A Sufficient Condition for the Lack of Cellular Convection in a Dissipative Plasma Column

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A one-dimensional boundary-value problem of dissipative plasma equilibrium in a cylinder is formulated and analytically solved. As regards the data, axial symmetry and uniformity along the axis of the cylinder are assumed; as regards the solution(s), a given periodicity along the axis of the cylinder is imposed. Viscous stresses and resistance are the dissipation processes taken into account, while a particle source and an externally driven electric field sustain the pressure gradient in the plasma. Plasma density, coefficients of viscosity and resistivity are given smooth functions of the radius. After analytically solving the boundary-value problem, a functional setting of the equations is established and a problem for weak solutions is formulated. The main achievement of the analysis is a rigorous uniqueness and *nonlinear* stability result for the analytical solution found; since such a solution describes a merely radial flow of the plasma across nested magnetic surfaces, what is derived is a sufficient condition for the lack of cellular convection. Finally, the significance of the physical model introduced in this paper, and herein theoretically analysed, is pointed out in view of possible computational work which might yield valuable insight.

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1. Introduction

In research on controlled thermonuclear fusion, the one-fluid magnetohydrodynamic (MHD) model is commonly adopted to study the containment of the plasma. Since the dissipative processes active in a laboratory plasma (viscous stresses, resistance, friction), as well as the inertia of the plasma, are typically much weaker than the electromagnetic force and the pressure gradient, the above-mentioned model is most often used in the ideal plasma approximation (Freidberg 1987) in which the dissipative terms and the inertia are disregarded.

However, the MHD equations that include inertia, viscosity and resistivity have the same mathematical structure as the Navier–Stokes equations (Sermange

& Temam 1983; Spada & Wobig 1992); therefore, some insight into the macroscopic behaviour of the plasma may be obtained from the extensive mathematical and experimental literature concerning ordinary hydrodynamics.

The nonlinearity of the Navier–Stokes equations may give rise to bifurcation phenomena and onset of instability with consequent production of striking flow patterns. The two phenomena most often considered are Taylor–Couette flow and Rayleigh–Bénard convection, to the theoretical and experimental investigation of which much effort has been and is being devoted (Chandrasekhar 1981). We recall that instability appears (or, mathematically speaking, the solution of the Navier–Stokes equations undergoes a bifurcation) when the external forces acting on the system become strong enough and/or the dissipation processes become weak enough.

Thus, the conjecture that similar phenomena may also take place in a laboratory plasma seems well grounded; they may be expected to occur when the source which sustains the equilibrium pressure gradient is large enough and/or resistivity and viscosity coefficients are small enough. As we have recalled above, in typical experimental conditions encountered in research on controlled thermonuclear fusion, the dissipation processes are *very* "weak". Indeed, experimental observations seem to indicate that the macroscopic behaviour of the plasma, e.g. in a stellarator, may have the nature of cellular convection or even turbulent convection.

In this paper, we analyse a dissipative MHD model of plasma equilibrium whose equations account for inertia, viscosity and resistivity; the pressure gradient is sustained by a plasma source and by an externally driven electric field. The plasma is assumed to be enclosed in an infinite cylinder along the axis of which all physical quantities are assumed to be periodic. Plasma density, resistivity, viscosity coefficients and plasma source are given smooth functions of the radius only. An axisymmetric boundary-value problem is formulated and an explicit analytic solution is found which describes the simple radial diffusion of the plasma. The original problem is rewritten in terms of new unknowns (difference between the old unknowns and the solution found). A functional setting of the latter equations is established and a problem for weak solutions is rigorously formulated; the zero element of the function space is obviously a solution for all values of the data. The main result of the analysis is a sufficient condition under which the above-mentioned trivial solution is unique and stable. Such a condition is discussed, paying particular attention to the effect of the presence in the model of the density gradient, viscosity gradient and resistivity gradient, which makes the model analysed in this paper quite general; it generalizes the model recently analysed by Spada & Wobig (1992).

This paper is organized as follows. In §2, we give an account of the model and the formulation of a boundary-value problem whose unknowns are the scalar pressure, the flow velocity field and the magnetic field. In §3, our functional setting of the equations is introduced; suitable function spaces are defined and a problem for weak solutions is established generalizing the techniques of mathematical hydrodynamics. In §4, we obtain and discuss the uniqueness and stability result. Finally, in §5 we concisely summarize our main results and point out the questions that seem to deserve further consideration.

