

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK
GARCHING BEI MÜNCHEN

ENERGY METHODS in DISSIPATIVE
MAGNETOHYDRODYNAMICS

H. TASSO

IPP 6/321

April 1994

hydrodynamic
magnetized
nonlinear
numbers is
Nonlinear
fully for all
requires, how
anis. Finally
and illustrated

to appear in "Proceeding
of Continuum Mechani

*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem
Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über
die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

Energy methods in dissipative magnetohydrodynamics*

H. Tasso

Max-Planck-Institut für Plasmaphysik

Euratom Association

85748 Garching bei München, Fed. Rep. of Germany

April 6, 1994

Abstract

A brief summary of energy methods for linear stability in dissipative magnetohydrodynamics is given. In this case, the methods are equally efficient for fixed and free boundary problems. Linear asymptotic stability has implications in nonlinear stability, at least for a modest but finite level of perturbations.

Sufficient conditions for nonlinear stability of dissipative magnetohydrodynamics flows are obtained and applied to the time dependent magnetized Couette flow. The fluid has a plate as boundary and nonlinear stability is unconditional. The range of stable Reynolds numbers is rather modest i.e. of the order of $2\pi^2 \approx 20$.

Nonlinear stability of force free fields can be treated very successfully for all values of dissipation and all levels of perturbations. It requires, however, the presence of perfectly conducting fixed boundaries. Finally, a special inertia-caused Hopf bifurcation is identified and illustrated by an appropriate example.

*to appear in "Proceedings of Workshop on Energy Methods in Free Boundary Problems of Continuum Mechanics", Oviedo, Spain, March 21-23, 1994

1 Introduction

Energy or more generally Lyapunov's methods are an elegant tool in the study of stability of fluids [1]. Though their efficiency in the stability of shear flows was rather limited, the applications in the area of magnetohydrodynamics (MHD) were not only clear and elegant, but led to important tools like the 'Energy Principle' of ideal MHD [2] and subsequent work for dissipative MHD [3]. This is certainly due to the nontrivial stability problems of MHD equilibria without or with small flow.

Energy methods for fluids with free boundaries need a special attention, however, in Hydrodynamics (HD) as well as in MHD. The 'Energy Principle' of MHD [2] is derived for a free plasma vacuum interface, and the dissipative 'Energy Principle' [3] can be made so by making the resistivity infinite outside the plasma core. This happy situation is valid only for the linearized stability problem.

Energy methods for nonlinear stability of free boundary problems seem to be very hard to establish. None is known to me in MHD. Even worse is the situation of compressible MHD. Compressibility causes large difficulties in terms of nonvanishing surface integrals even for the fixed boundary problem. This paper is organized as follows: Section 2 summarizes briefly the status of energy methods in linearized resistive MHD. Nonlinear stability of dissipative MHD flows with fixed boundaries is the subject of section 3. Section 4 specializes on the important case of MHD equilibria without flows. Hopf bifurcations due to overstability are treated in section 5. Finally, conclusions and outlook are given in section 6.

2 Energy methods in linearized resistive MHD

First, the equations for MHD equilibrium and the linearized resistive MHD dynamics around equilibrium are given. Then, a sufficient stability condition with respect to purely growing modes is derived (see [4]). Limiting cases of physical interest and implications for nonlinear stability are also discussed.

Resistive MHD equilibria generally have a flow which, for simplicity, we neglect in the equation of motion, but which we keep in Ohm's law. The

equilibrium equations are given by

$$\mathbf{J} \times \mathbf{B} = \nabla P_0, \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2)$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta_0 \mathbf{J}. \quad (3)$$

As usual \mathbf{B} is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$, \mathbf{E} is the curl-free electric field, \mathbf{V} is the flow velocity due to resistivity η_0 and P_0 is the pressure. The 'existence' of magnetic surfaces is assumed and the resistivity is taken as constant on these surfaces. The equations of the linearized perturbations are

$$\rho \ddot{\xi} + \nabla P_1 - \mathbf{j} \times \mathbf{B} - \mathbf{J} \times \mathbf{b} = 0, \quad (4)$$

$$\mathbf{e} + \dot{\xi} \times \mathbf{B} + \mathbf{V} \times \mathbf{b} - \eta_1 \mathbf{J} - \eta_0 \mathbf{j} = 0, \quad (5)$$

$$\nabla \times \mathbf{e} = -\dot{\mathbf{b}}, \quad (6)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (7)$$

$$\mathbf{j} = \nabla \times \mathbf{b}, \quad (8)$$

$$\mathbf{B} \cdot \nabla \eta_1 + \mathbf{b} \cdot \nabla \eta_0 = 0, \quad (9)$$

$$P_1 = -\gamma P_0 \nabla \cdot \xi - \xi \cdot \nabla P_0, \quad (10)$$

where ρ is the mass density, P_1 , \mathbf{j} , \mathbf{b} , \mathbf{e} and η_1 are the perturbations of, respectively, pressure, current, magnetic field, electric field and resistivity. The boundary conditions are $\mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \xi = 0$, where \mathbf{n} is the normal to a perfectly conducting wall.

