this drastically simplifies to

$$\frac{\partial}{\partial \theta} \left(\dot{G}_1 f_2 \right) = R_3 = \frac{N_3}{D_3} , \tag{46}$$

where

$$N_3 = u^2 \sigma_0'(s) - u u'(s) \sigma_0 - \sigma_0'(s) , \tag{47}$$

and

$$D_3 = [1 - u \cos(2\theta)]^2 . \tag{48}$$

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The requirement that the configuration, i.e. here that f_2 and G_1 be periodic in θ, implies $\int_0^{2\pi} R_3 \, d\theta = 0$. The numerator N_3 is independent of θ. and since

$$I_{20} \equiv \int_0^{2\pi} \frac{d\theta}{[1 - u\cos(2\theta)]^2} \neq 0 , \qquad |u| < 1 . \tag{49}$$

it follows that

$$N_3 = u^2 \sigma_0'(s) - uu'(s)\sigma_0 - \sigma_0'(s) = 0 . \tag{50}$$

This differential equation for σ_0 as a function of u can be integrated and gives

$$\sigma_0^2(s) = c_1 \left[1 - u^2(s)\right] , \tag{51}$$

where $c_1 > 0$ is an arbitrary constant.

Going back to the cross sections $F = const$ of Eq. (40), one now finds them to be given by

$$\frac{-4c_1}{P_1} F = \frac{x^2}{1 + u(s)} + \frac{y^2}{1 - u(s)} . \tag{52}$$

For $u'(s) \neq 0$ the shape of the cross sections varies along the axis but the "amplitude" stays fixed. $u > 0$ corresponds to horizontally elongated ellipses. The constant c_1 is a measure of the poloidal magnetic flux density.

A further consequence of Eqs. (46) and (50) is

$$\frac{\partial G_1}{\partial \theta} = \frac{h_1(s)}{f_2(\theta, s)} , \tag{53}$$

where $h_1(s)$ is an arbitrary periodic function. The same periodicity argument as before gives $h_1(s) = 0$, and therefore

$$G_1 = G_1(s) \tag{54}$$

is left as an arbitrary periodic function of s alone. This also implies that the field lines are still plane to this order.

D. "Force balance" equation, order $N = 4$

If the results obtained so far for f_2, f_3, and G_1 are inserted into the order $N = 4$ contributions of the FB equation (23), the following equation results

$$\frac{P_1}{32(u^2 - 1)^2 c_1} [N_{41} \cos\theta + N_{43} \cos(3\theta)] = 0 , \tag{55}$$

where

$$N_{41} = 3[16(f_{33}'u + u'f_{33})(u^2 - 1)c_1 - (2u^2 + u + 1)(u^2 - 1)\kappa'P_1$$

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$$+ (u^2 + 6u + 1)u'\kappa P_1] \tag{56}$$

and

$$N_{43} = -[(3u^2 + 2u + 3)u'\kappa P_1 - (u + 3)(u^2 - 1)\kappa' P_1 u$$

$$+ 16(u^2 - 1)^2 c_1 f'_{33}] . \tag{57}$$

In the following it will be assumed that the pressure profile $P(F)$ is not "degenerate" on the axis, i.e. $P_1 \neq 0$. Since the coefficients of both modes, $m = 1$ and $m = 3$, must vanish, it follows that

$$N_{41} = N_{43} = 0 . \tag{58}$$

We thus have two linear inhomogeneous equations for the unknowns $f_{33}(s)$ and $f'_{33}(s)$. Their solutions are (postponing the case $u'(s) = 0$)

$$f_{33}(s) = P_1 \frac{(3u + 1)u'\kappa - (u + 1)(u^2 - 1)\kappa'}{16(u^2 - 1)c_1 u'} , \tag{59}$$

$$f'_{33}(s) = -P_1 \frac{(3u^2 + 2u + 3)u'\kappa - (u + 3)(u^2 - 1)\kappa' u}{16(u^2 - 1)^2 c_1} . \tag{60}$$

Equations (59) and (60) have to be compatible. If f_{33} is differentiated with respect to s, the result must agree with the expression given in Eq. (60). This gives the necessary and sufficient condition

$$\kappa' u'' (1 + u) - 2\kappa' u'^2 - \kappa'' u' (1 + u) = 0 . \tag{61}$$

Equation (61) is easily integrated. (Division by $\kappa' u' (1 + u)$ makes this evident.) This yields

$$u(s) = \frac{\kappa_v}{\kappa(s) - \kappa_c} - 1 , \tag{62}$$

where $\kappa_v \neq 0$ and κ_c are two otherwise arbitrary constants.

The ellipticity u is a *local* function of the curvature κ of the axis. $u(s)$ can be prescribed at will at two arbitrary locations $s = s_1$ and $s = s_2$ along the axis. The values there determine κ_v and κ_c. $u(s)$ is then fully determined for all s. As Eq. (62) indicates, it may happen, however, with unfortunately chosen parameters, that $u(s)$ approaches 1 at some position(s), in particular if κ comes close to the critical $\kappa = \kappa_c$, and the ellipse degenerates into a horizontal line.

There are two special cases which play a particular role here and in the following sections. In both cases the axis is a circle (or more generally, $\kappa(s)$ is constant

on a finite s interval). In the first case the ellipticity $u(s)$ of the poloidal cross section is *not* constant, $u' \neq 0$, while in the second case it *is*.

In the first case Eqs. (59)-(61) are applicable. With $\kappa' \equiv 0$ the compatibility constraint (62) is automatically satisfied for *arbitrary* $u(s)$. (Formally, in Eq. (62) this case corresponds to $\kappa = \kappa_c$, $\kappa_v = 0$ and $u = 0/0$.) In Eq. (59) u' drops out, yielding

$$f_{33}(s) = f_{33}^\kappa(s) = P_1 \frac{(3u+1)\kappa}{16(u^2-1)c_1}, \quad u(s) = u^\kappa(s) = \text{free}. \tag{63}$$

where the index κ is used as a reminder for $\kappa'(s) \equiv 0$. (Let us add that instead of $\kappa' \equiv 0$ the weaker local condition $\kappa' = \kappa'' = 0$ also leads to Eq. (63). We shall not dwell on such turning-points any further.)

