INVARIANT EXPRESSION FOR THE DIVERGENCE OF BRAGINSKII'S GYROVISCOUS TENSOR FOR LOCALIZED MODES

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ABSTRACT

The divergence of Braginskii's ion gyroviscous tensor is expressed in invariant form for localized modes. The only localization assumption is that the wavelength of the modes across the magnetic surfaces is shorter than the equilibrium scale length. Therefore, localized modes of both the first kind — singular modes — and the second kind — ballooning modes — fall within the scope of the description. General toroidal equilibria are considered as well as applications to simplified equilibrium models. Applying the invariant form to expressions appropriate to ballooning and singular modes recovers several previous results.

I. INTRODUCTION

Recently, considerable attention has been paid to Braginskii's two-fluid theory [1], e.g. for ballooning modes in Refs. [2] – [5], low-n singular modes in Refs. [6] and [7], edge-localized modes (peeling and Mercier) in Refs. [8] and [9], and drift waves in Ref. [10]. Using Braginskii's two-fluid equations to investigate collisional effects is valid only for a collision-dominated plasma in which the mean free path of the particles is shorter than any characteristic length scale in the problem of interest. This is usually violated in fusion plasmas [11, 12]. In the above studies, however, Braginskii's two-fluid theory is shown to yield results in good agreement with those parts of kinetic theory which are amenable to a fluid description. For this reason, only the ion gyroviscosity is considered here. Actually, one of the central and the most complicated aspects of applying Braginskii's two-fluid theory is the treatment of the viscous tensor. Note that the gyroviscous tensor given by Braginskii is in a form decomposed into Eulerian coordinates, while for fusion studies convenient coordinates could be, for example, magnetic coordinates [13]. This is the motivation for expressing the divergence of this tensor in invariant form. With this invariant form one can easily solve difficulties induced by complicated geometry.

There are several studies treating Braginskii's gyroviscous tensor. For ballooning modes the tensor was expressed in the WKB form in Ref. [2]. For low-n singular modes, an expression with certain special assumptions can be found in Ref. [6]. Reference [10] also contains an expression for the gyroviscous tensor. But, in fact, it keeps only those terms which contain second-order derivatives of the velocity of the ion fluid, and drops many terms which are significant for localized modes. Therefore, to the authors' knowledge, a general invariant expression for the divergence of Braginskii's gyroviscous tensor is still not available.

Because the gyroviscosity effect is important only for localized modes, attention is limited here to these modes. The only localization assumption made is that the wavelength of the modes across the magnetic surfaces is shorter than the equilibrium scale length. Therefore, the expression covers localized modes of both the first kind — singular modes — and the second kind — ballooning modes. The expression is given for general toroidal configurations. In addition, several simplified equilibrium models are discussed.

Applying expression for the divergence of Braginskii's gyroviscous tensor to both ballooning and singular modes recovers several previous results.

The remaining part of the report is organized as follows: In Sec. II, the invariant expression is derived; Sec. III presents applications and the discussion.

II. INVARIANT EXPRESSION

The divergence of Braginskii's ion gyroviscous tensor $\boldsymbol{\Pi}$ was obtained in Ref. [1]. With the introduction of standard subscript notations for vectors and tensors, it can be expressed as

$$\nabla \cdot \boldsymbol{\Pi} \;=\; \partial_\alpha(\eta_3 W_{3\alpha\beta} + \eta_4 W_{4\alpha\beta}), \tag{1}$$

where

$$\eta_3 \;=\; MP/2eB, \tag{2}$$

$$\eta_4 \;=\; MP/eB, \tag{3}$$

$$W_{3\alpha\beta} \;=\; \frac{1}{2}(\delta^{\perp}_{\alpha\mu}\varepsilon_{\beta\gamma\nu} + \delta^{\perp}_{\beta\nu}\varepsilon_{\alpha\gamma\mu})H_\gamma W_{\mu\nu}, \tag{4}$$

$$W_{4\alpha\beta} \;=\; (H_\alpha H_\mu \varepsilon_{\beta\gamma\nu} + H_\beta H_\nu \varepsilon_{\alpha\gamma\mu})H_\gamma W_{\mu\nu}, \tag{5}$$

$$W_{\mu\nu} \;=\; \partial_\nu V_\mu + \partial_\mu V_\nu - \frac{2}{3}\delta_{\mu\nu}\partial_\lambda V_\lambda, \tag{6}$$

where V_λ is the velocity of the ion fluid, M is the ion mass, e represents the ion charge, P denotes the pressure of the ion fluid, $\delta^\perp_{\alpha\beta} = (\delta_{\alpha\beta} - H_\alpha H_\beta)$, $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$, otherwise zero, $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor, and $\mathbf{H} \equiv \mathbf{B}/B$, with $\mathbf{B}$ being the magnetic field.