Let us express \mathbf{e} and \mathbf{b} in terms of the vector potential \mathbf{A} and take the gauge of zero scalar potential :

$$\mathbf{e} = -\dot{\mathbf{A}},$$

$$\mathbf{b} = \nabla \times \mathbf{A},$$

with the boundary condition $\mathbf{n} \times \mathbf{A} = 0$. We insert \mathbf{j} from eq.(5) into eq.(4) to obtain a system written in terms of $\Psi = \begin{pmatrix} \xi \\ \mathbf{A} \end{pmatrix}$:

$$N \ddot{\Psi} + P \dot{\Psi} + Q \Psi = 0, \quad (11)$$

where N , P and Q are given by, respectively,

$$N = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} \mathbf{B}/\eta_0 \times (\cdots \times \mathbf{B}) & (\cdots \times \mathbf{B}/\eta_0) \\ -(\cdots \times \mathbf{B}/\eta_0 & 1/\eta_0 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} \nabla(-\gamma P_0(\nabla \cdot \cdots)) & -\mathbf{J} \times (\nabla \times \cdots) \\ -\nabla(\cdots \cdot \nabla P_0) & -1/\eta_0 \nabla P_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \cdots \cdot \nabla \eta_0) \\ & +\mathbf{B}/\eta_0 \times (\mathbf{V} \times \nabla \times \cdots) \\ 0 & \nabla \times \nabla \cdots \\ & +\mathbf{J}/\eta_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \cdots \cdot \nabla \eta_0) \\ & -\mathbf{V}/\eta_0 \times \nabla \times \cdots \end{pmatrix}.$$

The first two matrix operators are symmetric and positive. The last operator Q is obviously not selfadjoint. For this reason we cannot find a Lyapunov functional which would lead to a necessary and sufficient condition for stability as in, for example, [5] or [6].

As shown in [4], one can, however, write a sufficient condition for stability against purely growing modes in the form

$$\delta W = (\Psi, Q\Psi) \geq 0, \quad (12)$$

where the scalar product is defined with purely real quantities. Only the symmetric part Q_S of Q survives in eq.(12), but if a symmetrized form for eq.(12) is wanted, it is easy to construct Q^+ , the adjoint of Q , by integration by parts, and use $Q_S = (Q + Q^+)/2$ instead of Q in eq.(12).

Criterion (12) implies volume integrations which can be reduced to integrations on the magnetic surfaces and integrations across them. The operator $(\mathbf{B} \cdot \nabla)^{-1}$ in eq.(12), which comes from integration of eq.(9), is singular across the rational surfaces ($1/x$ singularity). This singularity is physically prohibited by the breakdown of eq.(9) due to a finite heat conduction κ (κ_{\parallel} is assumed to be infinite and $\kappa_{\perp} = 0$ for eq.(9)). In fact, η_1 should not become infinite on the rational magnetic surfaces, but small. It is then natural to define the integrations across the surfaces in the sense of Cauchy principal parts (no delta functions) as in [6]. Note here that these singularities are not aggravated by the above-mentioned symmetrizing integrations by parts, because they occur on the surfaces.

Let us now write δW explicitly:

$$\begin{aligned}
\delta W = & \int d\tau (\gamma P_0 (\nabla \cdot \xi)^2 + (\xi \cdot \nabla P_0) \nabla \cdot \xi) \\
& + \int d\tau (\nabla \times \mathbf{A})^2 - \int d\tau \xi \times \mathbf{J} \cdot \nabla \times \mathbf{A} + \\
& + p \int d\tau \mathbf{J} \cdot (\mathbf{A} - \xi \times \mathbf{B}) (\mathbf{B} \cdot \nabla)^{-1} (1/\eta_0) (\nabla \eta_0 \cdot \nabla \times \mathbf{A}) \\
& - \int d\tau (\mathbf{A} - \xi \times \mathbf{B}) \cdot \mathbf{V} \times (\nabla \times \mathbf{A}) 1/\eta_0.
\end{aligned} \tag{13}$$

If we choose in δW the MHD test function $\mathbf{A} = \xi \times \mathbf{B}$, then δW reduces to δW_{MHD} (see [2]). In the tokamak scaling (large axial wavelength and magnetic fields) and for $\mathbf{J} = \mathbf{e}_z J$, $\eta_0 J = ct$, $\xi = \mathbf{e}_z \times \nabla U$, $\mathbf{V} = 0$, δW reduces to the necessary and sufficient condition found in [6] for fixed boundary. If, in addition, we make the resistivity infinite outside a certain plasma core, we obtain the equivalent of a vacuum in that region, and a full 'Energy Principle' with free boundary is obtained.