The second case is originally defined as the case postponed so far, namely $u'(s) = 0$ locally or on a finite s interval. Resuming this case with *constant* ellipticity u of the poloidal cross section – so far for axes with arbitrary $\kappa(s)$ – we have to reconsider the equations $N_{41} = N_{43} = 0$ with $u' = 0$. One obtains the relations

$$\kappa' = f_{33}' = 0 \tag{64}$$

instead of Eqs. (59)-(62). Thus $u' = 0$ is possible only if at the same position $\kappa' = 0$ also. The result is again particularly relevant to configurations with circular axis: Eq. (64) implies that in this case, where the ellipticity is constant, the coefficient f_{33} is also constant with respect to s, and that its numerical value is free:

$$f_{33} = f_{33}^{\kappa u} = const = \text{free}, \quad u = u^{\kappa u} = const = \text{free}. \tag{65}$$

The case we are dealing with here, $\kappa(s) = const$, $u(s) = const$, implies $\partial f_2/\partial s = 0$. Pulling together all relevant equations, one finds that $\partial f_3/\partial s = 0$ as well. Axisymmetry of the cross section is thus maintained up to the third order.

As regards the three expressions for f_{33}, namely the ones in Eqs. (59), (63), and (65), f_{33}^κ follows from putting $\kappa' = 0$ in f_{33}, but $f_{33}^{\kappa u}$ is genuinely different. This leads to problems of presentation: f_{33} enters into many quantities to be derived below. It would take too much space to present every result in its concominant two versions as well. We therefore restrict the presentation (but not the discussion) as far as possible to one case, which in the upcoming section is the case with $u' \neq 0$.

We proceed with the investigation of the next quadruple of equations from Tables I and II.

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V. Equations in group $M = 2$

A. "Grad-Shafranov" equation, order $N = 3$

The GS equation (20) to order $N = 3$ gives the differential equation

$$\frac{\partial^2 f_4}{\partial \theta^2} + 16 f_4 = \frac{1}{\sigma_0^2}\left[-2\frac{\partial^2 f_2}{\partial \theta^2} f_2 G_1' \sigma_0 - \left(\frac{\partial f_2}{\partial \theta}\right)^2 G_1' \sigma_0 - 12 f_2^2 G_1' \sigma_0 - 2 f_2 P_2 \right.$$

$$\left. + \frac{\partial^2 f_3}{\partial \theta^2}\cos\theta\,\kappa\sigma_0^2 + \frac{\partial f_3}{\partial \theta}\sin\theta\,\kappa\sigma_0^2 - 3\cos^2\theta\,\kappa^2 P_1 + 6\cos\theta\,\kappa f_3 \sigma_0^2 \right] . \qquad (66)$$

If the general forms for f_2 and f_3 are inserted, the right-hand side becomes a Fourier series with mode numbers zero and two. As in order $N = 2$, a cancellation of the highest Fourier mode in the source terms of Eq. (66) takes place and prevents secularity of the solution. This cancellation can be confirmed analytically to all (odd) orders e.g. for terms with G_n' by evaluation of the first square brackets in Eq. (32). Integration of Eq. (66) gives

$$f_4(\theta, s) = f_{40}(s) + f_{42}(s)\cos(2\theta) + f_{44}(s)\cos(4\theta) , \qquad (67)$$

where

$$f_{40}(s) = \frac{[-2(3 + u^2)G_1' P_1 - (7 + u)(1 - u^2)c_1 \kappa^2 \sigma_0 + 4P_2\sigma_0]P_1}{128(1 - u^2)^2\, c_1^2\, \sigma_0} \qquad (68)$$

and

$$f_{42}(s) = \frac{-(3 + u)\kappa^2\, P_1 - 16(1 - u^2)c_1 f_{33}\kappa}{64(1 - u^2)c_1}$$

$$+ \frac{2G_1' P_1 - P_2\sigma_0}{24(1 - u^2)^2\, c_1^2\, \sigma_0}P_1 u . \qquad (69)$$

f_{44} is undetermined. G_1', by virtue of Eq. (54), now signifies dG_1/ds.

At this stage a simple test of the REDUCE program developed so far is possible: in axisymmetry and with purely poloidal fields an exact fourth-order polynomial solution of Eq. (20) for $F(r, \theta)$ exists [3]. It must be contained as a special case in the general form of our f_2, f_3, f_4 and G_0, G_1. In Appendix A this is confirmed. In addition, this solution must solve the GS equation (20) exactly, i.e. without an expansion around the axis. This test has also been successfully passed. (The FB equation (23), by virtue of symmetry, transforms to Eq. (24) and is trivially satisfied.)

B. "Grad-Shafranov" equation, order $N = 4$

To order $N = 4$ the GS equation (20) gives:

$$\frac{\partial^2 f_5}{\partial \theta^2} + 25 f_5 = S_5 , \tag{70}$$

where S_5 depends on f_2 - f_4 and G_1'. If f_2 - f_4 are inserted, S_5 becomes a Fourier series with mode numbers one and three. Straightforward integration as in previous orders gives

$$f_5(\theta, s) = f_{51}(s) \cos \theta + f_{53}(s) \cos(3\theta) + f_{55}(s) \cos(5\theta) , \tag{71}$$

where (for the case of nonconstant ellipticity)

$$f_{51}(s) = \frac{4(3u^2 - 6u + 11)u'\kappa + 3(1+u)(1-u^2)\kappa'u}{384(1-u^2)^3 c_1^3 u'} G_1' P_1^2 \sigma_0$$

$$+ \frac{-16(7-3u)u'\kappa P_2 + 3(1+u)(1-u^2)^2 c_1 \kappa'\kappa^2}{1536(1-u^2)^2 c_1^2 u'} P_1 . \tag{72}$$

and

$$f_{53}(s) = \frac{(3u^2 - 88u - 15)u'\kappa - 15(1+u)(1-u^2)\kappa'}{768(1-u^2)^3 c_1^3 u'} G_1' P_1^2 \sigma_0$$

$$+ \frac{(3+22u)u'\kappa + 3(1+u)(1-u^2)\kappa'}{384(1-u^2)^2 c_1^2 u'} P_1 P_2$$

$$+ \frac{2(3+5u)u'\kappa + 5(1+u)(1-u^2)\kappa'}{1024(1-u^2)c_1 u'} \kappa^2 P_1 - \frac{c_1^3 f_{44}\kappa}{4c_1^3} . \tag{73}$$

For $u' = 0$ the coefficients f_{51} and f_{53} are similar, except that u' in the denominator cancels, and they contain $f_{33} = f_{33}^{\kappa u} = const$ explicitly. f_{55} is undetermined.