Equation (1) needs to be linearized for linear stability analyses. In linearization Eq. (1) one of the quantities V_μ, P and B_μ must be considered as perturbed. Note that there is one derivative operator upon V_μ in $W_{\mu\nu}$. Therefore, the lowest-order terms for localized modes in Eq. (1) contain terms having second-order derivatives of perturbed velocity. But, as will be shown, the terms containing cross-field second-order derivatives of the perturbed velocity reduce to a gradient of a scalar, and hence contribute only the same-order effects as the terms containing first-order derivatives of the perturbed velocity for the parallel momentum equation and the vorticity equation, which is obtained by operating $\nabla\cdot(\mathbf{B}/B^2\times\cdots)$ on the momentum equation. Therefore, the calculation should keep the next-to-lowest-order terms, i.e. the terms which are of the same order as the terms containing first-order derivatives of the perturbed velocity. In the linearization, upper-case letters are employed to denote the equilibrium quantities (except η_3 and η_4) and lower-case letters to denote the corresponding perturbed quantities; boldface is introduced to denote vectors.

We are going to prove that in the linearization of Eq. (1) for localized modes it is sufficient to consider v_μ as a perturbed quantity. To do so, we introduce some notations. $\lambda_\perp$ and $\lambda_\wedge$ are introduced to denote the wavelengths of the modes across the magnetic surfaces and on the surfaces perpendicular to the field lines, respectively. R is used to represent the curvature radius of the field lines, and a the scale length of the equilibrium plasma pressure. It is assumed that the wavelength of the modes parallel to the field lines is longer than or of the order of R, the frequency of the modes is of the same order as the diamagnetic drift frequency of the modes ω_*, and $\lambda_\wedge$ is of the order of or larger than $\lambda_\perp$. We are interested in unstable localized modes. Therefore, the compressional Alfvén mode is assumed to be suppressed, so that

$$\nabla \cdot \boldsymbol{\xi} \sim \boldsymbol{\kappa} \cdot \boldsymbol{\xi}, \tag{7}$$

which leads to

$$\frac{\xi_\wedge}{\lambda_\wedge} \sim \frac{\xi_\perp}{\lambda_\perp}, \tag{8}$$

where $\boldsymbol{\xi}$ is the displacement of the field lines with $\xi_\wedge$ and $\xi_\perp$, these being its components on and across the magnetic surfaces, respectively. The perturbed velocity of the ion fluid $\mathbf{v}$ is ordered to be $\omega_* \boldsymbol{\xi}$. The perturbed parallel velocity $v_\parallel$ is assumed to be of the same order as or less than $\omega_* \xi_\wedge$. The equilibrium velocity V_γ is assumed to be of the order of the equilibrium diamagnetic velocity. To prove this assertion, it is sufficient to use the terms containing $(\partial_\perp P)(\partial_\perp v_\wedge)$, with $\partial_\perp$ being the derivative across the magnetic surfaces, as representative of terms containing $\partial_\mu v_\nu$ and to compare them with terms containing the perturbed pressure p and magnetic field b_μ. Thus, one has the following ordering sequences:

$$\frac{(\partial_\perp p)(\partial_\perp V_\gamma)}{(\partial_\perp P)(\partial_\perp v_\wedge)} \sim \frac{p}{P}\frac{\lambda_\wedge}{\xi_\wedge}, \tag{9}$$

$$\frac{P(\partial_\perp b_\mu)(\partial_\perp V_\gamma)}{B(\partial_\perp P)(\partial_\perp v_\wedge)} \sim \frac{b_\mu}{B}\frac{\lambda_\wedge}{\xi_\wedge}. \tag{10}$$