It is more convenient to treat δW in Hamada-like coordinates especially for the term $(\mathbf{B} \cdot \nabla)^{-1}$, which also appears in [6]. The symmetrization of Q , if desired, can be done either analytically in the same coordinates by integration by parts or after discretization in the case of numerical evaluation by computing the adjoint matrix.

The equilibrium quantities in eq.(13) should satisfy equations (1)-(3). To determine the contribution of the last integral in eq.(13), one requires a knowledge of unavoidable [7] Pfirsch-Schlüter-like flows, which are important especially for stellarators. The flow in a tokamak can probably be neglected if the aspect ratio is large enough and the poloidal currents are weak. One can then take $\nabla \times \eta_0 \mathbf{J} \approx \mathbf{0}$ as in [6].

The main advantage of (13) is that it can be numerically evaluated by spectral methods well known in ideal MHD stability and recently extended to MHD stability of stellarator equilibria. A second positive aspect is that this approach to resistive MHD stability is the only one which takes real geometry into account together with the complex flows it generates, and in an exact way at that.

One can 'upgrade' conditions (12)-(13) for two limiting cases of physical interest : 1) for $Q_a \approx \epsilon$ small, which relates to the 'tokamak scaling'. The condition becomes necessary and sufficient for stability with respect to all

modes (not the purely growing only). 2) $N = 0$, or neglecting inertia in which case the conditions (12)-(13) become sufficient for stability with respect to all modes. In addition, simplified versions of (13) can be obtained through physical or natural choices of the test function space. These points are discussed in detail in [3].

Before we proceed to the full nonlinear stability problem, let us mention that linear asymptotic stability of dissipative systems imply nonlinear stability for a restricted level of perturbations. In other words, linear stability of such systems provides already some finite basin of attraction in functional space. The boundaries delivered by the estimates theorems may be, however, much smaller than actual boundaries as will be seen in the next sections. For a discussion of such estimates for fluids see [8].

3 Nonlinear MHD stability with flow

Following the papers [1] , [9] , [10] , [11] , it is possible to formulate a sufficient condition for nonlinear stability in HD and MHD . It turns out that, in case it is satisfied, the system is unconditionally stable i.e. for all levels of perturbations in functional space. The condition is, however, satisfied for rather small Reynolds numbers of the order of 20. Let us give here a derivation, which, though not rigorous in the pure mathematical sense, is basic and compact. For incompressible fluids and in particular in HD and MHD the nonlinear terms in the equation of motion are of the quasilinear type and dissipation is present in the form of material viscosity or resistivity. More precisely if \underline{u} is a many components vector field in an L^2 function space representing the frame of the fluid motion, \underline{u} will obey an equation of the form

$$\dot{\underline{u}} = A(\underline{u})\underline{u} + D\underline{u}, \quad (14)$$

where $A(\underline{u})$ is a nonlinear operator depending linearly upon \underline{u} and D is a linear negative definite operator if $\underline{u} = 0$ at the boundary. A simple example is

$$A(\underline{u})\underline{u} = \underline{u} \cdot \nabla \underline{u}, \quad D\underline{u} = \nabla^2 \underline{u}. \quad (15)$$

We assume further that

$$(\underline{u}, A(\underline{u})\underline{u}) = 0, \quad (16)$$

where the scalar product is given by

$$(\underline{a}, \underline{b}) = \int \underline{a} \cdot \underline{b} \, d\tau, \quad (17)$$

the integration being done over the volume occupied by the fluid.

To study the nonlinear stability we split \underline{u} in

$$\underline{u} = \underline{u}_0 + \underline{u}_1, \quad (18)$$

where \underline{u}_1 is a finite perturbation zero at the boundary and \underline{u}_0 satisfies

$$\dot{\underline{u}}_0 = A(\underline{u}_0)\underline{u}_0 + D\underline{u}_0. \quad (19)$$

The equation for \underline{u}_1 is then

$$\dot{\underline{u}}_1 = A(\underline{u}_1)\underline{u}_1 + L\underline{u}_1, \quad (20)$$

with

$$L\underline{u}_1 = A(\underline{u}_0)\underline{u}_1 + A(\underline{u}_1)\underline{u}_0 + D\underline{u}_1. \quad (21)$$

L is a linear operator on \underline{u}_1 which in cases like (15) will remain negative definite if $A(\underline{u}_0)$ and \underline{u}_0 are small enough. Taking the scalar product of \underline{u}_1 with equation (20) we obtain

$$\frac{1}{2} (\underline{u}_1, \dot{\underline{u}}_1) = (\underline{u}_1, L\underline{u}_1) \quad (22)$$

by virtue of (16). Since all considered quantities are real we have

$$(\underline{u}_1, L\underline{u}_1) = (\underline{u}_1, L_s \underline{u}_1), \quad (23)$$

where L_s is the symmetric part of L . Nonlinear stability is then warranted by Lyapunov methods if

$$(\underline{u}_1, L_s \underline{u}_1) < 0, \quad (24)$$

for all \underline{u}_1 satisfying $(\underline{u}_1, \underline{u}_1) = \text{finite}$ and $\underline{u}_1 = 0$ at the boundary. Expression (24) is a sufficient condition for nonlinear stability. The stability problem is now reduced to the minimization of the hermetian form $(\underline{u}_1, L_s \underline{u}_1)$. This can always be done for any flow ultimately numerically using standard hermiteans eigenvalues techniques.