C. "Force balance" equation, order $N = 5$

From the FB equation (23) one obtains to order $N = 5$ a differential equation for G_2 of the form:

$$4\sigma_0 f_2^3 \frac{\partial}{\partial \theta} \left(\dot{G}_2 f_2 \right) = N_5 . \tag{74}$$

If f_2 - f_5 and Eq. (54) are inserted, this becomes

$$\frac{\partial}{\partial \theta} \left(\dot{G}_2 f_2 \right) = R_5 = \frac{N_5}{D_5} , \tag{75}$$

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where

$$N_5 = n_{50}(s) + n_{52}(s) \cos(2\theta) + n_{54} \cos(4\theta) \tag{76}$$

and

$$D_5 = 4\sigma_0 f_2^3 = 4\sigma_0 f_{20}^3 [1 - u\cos(2\theta)]^3 . \tag{77}$$

The coefficients n_{50} - n_{54} are linear and homogeneous in u', κ', $(\kappa')^2/u'$, G_1'' and f_{44}''. In order to avoid too much inflation of the text by formulae, whose structure but not whose details are relevant, such as here, they are not shown. For the same reason further intermediate results to be derived below are also not rendered explicitely, in particular, since with increasing order the number of terms strongly increases.

Periodicity of G_2 and f_2 in θ requires that

$$\int_0^{2\pi} R_5 \, d\theta = 0 . \tag{78}$$

In contrast to Eq. (47) N_5 depends on θ. The integrals

$$I_{mn}(u) \equiv \int_0^{2\pi} \frac{\cos(n\theta) \, d\theta}{[1 - u\cos(2\theta)]^m} , \tag{79}$$

which can also be looked up in integral tables, are found by using the algebraic computer language MATHEMATICA to be

$$I_{30} = (2 + u^2) c(u) , \quad I_{32} = 3u\, c(u) , \quad I_{34} = 3u^2\, c(u) . \tag{80}$$

The common factor $c(u) = \pi(1 - u^2)^{-5/2}$ drops out of the homogeneous equation (78), which assumes the form

$$u^2 f_{44}'(s) + 2uu' f_{44} = u'(s)\, m_1(u) + m_2(u)\, G_1'' + u'\, m_3(u)\, G_1' . \tag{81}$$

In its derivation Eq. (62) has been used in order to express κ' in terms of u'. $m_1(u)$ is a rational function of u with denominator $768(1 + u)(1 - u^2)^3 c_1^2$. Its numerator is a polynomial of seventh degree whose coefficients depend on κ_v, κ_c, c_1 and P_1, P_2. The function $m_2(u)$ is

$$m_2(u) = -\frac{P_1^2(3u^4 + 3u^2 - 2)}{192\, c_1^{5/2}\, (1 - u^2)^{5/2}} . \tag{82}$$

The factor $\sqrt{1 - u^2}$ in the denominator originates from a factor σ_0 and Eq. (51). It turns out that $dm_3/du = m_2$, so that Eq. (81) can be integrated exactly with respect to s. One obtains

$$u^2 f_{44} = I_4(u) + m_2(u)\, G_1' . \tag{83}$$

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$I_4(u) = \int u'(s) m_1(u) \, ds = \int m_1(u) \, du$ is a "simple" rational function of u (no logarithms present). It contains an arbitrary integration constant c_{I4}.

Equation (83) determines one of the two functions $f_{44}(s)$ and $G'_1(s)$ if the other one is specified as an arbitrary (periodic) function of s. If $u = u_0 = 0$ somewhere along the axis at discrete positions s_0, with $u'(s_0) \neq 0$, $G'_1(s_0)$ is determined and $f_{44}(s_0)$ is free. If $u^2 = u_{1,2}^2 = 0.5(\sqrt{11/3} - 1) \approx 0.4574$ at discrete positions $s_{1,2}$, with $u'(s_{1,2}) \neq 0$, such that $m_2 \sim 3u^4 + 3u^2 - 2 = 0$, then $f_{44}(s_{1,2})$ is determined and G'_1 is free.

We solve Eq. (83) for f_{44} and insert it into Eq. (75). It follows that

$$\frac{\partial}{\partial \theta} \left(\dot{G}_2 f_2 \right) = \mathcal{R}_5 = \frac{\hat{\mathcal{R}}_5}{f_2^3(\theta, s)} , \tag{84}$$

with

$$\hat{\mathcal{R}}_5 = \mathcal{R}_{50}(s) + \mathcal{R}_{52}(s) \cos(2\theta) + \mathcal{R}_{54} \cos(4\theta) . \tag{85}$$

The coefficients $\mathcal{R}_{5m}$ are of the same type as the cofficients n_{5m}. The second equality in Eq. (84) explicitly shows the θ dependence of the denominator. Equation (84) is to be integrated with respect to θ. This involves indefinite integrals $\mathcal{I}_{mn}$ of the type

$$\mathcal{I}_{mn}(u) \equiv \int \frac{\cos(n\theta) \, d\theta}{[1 - u \cos(2\theta)]^m} . \tag{86}$$