2. The model

In the first place, let us introduce the following subsets of the space $\mathbf{R}^3$ (with

$a, L > 0$):

$$D = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 < a^2\}$$

$$\Gamma = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 = a^2\}$$

$$\Omega = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 < a^2, 0 < z < L\}$$

$$\partial\Omega^{(l)} = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 = a^2, 0 \leq z \leq L\}$$

$$\partial\Omega^{(0)} = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 \leq a^2, z = 0\}$$

$$\partial\Omega^{(L)} = \{(x,y,z) \in \mathbf{R}^3, x^2 + y^2 \leq a^2, z = L\}$$

We assume that the plasma is enclosed in the fixed, infinite cylinder D, whose radius is a and whose boundary is Γ. Also, we shall assume that all physical quantities are periodic along the z direction with a periodicity length equal to L; thus, Ω is the fundamental region of periodicity and $\partial\Omega^{(l)}$ is its lateral surface, $\partial\Omega^{(0)}$ is its bottom, $\partial\Omega^{(L)}$ is its top.

Moreover, let $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ be the unit vectors along the x, y, z Cartesian axes; and let $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ be the orthogonal unit vectors associated with the cylindrical co-ordinate directions r, ϕ, z.

We shall be concerned with steady states of the plasma whose behaviour is described by the following one-fluid, dissipative MHD equations:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} + S\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B} + \partial_k\left[\nu\left(\partial_k\mathbf{v} + \nabla v_k\right)\right] + \nabla\left[\left(\zeta - \frac{2}{3}\nu\right)\nabla\cdot\mathbf{v}\right] \tag{1}$$

$$\eta\mathbf{j} = \mathbf{E} + \mathbf{v}\times\mathbf{B} + \epsilon\mathbf{e}_z \tag{2}$$

$$\mathbf{j} = \nabla\times\mathbf{B} \tag{3}$$

$$\nabla\cdot(\rho\mathbf{v}) = S \tag{4}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{5}$$

Here, ρ is the plasma density, ν and ζ are two coefficients of viscosity, η is the resistivity, $\mathbf{v}$ the flow velocity field, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the current density field, p the scalar pressure, $\mathbf{E} + \epsilon \mathbf{e}_z$ the electric field where $\mathbf{E} = -\nabla\varphi$ and $\epsilon\mathbf{e}_z$ is externally driven, S a particle source which together with ϵ sustains the pressure gradient in the plasma. Moreover, $v_k = \mathbf{v} \cdot \mathbf{e}_k$ and $\partial_k = \frac{\partial}{\partial k}$ $(k = x, y, z)$, repeated indices are summed, and $\mu_0 = 1$ throughout this paper.

The third term and the fourth term on the right-hand side of equation (1) are the viscous force per unit volume (Landau & Lifshitz 1959, p. 48). As a matter of fact, such a viscosity does not well describe the viscous stresses of a magnetized plasma. A more accurate expression, if the plasma may be assumed to be collision dominated, would be given by the Braginskii viscosity (Braginskii 1965); in this connection, Spada & Wobig (1992) have recently carried out a rigorous analysis of the Braginskii viscosity in a suitable functional framework. On the grounds of their results, the following analysis might be generalized in a straightforward way by adopting the Braginskii viscosity; the only reason why we choose the viscosity appearing in equation (1) is to maintain the analysis simple with respect to this issue.

According to our model assumptions, equations (1)–(5) are eleven scalar equations for the eleven scalar unknowns p, φ, $\mathbf{v}$, $\mathbf{B}$, $\mathbf{j}$. We assume that ρ, ν, ζ, η, S, ϵ are *given* quantities and, specifically, that

$$\epsilon = \text{constant}$$

$$\rho = \rho(r), \nu = \nu(r), \zeta = \zeta(r), \eta = \eta(r), S = S(r)$$

$$\rho, \nu, \zeta, \eta, S \in \mathcal{C}^\infty([0, a])$$

Moreover, we assume that

$$\rho(r) > 0, \nu(r) > 0, \zeta(r) > 0, \eta(r) > 0 \qquad \text{for all } r \in [0, a]$$

As a matter of fact, the density (ρ) and the pressure (p) are linked by the equation of state. Hence, the assumption that ρ is a given quantity reduces the self-consistency of the model; but to realize that the following analysis of equilibria where ρ is given is significant, it is enough to recollect the case of stellarator equilibria: the experiments show that electron cyclotron heated stellarator plasmas typically have very flat density profiles (and very peaky temperature profiles), and, as to such equilibria, ρ might be assumed given and uniform. As Spada & Wobig (1992) have recently pointed out, if ρ is a given quantity, the mathematical structure of the equations with which we are dealing is in substance that which one encounters in the well established theory of viscous incompressible flow. Similar remarks hold for the resistivity and the coefficients of viscosity as well, though a self-consistent account of their dependence upon other quantities does not seem possible at all outside a computational treatment.