3.1 Application to MHD Couette flows

Let us illustrate the above procedure by studying the nonlinear stability of a time-dependent MHD flow generalising the time-dependent planar Couette flow. It consists of a fluid bounded by two horizontal plates, the first plate at $z = 0$ and the second at $z = h$, with velocity parallel to the magnetic field and both depending only on one coordinate (z) and the time (t):

$$\mathbf{v} = v(z, t) \hat{e}_y, \quad (25)$$

$$\mathbf{B} = B(z, t) \hat{e}_y, \quad (26)$$

satisfying the equations

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial z^2} = 0, \quad (27)$$

$$\frac{\partial B}{\partial t} - \eta \frac{\partial^2 B}{\partial z^2} = 0, \quad (28)$$

$$\frac{\partial p}{\partial z} + B \frac{\partial B}{\partial z} = 0, \quad (29)$$

where ν and η are viscosity and resistivity respectively. For simplicity special solutions of these equations can be taken as

$$\mathbf{v} = \frac{v_0}{\sin \sqrt{\frac{\alpha}{\nu}} h} e^{-\alpha t} \sin \sqrt{\frac{\alpha}{\nu}} z \hat{e}_y, \quad (30)$$

$$\mathbf{B} = \frac{B_0}{\sin \sqrt{\frac{\alpha}{\eta}} h} e^{-\alpha t} \sin \sqrt{\frac{\alpha}{\eta}} z \hat{e}_y, \quad (31)$$

$$p = -\frac{B^2}{2} + f(t), \quad (32)$$

with the following boundary conditions:

$$v(0, t) = B(0, t) = 0, \quad (33)$$

$$v(h, t) = v_0 e^{-\alpha t}, \quad (34)$$

$$B(h, t) = B_0 e^{-\alpha t}. \quad (35)$$

and $f(t)$ fixed by the boundary conditions on p . In the limit $\alpha \rightarrow 0$ this system reduces to a stationary MHD flow. For $B_0 \rightarrow 0$ we have the time dependent Couette flow and when both α and $B_0 \rightarrow 0$, we obtain the stationary Couette flow.

Without going into the details of the calculations and notations, which can be found in [11], we give an outline of the successive steps leading to the final result. We obtain first the operator L from (21) and the nonlinear MHD equations, then extract the symmetric part of L , L_s and apply criterion (24) by maximizing the left hand side of (24). The final results can be summarized as follows

- For $\nu < \eta$ the system is stable if

$$Re \frac{\sqrt{\frac{\alpha}{\nu}} h}{\sin \sqrt{\frac{\alpha}{\nu}} h} + S \frac{\sqrt{\frac{\alpha}{\eta}} h}{\sin \sqrt{\frac{\alpha}{\eta}} h} < 2\pi^2, \quad (36)$$

where

$$Re = \frac{v_0 h}{\nu}, \quad (37)$$

$$S = \frac{B_0 h}{\nu}. \quad (38)$$

- For $\eta < \nu$ the system is stable if

$$Re_m \frac{\sqrt{\frac{\alpha}{\nu}} h}{\sin \sqrt{\frac{\alpha}{\nu}} h} + S_m \frac{\sqrt{\frac{\alpha}{\eta}} h}{\sin \sqrt{\frac{\alpha}{\eta}} h} < 2\pi^2, \quad (39)$$

where

$$Re_m = \frac{v_0 h}{\eta}, \quad (40)$$

$$S_m = \frac{B_0 h}{\eta}. \quad (41)$$

In the limit $\alpha \rightarrow 0$ (steady MHD flow) we obtain

$$Re + S < 2\pi^2 \quad \text{for } \nu < \eta \quad (42)$$

and

$$Re_m + S_m < 2\pi^2 \quad \text{for } \eta < \nu. \quad (43)$$

For the time-dependent Couette flow ($B_0 \rightarrow 0$), we have

$$Re \frac{\sqrt{\frac{\alpha}{\nu}} h}{\sin \sqrt{\frac{\alpha}{\nu}} h} < 2\pi^2, \quad (44)$$

and for the stationary Couette flow ($\alpha, B_0 \rightarrow 0$)

$$Re < 2\pi^2. \quad (45)$$

The critical value $2\pi^2 \simeq 19.7$ for the Reynolds number calls for some comments. The extremalization for the Couette flow in HD has been done in the literature by constraining the variations to be divergence-free (see e.g. [12]). As a consequence of that the critical value of 20.7 is found. This gain of 5% in the critical value is paid by a very sophisticated derivation which would not be tractable in the case of the generalized unsteady MHD Couette flow considered here. This justifies our procedure, which allows compressible test functions for the extremalization.