REDUCE finds the following expressions for the required $\mathcal{I}_{30}$, $\mathcal{I}_{32}$ and $\mathcal{I}_{34}$:

$$\mathcal{I}_{30}(u) = -\frac{[3\sin(4\theta)u + 2\sin(2\theta)u^2 - 8\sin(2\theta)] \, u}{8(1 - u^2)^2 f_{2u}^2}$$

$$+ \frac{2 + u^2}{2(1 - u^2)^{5/2}} \, a(u, \theta) , \tag{87}$$

$$\mathcal{I}_{32}(u) = -\frac{[(2u^2 + 1)\sin(4\theta)u - 2(u^2 + 2)\sin(2\theta)]}{8(1 - u^2)^2 f_{2u}^2}$$

$$+ \frac{3u}{2(1 - u^2)^{5/2}} \, a(u, \theta) , \tag{88}$$

$$\mathcal{I}_{34}(u) = -\frac{[(5u^2 - 2)\sin(4\theta) - 2(u^2 + 2)u\sin(2\theta)]}{8(1 - u^2)^2 f_{2u}^2}$$

$$+ \frac{3u^2}{2(1 - u^2)^{5/2}} \, a(u, \theta) . \tag{89}$$

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Here $f_{2u} = 1 - u\cos(2\theta)$, and

$$a(u, \theta) = \arctan\left(\sqrt{\frac{1+u}{1-u}}\,\tan\theta\right). \tag{90}$$

All $\mathcal{I}_{mn}$ consist of two parts: a purely periodic part, with the sin terms in the numerator, and a secular part, with arctan. If Eq. (84) is integrated with respect to θ, the secular terms on the r.h.s. cancel exactly. This is a consequence of Eq. (78), which forbids a net secular contribution. Owing to f_{2u}^2 in the denominators of Eqs. (87)-(89) and Eq. (39) the integrated Eq. (84) assumes the form

$$\dot{G}_2 f_2 = T_5 = \frac{\hat{T}_5}{f_2^2(\theta, s)}, \tag{91}$$

where the second equality explicitly shows the θ dependence of the denominator. $\hat{T}_5$ has the structure

$$\hat{T}_5 = \hat{T}_{52}\sin(2\theta) + \hat{T}_{54}\sin(4\theta), \tag{92}$$

where the coefficients $\hat{T}_{52}$, $\hat{T}_{54}$ are again rational functions of u, homogeneous and linear in u', κ', $(\kappa')^2/u'$, G_1''.

On the right-hand side of Eqs. (91) an arbitrary function of s, say $h_2(s)$ should have been added. The periodicity condition that $\int_0^{2\pi} \dot{G}_2 d\theta = 0$ determines h_2:

$$h_2(s) = -\frac{<T_5/f_2>}{<1/f_2>}. \tag{93}$$

where $< \cdots >$ stands for $\int_0^{2\pi} \cdots d\theta$. Since T_5 is antisymmetric and f_2 is symmetric with respect to θ, it follows that $h_2 \equiv 0$.

Before we come to a summary of this subsection, let us remember (see Section IV) that configurations with circular magnetic axis merit special attention. In the first case, with non-constant ellipticity $u(s)$, see Eq. (63), all results obtained so far in this subsection remain valid in the limit $\kappa'(s) \to 0$, or else, if no κ' is present, by putting $\kappa_v = \kappa_c = \kappa$. The ellipticity $u(s)$ remains a free function.

In the second case, with $u'(s) = 0$, see Eq. (65), the coefficients n_{5i}, $i = 0,\ 2,\ 4$ in Eq. (76) have to be rederived. It turns out that the free constant $f_{33} = f_{33}^{\kappa u}$ drops out, and moreover they are identical to their general ($\kappa' \neq 0$) form if one puts $\kappa' = 0$ first and then $u' = 0$ afterwards. In Eq. (81) all terms with factor u' vanish in this case, so that Eq. (83) holds, with $I_4(u) = c_{I4} = const$. Similarly, in Eq. (91) T_5 remains valid in the limit $\{ \kappa' = 0$, then $u' = 0 \}$.

Summarizing the results of this subsection, one finds that the functions $f_{44}(s)$ and $\dot{G}_2(\theta, s)$ are determined. Looking back to analogous Subsection IV.C, we

see the parallels: $f_{22}(s)$ and $\dot{G}_1(\theta, s)$ are determined there, namely in Eq. (51) (reading it from right to left) and in Eq. (54), which can be written in the form $\dot{G}_1(\theta, s) = 0$.

The field lines, to this order, could be non-plane since $\partial G_2 / \partial \theta$ is not a priori identically zero.

D. "Force balance" equation, order $N = 6$

The results obtained so far for f_2, f_3, f_4, f_5 and G_1, G_2 will be inserted here in the order $N = 6$ contributions of the FB equation (23). Extrapolating from the previous subsection and Section IV.D, we expect that $f_{55}(s)$ will be determined, and an extra condition on the so far free functions might be imposed. The number of free functions, alas, is not particularly large anymore: $\kappa(s)$ is free, and for $\kappa(s) = const$, by way of compensation, $u(s)$ is free. These two functions therefore are in danger of being restricted.

To order $N = 6$ one obtains from Eq. (23)

$$\sigma_0 f_2^2 \left(-10 f_2 \dot{f}_2 \kappa \cos\theta + 11 f_3 \dot{f}_2 + 6 f_2 \dot{f}_3 + 4 f_2^2 \kappa \sin\theta \right) \dot{G}_2$$

$$+ 4\sigma_0 f_2^3 \left(5 f_3 - 3 f_2 \kappa \cos\theta \right) \ddot{G}_2 + V_6(\theta, s) = 0 , \tag{94}$$

where $V_6(\theta, s)$, even if expressed in its original form as a function of f_2 - f_5 and G_1, is a rather lengthy expression.

Since $\partial(\dot{G}_2 f_2) / \partial\theta$ and $\dot{G}_2 f_2$ are both available from Eqs. (84) and (91), respectively, it is advantageous to regroup the first two terms of Eq. (94) accordingly. It follows that

$$\sigma_0 \left(2 f_2 \dot{f}_2 \kappa \cos\theta - 9 f_3 \dot{f}_2 + 6 f_2 \dot{f}_3 + 4 f_2^2 \kappa \sin\theta \right) f_2 \cdot \dot{G}_2 f_2$$

$$+ 4\sigma_0 \left(5 f_3 - 3 f_2 \kappa \cos\theta \right) f_2^2 \frac{\partial}{\partial\theta} \left(\dot{G}_2 f_2 \right) + V_6(\theta, s) = 0 . \tag{95}$$

If Eqs. (84) and (91) are inserted, some of the factors f_2 in the numerators and denominators are cancelled. With previous results for f_2 - f_5 one obtains from Eq. (95)

$$\frac{P_1^2}{98304(1 - u^2)^4 \, c_1^3 \, \sigma_0 u'^2 \, u^3 \, f_2} \sum_{m=1,3,5,7} N_{6m} \cos(m\theta) = 0 . \tag{96}$$

The coefficients N_{6m}, $m = 1, 3, 5, 7$ are fairly lengthy expressions. They contain first and second derivatives of $\kappa(s)$ and $u(s)$, but the latter only in the combination $\kappa'' u' - \kappa' u''$, which can be expressed in terms of κ', u' and u by means of Eq. (61).