To proceed, the ideal-MHD equations are employed to estimate the ordering for the perturbed plasma pressure p and magnetic field b_μ. Noting that the ideal MHD adiabatic law reads $p = -\boldsymbol{\xi} \cdot \nabla P - \gamma P \nabla \cdot \boldsymbol{\xi}$ and utilizing Eqs. (7) and (8), one has p/P being either of order $(\lambda_\perp/\lambda_\wedge)(\xi_\wedge/a)$ or of order $\xi_\wedge/R$. It can also be proved that the quantities b_μ/B are either of order $(\lambda_\perp/\lambda_\wedge)(\xi_\wedge/a)$ or of order $\xi_\wedge/R$ by noting that $\mathbf{b} = \nabla \times \boldsymbol{\xi} \times \mathbf{B}$ and by employing Eqs. (7) and (8). Therefore, the ordering analyses in Eqs. (9) and (10) indicate that, when $\lambda_\wedge/R$ is of order $\lambda_\perp/a$, it is possible to ignore

p and b_μ contributions in the linearization. Alternatively, one has $(\lambda_\wedge/R) \gg (\lambda_\perp/a)$, which leads to $(\lambda_\perp/\lambda_\wedge) \ll (a/R)$. This indicates that the modes in this case are of the first kind [14]. For localized modes of the first kind with $\lambda_\wedge/R \gg (\lambda_\perp/a)$, the frequency of the modes can be proved to be lower than that of the ion acoustic wave along the magnetic field lines if the ion gyroradius is smaller than $\xi_\perp$, which is a prerequisite for using Braginskii's two-fluid theory. In this low-frequency regime, it can be proved that the plasma tends to be isobaric along the magnetic field lines and the Alfvén wave is suppressed in such a way that p/P and b_μ/B are both of order $(\lambda_\perp/\lambda_\wedge)(\xi_\wedge/a)$ [7]. Therefore, the ordering analyses in Eqs. (9) and (10) indicate that p and b_μ contributions can also be ignored in this case. As a result, V_μ in Eq. (1) can be considered as the only quantity to be perturbed.

$\boldsymbol{\pi}$ is used to denote the linearized gyroviscous tensor. Linearization of Eq. (1) yields

$$
\begin{aligned}
\nabla\cdot\boldsymbol{\pi} &= (\partial_\alpha\eta_3)w_{3\alpha\beta} + (\partial_\alpha\eta_4)w_{4\alpha\beta} + \eta_3\partial_\alpha w_{3\alpha\beta} + \eta_4\partial_\alpha w_{4\alpha\beta} \\
&= \frac{1}{2}(\partial_\alpha\eta_4)w_{3\alpha\beta} + (\partial_\alpha\eta_4)w_{4\alpha\beta} \\
&\quad + \frac{1}{4}\eta_4\partial_\alpha\left[(\delta_{\alpha\mu}\varepsilon_{\beta\gamma\nu} + \delta_{\beta\nu}\varepsilon_{\alpha\gamma\mu})H_\gamma w_{\mu\nu}\right] \\
&\quad + \frac{3}{4}\eta_4\partial_\alpha\left[(H_\alpha H_\mu\varepsilon_{\beta\gamma\nu} + H_\beta H_\nu\varepsilon_{\alpha\gamma\mu})H_\gamma w_{\mu\nu}\right]. \qquad (11)
\end{aligned}
$$

Equation (11) is now evaluated term by term, it being kept in mind that only terms containing derivatives of the velocity need be preserved. These calculations are tedious, the results being as follows:

$$
\begin{aligned}
&\frac{1}{2}(\partial_\alpha\eta_4)w_{3\alpha\beta} + (\partial_\alpha\eta_4)w_{4\alpha\beta} \\
=\ &\frac{1}{4}(\partial_\mu\eta_4)\varepsilon_{\beta\gamma\nu}H_\gamma(\partial_\nu v_\mu + \partial_\mu v_\nu) + \frac{1}{4}(\partial_\alpha\eta_4)\varepsilon_{\alpha\gamma\mu}H_\gamma(\partial_\beta v_\mu + \partial_\mu v_\beta)
\end{aligned}
$$

$$+\frac{3}{4}(\mathbf{H}\cdot\nabla\eta_4)\varepsilon_{\beta\gamma\nu}H_\gamma\partial_\nu v_\parallel+\frac{3}{4}(\partial_\alpha\eta_4)\varepsilon_{\alpha\gamma\mu}H_\beta H_\gamma\partial_\mu v_\parallel$$