The sufficient condition (24) is general and robust, but also too stringent. It is fulfilled in HD and MHD only if the Reynolds and magnetic Reynolds numbers are small enough. Since viscosity and resistivity especially for hot plasmas are small, condition (24) would allow only a very low level of electrical currents and flows. Linear stability analysis and experimental evidence, however, seem to show that in some cases, values for currents and flows far beyond those allowed by condition (24) occur without any sign of gross instabilities. In the next section force free fields will be shown unconditionally stable for any value of the resistivity.

4 Nonlinear stability of MHD equilibria without flow

The stability of complex systems such as fluids or plasmas is usually investigated in the linearized case. Obviously, the linearization is done in order to simplify the analysis and obtain a first insight into the problem. This is, however, by no means sufficient for practical stability for the following reasons : If a system is linearly stable, it implies stability only for infinitesimal

perturbations. If it is linearly unstable, it may saturate at a low or high level in the nonlinear regime. Since for practical situations the perturbations are finite and the saturation levels critical, the study of nonlinear stability, especially for fluids and plasmas, becomes an important and sometimes crucial issue.

In hydrodynamics (HD) the planar Couette flow and the Poiseuille flow in a circular pipe are both linearly stable for all Reynolds numbers (see [12] and [13]). In practical situations turbulence occurs at Reynolds numbers larger than roughly one thousand. It is attributed to nonlinear instabilities or instabilities due to finite perturbations. This view was lent support by simple amplitude expansions [13] and numerical calculations [14].

In HD and magnetohydrodynamics (MHD) exact sufficient criteria for nonlinear stability exist (see [1], [10] and [11]). Such criteria are powerful and robust, and provide nonlinear stability for arbitrary perturbation levels. In other words, they ensure so-called unconditional stability, which in a certain sense is too good and is not needed for practical stability, since the perturbations can be assumed to be limited in an experiment, especially if one wants to avoid strong vibrations etc. Accordingly, the critical Reynolds numbers delivered by these criteria are too low, of the order of 5 (see [1]) and 20 (see [11]).

Unfortunately, no rigorous criteria are available in HD in the range of Reynolds numbers larger than roughly 20. This lack of knowledge is precisely in the range where the nonlinear stability margin will probably depend upon the perturbation level. This is equivalent to saying that what is missing is a knowledge of the basin of attraction of the unperturbed solution in functional space. For very low Reynolds numbers 5 to 20 the basin of attraction is infinite and for very large Reynolds numbers it is probably infinitesimal or very small.

Fortunately, the situation is not as bad in MHD. It is possible there to find unperturbed equilibria with zero flow which are unconditionally stable for all magnetic Reynolds numbers. A first example is the case of so-called force-free fields, whose nonlinear stability was recently analyzed by the author [15].

4.1 Nonlinear stability of force-free fields

Let us assume as unperturbed solution

$$\mathbf{j} = \lambda \mathbf{B} \quad (46)$$

with $\lambda = ct$, bounded by a perfectly conducting wall.

The equations of motion are those of incompressible MHD with a material resistivity η constant in space and time. For any finite perturbations \mathbf{v} of the velocity field with $\mathbf{n} \cdot \mathbf{v} = 0$ at the boundary and \mathbf{A} of the vector potential with $\mathbf{n} \times \mathbf{A} = 0$ at the boundary the equations of motion are [15]

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_1 \times \mathbf{B}_1, \quad (47)$$

with $\nabla \cdot \mathbf{v} = 0$,

$$\dot{\mathbf{A}} = \mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1) - \eta \mathbf{j}_1, \quad (48)$$

$$\dot{\mathbf{B}}_1 = \nabla \times (\mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1) - \eta \mathbf{j}_1). \quad (49)$$

Taking the scalar product of (47) with \mathbf{v} and that of (49) with \mathbf{B}_1 , adding and integrating over the volume, we obtain

$$\frac{\partial}{\partial t} \int \frac{d\tau}{2} (v^2 + \mathbf{B}_1^2) = \lambda \int d\tau \mathbf{v} \times \mathbf{B}_0 \cdot \mathbf{B}_1 - \int d\tau \eta \mathbf{j}_1^2. \quad (50)$$

Many quadratic and cubic terms integrate to zero because of the boundary condition being taken as perfectly conducting. Taking the scalar product of (48) with \mathbf{B}_1 , we can solve for $\mathbf{v} \times \mathbf{B}_0 \cdot \mathbf{B}_1$, and, inserting into (50), we obtain

$$\frac{\partial}{\partial t} \int \frac{d\tau}{2} (v^2 + \mathbf{B}_1^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) = -\eta \int d\tau (\mathbf{j}_1^2 - \lambda \mathbf{B}_1 \cdot \mathbf{j}_1) \quad (51)$$

or

$$\frac{d}{dt} \int d\tau \frac{1}{2} (v^2 + (\nabla \times \mathbf{A})^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) = -\eta \int d\tau ((\nabla \times \nabla \times \mathbf{A})^2 - \lambda \nabla \times \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{A}). \quad (52)$$

