MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Paraxial WKB solution of a scalar wave equation

G. V. Pereverzev

IPP 4/260 April 1993

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Paraxial WKB solution of a scalar wave equation

G. V. Pereverzev

I. V. Kurchatov Institute of Atomic Energy, Moscow, Russia

Abstract

An asymptotic method of solving a scalar wave equation in inhomogeneous media is developed. This method is an extension of the WKB method to the multidimensional case. It reduces a general wave equation to a set of ordinary differential equations similar to that of the eikonal approach and includes the latter as a particular case. However, the WKB method makes use of another kind of asymptotic expansion and, unlike the eikonal approach, describes the wave properties, i.e. diffraction and interference. At the same time, the three-dimensional WKB method is more simple for numerical treatment because the number of equations is less than in the eikonal approach. The method developed may be used for a calculation of wave fields in problems of RF heating, current drive and plasma diagnostics with microwave beams.

This work was performed during a stay of the author at
Max-Planck-Institut für Plasmaphysik,
Garching bei München

1 Introduction

The ray method or geometric optics is the most powerful and widespread method of solving the wave equation in the short-wavelength limit. It is used in numerous applications of optics, seismology, physics of fluids and solids, quantum mechanics, plasma physics and many other fields [1-3]. The ray method makes use of an asymptotic expansion of the solution sought and reduces the wave equation to an infinite set of coupled equations for successive terms of the expansion. The zero order term is known as the eikonal approach. It describes the phase behaviour in space which represents the most rapid variation of the wave field. It also gives rise to the ray or geometric-optics description in the narrow sense of the latter term. The first order equation describes the field amplitude evolution along the ray trajectory. This equation has the important corollary that the flux energy is directed along the rays. Only the leading order equation and this consequence of the first order are mainly used in practice. This reduced approach is usually called the ray tracing or geometric-optics method. It describes the corpuscular or ray properties of a propagating wave package.

In many cases of practical interest this method provides almost the only possibility of obtaining a solution. Therefore, in plasma phisics geometric optics is widely used, but, in contrast to conventional optics it is in most of cases far from being really justified and is sometimes clearly irrelevant. For example, as discussed in [4] description of ballooning instabilities of tokamak plasmas is impossible without modification of the eikonal representation. In many other cases of practical interest a wave package is localized in a small enough region and then the wave phenomena which exhibit energy flow transverse to the ray cannot be neglected. Thus in [5] it was shown that diffraction is significant in the lower-hybrid current drive problem. In [6] it was shown that sometimes the ray approximation fails even in the electron cyclotron range of frequencies.

The energy flow transverse to the rays, which has to be included into consideration, appears only on account of the higher order terms of the asymptotic expansion. However, the higher order equations of the ray method are very seldom used in practice because of their complexity. The wave properties of the propagating oscillations are usually studied by the quasi-optic approach. This was first introduced as the parabolic-equation technique by Fock and Leontovich [7]. This technique retains the wave description across the ray direction while using the eikonal approach along it. Profound developments of the parabolic-equation technique were made by Babič and Buldyrev [8] and Maslov [9]. Some applications of the parabolic equation to plasma physics are described in the review paper [10]. The same physical ideas are used in the concept of the complex eikonal by Choudhary and Felsen [11]. Mathematically, the ray method and the parabolic equation differ in that they use different kinds of asymptotic expansion with respect to the small parameter $\lambda/L \ll 1$, which is the ratio of a characteristic wavelength λ to a characteristic medium inhomogeneity size L.

In this paper a method is developed which can be regarded as an application of the parabolic equation to the propagation of narrow wave beams or to eigenfunctions of the "bouncing ball" type [8]. Whereas the ray method uses an asymptotic expansion with respect to the integer powers of the small parameter λ/L, in our case the half-integer powers of the same parameter are used. Accordingly, the method considered here can also be viewed as an extension of WKB technique to the multidimensional case. For the reasons presented below we call this approach the paraxial WKB (pWKB) method. The pWKB method includes conventional ray tracing as a particular case, but, it leads to a final set of equations which is different from that of other methods. The approach combines the simplicity of ray tracing with a description of the wave properties, i.e. diffraction and interference. Moreover, in spite of its broader applicability the pWKB method is even more suited to numerical treatment because fewer equations have to be solved in the pWKB method than in the ray tracing technique.

In Section 2 of this paper the main wave equation to be solved is formulated. The ray tracing technique and its relation to the pWKB method is briefly discussed. The derivation of the main equations of the pWKB method is described in Section 3. It comprises two successive steps: asymptotic expansion with respect to the small parameter $\sqrt{\lambda/L} \ll 1$ and then paraxial expansion into Taylor series. The first step gives the trajectory of the centre of gravity of a wave packet. This trajectory coincides with the geometric-optics ray and is described by the conventional Hamiltonian set of equations. Another small parameter $\Lambda/L \ll 1$, where Λ is the wave beam width, is used in the subsequent step. As a result, the second order partial differential wave equation is reduced to a set of first order ordinary differential equations in terms of the same Hamiltonian function as in the geometric optics. This set of equations is discussed in detail in Section 4. It describes the geometrical frame of the wave package characterized by such average quantities as the central ray trajectory, the beam width and the wave front curvature. This frame does not depend upon the amplitude distribution across the beam and is common to a family of wave beams of different transverse structure. The equation for the amplitude is formulated and solved in quadratures in Section 5. It is shown in Section 6 that the pWKB method gives a general solution to the wave equation. Finally, in the Appendix the pWKB technique is described by means of the advanced apparatus of the tensor calculus, which allows very compact and therefore very transparent representation of the pWKB method.

2 Eikonal approximation of the ray method

2.1 Wave equation and short wavelength ordering

Let us consider a field of monochromatic waves $e^{-i\omega t}\Phi(\vec{r})$ and suppose that the wave amplitude is described by a scalar time-independent wave equation of the form

$$\operatorname{div}(\hat{\varepsilon}\nabla\Phi) + \frac{\omega^2}{c^2}N^2(\vec{r})\Phi = 0. \tag{1}$$

Equation (1) represents a rather general form of wave equation and covers a number of cases of practical interest. For instance, if the tensor $\hat{\varepsilon}$ is unitary, then Eq. (1) is a Helmholtz equation:

$$\Delta\Phi + \frac{\omega^2}{c^2}N^2(\vec{r})\Phi = 0. \tag{2}$$

If $N^2(\vec{r}) \equiv 0$ and $\varepsilon^{\alpha}_{\beta}(\vec{r})$ are components of the dielectric tensor $\hat{\varepsilon}$, then Eq. (1) describes electrostatic oscillations of cold plasmas and takes the form

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial X^\alpha}\sqrt{g}g^{\alpha\gamma}\varepsilon^{\beta}_{\gamma}\frac{\partial\Phi}{\partial X^\beta} = 0. \tag{3}$$

Here and in what follows we adopt the summation convention: repeated Greek indices $\alpha, \beta, \gamma, \ldots$ are to be summed from 1 to 3. In addition, $g^{\alpha\beta}$ are the components of the contravariant fundamental tensor $g^{\alpha\beta} = \nabla X^\alpha \cdot \nabla X^\beta$ in the curvilinear coordinate system $\{X^\alpha = X^\alpha(\vec{r})\}$. g is inverse to the determinant of the matrix $g^{\alpha\beta}$, $g = 1/\det|g^{\alpha\beta}|$.

In general, all the coefficients of Eq. (1) are space dependent functions. We seek a solution of Eq. (1) taking advantage of the large parameter

$$\kappa = \omega L/c \gg 1, \tag{4}$$

where $L = \min\left\{|\nabla \ln |\varepsilon^{\beta}_{\alpha}||^{-1},\ |\nabla \ln |N||^{-1}\right\}$ is a characteristic medium inhomogeneity length. In what follows we mainly use the dimensionless space variables $x^\alpha = X^\alpha/L$.

2.2 Debye asymptotic expansion, eikonal approximation

The conventional technique of solving Eq. (1) in the short-wavelength limit $\kappa \gg 1$ is the ray approach based on an expansion of the solution in an asymptotic series with respect to integer powers of κ^{-1} (Debye expansion):

$$\Phi(\vec{r}) = \exp(i\kappa S(\vec{r}))\sum_{n=0}^{\infty}\frac{\Phi_n(\vec{r})}{(i\kappa)^n}. \tag{5}$$

Substitution of this expansion in the wave equation (1) gives to the lowest order the eikonal equation, which determines the eikonal function $S = S(\vec{r})$:

$$H \stackrel{\text{def}}{=} \frac{1}{2}\varepsilon^{\alpha\beta}S_\alpha S_\beta - N^2 = 0. \tag{6}$$

Here $S_\alpha = \partial S/\partial x^\alpha$ are covariant components of the vector $L\nabla S$ and the tensor

$$\varepsilon^{\alpha\beta} = g^{\alpha\gamma}\varepsilon^{\beta}_{\gamma} + g^{\beta\gamma}\varepsilon^{\alpha}_{\gamma} \tag{7}$$

is real when the tensor $\varepsilon^{\alpha}_{\beta}$ is Hermitean (the proof is given in Section 5). The Hamiltonian function $H = H(x^\alpha, S_\alpha)$ as introduced by Eq. (6) will play a principal part in our subsequent investigations.

2.3 Ray tracing

The eikonal equation (6) is a partial differential equation of the Hamilton-Jacobi type. It may be solved by reduction to the Hamiltonian set of ordinary differential equations

$$\frac{dq^\alpha}{dt} = \frac{\partial H}{\partial S_\alpha}, \tag{8}$$

$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha}, \tag{9}$$

where the 6 functions

$$x^\alpha = q^\alpha(t), \qquad S_\alpha = p_\alpha(t) \tag{10}$$

give the parametric representation of a ray trajectory in 6D phase space. The set of equations (8)-(9) requires a subsidiary initial condition, which can be given by prescribing the shape of the phase front $S(\vec{r}) = \text{Const}$. Then the eikonal function S can be determined along all trajectories (10) by integration of the equation

$$\frac{dS}{dt} = p_\alpha \frac{dq^\alpha}{dt}. \tag{11}$$

The set of equations (8), (9), (11) is widely used in numerous applications. It is known as the eikonal approach or the ray tracing technique. The next approximation of the ray approach gives the transport equation for the wave amplitude [1]. In accordance with this equation the wave energy propagates strictly along the ray trajectories. No energy flow across rays exists within the accuracy of up to $O(\kappa^{-1})$ of the ray approach. Therefore, such wave phenomena as diffraction and interference are described in the next approximations only. However, higher order equations of the ray method are practically never used because of their complexity.

It follows from the ray consideration that the Hamiltonian function H determines the ray or corpuscular properties of wave packet propagation. It is far from obvious that the Hamiltonian also affords information about the wave properties of the solution of Eq. (1). It will be further shown that, whereas the first derivatives of the Hamiltonian describe propagation of the maximum of the wave packet, the second derivatives describe diffractive broadening of the wave packet. To this end, a new sort of asymptotic expansion differing from Eq. (5) is introduced in the next section.

3 Paraxial WKB approach

3.1 Short-wavelength asymptotic expansion

The physical basis of the method discussed here is the concept of taking the functions of the parabolic cylinder as a basis for expanding an approximate solution rather than a Fourier series with respect to plane waves as in the eikonal approach. It will be seen that, mathematically, this results in an asymptotic expansion with respect to half-integer powers $\kappa^{-n/2}$ of the small parameter κ^{-1}, rather than integer powers κ^{-n} as in the Debye expansion (5). The asymptotic expansion considered here is of the same type as in the quasiclassical or WKB approach [12]. Therefore, the method obtained can be regarded as an extension of the WKB method to the multidimensional case. We call it the paraxial WKB (pWKB) method because the essential part of its derivation is based on a paraxial expansion.