$$=\frac{1}{4}\mathbf{H}\times\nabla(\nabla\eta_4\cdot\mathbf{v})+\frac{1}{4}\nabla\eta_4\cdot\nabla(\mathbf{H}\times\mathbf{v})$$

$$+\frac{1}{4}\{(\nabla\eta_4\times\mathbf{h})\times(\nabla\times\mathbf{v})+(\nabla\eta_4\times\mathbf{H}\cdot\nabla)\mathbf{v}\}+\frac{1}{4}(\nabla\eta_4\times\mathbf{H}\cdot\nabla)\mathbf{v}$$

$$+\frac{3}{4}(\mathbf{H}\cdot\nabla\eta_4)(\mathbf{H}\times\nabla v_\parallel)+\frac{3}{4}\mathbf{H}(\nabla\eta_4\cdot\mathbf{H}\times\nabla v_\parallel)$$

$$=\frac{1}{2}\nabla\eta_4\cdot\nabla(\mathbf{H}\times\mathbf{v}_\perp)+\frac{1}{2}(\nabla\eta_4\times\mathbf{H}\cdot\nabla)\mathbf{v}_\perp$$

$$-\frac{1}{4}(\mathbf{H}\cdot\nabla\eta_4)\nabla\times\mathbf{v}_\perp+(\mathbf{H}\cdot\nabla\eta_4)(\mathbf{H}\times\nabla v_\parallel)$$

$$+\mathbf{H}\left[(\nabla\eta_4\cdot\mathbf{H}\times\nabla v_\parallel)+\frac{1}{4}\nabla\eta_4\cdot(\nabla\times\mathbf{v}_\perp)\right],\qquad(12)$$

$$\partial_\alpha(\delta_{\alpha\mu}\varepsilon_{\beta\gamma\nu}H_\gamma w_{\mu\nu})$$

$$=\varepsilon_{\beta\gamma\nu}(\partial_\mu v_\nu+\partial_\nu v_\mu)\partial_\mu H_\gamma+\varepsilon_{\beta\gamma\nu}H_\gamma\partial_\mu(\partial_\mu v_\nu+\partial_\nu v_\mu-\frac{2}{3}\delta_{\mu\nu}\nabla\cdot\mathbf{v})$$

$$=\varepsilon_{\beta\gamma\nu}\partial_\mu v_\nu\partial_\mu H_\gamma-\nabla\times[(\mathbf{v}\cdot\nabla)\mathbf{H}]+\mathbf{H}\times\nabla^2\mathbf{v}+\frac{1}{3}\mathbf{H}\times\nabla(\nabla\cdot\mathbf{v})$$

$$=\frac{1}{2}\nabla^2(\mathbf{H}\times\mathbf{v})+\frac{1}{2}\mathbf{H}\times\nabla^2\mathbf{v}-\nabla\times[(\mathbf{v}\cdot\nabla)\mathbf{H}]+\frac{1}{3}\mathbf{H}\times\nabla(\nabla\cdot\mathbf{v}),\qquad(13)$$

$$\partial_\alpha(\delta_{\beta\nu}\varepsilon_{\alpha\gamma\mu}H_\gamma w_{\mu\nu})$$

$$=\varepsilon_{\alpha\gamma\mu}(\partial_\mu v_\beta+\partial_\beta v_\mu)\partial_\alpha H_\gamma+\varepsilon_{\alpha\gamma\mu}H_\gamma\partial_\alpha(\partial_\mu v_\beta+\partial_\beta v_\mu-\frac{2}{3}\delta_{\beta\mu}\nabla\cdot\mathbf{v})$$

$$= (\nabla \times \mathbf{H} \cdot \nabla)\mathbf{v} + \nabla(\nabla \times \mathbf{H} \cdot \mathbf{v}) + H_\gamma \partial_\beta(\varepsilon_{\alpha\gamma\mu}\partial_\alpha v_\mu) + \frac{2}{3}(\mathbf{H} \times \nabla)\nabla \cdot \mathbf{v}$$

$$= \mathbf{H} \times \nabla^2\mathbf{v} + (\nabla \times \mathbf{H} \cdot \nabla)\mathbf{v} + \nabla(\nabla \times \mathbf{H} \cdot \mathbf{v})$$

$$- (\mathbf{H} \cdot \nabla)\nabla \times \mathbf{v} - \frac{1}{3}(\mathbf{H} \times \nabla)\nabla \cdot \mathbf{v}, \tag{14}$$