Since (52) also holds for the linearized case, which was discussed a long time ago in [16], we reproduce the proof given there for the sufficiency of

$$\int d\tau \frac{1}{2} ((\nabla \times \mathbf{A})^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) \geq 0 \quad (53)$$

for nonlinear stability. Note first that $\mathbf{n} \times \mathbf{A} = 0$ implies $\mathbf{n} \cdot \nabla \times \mathbf{A} = 0$ at the boundary, so that if (53) is satisfied for $\mathbf{n} \times \mathbf{A} = 0$, then the right-hand side of (52) will be satisfied for $\mathbf{n} \cdot \nabla \times \mathbf{A} = 0$. By means of the Lyapunov theorems the expression under the time derivative of the left-hand side of (52) is a Lyapunov function if (53) is verified. Condition (53) is sufficient for stability independently of the values of the resistivity and viscosity. As mentioned above, there is nothing like this in HD.

4.2 Two-dimensional perturbations

A less spectacular example in MHD is the nonlinear stability of a straight z-pinch or tokamak surrounded by perfectly conducting walls. Here it is possible to prove nonlinear stability with respect to 2-dimensional perturbations if the current density is homogeneous, the velocity of the unperturbed fluid \mathbf{v}_0 being zero. The equilibrium is given by

$$\Delta \Psi = J_0 = -P'(\Psi), \quad (54)$$

$$\mathbf{j}_0 = \mathbf{e}_z J_0, \quad (55)$$

$$\mathbf{v}_0 = 0. \quad (56)$$

Ψ denotes the flux of the poloidal magnetic field, J_0 is the current density in the z direction and $P(\Psi)$ is the pressure as a function of Ψ . A constant magnetic field B_z in the z direction could be added without changing the shape of Ψ , which is determined by (54) for any given boundary condition on Ψ .

The MHD equations of motion for an incompressible fluid with mass density equal to unity are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_1 - \nabla P_1 + \mu \Delta \mathbf{v}, \quad (57)$$

$$-\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times (\mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1)) + \eta \nabla \times \mathbf{j}_1, \quad (58)$$

where \mathbf{v} and \mathbf{B}_1 are finite perturbations of the velocity and the magnetic fields having $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{B}_1 = 0$ at the boundary. Taking the scalar product of (57) with \mathbf{v} and that of (58) with \mathbf{B}_1 , adding and integrating over the volume, we obtain

$$\frac{d}{dt} \frac{1}{2} \int (\mathbf{v}^2 + \mathbf{B}_1^2) d\tau = \int \mathbf{v} \cdot \mathbf{j}_0 \times \mathbf{B}_1 d\tau - \mu \int (\nabla \times \mathbf{v}^2) d\tau - \eta \int (\nabla \times \mathbf{B}_1^2). \quad (59)$$

Many quadratic and cubic terms integrate to zero because of the boundary conditions. The right-hand side of (59) would be negative if the first integral on the right-hand side of (59) vanished. We now prove that this is the case if $\mathbf{j}_0 = ct$. Introducing the vector potential, we have

$$\int \mathbf{v} \times \mathbf{j}_0 \cdot \nabla \times \mathbf{A} d\tau = \int \mathbf{A} \cdot (\mathbf{j}_0 \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{j}_0) d\tau = 0. \quad (60)$$

The last equality is due to the two-dimensionality of \mathbf{v} and the assumption of a constant vector \mathbf{j}_0 . This means that the expression under the time derivative on the left-hand side of (59) is a Lyapunov function, from which nonlinear stability follows. Again this condition is independent of the Reynolds and magnetic Reynolds numbers, in contrast to, for example, [11], but the stability is unconditional.

For the above two examples we were able to obtain sufficient and unconditional stability conditions without limitations on the Reynolds numbers. Our examples are of course special and contain an important ingredient $\mathbf{v}_0 = 0$, i.e. no flow in the unperturbed state. This is nontrivial only in the MHD cases. In contrast, the criteria [1], [10] and [11] are very general but imply severe limitations on the Reynolds numbers. The examples given in this note and the general criteria [1], [10] and [11] have one thing in common, viz-they all deal with unconditional stability.

As mentioned at the beginning, this is not necessary for practical stability. If we want to get rid of unconditional stability in our proofs, we have to deal with finite basins of attraction in functional spaces. Practical stability is tied to this very difficult problem.

5 A manifest Hopf bifurcation

In this section we consider the case for which condition (12) is satisfied and prove that, if the inertial term can cause some additional overstability, the modes appearing in this way meet the requirements of the centre manifold theorem [17]. This means that they can be stabilized nonlinearly through a Hopf bifurcation, resulting in a limit cycle or nonlinear periodic oscillation. If $N \neq 0$ in equation (11), inertia-caused overstable modes can occur: In a special example [18], the overstability occurs only in the compressible case, primarily at the magnetoacoustic resonance.