Next, as we said we impose that

$$\text{the unknowns are periodic in } z \text{ with a periodicity length equal to } L \tag{6}$$

We proceed adding to the problem we are formulating the prescription of a *given* value of longitudinal magnetic flux:

$$\int_{\partial\Omega^{(0)}} \mathbf{B}\cdot\mathbf{e}_z \,\mathrm{d}\sigma = F \tag{7}$$

We supplement our problem with the following boundary conditions:

$$\mathbf{v} = v_0\mathbf{e}_r \quad \text{on } \Gamma, \quad \text{with } v_0 = \text{constant} \tag{8}$$

$$\mathbf{B}\cdot\mathbf{e}_r = 0 \quad \text{on } \Gamma \tag{9}$$

$$\mathbf{E}\times\mathbf{e}_r = \mathbf{0} \quad \text{on } \Gamma \tag{10}$$

Conditions (9)–(10) reflect the assumption that the boundary is a perfectly conducting wall. As regards condition (8), note that the data must be assumed to fulfil the compatibility condition $a\rho(a)v_0 = \int_0^a rS(r)\,\mathrm{d}r$.

Note that the boundary conditions (8)–(10) are uniform along the axis of the plasma column. As is shown by a great many theoretical and experimental results obtained in ordinary hydrodynamics (Chandrasekhar 1981), this uniformity does not prevent the solution(s) of the boundary-value problem from having a (non-trivial) periodicity along the direction at issue.

We have now formulated our stationary boundary-value problem. Let us look for a solution such that

$$p = p(r)$$

$$\mathbf{v} = v_r(r)\mathbf{e}_r$$

$$\mathbf{B} = B_\phi(r)\mathbf{e}_\phi + B_z(r)\mathbf{e}_z$$

Requiring also that the particle source by itself drives no longitudinal current (which turns out to be equivalent to imposing the regularity condition $rB_\phi(r) \to 0$ as $r \to 0$), we obtain:

$$\tilde{\mathbf{v}} = \frac{1}{r\rho(r)}\left[\int_0^r r' S(r')\,\mathrm{d}r'\right]\mathbf{e}_r \equiv \tilde{v}_r(r)\mathbf{e}_r \tag{11}$$

$$\begin{aligned}\tilde{\mathbf{B}} = & \frac{\epsilon}{r}\left\{\int_0^r \frac{r'}{\eta(r')}\exp\left[\int_{r'}^r \frac{\tilde{v}_r(r'')}{\eta(r'')}\,\mathrm{d}r''\right]\mathrm{d}r'\right\}\mathbf{e}_\phi \\ & + \frac{F}{2\pi\int_0^a \hat{r}\exp\left[\int_0^{\hat{r}}\frac{\tilde{v}_r(r'')}{\eta(r'')}\,\mathrm{d}r''\right]\mathrm{d}\hat{r}}\exp\left[\int_0^r \frac{\tilde{v}_r(r')}{\eta(r')}\,\mathrm{d}r'\right]\mathbf{e}_z \\ \equiv & \tilde{B}_\phi(r)\mathbf{e}_\phi + \tilde{B}_z(r)\mathbf{e}_z\end{aligned} \tag{12}$$

$$\begin{aligned}\tilde{\mathbf{j}} = & -\frac{\tilde{v}_r(r)}{\eta(r)}\frac{F}{2\pi\int_0^a \hat{r}\exp\left[\int_0^{\hat{r}}\frac{\tilde{v}_r(r'')}{\eta(r'')}\,\mathrm{d}r''\right]\mathrm{d}\hat{r}}\exp\left[\int_0^r \frac{\tilde{v}_r(r')}{\eta(r')}\,\mathrm{d}r'\right]\mathbf{e}_\phi \\ & + \frac{\epsilon}{\eta(r)}\mathbf{e}_z + \frac{\epsilon}{r}\frac{\tilde{v}_r(r)}{\eta(r)}\left\{\int_0^r \frac{r'}{\eta(r')}\exp\left[\int_{r'}^r \frac{\tilde{v}_r(r'')}{\eta(r'')}\,\mathrm{d}r''\right]\mathrm{d}r'\right\}\mathbf{e}_z \\ \equiv & \tilde{j}_\phi(r)\mathbf{e}_\phi + \tilde{j}_z(r)\mathbf{e}_z\end{aligned} \tag{13}$$

$$\tilde{\mathbf{E}} = \mathbf{0} \tag{14}$$

$$\begin{aligned}\tilde{p}'(r) =& -\rho(r)\tilde{v}_r(r)\tilde{v}_r'(r) - S(r)\tilde{v}_r(r)\\ & - \left\{\frac{1}{r}\tilde{B}_\phi(r)\left[r\tilde{B}_\phi(r)\right]' + \tilde{B}_z(r)\tilde{B}_z'(r)\right\}\\ & + 2\left\{[\nu(r)\tilde{v}_r'(r)]' + \frac{1}{r}\nu(r)\tilde{v}_r'(r) - \frac{1}{r^2}\nu(r)\tilde{v}_r(r)\right\}\\ & + \left\{\frac{1}{r}\left[\zeta(r) - \frac{2}{3}\nu(r)\right][r\tilde{v}_r(r)]'\right\}'\end{aligned} \tag{15}$$

The equilibrium (11)–(15) is uniform along the z direction and describes a merely radial flow of the plasma across nested magnetic surfaces.