A particular solution to the wave equation (1) will be sought in the form

$$\Phi_{lm}(\vec{r}) = A^{lm}\varphi_l(\sqrt{\kappa}u)\varphi_m(\sqrt{\kappa}v)\exp(i\kappa S), \qquad \text{(no sum on } l \text{ and } m), \tag{12}$$

where $S = S(\vec{r})$, $u = u(\vec{r})$, $v = v(\vec{r})$, $A^{lm} = A^{lm}(\vec{r})$ are unknown functions to be determined. The first three of them are also assumed to be real. $\varphi_l(\xi)$ are the functions of the parabolic cylinder which satisfy the equation

$$\varphi_l''(\xi) + (2l+1)\varphi_l(\xi) - \xi^2\varphi_l(\xi) = 0 \tag{13}$$

and can be expressed in terms of the Hermite polynomials $H_l(\xi)$:

$$\varphi_l(\xi) = e^{-\xi^2/2}H_l(\xi)/\sqrt{\pi^{1/2}2^l l!}. \tag{14}$$

Substitution from Eq. (12) into the wave equation (1) results in

$$\begin{aligned} &-A^{lm}\varphi_l\varphi_m\left[\tfrac{1}{2}\varepsilon^{\alpha\beta}\left(S_\alpha S_\beta - u^2u_\alpha u_\beta - v^2 v_\alpha v_\beta\right) - N^2\right] \\ &\quad + i\kappa^{-1/2}A^{lm}\left(\varphi_l'\varphi_m\varepsilon^{\alpha\beta}S_\alpha u_\beta + \varphi_l\varphi_m'\varepsilon^{\alpha\beta}S_\alpha v_\beta\right) \\ &\quad + i\kappa^{-1}\varphi_l\varphi_m\mathcal{L}[A^{lm}] + \kappa^{-1}A^{lm}\varphi_l'\varphi_m'\varepsilon^{\alpha\beta}u_\alpha v_\beta + O(\kappa^{-3/2}) = 0. \end{aligned} \tag{15}$$

Here

$$\begin{aligned} \mathcal{L}[A^{lm}] \overset{\text{def}}{=}\; & \varepsilon^{\alpha\beta}S_\beta\frac{\partial A^{lm}}{\partial x^\alpha} + \frac{A^{lm}}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}g^{\alpha\gamma}\varepsilon^\beta_\gamma S_\beta\right) \\ & + iA^{lm}\varepsilon^{\alpha\beta}\left[\left(l+\tfrac{1}{2}\right)u_\alpha u_\beta + \left(m+\tfrac{1}{2}\right)v_\alpha v_\beta\right] \end{aligned} \tag{16}$$

and the subscripts α, β denote partial derivatives with respect to the correspondent spatial coordinate, i.e.

$$S_\alpha = \frac{\partial S}{\partial x^\alpha}, \qquad u_\alpha = \frac{\partial u}{\partial x^\alpha}, \qquad v_\alpha = \frac{\partial v}{\partial x^\alpha}. \tag{17}$$

To satisfy Eq. (15), it is not sufficient to annul the coefficients for different powers of κ because the large parameter κ is also included in the argument of the functions φ_l. Therefore, we require

$$\varepsilon^{\alpha\beta}\left(S_\alpha S_\beta - u^2 u_\alpha u_\beta - v^2 v_\alpha v_\beta\right) = 2N^2, \tag{18}$$

$$\varepsilon^{\alpha\beta} S_\alpha u_\beta = \varepsilon^{\alpha\beta} S_\alpha v_\beta = \varepsilon^{\alpha\beta} u_\alpha v_\beta = 0, \tag{19}$$

$$\mathcal{L}[A^{lm}] = 0. \tag{20}$$

The significant feature of the derivation is the assumption that all the functions $S(\vec{r})$, $u(\vec{r})$, $v(\vec{r})$, $A^{lm}(\vec{r})$ have the same characteristic size of spatial variation as the medium inhomogeneity length L:

$$L = \min\left(\frac{|\varepsilon^\beta_\alpha|}{|\nabla\varepsilon^\beta_\alpha|}, \frac{|N|}{|\nabla N|}, \frac{|A^{lm}|}{|\nabla A^{lm}|}, \frac{|S|}{|\nabla S|}, \frac{|u|}{|\nabla u|}, \frac{|v|}{|\nabla v|}\right). \tag{21}$$

So a faster scale of variation is included in Eq. (15) via the functions of the parabolic cylinder φ_l only. It is also seen that in accordance with Eq. (13) φ_l'' (and hence φ_l') is of the same order as φ_l. However, $\partial\varphi_l/\partial x^\alpha$ is of order $\kappa^{1/2}\varphi_l$. During the derivation of Eq. (15) with allowance for Eq. (13) the quantity φ_l'' was split into two terms. With this formal ordering the last term on the left-hand side of Eq. (13) is attributed to the leading order in Eq. (15), while the second term of Eq. (13) appears in the second order of Eq. (15). Such a partition seems to be artificial, but it has strong physical reasons, which will be discussed in Section 5. Here we note only that Eqs. (18)-(19) do not depend on l and m and determine the Gaussian backbone, which gives the coarse structure of the wave packet, while Eq. (20) describes the amplitude distribution over the beam cross-section in more detail.

It is instructive to rewrite this set of equations in terms of the Hamiltonian function H given by Eq. (6). To this end, we introduce the notations

$$H_U = u_\alpha \frac{\partial H}{\partial S_\alpha}, \qquad H_V = v_\alpha \frac{\partial H}{\partial S_\alpha},$$
$$H_{UU} = u_\alpha u_\beta \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta}, \qquad H_{UV} = u_\alpha v_\beta \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta}, \qquad H_{VV} = v_\alpha v_\beta \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta}. \tag{22}$$

Equations (18)-(19) now take the form

$$H = \frac{1}{2}u^2 H_{UU} + \frac{1}{2}v^2 H_{VV}, \tag{23}$$

$$H_U = H_V = 0, \tag{24}$$

$$H_{UV} = 0. \tag{25}$$

The system (23)-(25) constitutes not merely a redesignation of Eqs. (18)-(19). It is valid for more general Hamiltonians than that given by Eq. (6). In particular, a much more complicated case of electromagnetic waves described by a set of Maxwell equations can also be represented in the form (23)-(25). The treatment presented in the following makes use only of the general form of Eqs. ((23)-(25) and does not use the specific expression for the Hamiltonian function unless a statement to the contrary is explicitly made.

The system (20), (23)-(25) is a system of 5 equations in 4 unknown functions $S = S(\vec{r})$, $v = v(\vec{r})$, $u = u(\vec{r})$ and $A^{lm} = A^{lm}(\vec{r})$. Nevertheless, the system is consistent and is, moreover, underdetermined; a solution exists and still allows a great deal of arbitrariness. The wave amplitude A^{lm} is included solely in Eq. (20). This equation is somewhat different from that describing the transport of the wave amplitude in the ray approach. However, Eq. (20) can also be reduced to a linear ordinary differential equation and then solved in a similar way. This will be considered in Section 5.

The set of equations (23)-(24) is the same as obtained in [11]. The effective method of solving this set of equations is described there. In this method the property of Eq. (30) is used and the so-called orthogonal trajectories formed by curves $S(\vec{r}) = \mathrm{Const}$ are found. In what follows, however, we shall use another technique which is more appropriate to the case under consideration.

3.2 Reference ray

Note that the set of equations (23)-(25) has the solution

$$x^\alpha = q^\alpha(t), \qquad S_\alpha = p_\alpha(t), \qquad v(\vec{r}) \equiv u(\vec{r}) \equiv 0, \tag{26}$$

where the functions $q^\alpha(t)$ and $p_\alpha(t)$ are determined by the Hamiltonian set of equations (8)-(9). On substitution from Eqs. (26) Eqs. (24)-(25) are fulfilled identically and Eq. (20) coincides with the amplitude transport equation of the eikonal approach. This means that the eikonal approach is included in the solution (12) as a particular case when the initial conditions given on some spatial surface Σ are consistent with Eqs. (26). In the general case, $u(\vec{r}) \not\equiv 0$ and $v(\vec{r}) \not\equiv 0$ and Eqs. (26) do not give a solution of system (23)-(25) any longer. However, the first two of Eqs. (26) still represent an approximate solution of Eqs. (23)-(25) in the vicinity of the spatial curve $u(\vec{r}) = v(\vec{r}) = 0$. It is clear that this curve is a characteristic (ray) of Eq. (1). So each ray generates a family of solutions such that $v = u = 0$ on the ray, but $u \neq 0$ and $v \neq 0$ outside this ray. To illuminate the difference between such solutions and the eikonal approach, let us consider the behaviour of the wave amplitude $|\Phi|$ of Eq. (9). To this end, we calculate

$$\nabla|\Phi_{lm}| = \nabla\left[|A^{lm}|\varphi_l(\sqrt{\kappa}u)\varphi_m(\sqrt{\kappa}v)\right] = \sqrt{\kappa}|A^{lm}|\left[\varphi_l'\nabla u + \varphi_m'\nabla v\right] + o(\kappa^{1/2}). \tag{27}$$

Equation (27) shows that ∇u and ∇v are the directions of the fastest decay of the wave amplitude $|\Phi_{lm}|$. Let us now introduce a quantity

$$V^\alpha = \frac{\partial H}{\partial S_\alpha}. \tag{28}$$

Equation (28) gives contravariant components of the vector $\vec{V}$, which is collinear to a vector of the group velocity. In accordance with Eq. (8) $\vec{V}$ is tangential to the ray

$$\begin{cases} x^\alpha = q^\alpha(t), \\ S_\alpha = p_\alpha(t). \end{cases} \tag{29}$$

Equation (29) coincides in form with Eq. (10). However, the difference is that Eq. (10) describes a manifold of all geometric-optics rays and Eq. (29) gives a single ray which coincides with a skew curve $u(\vec{r}) = v(\vec{r}) = 0$. Moreover, in contrast to Eq. (26) we assume that outside this curve $u(\vec{r}) \neq 0$ and $v(\vec{r}) \neq 0$. Equation (24) can now be written as

$$H_U = \vec{V}\nabla u = 0, \qquad H_V = \vec{V}\nabla v = 0. \tag{30}$$

Consequently, both vectors ∇u and ∇v are orthogonal to the vector of the group velocity $\vec{V}$. Equation (27) shows that Eq. (12) describes a wave beam with exponential decay outside the skew curve (29). This curve is, therefore, the spatial axis of the wave beam. To be more precise, this curve describes the trajectory of the centre of gravity of the wave packet amplitude [13] and plays a basic part in the following consideration. It is called **the reference ray** and denoted as $\Re$. By virtue of Eq. (27) the wave solution (12) is located in the vicinity of the reference ray (29). It also follows that the wave energy propagates along this ray and the vector $\vec{V}$ retains the same meaning as in the eikonal approach.

3.3 Wave packet description

Let us introduce the characteristic length of the amplitude decay $\Lambda = \min(|\Phi/\nabla\Phi|)$. It is physically evident that the case $\lambda \approx L$ as well as the case $\lambda \approx \Lambda$ have to be treated numerically when exact solution cannot be found. Therefore, the inequality $\lambda \ll \Lambda, L$ can be regarded as a natural restriction on any asymptotic approach.

We now discuss the relation between the other two quantities, Λ and L. The ray method is implicitly based on the idea that the plane wave as exact solution to the wave equation in a homogeneous medium also remains a reasonable approximation for inhomogeneous media. At first sight, this can indeed be expected to be the case for weakly inhomogeneous media at least. It is in fact never the case, because a plane wave has an infinite localization size $\Lambda = \infty$ which is always more than any finite length of the medium inhomogeneity L. To overcome this contradiction, the amplitude factor (Φ_n in Eq. (5)) is used in the ray method to describe a wave packet of finite size. This allows one to improve the situation, but the ray expansion is still restricted to consideration of nearly plane waves and, consequently, requires that the wave amplitude $\Phi_0(\vec{r})$ in Eq. (5) vary only slowly in space over a length of order L [1] or, in our notation, $\Lambda \approx L$. In other words, the ray method makes no distinction between characteristic lengths of medium L and wave amplitude Λ variations, viewing both lengths as the same quantity.