Let us now consider the case for which (12) is satisfied but (11) is overstable for $N \neq 0$. Any overstable mode of (11) is given by

$$\xi = \Psi e^{(i\omega + \gamma)t}, \quad (61)$$

where ω and γ are real and satisfy

$$(\gamma^2 - \omega^2)(\Psi, N\Psi) + \gamma(\Psi, P\Psi) + (\Psi, Q_s\Psi) = 0, \quad (62)$$

$$2\gamma\omega(\Psi, N\Psi) + \omega(\Psi, P\Psi) + (\Psi, Q_a\Psi) = 0. \quad (63)$$

We see from (62) and (63) and, generally, from the reality of the operators in (11) that $\xi^* = \Psi^* e^{(-i\omega + \gamma)t}$ is also an eigenmode of (11). It follows that the modes due to the inertia operator N always come in pairs with opposite sign of the real frequencies but the same growth rate, all other modes being damped because of (12). These features are precisely the principal ingredients of the centre manifold theorem [17]. In summary, if (12) is satisfied, inertia-caused overstability can lead to a Hopf bifurcation resulting in a periodic nonlinear oscillation.

Let us illustrate the occurrence of the overstability and the Hopf bifurcation by choosing the operators N and P of (11) proportional to the identity.

$$nI\ddot{\Psi} + pI\dot{\Psi} + (Q_s + Q_a)\Psi = 0, \quad (64)$$

where n and p are positive numbers.

If the eigenvalue problem for $Q_s + Q_a$ can be solved, we have

$$(Q_s + Q_a)\Psi_m = \lambda_m \Psi_m \quad (65)$$

with

$$\lambda_m = \lambda_{mR} + i\lambda_{mI}. \quad (66)$$

λ_m and Ψ_m are, in general, complex, though (64) involves only real quantities.

If we make the ansatz $\Psi = \Psi_m e^{\omega_m t}$, the eigenvalues of (64) ω_m are related to the λ_m by the following equation

$$n\omega^2 + p\omega + \lambda_R + i\lambda_I = 0, \quad (67)$$

valid for each pair of eigenvalues λ_m and ω_m .

Let us split ω in real and imaginary parts, then (67) can be written as a system

$$n(\omega_R^2 - \omega_I^2) + p\omega_R + \lambda_R = 0, \quad (68)$$

$$\omega_I(2n\omega_R + p) + \lambda_I = 0. \quad (69)$$

Inserting in (68) the value of ω_I obtained from (69), we have

$$n\omega_R^2 + p\omega_R + \lambda_R = \frac{n\lambda_I^2}{(2n\omega_R + p)^2}. \quad (70)$$

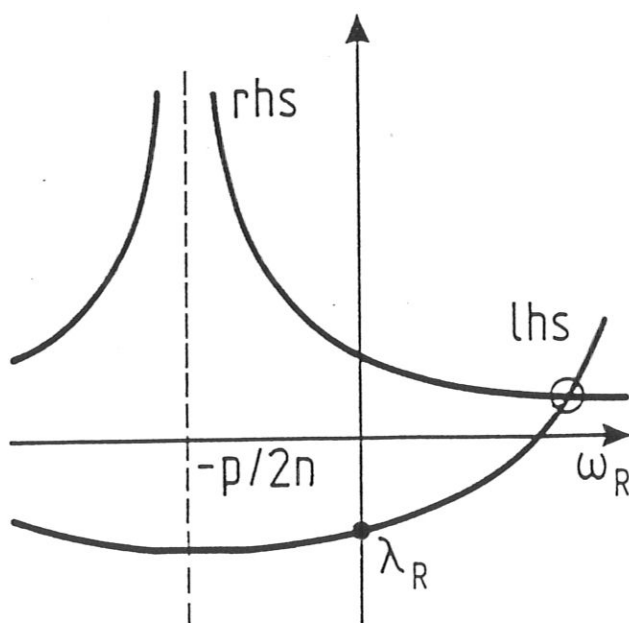


Fig. 1 $\lambda_R < 0$

Since n and p are positive, it is easy to make schematic plots of the left hand side (lhs) and right hand side (rhs) of (70) (see Figures 1 and 2). Let us consider two cases : first, some $\lambda_R < 0$ (Fig. 1), and second, all $\lambda_R > 0$ (Fig. 2), which follows from the validity of (12).