We assume that the above radial force balance is a good approximation as compared with that occurring in magnetic confinement devices such as the tokamak in their supposed way of operation. This is the reason why we analyse, in this paper, a model of MHD equilibrium where the plasma is assumed to be enclosed in a cylinder: we believe that toroidicity and three-dimensional effects of plasmas in magnetic confinement devices such as the tokamak and the stellarator are far less important than the radial force balance as far as the above-mentioned bifurcation and instability phenomena are concerned. Also, we recall that, from the point of view of physics, the cause of the instability with which we are concerned in this paper is the unfavourable curvature of the magnetic field lines.

Next, we return to the problem we have formulated and rewrite it eliminating $\mathbf{j}$ by means of equation (3) and φ by applying the curl operator to equation (2). Moreover, we introduce the following new unknowns:

$$p_* = p - \tilde{p}$$

$$\mathbf{v}_* = \rho(\mathbf{v} - \tilde{\mathbf{v}})$$

$$\mathbf{B}_* = \mathbf{B} - \tilde{\mathbf{B}}$$

Note that $\mathbf{v}_*$ is a mass flux. Carrying out some trivial calculations and using the identities

$$\begin{aligned}(\nabla\times\mathbf{a})\times\mathbf{a}' &= (\mathbf{a}'\cdot\nabla)\mathbf{a} - a_k'\nabla a_k\\ &= (\mathbf{a}'\cdot\nabla)\mathbf{a} - \nabla(\mathbf{a}\cdot\mathbf{a}') + a_k\nabla a_k'\end{aligned}$$

$$\nabla\times(\mathbf{a}\times\mathbf{a}') = \mathbf{a}\nabla\cdot\mathbf{a}' - \mathbf{a}'\nabla\cdot\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{a}' + (\mathbf{a}'\cdot\nabla)\mathbf{a}$$

$$(\mathbf{a} \times \mathbf{a}') \times \mathbf{a}'' = \mathbf{a}'\mathbf{a} \cdot \mathbf{a}'' - \mathbf{a}\mathbf{a}' \cdot \mathbf{a}''$$

we obtain the following final problem (*we henceforward omit the asterisk*):

$$\begin{aligned} -\partial_k\left(\frac{\nu}{\rho}\partial_k \mathbf{v}\right) = & -(\mathbf{v}\cdot\nabla)\frac{\mathbf{v}}{\rho} - \rho(\tilde{\mathbf{v}}\cdot\nabla)\frac{\mathbf{v}}{\rho} - (\mathbf{v}\cdot\nabla)\tilde{\mathbf{v}} - \frac{S}{\rho}\mathbf{v} \\ & + (\mathbf{B}\cdot\nabla)\mathbf{B} + \left(\tilde{\mathbf{B}}\cdot\nabla\right)\mathbf{B} + (\mathbf{B}\cdot\nabla)\tilde{\mathbf{B}} \\ & + \left[\partial_k\left(\frac{\nu}{\rho}\right)\right]\nabla v_k - \partial_k\left(\frac{\nu}{\rho^2}\mathbf{v}\partial_k\rho\right) - \partial_k\left(\frac{\nu}{\rho^2}v_k\nabla\rho\right) \\ & - \nabla\left(p + \frac{|\mathbf{B}|^2 + 2\mathbf{B}\cdot\tilde{\mathbf{B}}}{2}\right) \end{aligned} \tag{16}$$

$$\begin{aligned} \nabla\times(\eta\nabla\times\mathbf{B}) = & \left[\left(\tilde{\mathbf{B}} + \mathbf{B}\right)\cdot\nabla\right]\frac{\mathbf{v}}{\rho} - \frac{1}{\rho}(\mathbf{v}\cdot\nabla)\left(\tilde{\mathbf{B}} + \mathbf{B}\right) \\ & + (\mathbf{B}\cdot\nabla)\tilde{\mathbf{v}} - \mathbf{B}\nabla\cdot\tilde{\mathbf{v}} - (\tilde{\mathbf{v}}\cdot\nabla)\mathbf{B} \end{aligned} \tag{17}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{18}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{19}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma \tag{20}$$

$$\mathbf{B}\cdot\mathbf{e}_r = 0 \quad \text{on } \Gamma \tag{21}$$

$$\eta(\nabla\times\mathbf{B})\times\mathbf{e}_r = v_0\mathbf{B} \quad \text{on } \Gamma \tag{22}$$