As already mentioned in the Introduction, this requirement is rather restrictive. In the solution (12) the same restriction as in the eikonal approach is imposed on the function A^{lm} rather than on the wave amplitude. The latter varies, as described, mainly by the exponential factor in the functions of the parabolic cylinder. It is seen from Eq. (27) that Λ is intermediate between a medium inhomogeneity length L and wavelength $\lambda = c/\omega$ so that $\Lambda \approx \sqrt{L\lambda}$. As known from classical optic, this is just the threshold where diffraction becomes significant and prevents further localization of a wave package. It follows that the situation $\Lambda \ll \sqrt{L\lambda}$ can hardly be realized because of diffractive broadening. On the other hand, the case $\Lambda \approx L \gg \sqrt{L\lambda}$ is not excluded from Eq. (27) and hence from the solution (12) because $|\nabla u|$ as well as $|\nabla v|$ can be small and even zero. This means that the pWKB method is valid in the vast majority of practical problems; in particular, the eikonal approach is included in the pWKB solution as a specific case.

We remark in conclusion that in the ray method the effect of diffraction is described with terms of order κ^{-2}, whereas the pWKB method uses terms of order κ^{-1}. It is likely that the latter ordering is inherent to wave phenomena, which results, on the one hand, in better convergence of asymptotic series and, on the other hand, in a wider applicability of the pWKB technique.

3.4 Paraxial expansion

We now take advantage of the exponential factor present in the functions of the parabolic cylinder (14) and hence in all terms of Eq. (15). This causes fast decay of a wave amplitude outside $\Re$ and suggests that it is superfluous to know the solution (12) in the whole space with equal accuracy. Therefore, we seek an approximate solution of this equation in the vicinity of the reference ray $u(\vec{r}) = v(\vec{r}) = 0$ in the form of power series involving powers of u and v (paraxial expansion). The estimate

$$\left(\sqrt{u^2+v^2}\right)^l \exp\left(-\frac{\kappa(u^2+v^2)}{2}\right) \le \left(\frac{l}{e\kappa}\right)^{l/2}$$

shows that increasing the powers of u or v by unity is equivalent to transition to the next order in the expansion with respect to $\kappa^{-1/2}$. This means that in the leading order of Eq. (15), i.e. in Eq. (23), we need to retain terms including $u^0, v^0, u^1, v^1, u^2, v^2$, and uv. The subsequent terms of the paraxial expansion are of order $O(\kappa^{-3/2})$, which is already suppressed in Eq. (15). The procedure results in the following equations:

$$H|_{\Re} = 0, \qquad D_u H|_{\Re} = D_v H|_{\Re} = 0, \tag{31}$$

$$D^2_{uu} H\Big|_{\Re} = H_{UU}\Big|_{\Re}, \qquad D^2_{vv} H\Big|_{\Re} = H_{VV}\Big|_{\Re}, \qquad D^2_{vu} H\Big|_{\Re} = 0. \tag{32}$$

The operators D_u and D_v used above denote partial derivatives with respect to u and v, which are calculated with allowance for both explicit and implicit dependences of $H[u, v, t,\, x^\alpha(u, v, t),\, S_\alpha(u, v, t),\, u_\alpha(u, v, t),\, v_\alpha(u, v, t)]$ and other functions of u, v and

t, respectively. In the next order $O(\kappa^{-1/2})$ equation (24) it is necessary to keep terms including u^0, v^0, u^1 and v^1:

$$H_U|_{\Re} = H_V|_{\Re} = 0, \tag{33}$$

$$D_u(H_U)|_{\Re} = D_v(H_U)|_{\Re} = D_u(H_V)|_{\Re} = D_v(H_V)|_{\Re} = 0. \tag{34}$$

Finally, in the second order of the expansion, $O(\kappa^{-1})$, it is sufficient to retain zero order terms of the expansion with respect to powers of u and v only:

$$H_{UV}|_{\Re} = 0, \tag{35}$$

$$\mathcal{L}[A^{lm}]\Big|_{\Re} = 0. \tag{36}$$

In accordance with the procedure discussed all unknown functions, namely, $S(\vec{r})$, $u(\vec{r})$, $v(\vec{r})$, $A^{lm}(\vec{r})$, also have to be expanded in power series and only a restricted number of terms have to be retained in the expansions. Inspection of Eqs. (31)-(32) shows that the unknown eikonal $S = S(\vec{r})$ has to be determined up to second order terms of the paraxial expansion, viz.

$$S(\vec{r}) = S\Big|_{\Re} + p_\alpha(x^\alpha - q^\alpha) + \frac{1}{2}S_{\alpha\beta}\Big|_{\Re}(x^\alpha - q^\alpha)(x^\beta - q^\beta) + \cdots \tag{37}$$

It is understood here that the reference ray $\Re$ is described by Eq. (29) and $S_{\alpha\beta}$ is defined as

$$S_{\alpha\beta} = \frac{\partial^2 S}{\partial x^\alpha \partial x^\beta}. \tag{38}$$

In practice, however, it is more convenient to use the same expansion for the covariant components of the vector ∇S:

$$S_\alpha(\vec{r}) = p_\alpha + S_{\alpha\beta}|_{\Re}(x^\beta - q^\beta) + \cdots \tag{39}$$

or

$$S_\alpha(\vec{r}) = p_\alpha + \frac{\partial S_\alpha}{\partial u}\Big|_{\Re} u + \frac{\partial S_\alpha}{\partial v}\Big|_{\Re} v + \cdots \tag{40}$$

The functions u and v are involved in the higher order equations (33)-(35) and it is sufficient to retain two terms in the paraxial expansions. However, in view of the vanishing of the functions $u(\vec{r})$ and $v(\vec{r})$ on $\Re$ only one term remains in the Taylor series

$$u(\vec{r}) = u_\alpha\Big|_{\Re}(x^\alpha - q^\alpha) + \cdots, \qquad v(\vec{r}) = v_\alpha\Big|_{\Re}(x^\alpha - q^\alpha) + \cdots, \tag{41}$$

which is equivalent to

$$x^\alpha(u, v, t) = q^\alpha(t) + \frac{\partial x^\alpha}{\partial u}\Big|_{\Re} u + \frac{\partial x^\alpha}{\partial v}\Big|_{\Re} v + \cdots \tag{42}$$

Finally, as can be seen from Eq. (36), only the zero order term has to be retained for the function $A^{lm} = A^{lm}(t)$.

The functions $q^\alpha(t)$ and $p_\alpha(t)$ have already been determined as a solution to the Hamiltonian set of equations (8)-(9). Similarly, the first term of the expansion (37), which determines the eikonal S along $\Re$, is given by Eq. (11). The functions $S_{\alpha\beta}$, u_α, v_α, A^{lm} are evaluated on the reference ray solely and they will be viewed further as functions of the only argument t. After these functions are found they can be substituted into Eqs. (37), (41) and then into Eq. (12) thus solving the problem under consideration. The functions obey the set of ordinary differential equations which is a corollary to Eqs. (31)-(36) and will be discussed in the next Section.

4 Beam tracing

4.1 Ray coordinates

Let the set of three functions

$$\begin{cases} u = u(\vec{r}) \\ v = v(\vec{r}) \\ t = t(\vec{r}) \end{cases} \tag{43}$$

be a new coordinate system in space. In accordance with Kravtsov and Orlov [1] we call the coordinate system (43) the ray coordinates. The physical meaning of the ray coordinate system is that in its frame the solution (12) represents a straight plane beam of constant width. The generic notations $w^1 = u$, $w^2 = v$, $w^3 = t$ will also be used equivalently to those of Eq. (43).

The ray coordinates so far obey the conditions: (i) the Jacobian of the coordinate transformation $\{x^\alpha\} \to \{u, v, t\}$ does not vanish; (ii) the coordinate line $u(\vec{r}) = v(\vec{r}) = 0$ coincides with the reference ray $\Re$, which is a solution to the Hamiltonian set of equations (8), (9). The sense of the first condition is obvious. In view of Eqs. (8), (28) the second one means that the vector $\vec{V}$ of the group velocity tangential to the reference ray is one of the basis vectors of the ray coordinate system. The vectors ∇u and ∇u are reciprocal vectors orthogonal to $\vec{V}$. Therefore, Eqs. (33) are valid for any choice of $u(\vec{r})$ and $v(\vec{r})$ satisfying the condition (ii). Equations (31) are fulfilled due to Eqs. (8)-(9). In turn, the fulfilment of Eqs. (31), (33) means that Eq. (15) is satisfied up to first order inclusively with an accuracy of $o(\kappa^{-1/2})$. The remaining second order equations (32), (34), (35) are discussed in the rest of this section.

4.2 First form of the beam tracing equations

We now show that the higher order equations (32), (34), (35) determine the higher order terms u_α, v_α, $S_{\alpha\beta}$ in the expansions (37), (39)-(42). First of all, note that Eqs. (33) are valid for any t. It follows that $D_t(H_U)|_\Re = D_t(H_V)|_\Re = 0$. Together with Eqs. (34) this means that $D_{x^\alpha}(H_U)|_\Re = D_{x^\alpha}(H_V)|_\Re = 0$ for any $\alpha = 1, 2, 3$. Calculating these derivatives by means of Eqs. (8), (22), (24), (28), we have

$$\begin{aligned}\frac{du_\alpha}{dt} &= -\left(\frac{\partial^2 H}{\partial x^\alpha \partial S_\beta} + \frac{\partial^2 H}{\partial S_\beta \partial S_\gamma} S_{\alpha\gamma}\right) u_\beta, \\ \frac{dv_\alpha}{dt} &= -\left(\frac{\partial^2 H}{\partial x^\alpha \partial S_\beta} + \frac{\partial^2 H}{\partial S_\beta \partial S_\gamma} S_{\alpha\gamma}\right) v_\beta.\end{aligned} \tag{44}$$

In accordance with the remark made at the end of the previous section the full derivatives appear on the left-hand sides of these equations, showing that these quantities are calculated along $\Re$. Moreover, Eqs. (44) as well as all the equations in the following are considered along the reference ray solely, and so the subscript $\Re$ is omitted here and in what follows. Note also that here and throughout the paper the commonly used notations for the derivatives of the Hamiltonian are retained. That is, $\partial H/\partial S_\alpha$ denotes a partial derivative with respect to S_α with x^α fixed, while $\partial H/\partial x^\alpha$ is computed with S_α kept constant. We use the notation $D_{x^\alpha}[H(x^\alpha, S_\beta)]$ for the "full" partial derivative, which is computed with allowance for the dependence $S_\beta(x^\alpha)$ along a ray.

By making use of

$$D^2_{x^\alpha x^\beta}(H) = D_{x^\alpha}\left[\frac{\partial w^\nu}{\partial x^\beta}(D_{w^\nu} H)\right] = \frac{\partial w^\nu}{\partial x^\beta} D_{x^\alpha}(D_{w^\nu} H) = \frac{\partial w^\mu}{\partial x^\alpha}\frac{\partial w^\nu}{\partial x^\beta} D^2_{w^\mu w^\nu}(H)$$

and $D_{w^\alpha} H = 0$, $D_t(D_{w^\alpha} H) = 0$, all three equations (32) may be replaced with

$$D^2_{x^\alpha x^\beta} H = u_\alpha u_\beta H_{UU} + v_\alpha v_\beta H_{VV},$$

from which, by direct differentiation, we obtain

$$\begin{aligned}\frac{dS_{\alpha\beta}}{dt} + \frac{\partial^2 H}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 H}{\partial x^\alpha \partial S_\gamma} S_{\beta\gamma} + \frac{\partial^2 H}{\partial x^\beta \partial S_\gamma} S_{\alpha\gamma} + \frac{\partial^2 H}{\partial S_\gamma \partial S_\delta} S_{\alpha\gamma} S_{\beta\delta} \\ = u_\alpha u_\beta H_{UU} + v_\alpha v_\beta H_{VV}.\end{aligned} \tag{45}$$

The set of ordinary differential equations (44), (45) for the quantities u_α, v_α, $S_{\alpha\beta}$ can be readily integrated along the reference ray, thus solving the problem of determining the functions $S = S(\vec{r})$, $v = v(\vec{r})$, $u = u(\vec{r})$ within the accuracy required. Initial conditions for the set of equations (44), (45) are discussed later in this section. In particular, it will be shown that the fulfilment of Eq. (35) still unused is ensured by a special choice of initial conditions for u_α and v_α.