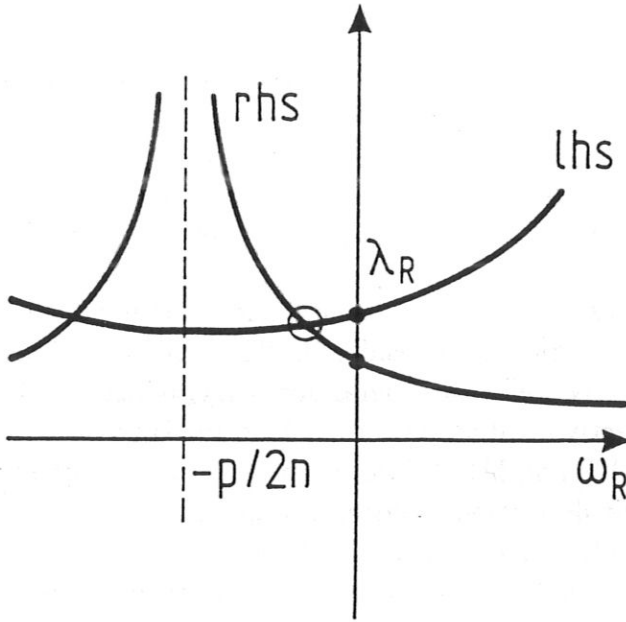


Fig. 2 $\lambda_R > 0$

We see that, if some $\lambda_R < 0$, the system (68, 69) has always a positive root ω_R (see Fig. 1), which means instability. Let us note that a violation of (12) for a Ψ which is not representative of the eigenfunction, does not necessarily imply that $\lambda_R < 0$.

In case $\lambda_R > 0$, the crossing point leads to an instability only if $\lambda_R < \frac{n\lambda_I^2}{p^2}$ (see Fig. 2). As mentioned above and in [19] the unstable crossing cannot occur for $n \approx 0$ or λ_I very small or p large. Starting from a small n , for $\lambda_R > 0$ and λ_I and p fixed, we obtain an overstability by increasing the value

of n until

$$\lambda_R < \frac{n\lambda_I^2}{p^2}, \quad (71)$$

as explained in [20].

6 Conclusions

Energy methods appear as a powerful tool in MHD stability in the linear case as well as in the nonlinear one. The energy methods of the linear case are efficient for fixed and free boundaries. The linear stability problem is essentially reducible to a Hermitean form, which can be minimized analytically or numerically. For general geometries, codes similar to those considered in [21] lead to many applications.

At present, energy methods are also efficient for nonlinear stability as long as no free boundaries are considered. The ability to find statements, especially for force free fields in general geometry, is possible if the fluid is bounded by a perfectly conducting wall. The same is true for the nonlinear stability of flows in HD and MHD. The assumption of incompressibility seems also essential for the derivation of criteria in nonlinear stability. This situation is, hopefully, only due to technical reasons, perhaps because the known criteria are unconditional. Unconditional stability seems to be theoretically more accessible, though it is not needed experimentally.

References

- [1] J. Serrin. *Handbuch der Physik (Springer Verlag, 1959) vol. VIII/1 p. 125*
- [2] I.B. Bernstein, E.A. Frieman, M.D. Kruskal, R.M. Kulsrud. *Proc. Roy. Soc. A 244, 17 (1958)*
- [3] H. Tasso. *to be published in Annals Physics*
- [4] H. Tasso. *Phys. Let. A 147, 28 (1990)*
- [5] H. Tasso. *Plasma Phys. 17, 1131 (1975)*

- [6] H. Tasso, J.T. Virtamo. *Plasma Phys.* 22, 1003 (1980)
- [7] H. Tasso. *Lectures on Plasma Physics IFUSP/P-181 Sao Paulo (1979)*
- [8] J. Neustupa. *Workshop on 'Recent Progress in the Theory of the Navier-Stokes Equations and Related Topics', Bayreuth, (July 1992)*
- [9] S. Rionero. *Ann. Math. Pura Appl.* LXXVI, 75 (1967); LXXVIII, 339 (1968)
- [10] G.P. Galdi, S. Rionero. *'Weighted Energy Methods in Fluid Dynamics and Elasticity', Lecture Notes in Math., Springer Verlag (1985)*
- [11] H. Tasso, S.J. Camargo. *Nuovo Cimento* 107 B, 733 (1992)
- [12] P.G. Drazin, W.H. Reid. *'Hydrodynamic Stability' Cambridge University Press (1981)*
- [13] J.T. Stuart. in *'Transition and Turbulence', Editor R.E. Meyer, Academic Press, 1981*
- [14] S.A. Orszag, A.T. Patera. in *'Transition and Turbulence', Editor R.E. Meyer, Academic Press, 1981*
- [15] H. Tasso. *Phys. Let. A* 169, 396 (1992)
- [16] H. Tasso. *Lecture at 'Theoretical and Computational Plasma Physics', Trieste, p. 321, IAEA, Vienna 1978. See also IPP 6/151, Dec. 1976*
- [17] J.E. Marsden, M. Mc Cracken. *'The Hopf Bifurcation and its Applications', Springer Verlag, New York, 1976*
- [18] H. Tasso, M. Cotsaftis. *J. Nucl. En. (part C, Plasma Phys.)* 7, 29 (1965)
- [19] H. Tasso. *Phys. Let. A* 161, 289 (1991)
- [20] H. Tasso. *Phys. Let. A* 180, 257 (1993)
- [21] W. Kerner, H. Tasso. *Phys. rev. Let.* 49, 654 (1982)