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All coefficients N_{6m} are then found to be proportional to u'^2 so that u'^2 in the denominator of Eq. (96) cancels. The factor u^3 there originates from the insertion of $f_{44}(s)$, see Eq. (83). At isolated points where $u = 0$ Eq. (83) should be solved for G'_1 instead of f_{44}, as mentioned in Section V.C above, in order to avoid this spurious singularity.

The present discussion refers to the cases with noncircular or circular axis, both with $u' \neq 0$, see Section IV. For $\kappa' = u' = 0$ the discussion will be resumed below.

All N_{6m}, $m = 1, 3, 5, 7$ have the same form, namely

$$N_{6m} \equiv u'^2 \, \hat{N}_{6m}$$

$$= u'^2 \, (N_{6ma} + N_{6mb} \, f_{55} + N_{6mc} \, f'_{55} + N_{6md} \, G'_1 + N_{6me} \, G''_1) \, . \tag{97}$$

i.e. they are *linear* in $f_{55}(s)$, $f'_{55}(s)$, $G'_1(s)$ and $G''_1(s)$. The N_{6ma} - N_{6me} depend on κ, κ', u and u'.

From Eq. (96) it follows that the four equations

$$\hat{N}_{6m} = 0 \, , \qquad m = 1, 3, 5, 7 \, , \tag{98}$$

must hold. There are thus four linear inhomogeneous equations for four unknowns, f_{55}, f'_{55}, G'_1 and G''_1.

We treat first the most general case, namely configurations with noncircular axis, $\kappa'(s) \neq 0$. In this case Eq. (62) relates $\kappa(s)$ and $u(s)$. Their derivatives are related by the equation

$$\kappa' = \frac{-\kappa_v}{(1+u)^2} \, u' \, . \tag{99}$$

All four unknown functions $(X_1, X_2) = (f_{55}, G'_1)$ and $(X_3, X_4) = (f'_{55}, G''_1)$ can then be expressed in terms of u, u' (and the constants c_1, κ_v, κ_c, c_{I4}, P_1, P_2). REDUCE finds rather involved expressions of the form

$$X_m = \frac{A_m(u)}{B_m(u)} \, , \qquad m = 1, 2 \, , \tag{100}$$

and

$$X_m = \frac{A_m(u)}{B_m(u)} \, u' \, , \qquad m = 3, 4 \, . \tag{101}$$

Here, A_m and B_m are polynomials of up to eleventh degree in u. In addition, B_2 and B_4 are proportional to $\sqrt{1 - u^2}$.

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Two compatibility conditions have to be satisfied, namely $X_1'(s) = X_3(s)$, and $X_2'(s) = X_4(s)$. Both conditions assume the form

$$C_n \equiv \frac{D_n(u)}{E_n(u)} u' = 0 \; , \quad n = 1, \, 2 \; . \tag{102}$$

Both the numerators D_n and the denominators E_n are again polynomials in u and, in addition, E_2 is proportional to $\sqrt{1 - u^2}$. Although the details of the numerators do not matter much, as will become clear presently, they are at least rudimentarily shown in Appendix B in order to keep the discussion less abstract. A short discussion with regard to the zeros of the denominators B_m and E_n is given in Appendix C.

Equations (102) can, in principle, be satisfied in four cases: the case $u' = 0$, the two mixed cases $u' = D_1(u) = 0$ and $u' = D_2(u) = 0$ and the case $D_1(u) = D_2(u) = 0$. By assumption, we have $u' \neq 0$ (in agreement with the noncircular axis case $\kappa' \neq 0$, see Eq. (99)). This already eliminates three of the four cases. The fourth case is $D_1(u) = D_2(u) = 0$, where $D_1(u)$ and $D_2(u)$ are polynomials of eleventh and ninth degree, respectively, which depend on the parameters κ_v, κ_c, c_1, c_{I4}, P_1, see Appendix B. Depending on these values, the condition $D_1(u) = D_2(u) = 0$ may or may not have one or more real solutions u. In any case, countably many *fixed* values of u are obtained. This, however, is in contradiction to the assumption $u'(s) \neq 0$ for a continuum of s values, which requires that a *continuum* of u values should satisfy Eqs. (102). This eliminates the fourth case as well.

In consequence, it has been proved that the assumption of an up-down symmetric MHD equilibrium with *noncircular* plane axis leads to a contradiction.

We shall proceed with another type of configurations, namely those with *circular* axis but again with an ellipticity $u(s)$ which varies along the axis.

We know that $u(s)$ is then arbitrary so far and Eqs. (62), (99) do not hold any more, as discussed in Section IV.D. For $u' \neq 0$, however, the system of equations (98) is still valid. With $\kappa' = \kappa'' = 0$ its solution $(X_1, X_2, X_3, X_4) = (f_{55}, G_1', f_{55}', G_1'')$ is obtained just as in Eqs. (100) and (101), except that the polynomials $A_m(u)$, $B_m(u)$ are less involved and depend on $\kappa = const$. The same holds for the compatibility conditions of the type of equations (102). They are as follows:

$$C_1 \equiv \frac{(8775u^5 - 6275u^4 + 11950u^3 - 23610u^2 - 12309u + 20061)P_1}{(5u^2 - 120u + 27)^2(1 + u)(1 - u)^2 c_1}$$

$$\times \; \kappa^3 u' = 0 \; . \tag{103}$$

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$$C_2 \equiv \frac{(65u - 87)(1 + u)^2(1 - u)c_1^{3/2}u^2}{(5u^2 - 120u + 27)^2\sqrt{1 - u^2}P_1}\kappa^2 u' = 0 . \tag{104}$$

In the truly toroidal case, $\kappa \neq 0$, and on the assumption that $u' \neq 0$ Eqs. (103) and (104) can never be satisfied both together. (This conclusion holds for $5u^2 - 120u + 27 = 0$ as well.)