$$\int_{\partial\Omega^{(0)}} \mathbf{B}\cdot\mathbf{e}_z \, d\sigma = 0 \tag{23}$$

$$p, \mathbf{v}, \mathbf{B} \text{ are periodic in } z \text{ with a periodicity length equal to } L \tag{24}$$

$$\mathbf{v}\cdot\mathbf{e}_r = 0 \tag{25}$$

and $rB_\phi(r,\phi,z) \to 0$ as $r \to 0$, $\phi \in [0, 2\pi)$, $z \in [0, L)$. Note that we have added to our problem the condition (25) which imposes that the radial component of the flow velocity field must be given by expression (11). Condition (25) restricts the class of equilibria that we consider and reduces the significance of the results we shall obtain from the physical point of view. The reason why we impose condition (25) is technical and is connected with the non-uniformity of the plasma density: we need that $\nabla \cdot (\mathbf{v}/\rho) = 0$ in order to obtain a *nonlinear* stability result. The mathematical structure of the equations does seem to be deeply affected by the non-uniformity of the plasma density. In a forthcoming paper, we shall also present a *linear* stability result for the above boundary-value problem where condition (25) is not imposed. Finally, we stress that, if the plasma density is assumed to be uniform, we could obtain a *nonlinear* stability result for the above boundary-value problem without imposing condition (25).

3. Functional setting of the equations

Obviously, $p = 0$, $\mathbf{v} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ is a solution of problem (16)–(25) for all values of the data. This solution is expected to lose uniqueness and stability if the plasma equilibrium is driven strongly enough or the dissipation processes are weak enough. In order to obtain an uniqueness result for the above solution, we firstly establish a functional setting of the equations. We shall use the following spaces:

$$\mathcal{V}_1 = \{\mathbf{v} \in (\mathcal{C}^\infty (D \cup \Gamma))^3,$$

$$\mathbf{v} \text{ is periodic in } z \text{ with a periodicity length equal to } L,$$

$$\mathbf{v} \cdot \mathbf{e}_r = 0 \text{ in } D,\ \nabla \cdot \mathbf{v} = 0 \text{ in } D,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma\}$$

$V_1 =$ the closure of $\mathcal{V}_1$ with the norm defined by

$$\text{the scalar product } ((\mathbf{v}, \mathbf{v}'))_1 = \int_\Omega \frac{\nu}{\rho} (\partial_i \mathbf{v}) \cdot (\partial_i \mathbf{v}')\, \mathrm{d}^3 x$$

$$\mathcal{V}_2 = \{\mathbf{B} \in (\mathcal{C}^\infty (D \cup \Gamma))^3,$$

$$\mathbf{B} \text{ is periodic in } z \text{ with a periodicity length equal to } L,$$

$$\int_{\partial\Omega^{(0)}} \mathbf{B} \cdot \mathbf{e}_z\, \mathrm{d}\sigma = 0,\ \nabla \cdot \mathbf{B} = 0 \text{ in } D,\ \mathbf{B} \cdot \mathbf{e}_r = 0 \text{ on } \Gamma\}$$

$V_2 =$ the closure of $\mathcal{V}_2$ with the norm defined by

$$\text{the scalar product } ((\mathbf{B}, \mathbf{B}'))_2 = \int_\Omega \eta (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}')\, \mathrm{d}^3 x$$

The bilinear form $((\bullet,\bullet))_1$ actually defines a scalar product thanks to the Poincaré inequality; this scalar product provides the norm given by $\|\mathbf{v}\|_1 = \{((\mathbf{v},\mathbf{v}))_1\}^{1/2}$. The fact that the bilinear form $((\bullet,\bullet))_2$ actually defines a scalar product is a very technical point (Spada & Wobig 1992, and references therein); this scalar product provides the norm given by $\|\mathbf{B}\|_2 = \{((\mathbf{B},\mathbf{B}))_2\}^{1/2}$.

Finally, we introduce the product space

$$V = V_1 \times V_2$$

and equip it with the scalar product

$$((\Phi,\Phi')) = \frac{1}{\rho_\star}((\mathbf{v},\mathbf{v}'))_1 + ((\mathbf{B},\mathbf{B}'))_2 \quad \text{for all } \Phi = (\mathbf{v},\mathbf{B}),\ \Phi' = (\mathbf{v}',\mathbf{B}') \in V$$

where $\rho_\star \equiv \max_{r\in[0,a]} \rho(r)$. This scalar product provides the norm on V given by $\|\Phi\| = \{((\Phi,\Phi))\}^{1/2}$.

We proceed now by establishing a weak formulation of problem (16)–(25).