$$\partial_\alpha(H_\alpha H_\mu \varepsilon_{\beta\gamma\nu} H_\gamma w_{\mu\nu}) = \partial_\alpha(H_\alpha H_\mu H_\gamma \varepsilon_{\beta\gamma\nu}\partial_\nu v_\mu)$$

$$= (\nabla \cdot \mathbf{H} + \mathbf{H} \cdot \nabla)(\mathbf{H} \times \nabla v_\parallel), \tag{15}$$

$$\partial_\alpha(H_\beta H_\nu \varepsilon_{\alpha\gamma\mu} H_\gamma w_{\mu\nu})$$

$$= \varepsilon_{\alpha\gamma\mu}\partial_\alpha(H_\beta H_\nu H_\gamma)\partial_\mu v_\nu + H_\beta H_\gamma \varepsilon_{\alpha\gamma\mu}(\mathbf{H} \cdot \nabla)\partial_\alpha v_\mu + \varepsilon_{\alpha\gamma\mu} H_\beta H_\gamma(\partial_\alpha H_\nu)(\partial_\nu v_\mu)$$

$$= \mathbf{H}\varepsilon_{\alpha\gamma\mu} H_\gamma \partial_\alpha(H_\nu \partial_\mu v_\nu) + \mathbf{H}[\mathbf{H} \cdot (\nabla \times \mathbf{H} \cdot \nabla \mathbf{v})]$$

$$+ [(\mathbf{H} \times \nabla v_\parallel) \cdot \nabla]\mathbf{H} - \mathbf{H}\mathbf{H} \cdot [(\mathbf{H} \cdot \nabla)(\nabla \times \mathbf{v})] + \mathbf{H}\nabla \cdot [(\mathbf{H} \times \mathbf{v} \cdot \nabla)\mathbf{H}]$$

$$= \mathbf{H}\Big\{(\mathbf{H} \cdot \nabla \times \mathbf{v})(\nabla \cdot \mathbf{H}) - \mathbf{H} \cdot \nabla \times [(\mathbf{H} \cdot \nabla)\mathbf{v}] + \nabla \times \mathbf{H} \cdot \nabla v_\parallel + \nabla \cdot [(\mathbf{H} \times \mathbf{v} \cdot \nabla)\mathbf{H}]\Big\}$$

$$+ [(\mathbf{H} \times \nabla v_\parallel) \cdot \nabla]\mathbf{H}. \tag{16}$$

Inserting the results of Eqs. (12) – (16) into Eq. (1) finally yields

$$
\begin{aligned}
\nabla\cdot\boldsymbol{\pi} \;=\; & \frac{1}{2}\nabla\eta_4\cdot\nabla(\mathbf{H}\times\mathbf{v}_\perp)+\frac{1}{2}(\nabla\eta_4\times\mathbf{H}\cdot\nabla)\mathbf{v}_\perp \\
& -\frac{1}{4}(\mathbf{H}\cdot\nabla\eta_4)\nabla\times\mathbf{v}_\perp+(\mathbf{H}\cdot\nabla\eta_4)(\mathbf{H}\times\nabla v_\parallel) \\
& +\mathbf{H}\left[(\nabla\eta_4\cdot\mathbf{H}\times\nabla v_\parallel)+\frac{1}{4}\nabla\eta_4\cdot(\nabla\times\mathbf{v}_\perp)\right] \\
& +\frac{\eta_4}{4}\left\{\frac{1}{2}\nabla^2(\mathbf{H}\times\mathbf{v}_\perp)+\frac{3}{2}\mathbf{H}\times\nabla^2\mathbf{v}-\nabla\times[(\mathbf{v}_\perp\cdot\nabla)\mathbf{H}]\right. \\
& +(\nabla\times\mathbf{H}\cdot\nabla)\mathbf{v}_\perp+\nabla(\nabla\times\mathbf{H}\cdot\mathbf{v})-(\mathbf{H}\cdot\nabla)\nabla\times\mathbf{v}_\perp+3(\nabla\cdot\mathbf{H})(\mathbf{H}\times\nabla v_\parallel) \\
& +4\mathbf{H}\times\nabla(\mathbf{H}\cdot\nabla v_\parallel)+3[(\mathbf{H}\times\nabla v_\parallel)\cdot\nabla]\mathbf{H}+5(\mathbf{H}\cdot\nabla\mathbf{H})\times\nabla v_\parallel \\
& +\mathbf{H}\Big[3(\mathbf{H}\cdot\nabla\times\mathbf{v}_\perp)(\nabla\cdot\mathbf{H})-3\mathbf{H}\cdot\nabla\times(\mathbf{H}\cdot\nabla)\mathbf{v}_\perp \\
& \left.+7\mathbf{H}\times(\mathbf{H}\cdot\nabla\mathbf{H})\cdot\nabla v_\parallel+3\nabla\cdot(\mathbf{H}\times\mathbf{v}_\perp\cdot\nabla\mathbf{H})\Big]\right\}. \qquad (17)
\end{aligned}
$$