Apart from the independent task of solving Eq. (36), which is discussed in Section 5, the original problem for wave equation (1) is already solved. The procedure of solution is as follows. The set of ordinary differential equations (8), (9), (44), (45) is integrated, yielding the functions q^α, S_α, $S_{\alpha\beta}$, u_α, v_α of the single argument t. These functions are substituted in Eqs. (37), (39), (41) and then in Eq. (12).

It is worth noting that only the right-hand side of Eq. (45) contains terms which are new in comparison with the ray approach. It is clear that these terms are significant in the neighbourhood of a point where ∇u and/or ∇v, or u_α and/or v_α, become large. In accordance with Eq. (27) this means that the wave amplitude rapidly decreases outside the reference ray. Such a situation occurs, for instance, near caustics or focal points, where diffraction is expected to be significant. Consequently, the two terms on the right-hand side of Eq. (45) are responsible for describing the wave properties in our approach. Their smallness may be a quantitative measure of the applicability of the eikonal approach.

Physically, it is clear that the behaviour of the wave beam width, or, in other words, the beam convergence or divergence, is coupled with the curvature of the wave front. In agreement with this, the second derivatives $S_{\alpha\beta}$ representing the curvature and the beam width terms appear in the same Eq. (45).

4.3 Second form of the beam tracing equations

A different mode of attacking this problem gives an additional insight into and an alternative representation of the beam tracing equations. Let us regard the unknown ray coordinates (43) in the form $x^\alpha = x^\alpha(u, v, t)$ and use the expansions (40), (42) instead of (39), (41). To derive equations for evolution of the quantities $\partial x^\alpha / \partial w^\beta$ along $\mathfrak{R}$, let us use the identities

$$\frac{\partial x^\alpha}{\partial w^\gamma} \frac{\partial w^\gamma}{\partial x^\beta} = \delta^\alpha_\beta. \tag{46}$$

On differentiation of Eq. (46) with allowance for Eq. (44) (for details see Appendix) we have

$$\begin{aligned} \frac{d}{dt} \frac{\partial x^\alpha}{\partial u} &= \frac{\partial^2 H}{\partial S_\alpha \partial x^\beta} \frac{\partial x^\beta}{\partial u} + \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta} \frac{\partial S_\beta}{\partial u}, \\ \frac{d}{dt} \frac{\partial x^\alpha}{\partial v} &= \frac{\partial^2 H}{\partial S_\alpha \partial x^\beta} \frac{\partial x^\beta}{\partial v} + \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta} \frac{\partial S_\beta}{\partial v}. \end{aligned} \tag{47}$$

Here we can repeat word for word the comment on Eqs. (44) and consider all the terms in Eqs. (47) as functions of the single variable t. The counterpart equations for $\partial S_\alpha / \partial w^\beta$ follow directly from Eq. (32) and (47):

$$\begin{aligned} \frac{d}{dt} \frac{\partial S_\alpha}{\partial u} + \frac{\partial^2 H}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\beta}{\partial u} + \frac{\partial^2 H}{\partial x^\alpha \partial S_\beta} \frac{\partial S_\beta}{\partial u} &= u_\alpha H_{UU}, \\ \frac{d}{dt} \frac{\partial S_\alpha}{\partial v} + \frac{\partial^2 H}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\beta}{\partial v} + \frac{\partial^2 H}{\partial x^\alpha \partial S_\beta} \frac{\partial S_\beta}{\partial v} &= v_\alpha H_{VV}. \end{aligned} \tag{48}$$

The set of equations (47), (48) can be used instead of (44), (45). In this case, the expansions (40), (42) have to be used instead of (37), (39), (41). The set of the algebraic equations (46) may then be used to express the quantities u_α, v_α on the right-hand side of Eqs. (48) in terms of $\partial x^\alpha / \partial u$, $\partial x^\alpha / \partial v$ and V^α.

Equations (47), (48) are linear unless the right-hand-side terms in Eqs. (48) responsible for diffraction become insignificant. Equations (47) are nothing but the derivatives of Eq. (8) with respect to u and v, and the left-hand sides of Eqs. (48) coincide with the results of differentiating Eq. (9). The meaning of this coincidence is that, when diffraction is negligible, the beam tracing equations describe a pencil of ray trajectories adjacent to the reference ray. However, when the ray pencil becomes narrow enough and the transverse energy flow becomes significant, then the right-hand sides of Eqs. (45) and (48) come into play and the pWKB solution departs from the ray solution.

4.4 Initial conditions for the beam tracing equations

We still need to prove that Eq. (35) is consistent with Eqs. (44), (45) and (47), (48). We shall now show that this equation can be viewed as one of the initial conditions for the beam tracing equations. To this end, we introduce the symmetric matrix

$$B^{\alpha\beta} = \varepsilon^{\alpha\gamma} D_{x^\gamma}\left(\frac{\partial H}{\partial S_\beta}\right) + \varepsilon^{\beta\gamma} D_{x^\gamma}\left(\frac{\partial H}{\partial S_\alpha}\right) - \frac{\partial H}{\partial S_\gamma} D_{x^\gamma}(\varepsilon^{\alpha\beta}), \tag{49}$$

where the matrix $\varepsilon^{\alpha\beta}$ is determined by Eq. (7) and in general as

$$\varepsilon^{\alpha\beta} = \frac{\partial^2 H}{\partial S_\alpha \partial S_\beta}. \tag{50}$$

We also introduce the notations

$$u^\alpha = \varepsilon^{\alpha\beta} u_\beta \qquad \text{and} \qquad v^\alpha = \varepsilon^{\alpha\beta} v_\beta. \tag{51}$$

As shown in Appendix the ray direction V^α coincides with one of the principal directions determined by $B^{\alpha\beta}$. Hence it is always possible to take the vectors u_α and v_α at the initial point of the reference ray $t = 0$ as the two other eigenvectors of the matrix $B^{\alpha\beta}$, i.e.

$$B^{\alpha\beta} u_\beta\Big|_{t=0} = \lambda_{(u)} u^\alpha\Big|_{t=0} \qquad \text{and} \qquad B^{\alpha\beta} v_\beta\Big|_{t=0} = \lambda_{(v)} v^\alpha\Big|_{t=0}. \tag{52}$$

In view of Eqs. (51), (52) and symmetry of the matrix $B^{\alpha\beta}$ we have

$$\left(\lambda_{(u)} - \lambda_{(v)}\right) u_\alpha \varepsilon^{\alpha\beta} v_\beta\Big|_{t=0} = \left(\lambda_{(u)} u^\beta v_\beta - \lambda_{(v)} u_\alpha v^\alpha\right)\Big|_{t=0} = u_\beta v_\alpha \left(B^{\alpha\beta} - B^{\beta\alpha}\right)\Big|_{t=0} = 0. \tag{53}$$

If $\lambda_{(u)} \neq \lambda_{(v)}$, then it follows that $(u_\alpha \varepsilon^{\alpha\beta} v_\beta)\Big|_{t=0} = 0$, i.e. Eq. (35) is fulfilled at the original point of the ray $\Re$, viz. at $t = 0$. In the case of the multiple eigenvalue $\lambda_{(u)} = \lambda_{(v)}$ it is also possible to choose two different vectors ∇u and ∇v so that Eq. (35) is valid at the point $t = 0$. With a straightforward, though somewhat lengthy, calculation (see Appendix) we arrive at the relation

$$\frac{d}{dt} u_\alpha \varepsilon^{\alpha\beta} v_\beta = -u_\alpha v_\beta B^{\alpha\beta}. \tag{54}$$

We can then write

$$\left(\frac{d}{dt}u_\alpha\varepsilon^{\alpha\beta}v_\beta\right)\Big|_{t=0} = \left(-u_\alpha v_\beta B^{\alpha\beta}\right)\Big|_{t=0} = \left(-\lambda_{(u)}u_\alpha v_\beta\varepsilon^{\alpha\beta}\right)\Big|_{t=0} = 0. \tag{55}$$

One can consecutively show that all the higher order derivatives $d^n(u_\alpha\varepsilon^{\alpha\beta}v_\beta)/dt^n$ vanish at $t=0$, which finally proves that with the initial conditions (52) Eq. (35) is fulfilled identically along $\Re$.

We can now formulate the initial conditions for the set of beam tracing equations. Initial conditions for the eikonal S and its derivatives S_α, $S_{\alpha\beta}$ are determined if the shape of the phase front is given at the beginning of $\Re$. Like the initial conditions for the quantities $\partial S_\alpha/\partial u = S_{\alpha\beta}\partial x^\beta/\partial u$ and $\partial S_\alpha/\partial v = S_{\alpha\beta}\partial x^\beta/\partial v$, they depend on the choice of u_α and v_α. However, a great deal of uncertainty is still present in the initial conditions for. u_α and v_α. The only constraint imposed on these quantities so far is Eq. (52), which prescribes the directions of the vectors ∇u and ∇v. The lengths of these vectors are still arbitrary. We shall return to the choice of $|\nabla u|$ and $|\nabla v|$ in Section 6 and show that this freedom can be used to improve the asymptotic convergence to the exact solution.

4.5 Discussion of the beam tracing equations

The first remark to be made here is that each of the systems (44), (45) or (46)-(48), although giving a solution to the problem considered involves twice as many equations as are really necessary. Actually, with allowance for the symmetry of the second derivative $S_{\alpha\beta}$ Eqs. (44), (45) give a set of 12 equations. Only half of them are independent. The point is that in accordance with Eq. (30) the vectors ∇u and ∇v are normal to the reference ray direction $\vec{V}$ at any point of the ray. This together with Eq. (35) imposes three conditions on the components of the vectors ∇u and ∇v. A similar statement is valid for the quantities $S_{\alpha\beta}$. Namely, the three combinations $S_{\alpha\beta}V^\beta = \dot{S}_\alpha$ are already determined by Eq. (9), which shows that only 3 of 6 equations (45) or (48) are independent. Therefore, six algebraic conditions (19) and $S_{\alpha\beta}V^\beta = \dot{S}_\alpha$ can be used either to check the accuracy of a numerical solution or to reduce the number of differential equations to be solved.

The next remark is that in conventional ray tracing, as also in classical mechanics, the two Hamiltonian equations (8) and (9) are fully symmetric. This is not the case in our consideration. To highlight the difference, let us recall the procedure deriving Eqs. (8)-(9). Equation (8) merely introduces a new quantity $\dot{x}^\alpha = V^\alpha = \partial H/\partial S_\alpha$. Then Eq. (9) follows from this designation and vanishing of the first spatial derivatives of the Hamiltonian H. In the ray approach, we have $H(x^\alpha, S_\alpha) \equiv 0$ everywhere in space and, therefore, the higher order spatial derivatives of H vanish and, accordingly, all the equations obtained by differentiation of Eqs. (8) and (9) are also valid. In the pWKB approach, the Hamiltonian H together with its first derivatives vanishes along $\Re$ only (Eq. (31)), while second derivatives do not vanish at all (Eq. (32)). In both approaches Eq. (8) is viewed as a new designation and has to be fulfilled with all its spatial derivatives. In particular, the sets of equations (44) and (47) could be formally derived by such differentiation. Vanishing of the first derivatives of H yields in Eq. (9) in the same manner for both approaches. However, higher derivatives of H do not vanish in the pWKB approximation and it can be conluded that the differentiations which could be done on Eq. (8) are not allowed for Eq. (9), because the derivatives of the two sides of Eq. (9) are no longer equal.