In consequence, it has also been proved that an equilibrium with circular axis and non-constant ellipticity along the axis does not exist.

It thus follows that within our premises nonaxisymmetric equilibria, if they exist, would have a circular axis, a constant ellipticity of the cross section, and, in view of Eq. (65), also a constant triangularity. Only higher-order terms, $f_{m\geq 4}(\theta, s)$, responsible for quadrangularity, etc. could vary along the axis. In addition, the magnetic field could vary if $G_1(s)$ and higher-order G_n are not constant, see Eqs. (15). In the next section it is shown to be plausible that equilibria with these two types of nonaxisymmetry do not exist, either.

VI. "Almost symmetric" equilibria

Here we continue the as yet unfinished case, namely configurations with circular magnetic axis and constant ellipticity u and triangularity f_{33} of the cross section along it.

The collected results with regard to the two basic functions F and G, which determine the magnetic field $\mathbf{B}$ and which have been Taylor-expanded in the distance from the axis, Eqs. (28) and (29), can be written as follows:

$$\frac{\partial f_2}{\partial s} = \frac{\partial f_3}{\partial s} = \frac{\partial^2 G_0}{\partial s^2} = \frac{\partial G_0}{\partial \theta} = \frac{\partial G_1}{\partial \theta} = 0 . \tag{105}$$

Independence of f_m and G_n of s implies axisymmetry, while independence of G_n of θ implies that the field lines are confined to planes, to the respective order. It will be shown to be plausible that Eq. (105) can be extended to f_m and G_n with larger and larger indices m and n, so that the remaining allowed nonaxisymmetry (and non-flatness) shrinks more and more. In the limit $m, n \rightarrow \infty$ this would rule out the last remaining bits of nonaxisymmetry for analytic MHD equilibria (within the assumed premises).

Reverting to the θ-averaged FB equation, Eq. (81), one finds for $u'(s) = 0$

$$192\sigma_0 c_1^2 (1 - u^2)^2 u^2 f_{44}'(s) + P_1^2 (3u^4 + 3u^2 - 2) G_1'' = 0 . \tag{106}$$

Three cases already encountered before merit separate discussion, namely a) $u = 0$, b) $3u^4 + 3u^2 - 2 = 0$, and c) neither a) nor b). Cases a) and b), respectively, give

$$G_1''(s) = 0 , \quad f_{44}'(s) \text{ arbitrary} , \tag{107}$$

$$f_{44}'(s) = 0 , \quad G_1''(s) \text{ arbitrary} . \tag{108}$$

The FB equation (95) has to be re-evaluated for the present case $\kappa' = u' = 0$. In case a), for example, one obtains analogously to Eq. (96)

$$\frac{1}{61 f_{2u}} \sum_{m=1,3,5,7} N_{6m}^a \cos(m\theta) = 0 , \tag{109}$$

with four coefficients N_{6m}^a which have to vanish. Omitting trivial nonvanishing factors, one obtains the following three independent relations:

$$f_{44}' f_{33} = f_{44}' \kappa = 271 f_{44}' \kappa + 96 f_{55}' = 0 . \tag{110}$$

In consequence, it follows that

$$f_{44}' = f_{55}' = 0 . \tag{111}$$

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The right-hand side of Eq. (91) then also vanishes. This implies $\dot{G}_2 = 0$. Together with the equations for the other f_{4m}, f_{5m} the final result for case a) is

$$\frac{\partial f_4}{\partial s} = \frac{\partial f_5}{\partial s} = \frac{\partial^2 G_1}{\partial s^2} = \frac{\partial G_2}{\partial \theta} = 0 . \tag{112}$$

The discussion of cases b) and c) is slightly longer but analogous and gives the same result, Eq. (112).

The results in Eq. (105) have thus been pushed up two orders of magnitude for f_m and one order for G_n. In order to find out whether this continues to even higher order, the next quadruple of equations, namely for $M = 3$, is also briefly investigated. A comment regarding *arbitrary* orders is also made.

A. "Grad-Shafranov" and "force balance" equations, in group $M = 3$

Just as in lower orders, it is straightforward to solve the GS equation, Eq. (20), in the orders $N = 5$ and $N = 6$ for $f_6(\theta, s)$ and $f_7(\theta, s)$. The result is completely analogous to Eqs. (67) and (71) for f_4 and f_5, respectively. The terms f_{60}, f_{62}, f_{64} and f_{71}, f_{73}, f_{75} are found in terms of lower-order f_{mn} and G_1, G_2 while $f_{66}(s)$ and $f_{77}(s)$ remain as yet undetermined.

The FB equation, Eq. (23), to order $N = 7$ is

$$\frac{\partial}{\partial \theta} \left(\dot{G}_3 f_2 \right) = R_7 = \frac{N_7}{D_7} , \tag{113}$$

where

$$N_7 = n_{70}(s) + n_{72}(s) \cos(2\theta) + n_{74} \cos(4\theta) + n_{76} \cos(6\theta) \tag{114}$$

and

$$D_7 = 4\sigma_0 f_2^4 = 4\sigma_0 f_{20}^4 [1 - u \cos(2\theta)]^4 . \tag{115}$$

Periodicity of G_3 in θ requires that $\int_0^{2\pi} R_7 \mathrm{d}\theta = 0$. This leads to

$$11520\sigma_0 c_1^3 (1-u^2)^3 u^3 f_{66}'(s) + P_1^2 (15u^6 + 15u^4 - 20u^2 + 8) G_2'' = 0 , \tag{116}$$

in complete analogy to Eq. (106).