III. APPLICATIONS AND DISCUSSION

The invariant form of Braginskii's gyroviscous tensor for localized modes has been given in Eq. (17). This invariant form makes it easy to evaluate the gyroviscosity effect on localized modes of different kinds in different geometries.

Note that the only two terms containing second-order perpendicular derivatives of the velocity of the ion fluid in Eq. (17) can be combined as follows:

$$
\begin{aligned}
& \frac{\eta_4}{8}\nabla^2\mathbf{H}\times\mathbf{v}+\frac{3\eta_4}{8}\mathbf{H}\times\nabla^2\mathbf{v} \\
= \;& \nabla\left(\frac{\eta_4}{2}\mathbf{H}\cdot\nabla\times\mathbf{v}\right)+(\text{terms containing only first-order derivatives of } \mathbf{v}).
\end{aligned}
$$

This shows that the terms containing second order derivatives in Eq. (17) can be reduced to a gradient of a scalar except for terms containing first-order derivatives. If only $\mathbf{B} \cdot (\nabla \cdot \boldsymbol{\pi})$ and $\nabla \cdot \frac{\mathbf{B}}{B^2} \times \nabla \cdot \boldsymbol{\pi}$ enter into the eigenmode equations, this gradient makes contributions of the same order as the terms containing first-order derivatives. That is why we make the calculations on the terms containing first order-derivatives of $\mathbf{v}$.

We first consider equilibria with $(|\nabla P|/P) \gg (\nabla B_\nu/B)$, which is assumed in many cylindrical and slab models. In this case, only the terms containing derivatives of P and second-order derivatives of the velocity of the ion fluid need be kept. The perpendicular component of the divergence of the gyroviscous tensor in Eq. (17) is reduced to

$$(\nabla \cdot \boldsymbol{\pi})_\perp = \frac{MP}{2eB}\nabla^2(\mathbf{H} \times \mathbf{v}) + \frac{M}{2eB}\nabla P \cdot \nabla(\mathbf{H} \times \mathbf{v}) + \frac{M}{2eB}(\nabla P \times \mathbf{H} \cdot \nabla)\mathbf{v}. \quad (18)$$

This result completely coincides with that in Ref. [15]. Note that Eq. (18) is also true of the edge localized modes [8, 9].

Next, we turn to the ballooning modes. Replacing the nabla operators in Eq. (17) by $i\mathbf{k}$, with $\mathbf{k}$ being the perpendicular wave vector, and introducing the perturbed electrostatic potential ϕ and the perturbed pressure of the ion fluid p to represent the perpendicular velocity [2],

$$\mathbf{v}_\perp = \frac{\mathbf{B} \times \nabla(Ne\phi + p)}{NeB^2},$$

with N being the density of the ion fluid, one can express the divergence of Braginskii's gyroviscous tensor for ballooning modes as follows:

$$\mathbf{B} \cdot (\nabla \cdot \boldsymbol{\pi}) = -i\omega^T_{*i} NMBv_\parallel + i(\omega_B + 3\omega_\rho)NMBv_\parallel + b_i \mathbf{B} \cdot \nabla(p + Ne\phi) + \frac{(p + Ne\phi)}{2B}\mathbf{B} \cdot \nabla(Bb_i) \quad (19)$$

and

$$\nabla \cdot \left[\frac{\mathbf{B}}{B^2} \times (\nabla \cdot \pi)\right]$$

$$= -i\omega_{*i}^T Ne\frac{b_i}{2}\left(\frac{e\phi}{T_i}+\frac{p}{P}\right) + i(\omega_\rho + 3\omega_B)Ne\frac{b_i}{2}\left(\frac{e\phi}{T_i}+\frac{p}{P}\right)$$