As shown in [5], the reference ray describes the trajectory of the centre of gravity of the solution (12). This trajectory coincides with the geometric-optics trajectory and does not depend on the transverse structure of the solution (12). In turn, this structure is described by the functions $u(\vec{r})$ and $v(\vec{r})$. The larger $|\nabla u|$ and $|\nabla v|$ are the narrower is the wave beam. Therefore, the quantity $(|\nabla u|^2 + |\nabla v|^2)^{-1/2}$ characterizes the width of the wave package and can be regarded as the second moment of the amplitude distribution across the beam axis. This quantity is determined by the beam tracing equations and it still describes the gross structure of the solution, i.e. the common backbone for a whole variety of particular solutions of type (12) differing in fine structure as described by the different transverse mode numbers l and m. We now proceed to the equation defining this fine structure of the amplitude distribution over the wave packet.

5 Transport of amplitude

Let us consider Eq. (20) describing the amplitude evolution along the reference ray. As follows from the discussion in Section 3.3, all functions included in this equation can be viewed as functions of a single variable t. First of all, we transform the quantities appearing in the second term on the right-hand side of Eq. (16):

$$g^{\alpha\gamma}\varepsilon_\gamma^\beta = \frac{1}{2}(g^{\alpha\gamma}\varepsilon_\gamma^\beta + g^{\beta\gamma}\varepsilon_\gamma^\alpha) + \frac{1}{2}(g^{\alpha\gamma}\varepsilon_\gamma^\beta - g^{\beta\gamma}\varepsilon_\gamma^\alpha) = \frac{1}{2}(\varepsilon^{\alpha\beta} + i\tilde{\varepsilon}^{\alpha\beta}). \tag{56}$$

If the dielectric tensor ε_α^β is Hermitian, then the tensors $\varepsilon^{\alpha\beta}$ and $\tilde{\varepsilon}^{\alpha\beta}$ are real. The statement is obvious for a coordinate system in which $g^{\alpha\beta}$ coincides with the Kronecker symbol. This is the case for a Cartesian coordinate system with one of the axes oriented along the magnetic field. It follows that the the tensors $\varepsilon^{\alpha\beta}$ and $\tilde{\varepsilon}^{\alpha\beta}$ are real in any coordinate system.

In view of Eqs. (8), (16), (28) and (56) Eq. (20) may be written as

$$\frac{d\ln(A^{lm})^2}{dt} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}(\sqrt{g}V^\alpha) + \frac{iS_\beta}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\tilde{\varepsilon}^{\alpha\beta}\right) + i(2l+1)H_{UU} + i(2m+1)H_{VV} = 0. \tag{57}$$

The skew-symmetry of the tensor $\tilde{\varepsilon}^{\alpha\beta}$ has been taken into account here. By virtue of the Liouville theorem [1] the second term in Eq. (57) can be represented as $d(\ln J)/dt$, where J is the Jacobian of the transformation from the original Cartesian coordinates to the ray coordinates $\{u, v, t\}$:

$$J = \sqrt{g}\frac{D(x^1, x^2, x^3)}{D(u, v, t)}. \tag{58}$$

The amplitude A^{lm} can now be obtained by integration along the reference ray

$$A^{lm}(t) = C^{lm}\sqrt{\frac{J_0}{J(t)}}\exp\left\{-i\left[\phi_g(t) + (2l+1)\phi_u(t) + (2m+1)\phi_v(t)\right]\right\}, \tag{59}$$

where C^{lm} are arbitrary constants, $J_0 = J|_{t=0}$ and

$$\phi_g = \frac{1}{2}\int_0^t \frac{S_\beta}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\tilde{\varepsilon}^{\alpha\beta}\right) dt, \qquad \phi_u = \frac{1}{2}\int_0^t H_{UU} dt, \qquad \phi_u = \frac{1}{2}\int_0^t H_{VV} dt. \tag{60}$$

The third term in Eq. (57) and, correspondingly, the term ϕ_g in the exponent of Eq. (59) describes the medium gyrotropy. It is present in the ray approach also. The last two terms on the left-hand side of Eq. (57) are new in comparison with the ray tracing description. The similar terms on the right-hand sides of Eqs. (45) and (48) have already been discussed in the previous section. However, while the terms in the beam tracing equations are responsible for the amplitude behaviour of the wave beam, the terms in Eq. (57) contribute to the phase of the wave. As distinct from the beam tracing equations, where the terms are real and their magnitude can be compared with other terms in the equations, here the terms are purely imaginary and their significance cannot be estimated on the basis of Eq. (57). So we shall resume discussion of them in the next section.

6 Solution of the wave equation

6.1 Partial solution

Making use of Eq. (12) and Eq. (59) we can write a particular solution of Eq. (1):

$$\Phi_{lm} = \sqrt{\frac{J_0}{J}}\varphi_l(\sqrt{\kappa}u)\varphi_m(\sqrt{\kappa}v)\exp\left\{i\kappa S - i\phi_g - i(2l+1)\phi_u - i(2m+1)\phi_v\right\}. \tag{61}$$

As discussed in Section 4, the unknown functions $u(\vec{r})$, $v(\vec{r})$ and $S(\vec{r})$ contained in Eq. (61) are to be found by integration of either set of ordinary differential equations (8), (9), (11) and (44), (45) or (47), (48) with subsequent use of the expansions (37), (41) or (40), (42), respectively. Equation (61) can be also viewed as a parametric representation of the solution. Then u, v and t have the sense of parameters and the functions $x^\alpha = x^\alpha(u, v, t)$ and $S = S(u, v, t)$ are determined by Eqs. (37) and (42).

Consider now the phase of the solution (61). Along the reference ray it can be written as

$$\operatorname{Arg}(\Phi_{lm})\Big|_{\Re} = \int_0^t \left[\kappa S_\alpha S_\beta - \left(l+\frac{1}{2}\right) u_\alpha u_\beta - \left(m+\frac{1}{2}\right) v_\alpha v_\beta\right] h^{\alpha\beta} dt - \phi_g. \tag{62}$$

As mentioned above, the term ϕ_g describes the medium gyrotropy. It does not contain the large factor κ and might seem insignificant. However, this is not correct. Actually, this term appears in a description of electrostatic oscillations of plasma. But in this case N^2 in Eq. (1) vanishes and as a consequence Eq. (11) reads $dS/dt = h^{\alpha\beta} S_\alpha S_\beta = 0$. Physically, this means that the geometric-optics phase (eikonal) of the electrostatic wave does not vary along a ray. The latter is not valid any longer if the higher order terms of the wave phase are taken into account. Moreover, on the zero background any contribution to the phase variation may be significant.

It is seen from Eq. (61) that l and m are the numbers of zeros of the functions of the parabolic cylinder φ_l and φ_m. So the numbers describe a transverse field variation supplementary to that given by the second term in the expansion (37). In other words, l and m represent the fraction of waves having wave vectors different from the carrier wave vector of the wave envelope $\vec{k} = (\omega/c)\nabla S|_{\Re} = \{\kappa p_\alpha(t)\}$. The two last terms in Eq. (61) describe the corresponding corrections to the phase behaviour.

6.2 General solution

We can now construct a general solution to the wave equation (1):

$$\begin{aligned}\Phi(\vec{r}) &= \sqrt{\frac{J_0}{J(t)}} \exp\left\{i\kappa S(u,v,t) - i\phi_g(t) - \tfrac{1}{2}\kappa(u^2+v^2)\right\} \\ &\times \sum_{l,m} C^{lm} \frac{H_l(\sqrt{\kappa}u) H_m(\sqrt{\kappa}v)}{\sqrt{l!m!\pi 2^{l+m}}} \exp\left\{-i(2l+1)\phi_u(t) - i(2m+1)\phi_v(t)\right\}.\end{aligned} \tag{63}$$

Actually, let us suppose that an arbitrary field distribution is prescribed on some surface Σ in space. Without any loss of generality, it can be assumed that this surface is described by the parametric equations $x^\alpha = x^\alpha(u,v,t)|_{t=0}$. Then

$$\Phi(\vec{r})\Big|_\Sigma = e^{i\kappa S_\Sigma(u,v)} \sum_{l,m} C^{lm} \varphi_l(\sqrt{\kappa}u)\varphi_m(\sqrt{\kappa}v). \tag{64}$$

From Eq. (64) we immediately find

$$S_\Sigma(u,v) \overset{\text{def}}{=} S(u,v,t)\Big|_{t=0} = \frac{1}{\kappa} \operatorname{Arg}[\Phi(\vec{r})]\Big|_\Sigma \tag{65}$$

and the coefficients of the expansion (37) are determined as

$$\begin{aligned} S\Big|_O = S_\Sigma\Big|_O, \qquad & p_\alpha\Big|_{t=0} = \left(\frac{\partial S_\Sigma}{\partial w^\beta}\frac{\partial w^\beta}{\partial x^\alpha}\right)\Big|_O, \\ S_{\alpha\beta}\Big|_{t=0} = & \left(\frac{\partial^2 S_\Sigma}{\partial w^\nu \partial w^\mu}\frac{\partial w^\nu}{\partial x^\alpha}\frac{\partial w^\mu}{\partial x^\beta} + \frac{\partial S_\Sigma}{\partial w^\gamma}\frac{\partial^2 w^\gamma}{\partial x^\alpha \partial x^\beta}\right)\Big|_O . \end{aligned} \tag{66}$$

All the quantities on both sides of these formulae are evaluated at the point $O = \Sigma \cap \Re$, which is the initial point of the reference ray $x^\alpha = q^\alpha|_{t=0}$ and the origin of the ray frame of references $u = v = t = 0$. Of couse, it is implied that this point is already chosen. In principle, the position of this point is arbitrary on Σ, but convergence of the asymptotic series depends on this choice. It was discussed in [5] and it is physically evident that this point should coincide with the centre of gravity of the field distribution over Σ.

An intermediate step is also to prescribe the directions of the vectors ∇u and ∇v according to the procedure described in Section 4.4. There still remains one uncertainty, namely the choice of $|\nabla u|$ and $|\nabla v|$. The situation is similar to this with the original point O. To illustrate what is meant, let us find the coefficients C^{lm}. From Eq. (64) we know the field amplitude distribution over Σ, whence

$$C^{lm} = \int_{-\infty}^{\infty} \varphi_l(\sqrt{\kappa}u)\,du \int_{-\infty}^{\infty} |\Phi(\vec{r})|\Big|_\Sigma \varphi_m(\sqrt{\kappa}v)\,dv. \tag{67}$$

As far as the functions of the parabolic cylinder form a complete set of basis functions it is possible to match any field distribution on the boundary surface Σ and we can conclude that Eq. (63) gives a general solution of the wave equation (1). This conclusion is not affected if one of the variables $w^1 = u$ and $w^2 = v$, or both, is changed by $\tilde{w}^i = \alpha w^i + \beta$ with arbitrary α and β. However, the rate of decrease of C^{lm} with growth of the numbers l, m essentially depends upon the choice of α and β. It will be shown below that the numbers l and m have to be small enough compared with κ, i.e. $l, m \ll \kappa$, which is relevant when the coefficients C^{lm} decrease fast enough. Hence the arbitrariness in the choice of the coordinate system can be used to reduce the number of terms in Eq. (63) and thus reduce the discrepancy between the exact and asymptotic solutions.

Acknowledgements

The author is grateful to M. Brambilla for valuable discussions and for the opportunity to familiarize himself with the manuscript of his new book. The author also wishes to express his appreciation to Technology Division and Tokamak Physics Division of Max-Planck-Institut für Plasmaphysik for help and support received during this work.