Again, three cases have to be discussed separately, namely a) $u = 0$, b) $15u^6 + 15u^4 - 20u^2 + 8 = 0$, and c) neither a) nor b). For cases a) and c), for example, Eq. (116) can be solved for G_2''. If it is inserted in Eq. (113) and the indefinite integrals $\mathcal{I}_{4n}(u)$, Eq. (86), are evaluated by REDUCE, one obtains

$$\dot{G}_3 f_2 = \frac{(11u^2 - 8)\sin(6\theta) - 6(3u^2 - 1)u\sin(4\theta) + 15u^2 \sin(2\theta)}{3(15u^6 + 15u^4 - 20u^2 + 8) f_{2u}^3 P_1^3 \sigma_0}$$

$$\times \, 64(u^2 - 1)^5 c_1^4 f_{66}' \ . \tag{117}$$

An arbitrary function $h_3(s)$ could be added to the r.h.s. of Eq. (117). Similarly to $h_2(s)$ in the previous section, it can be shown that the periodicity condition leads to $h_3(s) \equiv 0$.

Of the $M = 3$ quadruple of equations the FB equation to order $N = 8$ remains to be analyzed. Analogously to Eq. (96) it follows that

$$\sum_{m=1}^{m=17} N_{8m} \, \cos(m\theta) \ = \ 0 \ , \tag{118}$$

where a (nonvanishing) denominator was drawn into the coefficients. The summation extends over odd values of m and comprises eight Fourier modes. A complete discussion of all subcases is therefore rather laborious and has not been attempted. No basic difference to the order $M = 2$ case, however, is in sight.

The case a), $u = 0$, is simple enough to be analyzed in a few lines. The conditions $N_{8m} = 0$, $m = 1, \, 3, \, \cdots \, 17$, give three independent equations, namely

$$f_{66}' f_{33} \ = \ f_{66}' \kappa \ = \ 437 f_{66}' \kappa + 120 f_{77}' \ = \ 0 \ . \tag{119}$$

Hence, one finds that $f_{66}' = f_{77}' = 0$, and, with Eq. (117), that also $\dot{G}_3 = 0$. Altogether, one obtains

$$\frac{\partial f_6}{\partial s} \ = \ \frac{\partial f_7}{\partial s} \ = \ \frac{\partial^2 G_2}{\partial s^2} \ = \ \frac{\partial G_3}{\partial \theta} \ = \ 0 \ . \tag{120}$$

The results of Eq. (112) have thus been pushed up another two orders with respect to f_m and another order with respect to G_n (in the subcase $u = 0$ among the cases $u = const$ of the present section).

There has not been any fundamental difference in the discussion of the quadruple of equations for $M = 2$ and $M = 3$, once the conditions $\kappa'(s) = u'(s) = 0$ were applied. One expects, therefore, that the results for even larger M would be analogous. This would eliminate the last traces of nonaxisymmetry for configurations with circular axis and constant ellipticity along it.

It should be possible to prove by complete induction that the result from a quadruple M, namely $f_{2M}' = f_{2M+1}' = G_{M-1}'' = \dot{G}_M = 0$, holds for the quadruple $M + 1$ as well. In practice, this is found to be a rather lengthy procedure since four interconnected equations are involved. We therefore do not dwell on it here.

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VII. Discussion and conclusions

The existence of *nonaxisymmetric* toroidal MHD equilibria with mirror symmetry with respect to an equatorial plane was investigated. Owing to this symmetry all field lines are poloidally closed around the magnetic axis. Two types of non-axisymmetry have to be considered: a) a noncircular magnetic axis. b) a circular axis but variations of the geometry and/or magnetic field along the axis. It was proved that no equilibrium of type a) exists. As regards type b), it was shown that neither the ellipticity of the cross section, nor the triangularity, nor higher-order deformations up to and including the seventh order are allowed to vary. For still higher orders the same was shown to be plausible. Analoguous results hold for the magnetic field. Type b) equilibria are thus also practically ruled out. One has to conclude, therefore, that *axisymmetric* equilibria with poloidally closed field lines are in a sense singular: the slightest deformation away from symmetry destroys the equilibrium.

This result is unexpected: Lortz [22] has shown that mirror symmetry with respect to a *poloidal* plane – causing toroidally closed field lines – guarantees the existence of nonaxisymmetric MHD equilibria (provided the value of plasma beta is not too large). His result was indeed of such suggestive power that it is occasionally cited erroneously as proof of existence under any kind of mirror symmetry or for closed field lines quite generally; see, for example [14]. The difference between the present outcome and [22] might be that the equatorial symmetry splits the toroidal domain into two still toroidally connected domains, while the opposite symmetry produces two singly connected domains. Also, more formally, the proof in [22] (an iteration scheme which involves the plasma beta and which requires a vacuum field to start with) cannot be applied to the present case since a poloidal vacuum field without singularity on the magnetic axis does not exist.

The result does not support Grad's expectation [9] either that nonaxisymmetric equilibria *with all field lines closed* are more likely to exist than other ones because the field lines would be freer to interchange in such a way that condition (4) could be satisfied.

It proves a posteriori to be fortuitous that the expansion in the distance to the magnetic axis gave definitive conclusions about nonexistence. The more common outcome of such expansions is at most an iterative scheme that shows how to proceed from order n to order $n+1$, but which leaves open convergence or divergence in the limit $n \to \infty$; see e.g. [16]. As regards poloidal fields with more general geometry – without mirror symmetry and with torsion of the axis allowed

– it remains to be seen whether the method still works as well and whether the extra freedom gained might even suffice to make a poloidal equilibrium possible.

A final comment on the balance between analytic parts and the use of an algebraic computer language in the present investigation might be appropriate. If one goes up to the seventh order in an expansion, as is done here, an algebraic computer language must of course be applied – mostly REDUCE in our case. It turns out, however, that a "blind" expansion is rather useless. It is essential to know by analytic consideration the structure of the equations, for example which terms can be grouped together into partial derivatives. This alone makes the computer part workable and transparent.