$$+\frac{NeB}{2}\mathbf{B}\cdot\nabla\left(\frac{b_i v_\parallel}{B^2}\right), \qquad (20)$$

where

$$\omega_{*i}^T = \omega_*(1+\eta_i),$$

$$\omega_* = \frac{T_i}{NeB^2}\mathbf{B}\times\nabla N\cdot\mathbf{k},$$

$$\eta_i = \frac{\partial \ln T_i}{\partial \ln N},$$

$$b_i = \frac{k^2 M T_i}{e^2 B^2},$$

$$\omega_\rho = \left[\mathbf{k}\times\mathbf{B}\cdot\nabla\left(P+\frac{1}{2}B^2\right)\right]\frac{T_i}{eB^4},$$

$$\omega_B = (\mathbf{k}\times\mathbf{B}\cdot\nabla B)\frac{T_i}{eB^3},$$

and T_i is the temperature of the ion fluid. Equations (19) and (20) are exactly the results given in Ref. [2]. When the aspect ratio is large, only the first term in Eq. (20) remains, and then the result used in Ref. [5] is recovered.

For localized modes of the first kind, one has $\lambda_\perp \ll \lambda_\wedge$. This leads to $v_\perp = (v^S/|\nabla S|) \ll (v^u|\nabla S|/B) = v_\wedge$, where S is the volume inside a magnetic surface, u is defined by $\mathbf{B} = \nabla S \times \nabla u$, $v^S = \mathbf{v}\cdot\nabla S$ and $v^u = \mathbf{v}\cdot\nabla u$. Taking into account this behavior of the modes, one can, after laborious calculation, obtain [7]

$$\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}) = -i\omega_{*i}^{T}NMBv_{\parallel} - 2\frac{BMP}{eP'}(\mathbf{B}\cdot\nabla\sigma)\frac{\partial v_{\parallel}}{\partial x} - \frac{MP}{2eB}\mathbf{B}\cdot\nabla\left(\frac{|\nabla S|^2}{B}\right)\frac{\partial v^u}{\partial x} \tag{21}$$

and

$$\nabla\cdot\left[\frac{\mathbf{B}}{B^2}\times(\nabla\cdot\boldsymbol{\pi})\right] = i\omega_{*i}^{T}NM\frac{|\nabla S|^2}{B^2}\frac{\partial v^u}{\partial x} + \frac{MP}{eB^2P'}|\nabla S|^2(\mathbf{B}\cdot\nabla\sigma)\frac{\partial^2 v^u}{\partial x^2} - \frac{1}{2}\frac{MP}{eB^3}|\nabla S|^2\mathbf{B}\cdot\nabla\frac{\partial^2 v_{\parallel}}{\partial x^2} - \frac{1}{2}\frac{MP}{eB^3}(\mathbf{B}\cdot\nabla|\nabla S|^2)\frac{\partial^2 v_{\parallel}}{\partial x^2} + 2\frac{MP}{eB^4}(\mathbf{B}\cdot\nabla B)|\nabla S|^2\frac{\partial^2 v_{\parallel}}{\partial x^2}, \tag{22}$$

where $\sigma = \mathbf{J}\cdot\mathbf{B}/B^2$ and $x = S - S_0$ with S_0 labeling the reference rational magnetic surface. Noting that $\mathbf{B}\cdot\nabla(p + Ne\phi) = 0$ for the localized modes of the first kind, one can see that Eq. (19) coincides with Eq. (21), as well as Eq. (20) with Eq. (22) when the WKB ansatz is also made in Eq. (21) and Eq. (22).

In all these examples we see that directly using the invariant form in Eq. (17) is more convenient than using the form decomposed into Eulerian coordinates in Ref. [1]. Note that Eq. (17) is obtained under the sole localization assumption that the wavelength of the modes across the magnetic surfaces is shorter than the scale of the equilibrium. Therefore, Eq. (17) is more general than the results in Ref. [2] for ballooning modes and the results in Refs. [6] and [7] for localized modes of the first kind. Although we do not present any practical applications of Braginskii's two-fluid theory, because treating the gyroviscous tensor is one of the most complicated aspects of such applications, the tractable expression of the divergence of the tensor in Eq. (17) can be considered to have overcome one of the obstacles to applying Braginskii's two-fluid theory.

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