Appendix A. Geometric properties of ray trajectories

A.1 The Fermat principle for Eq. (1)

It is well known [12] that an alternative approach to geometric optics is possible on the basis of the Fermat principle. This gives variational formulation equivalent to the eikonal equation. On the basis of this formalism, some properties of ray trajectories which are used in this report can be derived in the most general way. For convenient reference, we present here a derivation according to [8]. It is shown that the ray trajectories are geodesics in some Riemannian space. To this end, we start with the first order partial differential eikonal equation

[6] $$H(x^\alpha, S_\beta) = H(x^\alpha, \frac{\partial S}{\partial x^\beta}) = 0. \tag{A.1}$$

Here and below the formula number given in square brackets on the left-hand side refers to the correspondent formula in the main part of the paper. Following Cauchy's method of characteristics, we consider the parametric representation of the coordinates $x^\alpha = x^\alpha(t)$ and wave vector components $S_\alpha = S_\alpha(t)$. Suppose that these six functions are given in 6D phase space $\{x^\alpha, S_\beta\}$ at some 5D hypersurface Σ such that $t = t_0$ at Σ. The initial values must satisfy the dispersion relation $H|_\Sigma = 0$. The requirement

$$\frac{dH}{dt} = \frac{\partial H}{\partial x^\alpha}\dot{x}^\alpha + \frac{\partial H}{\partial S_\alpha}\dot{S}_\alpha = 0 \tag{$A.2$}$$

then ensures that the eikonal equation $H \equiv 0$ is fulfilled along any trajectory in the 6D phase space passing through some point of Σ. We can require that

[8] $$\dot{x}^\alpha = \frac{\partial H}{\partial S_\alpha}, \tag{A.3}$$

then Eq. (A.2) is clearly valid if

[9] $$\dot{S}_\alpha = -\frac{\partial H}{\partial x^\alpha}. \tag{A.4}$$

The eikonal equation is thus satisfied along any trajectory of the Hamiltonian set of equations Eqs. (A.3)-(A.4).

Let us now introduce the notations

[28] $$V^\alpha \stackrel{\text{def}}{=} \frac{\partial H}{\partial S_\alpha} \tag{A.5}$$

and

$$F(x^\alpha, V^\beta) \stackrel{\text{def}}{=} S_\gamma V^\gamma - H(x^\alpha, S_\beta). \tag{$A.6$}$$

On differentiation of Eq. (A.6) with allowance for Eq. (A.3) and the dependences explicitly shown in Eq. (A.6) we obtain

$$\frac{\partial F}{\partial x^\alpha} = V^\beta \frac{\partial S_\beta}{\partial x^\alpha} - \frac{\partial H}{\partial x^\alpha} - \frac{\partial H}{\partial S_\beta}\frac{\partial S_\beta}{\partial x^\alpha} = -\frac{\partial H}{\partial x^\alpha}, \tag{$A.7$}$$

$$\frac{\partial F}{\partial V^\alpha} = S_\alpha + V^\beta \frac{\partial S_\beta}{\partial V^\alpha} - \frac{\partial H}{\partial S_\beta}\frac{\partial S_\beta}{\partial V^\alpha} = S_\alpha. \qquad (A.8)$$

Equation (A.4) may now be written as

$$\frac{d}{dt}\frac{\partial F}{\partial V^\alpha} - \frac{\partial F}{\partial x^\alpha} = 0. \qquad (A.9)$$

These are clearly Euler's equations for the Fermat functional:

$$I = \int F\, dt = \int S_\alpha\, dx^\alpha - H\, dt \qquad (A.10)$$

The general variational problem may now be enunciated in the form of the **Principle of Least Action:**

A ray trajectory is an extremal of the action integral

$$\mathcal{A} = \int S_\alpha\, dx^\alpha \qquad (A.11)$$

satisfying the subsidiary condition $H = 0$.

The action $\mathcal{A}$ can also be written in any of the following forms:

$$\mathcal{A} = \int S_\alpha \frac{\partial H}{\partial S_\alpha} dt = \int S_\alpha V^\alpha\, dt = \int S_\alpha \frac{dx^\alpha}{dt} dt = \int dS. \qquad (A.12)$$

The latter representation results in the **Fermat principle:**

The optical length between two points is minimal along the ray trajectory.

It can be shown, conversely, that the eikonal equation is a corollary to the Fermat principle, thus proving the equivalency of the two approaches. However, this is not done, because no use is made of it here.

A.2 Rays as geodesics in a Riemannian space

Another formulation of the Fermat principle states that a ray trajectory is a geodesic in a (in general, non-Euclidean) space with the arc element dS. We now consider metric properties of this Riemannian space. Starting at this point, we use a specific expression (6) for the Hamiltonian corresponding to the wave equation in the form of Eq. (1):

[6]
$$H = \frac{1}{2}\varepsilon^{\alpha\beta} S_\alpha S_\beta - N^2 = 0. \qquad \text{(A.13)}$$

First it is assumed that the quantity $N^2(\vec{r})$ is not equal to zero in some space region (this is not the case for Eq. (3) in the main part of the paper). Then we can transit to a new Hamiltonian $\mathcal{H}$ by dividing Eq. (A.) by $2N^2$:

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{H}{2N^2} = \frac{1}{4N^2}\varepsilon^{\alpha\beta} S_\alpha S_\beta - \frac{1}{2} = 0. \qquad (A.14)$$

Introducing a new matrix

$$h^{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{2N^2} \tag{A.15}$$

and its inverse $h_{\alpha\beta}$ such that $h_{\alpha\gamma}h^{\gamma\beta} = \delta^{\beta}_{\alpha}$, we can write

$$S^{\alpha} \stackrel{\text{def}}{=} \frac{\partial \mathcal{H}}{\partial S_{\alpha}} = h^{\alpha\beta} S_{\beta}, \qquad\qquad S_{\alpha} = h_{\alpha\beta} S^{\beta}, \tag{A.16}$$

$$\mathcal{H} = \frac{1}{2} h^{\alpha\beta} S_{\alpha} S_{\beta} - \frac{1}{2} = \frac{1}{2} h_{\alpha\beta} S^{\alpha} S^{\beta} - \frac{1}{2} = \frac{1}{2} S_{\alpha} S^{\alpha} - \frac{1}{2} = 0. \tag{A.17}$$

Accordingly, the Lagrangian function $\mathcal{F}$ takes the form

$$\mathcal{F} \stackrel{\text{def}}{=} S_{\alpha} S^{\alpha} - \mathcal{H} = \frac{1}{2} h_{\alpha\beta} S^{\alpha} S^{\beta} + \frac{1}{2}. \tag{A.18}$$

The significant property of the new Hamiltonian $\mathcal{H}$ is

$$\frac{dS}{dt} = S_{\alpha} \frac{\partial \mathcal{H}}{\partial S_{\alpha}} = S_{\alpha} S^{\alpha} = h_{\alpha\beta} S^{\alpha} S^{\beta} = 1, \tag{A.19}$$

which shows that the parameter of trajectory t coincides with the eikonal function S and that the two vectors of the group velocity $\vec{V}$ and phase velocity ∇S coincide, being unity vectors with the contravariant and covariant components $V^{\alpha} = S^{\alpha}$ and $V_{\alpha} = S_{\alpha}$, respectively.

We now derive an equation for a ray trajectory described by the Hamiltonian set of equations (A.3)-(A.4) with the Hamiltonian $\mathcal{H}$ and show that it is a geodesic in a Riemannian space with the metric $h_{\alpha\beta}$. To this end, we can start with the Fermat principle and consequently with Eq. (A.9) for the Lagrangian (A.18). However, it is more instructive to depart directly from the second of the Hamiltonian equations (A.4). Using also Eqs. (A.16) and (A.17), we have

$$\dot{S}_{\alpha} + \frac{\partial \mathcal{H}}{\partial x^{\alpha}} = h_{\alpha\beta} \dot{S}^{\beta} + S^{\beta} S^{\gamma} \frac{\partial h_{\alpha\beta}}{\partial x^{\gamma}} + \frac{1}{2} h_{\beta\nu} S^{\beta} h_{\gamma\mu} S^{\gamma} \frac{\partial h^{\nu\mu}}{\partial x^{\alpha}} = 0. \tag{A.20}$$

Using the following notations for Christoffel symbols:

$$\left\{ {\alpha \atop \beta\gamma} \right\} = h^{\alpha\delta} [\beta\gamma, \delta] = \frac{1}{2} h^{\alpha\delta} \left(\frac{\partial h_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\gamma\beta}}{\partial x^{\delta}} \right), \tag{A.21}$$

and by virtue of the obvious relations

$$h_{\alpha\gamma} \frac{\partial h^{\gamma\beta}}{\partial x^{\nu}} = -h^{\gamma\beta} \frac{\partial h_{\alpha\gamma}}{\partial x^{\nu}} \qquad \text{and} \qquad \frac{\partial h^{\beta\gamma}}{\partial x^{\alpha}} = -h^{\beta\nu} \left\{ {\gamma \atop \alpha\nu} \right\} - h^{\gamma\nu} \left\{ {\beta \atop \alpha\nu} \right\}, \tag{A.22}$$

we can rewrite Eq. (A.20) as

$$h_{\alpha\beta} \dot{S}^{\beta} + \frac{1}{2} S^{\beta} S^{\gamma} \left(\frac{\partial h_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\alpha}} \right) = 0. \tag{A.23}$$

Finally, by multiplying Eq. (A.23) by $h^{\alpha\nu}$ and changing indices we get the desired equation for the ray

$$\dot{S}^\alpha + S^\beta S^\gamma \left\{ {\alpha \atop \beta\gamma} \right\} = 0, \tag{A.24}$$

which coincides with the equation for a geodesic in a Riemannian space metrized by

$$dS^2 = h_{\alpha\beta} dx^\alpha dx^\beta = h_{\alpha\beta} S^\alpha S^\beta dt^2. \tag{A.25}$$

The concept of covariant differentiation [14] of the tensor calculus is now utilized. The covariant derivatives of a vector $\vec{B}$ are determined as

$$B_{\alpha|\beta} \overset{\text{def}}{=} \frac{\partial B_\alpha}{\partial x^\beta} - B_\gamma \left\{ {\gamma \atop \alpha\beta} \right\}, \qquad B^\alpha_{|\beta} \overset{\text{def}}{=} \frac{\partial B^\alpha}{\partial x^\beta} + B^\gamma \left\{ {\alpha \atop \beta\gamma} \right\} \tag{A.26}$$

for covariant B_α and contravariant B^α components, respectively. Furthermore, the intrinsic (or absolute) derivatives are

$$\frac{\delta B_\alpha}{\delta t} \overset{\text{def}}{=} S^\beta B_{\alpha|\beta}, \qquad \frac{\delta B^\alpha}{\delta t} \overset{\text{def}}{=} S^\beta B^\alpha_{|\beta}. \tag{A.27}$$

Noting that the covariant derivatives of the fundamental tensors vanish (Ricci theorem), we can write the result of differentiation of Eqs. (A.16) as

$$S_{\alpha|\beta} = S_{\beta|\alpha} = h_{\alpha\gamma} S^\beta_{|\gamma} \qquad \text{or} \qquad S^\alpha_{|\beta} = h^{\alpha\gamma} S^\beta_{|\gamma}. \tag{A.28}$$

Equation (A.24) then reads

[8,9]

$$\frac{\delta S^\alpha}{\delta t} = 0 \qquad \text{or} \qquad \frac{\delta S_\alpha}{\delta t} = 0. \tag{A.29}$$

These equations show that the vectors of the group and phase velocities $\vec{V}$ and ∇S form parallel vector fields along the ray trajectory.