Acknowledgement

Appendix A: Exact Shafranov solution

An exact solution of the MHD equations (1) in toroidal geometry, with purely poloidal magnetic fields $\mathbf{B} = \text{curl } \mathbf{A}$, sometimes called Hill's vortex, can be found, for example, in [3]:

$$A_{\hat{r}} = A_z = 0 , \qquad A_\phi = \frac{\Psi}{2\pi \hat{r}} .\tag{B.1}$$

where $\hat{r}, \phi, z$ are cylindrical coordinates, and the flux function Ψ is given by

$$\Psi(\hat{r}, z) = \Psi_0 \frac{\hat{r}^2}{R^4} \left(2R^2 - \hat{r}^2 - 4a^2 z^2 \right) .\tag{B.2}$$

Ψ_0 and a are constants. R is the major radius of the configuration. $dP/d\Psi = const$ is assumed.

In our formalism we have in axisymmetry $F(r,\theta) = f_2(\theta)r^2 + f_3(\theta)r^3 + f_4(\theta)r^4$. It implies $u(s) = const$ and $G_1(s) = const$. With the choice $c_1 = 1/(1-u^2)$ there results $\sigma_0 = 1$. With $P_2 = 0$ our solution contains four free constants: f_{20}, u, f_{33} and f_{44}. The remaining f_{mn} are fixed and given by Eqs. (44), (68) and (69). Total agreement with the Shafranov solution is achieved if we take $f_{33} = f_{33}^S$, $f_{44} = f_{44}^S$, where

$$f_{33}^S = \frac{1}{4}\kappa f_{20} (1 + 3u) ,\tag{B.3}$$

$$f_{44}^S = \frac{-1}{32}\kappa^2 f_{20} (3 + 5u) .\tag{B.4}$$

Surprisingly, f_{33}^S agrees with f_{33}^κ from Eq. (63).

After transforming to cylindrical coordinates $\hat{r}, z$ it follows that

$$F = \frac{1-u}{4\kappa^2} - \frac{1-u}{4}\kappa^2 \hat{r}^2 \left(\frac{2}{\kappa^2} - \hat{r}^2 - 4\frac{1+u}{1-u}z^2 \right) .\tag{B.5}$$

With the identifications $\Psi_0 = (u-1)\kappa^2 R^4/4$ and $a^2 = (1+u)/(1-u)$ the two fluxes Ψ and F agree (apart from a constant which reflects a different normalization on the magnetic axis).

Appendix B: Functions $D_1(u)$, $D_2(u)$

The functions $D_1(u)$, $D_2(u)$ are given by

$$
\begin{aligned}
D_1 = {} & [\, p_1(u)\kappa_v^5 - 4p_2(u)\kappa_c\kappa_v^4 + 2p_3(u)\kappa_c^2\kappa_v^3 + 4p_4(u)\kappa_c^3\kappa_v^2 \\
& -8p_5(u)\kappa_c^4\kappa_v - 64p_6(u)\kappa_c^5 \,]\, P_1 \\
& -256\,[\, p_7(u)\kappa_v^2 - 2p_8(u)\kappa_c\kappa_v - 660p_9(u)\kappa_c^2\,]\,(1+u)^2(1-u)c_1 c_{I4}\kappa_v .
\end{aligned} \tag{B.1}
$$

$$
\begin{aligned}
D_2 = {} & \Big\{ [\, 2q_1(u)\kappa_v^4 - 8q_2(u)\kappa_c\kappa_v^3 + q_3(u)\kappa_c^2\kappa_v^2 - 6q_4(u)\kappa_c^3\kappa_v \\
& -16q_5(u)\kappa_c^4\,]\, P_1 - 128\,[\, q_6(u)\kappa_v - 22q_7(u)\kappa_c\,]\,(1+u)^2(1-u)c_1 c_{I4}\kappa_v \Big\} \\
& \times 36(1-u)c_1^{3/2}u^2 ,
\end{aligned} \tag{B.2}
$$

where p_n, q_n are polynomials in u alone. Most of them are fairly long and it would not make much sense to display them all. It should suffice to give a few of them here:

$$
\begin{aligned}
p_1(u) = {} & 1800u^{11} - 46107u^{10} - 216738u^9 - 119759u^8 - 255664u^7 \\
& -1705590u^6 + 874796u^5 + 1091354u^4 + 290936u^3 - 352943u^2 \\
& -185946u - 108811
\end{aligned} \tag{B.3}
$$

$$
p_9(u) = (45u^5 + 55u^4 + 118u^3 + 46u^2 - 91u - 45)(1+u)^3 \tag{B.4}
$$

$$
q_1(u) = 5u^6 - 146u^5 - 103u^4 + 220u^3 - 973u^2 + 670u - 9 \tag{B.5}
$$

$$
q_7(u) = (5u + 3)(1+u)^2 \tag{B.6}
$$

Appendix C: Critical denominators

The denominators $B_m(u)$ and $E_n(u)$ in Eqs. (100)-(102) have the following common form H_i, $i = 1, 2, \cdots, 6$:

$$H_i(u) = a_i \, h^{b_i}(u) \, (1 + u)^{c_i} \, (1 - u)^{d_i} \, (\sqrt{1 - u^2})^{e_i} \, , \qquad \text{(C.1)}$$

where $h(u) = (3u^3 + 11u^2 - 231u + 65)\kappa_v + 2(5u^2 - 120u + 27)(u + 1)\kappa_c$, the a_i are constants, and, in order of occurrence, b_1-$b_4 = 1$, $b_5 = b_6 = 2$, the c_i and d_i vary from 0 to 5, and $e_1 = e_3 = e_5 = 0$, $e_2 = e_4 = e_6 = 1$. The denominators might vanish at up to three critical values u_c of u if the third order polynomial $h(u) = 0$ has real solution(s) with $u^2 < 1$. At these discrete values the conclusions to be derived below with respect to noncircular-axis cases would not hold, a priori.

Vanishing of $h(u)$ implies that the determinant of the system (98) is zero. According to the theory of linear systems of equations there is then either no solution or the coefficients of the unknowns have to satisfy a number of side conditions. These would only involve u and u', so that, in addition to u, also u' would be forced to take on special values (possibly nonreal). More over, at least one side condition, which we happened to look at, assumes the form

$$\hat{C} \equiv k(u)u' = 0 \, , \qquad \text{(C.2)}$$

where $k(u)$ is again a rational function of u. Its numerator and denominator, in general, differ from those in Eqs. (102). The same conclusions as with the numerators $E_n \neq 0$ can therefore be reached.

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