Let us assume that p, $\mathbf{v}$, $\mathbf{B}$ is a *smooth solution*. The first step is to multiply equation (16) by a test function $\mathbf{w} \in \mathcal{V}_1$ and integrate over Ω. It is easy to check that for all $\xi \in \mathcal{C}^\infty(D \cup \Gamma)$, ξ periodic in z with a periodicity length equal to L, we have $\int_\Omega (\nabla\xi)\cdot\mathbf{w}\,\mathrm{d}^3x = 0$.

In order to shorten the notation, we introduce a trilinear form on $\left(\mathbf{H}^1(\Omega)\right)^3$ (see Spada & Wobig 1992) by setting

$$b(\mathbf{a},\mathbf{a}',\mathbf{a}'') = \int_\Omega a_i(\partial_i a'_j)a''_j\,\mathrm{d}^3x$$

This form is *continuous*.

Thus, the previously mentioned projection of equation (16) yields the following (weak) equation:

$$\begin{aligned}((\mathbf{v},\mathbf{w}))_1 = &- b\left(\mathbf{v},\frac{\mathbf{v}}{\rho},\mathbf{w}\right) + b(\mathbf{B},\mathbf{B},\mathbf{w}) - b(\mathbf{v},\tilde{\mathbf{v}},\mathbf{w}) - \int_\Omega \frac{S}{\rho}\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}^3x \\ &- b\left(\rho\tilde{\mathbf{v}},\frac{\mathbf{v}}{\rho},\mathbf{w}\right) + b\left(\mathbf{B},\tilde{\mathbf{B}},\mathbf{w}\right) + b\left(\tilde{\mathbf{B}},\mathbf{B},\mathbf{w}\right) \\ &+ b\left(\frac{\nu}{\rho^2}\nabla\rho,\mathbf{w},\mathbf{v}\right) + b\left(\mathbf{v},\mathbf{w},\frac{\nu}{\rho^2}\nabla\rho\right) + b\left(\mathbf{w},\mathbf{v},\nabla\left(\frac{\nu}{\rho}\right)\right)\end{aligned} \tag{26}$$

which is obtained by carrying out some easy calculations, mainly integrations by parts. Note that p does not appear in equation (26).

Next, let us deal with equation (17) and remember we are assuming p, $\mathbf{v}$, $\mathbf{B}$ to be a smooth solution; by multiplying equation (17) by a test function $\mathbf{C} \in \mathcal{V}_2$ and integrating over Ω (Sermange & Temam 1983; Spada & Wobig 1992), we obtain the following (weak) equation:

$$\begin{aligned}((\mathbf{B},\mathbf{C}))_2 = &- b\left(\tilde{\mathbf{B}}+\mathbf{B},\mathbf{C},\frac{\mathbf{v}}{\rho}\right) + b\left(\frac{\mathbf{v}}{\rho},\mathbf{C},\tilde{\mathbf{B}}+\mathbf{B}\right)\\ &- b(\mathbf{B},\mathbf{C},\tilde{\mathbf{v}}) + b(\tilde{\mathbf{v}},\mathbf{C},\mathbf{B})\end{aligned} \tag{27}$$

To derive equation (27), we have carried out some easy calculations, again mainly integrations by parts, used the boundary condition (8) and the identity

$$b(\mathbf{a},\mathbf{a}',\mathbf{a}'') = -b(\mathbf{a},\mathbf{a}'',\mathbf{a}')$$

which holds if $\nabla\cdot\mathbf{a} = 0$ in D, $\mathbf{a}\cdot\mathbf{e}_r = 0$ on Γ, and $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}''$ are periodic in z with a periodicity length equal to L. Moreover, we have used the identity

$$\begin{aligned}\int_\Omega [\nabla\times(\eta\nabla\times\mathbf{B})]\cdot\mathbf{C}\,\mathrm{d}^3x &= ((\mathbf{B},\mathbf{C}))_2 - \int_{\partial\Omega^{(l)}} \eta\,[(\nabla\times\mathbf{B})\times\mathbf{e}_r]\cdot\mathbf{C}\,\mathrm{d}\sigma\\ &= ((\mathbf{B},\mathbf{C}))_2 - v_0\int_{\partial\Omega^{(l)}} \mathbf{B}\cdot\mathbf{C}\,\mathrm{d}\sigma\end{aligned}$$

whose derivation is easy and where the boundary condition (22) is used.