The condition $N^2 \neq 0$ used above is not necessary. Actually, it was used solely in transition from Eq. (A.13) to Eq. (A.14). The goal of this transformation was to obtain a Hamiltonian which satisfies the condition (A.19). The latter has the geometrical meaning that in the Riemannian space with the metric $h_{\alpha\beta}$ the arc length is described by the eikonal S. Therefore, to represent the ray equation in the form (A.24), the eikonal function S must be the parameter of the ray, otherwise the right-hand side of Eq. (A.24) would be non-zero and the ray trajectories would not be geodesics. Consequently, $dt = dS$ and dots in Eqs. (A.23), (A.24) denote derivatives with respect to S.

Let us assume that in some point of the ray trajectory we have $N^2 = 0$, as with electrostatic waves. Then Eq. (A.19) takes the form $dS/dt = 0$, but all operations performed in Eqs. (A.20)-(A.24) remain valid with one substantial exception. Now we have $S \equiv \text{Const}$ along the ray and it is impossible to use the eikonal S as a parameter of the ray trajectory. Nevertheless, any permitted parametrization of the ray may be chosen and vanishing of the quantity N^2 does not change either the Fermat principle or the conclusion that the ray trajectory is a geodesic in some Riemannian space. Transition to electrostatic oscillations of a cold plasma $H = h_{\alpha\beta}S^\alpha S^\beta = 0$ as a case of practical interest can be done simply by renumbering the ray coordinates [15].

Geometrically, $N^2 = 0$ means that the metric of this Riemannian space is not positive-definite. Actually, in this case $dS/dt = \varepsilon_{\alpha\beta}V^\alpha V^\beta = 0$ and the ray is a null-curve. It is known [16] that, if any portion of a geodesic is a null-curve, then the whole geodesic is a null-geodesic. This remark implies that the mixed trajectories such that on one part of them $N^2 = 0$ and on another $N^2 \neq 0$ do not exist or if a ray passes through a point $\vec{r}_0$ such that $N^2(\vec{r}_0) = 0$, then $N^2 \equiv 0$ in all points of the ray. In other words, if a ray is tangential to the phase front at one point of a trajectory, then the property holds at all points of the trajectory.

For the sake of simplicity it is assumed in this Appendix that the Hamiltonian has the form of Eq. (A.18). Note, however, that in Section 4 an account of the general case is given without any specification of the value N^2 or even of the form of the Hamiltonian.

A.3 Ray coordinates

Let us consider a single ray as a general member of the variety of rays satisfying Eq. (A.24). In Section 4 this ray was called the reference ray and denoted as $\Re$. Let us also assume that the ray $\Re$ is described by parametric equations $x^\alpha = q^\alpha(t)$ and introduce the ray-related coordinate system $\{u, v, S\}$ such that $u(\vec{r}) = v(\vec{r}) = 0$ on the ray $\Re$. We also use the notations

[43] $$w^1 = u, \qquad w^2 = v, \qquad w^3 = S \tag{A.30}$$

and mark with bars the components of all tensors in the ray coordinate system. It is assumed in what follows that the Latin indices take values 1 and 2 so that w^i can stand for either u or v but not for S. The matrices of the direct and inverse transformations between the coordinate systems $\{x^\alpha\}$ and $\{w^\alpha\}$ are denoted as

$$x^\alpha_\beta = \frac{\partial x^\alpha}{\partial w^\beta}, \qquad w^\alpha_\beta = \frac{\partial w^\alpha}{\partial x^\beta}. \tag{A.31}$$

The components of the fundamental tensors in the ray coordinate system are

$$\bar{h}^{\alpha\beta} = h^{\nu\mu} w^\alpha_\nu w^\beta_\mu, \qquad \bar{h}_{\alpha\beta} = h_{\nu\mu} x^\nu_\alpha x^\mu_\beta. \tag{A.32}$$

The beam tracing equations now read

[31,33,35] $$H = \bar{h}^{33} - 1 = 0, \qquad \bar{h}^{13} = \bar{h}^{23} = \bar{h}^{12} = 0, \tag{A.33}$$

[34] $$D_{w^i}(\bar{h}^{j3}) = 0, \tag{A.34}$$

[32] $$D^2_{w^i w^j}(\bar{h}^{33}) = \bar{h}^{ij}. \tag{A.35}$$

The reader is reminded here that in accordance with the sense of Eqs. (31)-(35) all the quantities in Eqs. (A.33)-(A.35) are evaluated on $\Re$, i.e. for $u = v = 0$. In the rest of this Appendix, if the opposite is not explicitly stated all terms in all equations are evaluated on the reference ray and the index $\Re$ will be suppressed.

Equations (A.33)-(A.35) represent the metric properties of the considered Riemannian space in the vicinity of the reference ray. However, the tensor $\bar{h}^{\alpha\beta}$ is unknown and the equations cannot serve for determining the ray coordinates. For this purpose the set of equations introduced in the next paragraph has to be used.

A.4 Derivation of the beam tracing equations

Here we briefly reproduce derivations of the main part of this paper, taking advantage of the tensor formalism. First of all, we have

$$D_{x^\alpha}[\mathcal{H}_U] = (S^\beta u_\beta)_{|\alpha} = S^\beta u_{\beta|\alpha} + S^\beta_{|\alpha} u_\beta = \frac{\delta u_\alpha}{\delta S} + S^\beta_{|\alpha} u_\beta = 0.$$

Whence, by virtue of $\bar{S}_{3|\alpha} = \bar{S}_{\alpha|3} = \bar{S}^3_{|\alpha} = \bar{S}^\alpha{}_{|}3 = 0$ we get

[44]
$$\frac{\delta w^i_\alpha}{\delta S} = -S^\beta_{|\alpha} w^i_\beta = -\bar{S}^i_{|j} w^j_\alpha. \qquad (A.36)$$

It is also assumed that here and below only Greek indices are taken into account in the intrinsic derivatives with respect to S, so that in such a differentiation the quantity w^i_α is viewed as a covariant vector and x^α_i as a contravariant vector rather than a mixed tensor of rank 2. To derive an equation for $S_{\alpha\beta}$, we use Eq. (35). By virtue of Eqs. (A.20), (A.24) and (A.29) we have

$$\begin{aligned}
D_{x^\alpha}[D_{x^\beta}(\mathcal{H})] &= D_{x^\alpha}\left[\frac{\partial \mathcal{H}}{\partial x^\beta} + S^\nu S_{\nu\beta}\right] = D_{x^\alpha}\left[S^\nu S_{\beta|\nu}\right] \\
&= \left(S^\nu S_{\beta|\nu}\right)_{|\alpha} + \left\{\begin{matrix}\nu \\ \alpha\beta\end{matrix}\right\}\frac{\delta S_\nu}{\delta S} = S^\nu_{|\alpha} S_{\beta|\nu} + S^\nu S_{\beta|\nu|\alpha} \qquad (A.37) \\
&= h^{\nu\mu} S_{\alpha|\mu} S_{\beta|\nu} + S^\nu S_{\beta|\nu|\alpha} = u_\alpha u_\beta \bar{h}^{11} + v_\alpha v_\beta \bar{h}^{22}.
\end{aligned}$$

The known formula of the tensor calculus [14] reads

$$S_{\beta|\nu|\alpha} - S_{\beta|\alpha|\nu} = -S^\mu R_{\mu\beta\alpha\nu}$$

where

$$R_{\mu\beta\alpha\nu} = h_{\mu\gamma} R^\gamma_{\beta\alpha\nu} = h_{\mu\gamma}\left(\frac{\partial}{\partial x^\alpha}\left\{\begin{matrix}\gamma \\ \beta\nu\end{matrix}\right\} - \frac{\partial}{\partial x^\nu}\left\{\begin{matrix}\gamma \\ \beta\alpha\end{matrix}\right\} + \left\{\begin{matrix}\lambda \\ \beta\nu\end{matrix}\right\}\left\{\begin{matrix}\gamma \\ \lambda\alpha\end{matrix}\right\} - \left\{\begin{matrix}\lambda \\ \beta\alpha\end{matrix}\right\}\left\{\begin{matrix}\gamma \\ \lambda\nu\end{matrix}\right\}\right)$$

is the covariant Riemann-Christoffel tensor. For the considered problem with the Hamiltonian of the form (A.14) the tensor $R_{\mu\beta\alpha\nu}$ represents the properties of the Riemannian space but does not depend on the specific ray trajectory. We introduce the tensors

$$K_{\alpha\beta} \stackrel{\text{def}}{=} S^\nu S^\mu R_{\mu\alpha\beta\nu} \quad \text{and} \quad K^\alpha_\beta \stackrel{\text{def}}{=} h^{\alpha\gamma} K_{\beta\gamma} = h^{\alpha\gamma} S^\nu S^\mu R_{\mu\beta\gamma\nu} \qquad (A.38)$$

with properties which follow directly from the general properties of the Riemann-Christoffel tensor:

$$K_{\alpha\beta} = K_{\beta\alpha}, \qquad S^\beta K_{\alpha\beta} = S^\beta K^\alpha_\beta = S_\beta K^\beta_\alpha = 0, \qquad \bar{K}_{3\alpha} = \bar{K}_{\alpha 3} = \bar{K}^\alpha_3 = 0. \qquad (A.39)$$

We can now continue the transformations in Eq. (A.37):

[45] $$\frac{\delta}{\delta S} S_{\alpha|\beta} + h^{\nu\mu} S_{\alpha|\mu} S_{\beta|\nu} - K_{\beta\alpha} = u_\alpha u_\beta \bar{h}^{11} + v_\alpha v_\beta \bar{h}^{22}. \qquad (A.40)$$

On differentiation of the obvious relation

[46] $$x_i^\alpha w_\alpha^j = \delta_i^j \qquad (A.41)$$

we have by virtue of Eq. (A.36)

$$w_\alpha^j \frac{\delta x_i^\alpha}{\delta S} + x_i^\alpha \frac{\delta w_\alpha^j}{\delta S} = w_\alpha^j \frac{\delta x_i^\alpha}{\delta S} - x_i^\alpha S^\beta_{|\alpha} w_\beta^j = w_\alpha^j \left(\frac{\delta x_i^\alpha}{\delta S} - S^\alpha_{|\beta} x_i^\beta \right) = 0.$$

Hence, using again Eq. (A.41), we can write Eq. (47) in any of the following forms:

[47] $$\frac{\delta x_i^\alpha}{\delta S} = S^\alpha_{|\beta} x_i^\beta = \bar{S}^j_{|i} x_j^\alpha = \frac{\delta S^\alpha}{\delta w^i}. \qquad (A.42)$$

Equation (A.42) as well as Eq. (A.36) are valid for any i, including $i = 3$, but $x_3^\alpha \equiv S^\alpha$ obeys more simple equation (A.29) and it makes sense to use Eqs. (A.36), (A.42) for $i = 1, 2$ only. In order to obtain Eq. (48), we multiply Eq. (A.40) by x_i^β:

$$\frac{\delta}{\delta S} \left(S_{\alpha|\beta} x_i^\beta \right) - S_{\alpha|\beta} \left(\frac{\delta x_i^\beta}{\delta S} - S^\beta_{|\gamma} x_i^\gamma \right) = x_i^\beta K_{\beta\alpha} + u_\alpha \delta_i^1 \bar{h}^{11} + v_\alpha \delta_i^2 \bar{h}^{22},$$

and by virtue of Eq. (A.42) this yields

[48] $$\frac{\delta}{\delta S} \left(S_{\alpha|\beta} x_i^\beta \right) = x_i^\beta K_{\beta\alpha} + u_\alpha \delta_i^1 \bar{h}^{11} + v_\alpha \delta_i^2 \bar{h}^{22}. \qquad (A.43)$$

A.5 Initial conditions

In line with the derivation made in Section 4.4 we calculate the derivative

$$\begin{aligned} \frac{d}{dS} \left(w_\alpha^i h^{\alpha\beta} w_\beta^j \right) &= \frac{\delta}{\delta S} \left(w_\alpha^i h^{\alpha\beta} w_\beta^j \right) = -h^{\alpha\beta} w_\beta^j S^\nu_{|\alpha} w_\nu^i - h^{\alpha\beta} w_\alpha^i S^\nu_{|\beta} w_\nu^j \\ &= -w_\alpha^i w_\beta^j \left(h^{\beta\nu} S^\alpha_{|\nu} + h^{\alpha\nu} S^\beta_{|\nu} \right) = -w_\alpha^i w_\beta^j \left(h^{\beta\nu} h^{\alpha\mu} S_{\mu|\nu} + h^{\alpha\nu} h^{\beta\mu} S_{\mu|\nu} \right) \\ &= -2 w_\alpha^i w_\beta^j h^{\alpha\nu} h^{\beta\mu} S_{\mu|\nu} = -2 w_\alpha^i w_\beta^j h^{\alpha\gamma} S^\beta_{|\gamma} = -2 w_\alpha^i w_\beta^j h^{\beta\gamma} S^\alpha_{|\gamma}. \end{aligned} \qquad (A.44)$$

The matrix $S^\beta_{|\alpha}$ on the right-hand side of Eq. (A.44) has the eigenvector S_α with zero eigenvalue

$$S^\beta_{|\alpha} S_\beta = 0. \qquad (A.45)$$

The two other eigenvectors of the matrix $S^{\beta}_{|\alpha}$ satisfy the equation

[53] $$S^{\beta}_{|\alpha} e^{(i)}_{\beta} = \lambda_{(i)} e^{(i)}_{\alpha} \qquad \text{(sum on } \beta \text{ only, } i \text{ fixed).} \tag{A.46}$$

With the proper choice of the two different eigenvectors $u_{\alpha} = e^{(1)}_{\alpha}$ and $v_{\alpha} = e^{(2)}_{\alpha}$ we can always fulfil the orthogonality condition at the initial point $S = 0$:

$$\bar{h}^{12}\Big|_{S=0} = \left(w^{1}_{\alpha} h^{\alpha\beta} w^{2}_{\beta}\right)\Big|_{S=0} = 0. \tag{A.47}$$

However, we need to prove that $\bar{h}^{12}(S) = 0$ for arbitrary S. To this end, we expand the quantity $\bar{h}^{12}(S)$ in the Taylor series

$$\bar{h}^{12}(S) = \bar{h}^{12}(0) + S\,\frac{d\bar{h}^{12}}{dS}\Big|_{S=0} + \frac{1}{2}S^{2}\,\frac{d^{2}\bar{h}^{12}}{dS^{2}}\Big|_{S=0} + \cdots \tag{A.48}$$

By virtue of Eqs. (A.32), (A.44) and (A.46) we get

[55] $$\frac{d\bar{h}^{12}}{dS}\Big|_{S=0} = -\left(\lambda_{(1)}\bar{h}^{12}\right)\Big|_{S=0} = 0 \tag{A.49}$$

and see that the second term on the right-hand side of Eq. (A.48) is zero. Repeating the procedure, it is possible to show that all terms on the right-hand side of Eq. (A.48) vanish. This proves that $\bar{h}^{12}(S) \equiv 0$ as required by Eq. (A.33).

A.6 Metric properties of the Riemannian space

We can now state that the fundamental tensor $\bar{h}^{\alpha\beta}$ in the ray coordinates is of diagonal form. It follows that the another fundamental tensor $\bar{h}_{\alpha\beta}$ possesses the same property. Similarly to the derivation of Eq. (A.44), we can obtain

$$\frac{d\bar{h}_{ij}}{dS} = \frac{d}{dS}\left(h_{\alpha\beta} x^{\alpha}_{i} x^{\beta}_{j}\right) = 2x^{\alpha}_{i} x^{\beta}_{j} S_{\alpha|\beta} = 2\bar{S}_{i|j} \tag{A.50}$$

and conclude that the tensor $\bar{S}_{i|j}$ and associated tensor $\bar{S}^{i}_{|j} = \bar{h}^{ik}\bar{S}_{i|k}$ are also diagonal. In view of Eq. (A.46) the matrix $\bar{S}^{\alpha}_{|\beta}$ has the form

$$\bar{S}^{\alpha}_{|\beta} = \begin{pmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.51}$$

Substituting Eq. (A.51) into Eqs. (A.36) and (A.42) we obtain

[44] $$\frac{\delta w^{i}_{\alpha}}{\delta S} = -\lambda_{(i)} w^{i}_{\alpha} \qquad \text{(no sum on } i\text{)} \tag{A.52}$$

and

[47] $$\frac{\delta x^{\alpha}_{i}}{\delta S} = \lambda_{(i)} x^{\alpha}_{i} \qquad \text{(no sum on } i\text{),} \tag{A.53}$$

respectively. Let us assume that in all subsequent formulae no summation is done on Latin indices. In view of Eqs. (A.36) and (A.44) we have

$$\frac{d\,\bar{h}^{ii}}{dS} = -2\lambda_{(i)}\bar{h}^{ii}. \tag{A.54}$$

Equations for the metric coefficients $\bar{h}_{ii}$, which are inverse to $\bar{h}^{ii}$, read

$$\frac{d\,\bar{h}_{ii}}{dS} = 2\lambda_{(i)}\bar{h}_{ii}. \tag{A.55}$$

As follows from Eq. (A.32), the quantities $\bar{h}^{ii}$ give the lengths of the vectors u_α and v_α. Inspection of Eq. (12) shows that characteristic widths $d_{(i)}$ of a wave beam in two orthogonal directions can be introduced as

$$d_{(i)} = \frac{1}{|\nabla w^i|} = \left(\bar{h}^{ii}\right)^{-1/2} = \left(\bar{h}_{ii}\right)^{1/2}. \tag{A.56}$$

These quantities satisfy the equation

$$\frac{d}{dS}d_{(i)} = \lambda_{(i)}d_{(i)}, \tag{A.57}$$

which immediately follows from Eq. (A.54) or Eq. (A.55).

Let us now introduce the unit base vectors of the ray coordinate system

$$\hat{u}_\alpha = d_{(1)}u_\alpha, \qquad \hat{v}_\alpha = d_{(2)}v_\alpha. \tag{A.58}$$

In view of Eqs. (A.36), (A.52) and (A.54) we obtain

$$\frac{\delta\,\hat{u}_\alpha}{\delta S} = \frac{\delta\,\hat{v}_\alpha}{\delta S} = 0. \tag{A.59}$$

In a parallel fashion we introduce the reciprocal vectors

$$\hat{x}_1^\alpha = \frac{x_1^\alpha}{d_{(1)}}, \qquad \hat{x}_2^\alpha = \frac{x_2^\alpha}{d_{(2)}}, \tag{A.60}$$

which obey the set of equations

$$\frac{\delta\,\hat{x}_1^\alpha}{\delta S} = \frac{\delta\,\hat{x}_2^\alpha}{\delta S} = 0. \tag{A.61}$$

Equations (A.59) and (A.61) are similar to Eq. (A.29) and mean that each of the vectors $\hat{u}_\alpha$, $\hat{v}_\alpha$, S_α, $\hat{x}_1^\alpha$, $\hat{x}_2^\alpha$ and S^α forms a parallel vector field along $\mathfrak{R}$, so that

$$\hat{w}_\alpha^i h^{\alpha\beta}\hat{w}_\beta^j = \delta^{ij} \qquad \text{and} \qquad \hat{x}_i^\alpha h_{\alpha\beta}\hat{x}_j^\beta = \delta_{ij}. \tag{A.62}$$

It can also be stated that the particular choice of the new coordinate system $\{\hat{u}, \hat{v}, S\}$ results in recovery of the Cartesian metric on the reference ray $\mathfrak{R}$.

Multiplying Eq. (A.43) by x_j^α and making use of Eq. (A.54), we obtain the equation for $\lambda_{(i)} = \bar{S}^i_{|i}$

$$\frac{d\lambda_{(i)}}{dS} + \lambda_{(i)}^2 = \bar{K}^i_i + \left(\bar{h}^{ii}\right)^2 . \tag{A.63}$$

The set of beam tracing equations comprises Eqs. (A.54), (A.60) and (A.63). Four equations (A.54) and (A.63) can be also written in the form of two complex equations

$$\frac{d}{dS}\left(\lambda_{(j)} + i\bar{h}^{jj}\right) + \left(\lambda_{(j)} + i\bar{h}^{jj}\right)^2 = \bar{K}^j_j \tag{A.64}$$

or as two second order equations

$$\frac{d^2}{dS^2} d_{(i)} - \bar{K}^i_i d_{(i)} = \frac{1}{d^3_{(i)}} . \tag{A.65}$$

The quantities $\bar{K}^i_i$ are given by

$$\bar{K}^i_i = S^\nu S^\mu R_{\nu\alpha\beta\mu} \hat{x}^\alpha_i \hat{x}^\beta_i . \tag{A.66}$$

The quantities $S_{\alpha\beta}$ are expressed via the solution of Eqs. (A.59) and (A.63) as

$$S_{\alpha\beta} = S_{\alpha|\beta} + S_\gamma \left\{ {\gamma \atop \alpha\beta} \right\}, \qquad S_{\alpha|\beta} = \lambda_{(1)} \hat{u}_\alpha \hat{u}_\beta + \lambda_{(2)} \hat{v}_\alpha \hat{v}_\beta . \tag{A.67}$$

References

1. Yu. A. Kravtsov and Yu. I. Orlov, Geometrical Optics of Inhomogeneous Media (Springer Series on Wave Phenomena; Vol. 6), Springer-Verlag, Berlin-Heidelberg 1990.
2. I. B. Bernstein and L. Friedland, in Handbook of Plasma Physics, Eds. M. N. Rosenbluth and R. Z. Sagdeev, Vol. 1, (1983) 367.
3. M. Brambilla and A. Cardinali, Plasma Physics, **24** (1982) 1187.
4. J. W. Connor, R. J. Hastie and J. B. Taylor, Proc. R. Soc. Lond. **A.365** (1979) 1.
5. G. V. Pereverzev, Nucl. Fusion **32** (1992) 1091.
6. E. Mazzucato, Phys. Fluids **B1** (1989) 1855.
7. V. A. Fock, Electromagnetic Diffraction and Propagation Problems (Pergamon, New York 1965).
8. V. M. Babič and V. S. Buldyrev, Short-Wavelength Diffraction Theory: Asymptotic Methods (Springer Series on Wave Phenomena; Vol. 4), Springer-Verlag, Berlin-Heidelberg 1991.
9. V. P. Maslov, Complex WKB Method in Nonlinear Equations, Nauka, Moscow, 1977 (in Russian).
10. A. G. Litvak, in Reviews of Plasma Physics, Consultants Bureau, 1986, Vol. 10, p. 193.
11. S. Choudhary and L. B. Felsen, Proceedings of the IEEE **62** (1974) 1530.
12. L. D. Landau and E. M. Lifshitz, Quantum mechanics, Pergamon Press, Oxford, 1965.
13. G. V. Pereverzev, JETP Letters, **44** (1986) 549.
14. I. S. Sokolnikoff, Tensor Analysis: Theory and Applications to Geometry and Mechanics of Continua, John Wiley & Sons, Inc., New York - London - Sydney 1967.
15. G. V. Pereverzev, 19th EPS Conference on Controlled Fusion and Plasma Physics, Innsbruck, 29 June – 3 July 1992, Contributed Papers, Vol. 16C, Part II, 1489.
16. D. Lovelock and H. Rund, Tensors, Differential Forms, and Variational Principles (Pure and Applied Mathematics), John Wiley & Sons, Inc., New York - London - Sydney - Toronto, 1975.
17. B. Spain, Tensor Calculus, Oliver and Boyd, Edinburgh and London, New York: Interscience Publisher, Inc. 1953