In order to shorten the notation, let us define the following mapping:

$$\begin{aligned}\mathcal{A}: V\times V &\longrightarrow \mathbf{R}\\ (\Phi,\Phi') &\longmapsto \mathcal{A}(\Phi,\Phi')\end{aligned}$$

$$\begin{aligned}\mathcal{A}(\Phi,\Phi') \equiv \frac{1}{\rho_\star}\Bigg[&- b\left(\mathbf{v},\frac{\mathbf{v}}{\rho},\mathbf{v}'\right) + b(\mathbf{B},\mathbf{B},\mathbf{v}') - b(\mathbf{v},\tilde{\mathbf{v}},\mathbf{v}')\\ &- \int_\Omega \frac{S}{\rho}\mathbf{v}\cdot\mathbf{v}'\,\mathrm{d}^3x - b\left(\rho\tilde{\mathbf{v}},\frac{\mathbf{v}}{\rho},\mathbf{v}'\right) + b\left(\mathbf{B},\tilde{\mathbf{B}},\mathbf{v}'\right) + b\left(\tilde{\mathbf{B}},\mathbf{B},\mathbf{v}'\right)\\ &+ b\left(\frac{\nu}{\rho^2}\nabla\rho,\mathbf{v}',\mathbf{v}\right) + b\left(\mathbf{v},\mathbf{v}',\frac{\nu}{\rho^2}\nabla\rho\right) + b\left(\mathbf{v}',\mathbf{v},\nabla\left(\frac{\nu}{\rho}\right)\right)\Bigg]\\ &- b\left(\tilde{\mathbf{B}}+\mathbf{B},\mathbf{B}',\frac{\mathbf{v}}{\rho}\right) + b\left(\frac{\mathbf{v}}{\rho},\mathbf{B}',\tilde{\mathbf{B}}+\mathbf{B}\right)\\ &- b(\mathbf{B},\mathbf{B}',\tilde{\mathbf{v}}) + b(\tilde{\mathbf{v}},\mathbf{B}',\mathbf{B})\end{aligned}$$

where $\Phi = (\mathbf{v}, \mathbf{B})$ and $\Phi' = (\mathbf{v}', \mathbf{B}')$. Note that the mapping $\mathcal{A}$ is manifestly linear in the second argument but nonlinear in the first one.

Now, we add equations (26) (multiplied by $1/\rho_\star$) and (27); thus, we obtain the following (weak) equation:

$$((\Phi, \Psi)) = \mathcal{A}(\Phi, \Psi) \tag{28}$$

where $\Phi = (\mathbf{v}, \mathbf{B})$ and $\Psi = (\mathbf{w}, \mathbf{C})$.

We can now establish the following weak formulation of problem (16)–(25) (Spada & Wobig 1992, and references therein):

PROBLEM (*weak solutions*). Under the previous hypotheses concerning the data, find $\Phi = (\mathbf{v}, \mathbf{B}) \in V$ such that equation (28) is satisfied for all $\Psi = (\mathbf{w}, \mathbf{C}) \in V$.

This problem might be reduced to solving a nonlinear equation, containing a completely continuous operator, in the space V. This step is advantageous if one wants to prove the existence of at least one weak solution (see, for example, Spada & Wobig 1992); but we already know that $\Phi = 0$ is a solution for all values of the data.

4. The uniqueness result

We conclude this study by dealing with the uniqueness of the trivial solution of equation (28). It is possible to obtain a *sufficient* condition for uniqueness in the following way. Suppose that $\Phi = (\mathbf{v}, \mathbf{B}) \in V$ is a weak solution. Let us define $\Phi_\rho = (\rho_\star \rho^{-1}\mathbf{v}, \mathbf{B})$; one can easily check that $\Phi_\rho \in V$. Therefore, we have

$$\mathcal{A}(\Phi, \Phi_\rho) = ((\Phi, \Phi_\rho)) \geq \|\Phi\|^2 - b\left(\frac{\nu}{\rho^2}\nabla\rho, \mathbf{v}, \frac{\mathbf{v}}{\rho}\right)$$

where we have carried out an easy estimate. Using the above estimate for $\|\Phi\|^2$, and carrying out a great many further estimates, we can obtain the final estimate

$\|\Phi\|^2 \leq \lambda \|\Phi\|^2$, where λ does not depend on Φ. The derivation of such a final estimate is a long calculation where, however, one must only apply the Cauchy–Schwarz inequality for sums and for integrals and the same techniques as those relevant to the derivation of equations (26) and (27); moreover, it is advantageous to use the identity

$$b(\mathbf{a},\mathbf{a}',\mathbf{a}'') - b(\mathbf{a}'',\mathbf{a}',\mathbf{a}) = \int_\Omega [(\nabla \times \mathbf{a}') \times \mathbf{a}] \cdot \mathbf{a}'' \, \mathrm{d}^3 x$$

Thus, if $\lambda < 1$, then $\|\Phi\| = 0$, viz., $\Phi = 0$: the trivial solution is unique. This condition is explicitly

$$\begin{aligned}
&(2\pi L)^{1/4} \rho_\star \left[\left(\int_0^a \frac{r\nu'^4}{\nu^2 \rho^6} \mathrm{d}r \right)^{1/4} + 2 \left(\int_0^a \frac{r\nu^2 \rho'^4}{\rho^{10}} \mathrm{d}r \right)^{1/4} \right] M_1 \\
&+ (2\pi L)^{1/2} \rho_\star \left\{ \left[\int_0^a r \left(\frac{S}{2\rho^2} + \frac{\nu \rho'^2}{\rho^4} \right)^2 \mathrm{d}r \right]^{1/2} \right. \\
&\qquad \left. + \left[\int_0^a \left(\frac{r\tilde{v}_r'^2}{\rho^2} + \frac{\tilde{v}_r^2}{r\rho^2} \right) \mathrm{d}r \right]^{1/2} \right\} M_1^2 \\
&+ (2\pi L)^{1/2} \rho_\star^{1/2} \left[\int_0^a \frac{r}{\rho^2} \left(\tilde{j}_\phi^2 + \tilde{j}_z^2 \right) \mathrm{d}r \right]^{1/2} M_1 M_2 \\
&+ (2\pi L)^{1/4} \left(\int_0^a \frac{r\tilde{v}_r^4}{\eta^2} \mathrm{d}r \right)^{1/4} M_2 < 1
\end{aligned} \tag{29}$$

Here, $M_1 = M_1(a, L; \nu/\rho)$ is the imbedding constant of the compact imbedding $V_1 \longrightarrow \mathbf{L}^4(\Omega)$ (Spada & Wobig 1992, and references therein); and $M_2 = M_2(a, L; \eta)$ is the imbedding constant of the compact imbedding $V_2 \longrightarrow \mathbf{L}^4(\Omega)$.

We point out that uniqueness and stability of equilibrium (11)–(15) are very related properties. Here, we only say that, if the deviation from equilibrium (11)–(15) may have a time dependence, one can easily obtain the estimate $\mathrm{d}\left(\|\Phi\|_{\mathrm{L}}^2\right)/\mathrm{d}t \leq 2(\lambda - 1)\|\Phi\|^2$, where $\|\Phi\|_{\mathrm{L}} \equiv \left[\int_\Omega \left(\rho^{-1}|\mathbf{v}|^2 + |\mathbf{B}|^2\right) \mathrm{d}^3 x\right]^{1/2}$.

As one could expect, in condition (29) the dissipation processes manifestly appear to be stabilizing whereas the source terms appear to be destabilising. (Note, however, that ν appears to be destabilising in the terms containing the plasma density gradient; this is another sign of the fact that the non-uniformity of the plasma density deeply affects the mathematical structure of the equations.) *Condition (29) shows the manner in which they are stabilizing or destabilising*, and, in one sense, contains all the physics of the boundary-value problem we have analysed in this paper.

5. Concluding remarks

In this paper, we have formulated and analytically solved a one-dimensional boundary-value problem of dissipative plasma equilibrium in a cylinder. Viscous stresses and resistance are the dissipation processes we have taken into account, while a particle source and an externally driven electric field sustain the pressure gradient in the plasma. Plasma density, coefficients of viscosity and resistivity are given smooth functions of the radius only. The radial force balance given by the analytical equilibrium (11)–(15) has been assumed to be a good approximation as compared with that occurring in magnetic confinement devices such as the tokamak in their supposed way of operation. Moreover, a functional setting of the equations has been established and a rigorous uniqueness and nonlinear stability result for the solution (11)–(15) has been obtained; it is expressed by condition (29).

Several questions seem to deserve further consideration and analysis.

As regards the results presented in this paper, it would be interesting to plot the analytical equilibrium (11)–(15) with parameters relevant to controlled fusion research. Also, condition (29) merits further analysis. We refer to a forthcoming paper for these questions.

If the plasma is driven strongly enough, or if the dissipation processes are weak enough, the equilibrium (11)–(15) loses uniqueness and stability and a new stable equilibrium bifurcates. It would be of great interest to understand at least the chief features of the bifurcating equilibrium. In this connection, we draw attention to the very nice results obtained by Montgomery and co-workers and presented in a recent series of papers (see, especially, Agim & Montgomery 1991; Shan *et al.* 1991). We intend to address this issue for the model that we have analysed in this paper.

Moreover, as we wrote after equation (25), in a forthcoming paper we shall also present a linear stability result obtained without imposing the stringent condition (25).

Finally, we believe that a thorough computational analysis of the boundary-value problem formulated and theoretically analysed in this paper would yield valuable insight and might help explain some aspects of the macroscopic behaviour of the plasma in typical experimental conditions encountered in research on controlled thermonuclear fusion. That would require a lot of work and time, as well as the co-operation of various competences; as we believe that it would be worthwhile, we hope to pursue this aim as well.

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