LYAPUNOV STABILITY OF LARGE SYSTEMS OF VAN DER POL-LIKE OSCILLATORS AND CONNECTION WITH TURBULENCE AND FLUCTUATIONS SPECTRA

H. TASSO

IPP 6/316 April 1993

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

8046 GARCHING BEI MÜNCHEN

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

LYAPUNOV STABILITY OF LARGE SYSTEMS OF VAN DER POL-LIKE OSCILLATORS AND CONNECTION WITH TURBULENCE AND FLUCTUATIONS SPECTRA

H. TASSO

IPP 6/316 April 1993

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Lyapunov Stability of Large Systems of van der Pol-like Oscillators and Connection with Turbulence and Fluctuations Spectra *

H. Tasso
Max-Planck-Institut für Plasmaphysik
Euratom Association
D-8046 Garching bei München, Federal Republic of Germany

April 26, 1993

Abstract

For a system of van der Pol-like oscillators, Lyapunov functions valid in the greater part of phase space are given. They allow a finite region of attraction to be defined. Any attractor has to be within the rigorously estimated bounds. Under a special choice of the interaction matrices the attractive region can be squeezed to zero. In this case the asymptotic behaviour is given by a conservative system of nonlinear oscillators which acts as attractor.

Though this system does not possess, in general, a Hamiltonian formulation, Gibbs statistics is possible due to the proof of a Liouville theorem and the existence of a positive invariant or 'shell' condition. The 'canonical' distribution on the attractor is remarkably simple despite nonlinearities. Finally the connection of the van der Pol-like system and of the attractive region with turbulence and fluctuation spectra in fluids and plasmas is discussed.

---

*Lecture at Spring College on Plasma Physics, ICTP Trieste, 17 May - 11 June 1993. Revised version of IPP 6/277.

1 Introduction

The purpose of the van der Pol equation [1] was to study the nonlinear oscillations of a L-Ccircuit driven by a triode. The tension at the grid was taken as a solution of the equation

$$\ddot{y} + (y^2 - 1)\dot{y} + y = 0. \tag{1}$$

The term $-\dot{y}$ represents the amplification of the triode while $y^2\dot{y}$ is due to its nonlinear characteristic curve (see for example [2]).

Due to standard theorems [2] of Poincaré, Bendixon, Levinson and Smith the existence of an attracting limit cycle to equation (1) is known. Practical calculations of the limit cycle are done by means of series expansions and numerical calculations. A typical phase plot is given in Fig. 1.

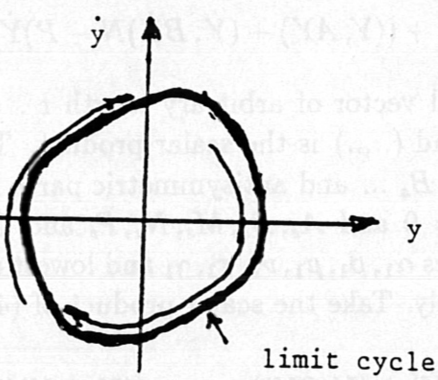

Fig. 1

Let us introduce here a modified van der Pol equation [3] for which the Lyapunov function and the limit cycle can be constructed easily. The modified equation is

$$\ddot{y} + (y^2 + \dot{y}^2 - 1)\dot{y} + y = 0. \tag{2}$$

Multiplying equation (2) by $\dot{y}$ one obtains

$$\frac{1}{2}\frac{\partial}{\partial t}(y^2 + \dot{y}^2) = -\dot{y}^2(y^2 + \dot{y}^2 - 1). \tag{3}$$

Due to Lyapunov stability theorems one obtains

Stability if $\dot{y}^2 + y^2 > 1$,
Instability if $\dot{y}^2 + y^2 < 1$,
$\dot{y}^2 + y^2 = 1$ being the equation of the limit cycle.

2 Stability of a system of van der Pol-like oscillators

The existence theorems [2] for limit cycles are restricted to the case of a single oscillator. They cannot be extended to general systems of oscillators, in particular due to the possibility of more complex attractors like 'strange attractors' [4, 5]. Systems of oscillators of the kind given by (2) turn out to be more tractable as shown by author's work [3] and as explained below. Consider the following system

$$\ddot{Y} + ((Y, AY) + (\dot{Y}, B\dot{Y})N - P)\dot{Y} + CY = 0, \tag{4}$$

where Y is a real vector of arbitrary length r . A, B, C, M, N, P are real $r \times r$ matrices and (..,..) is the scalar product. These matrices can be split in symmetric A_s, B_s ... and antisymmetric parts A_a, B_a ... .

Assume $C_a = 0$ and A_s, B_s, M_s, N_s, P_s and C be positive definite with largest eigenvalues $\alpha_1, \beta_1, \mu_1, \nu_1, \pi_1, \gamma_1$ and lowest eigenvalues $\alpha_0, \beta_0, \mu_0, \nu_0, \pi_0$ and γ_0 respectively. Take the scalar product of (4) by $\dot{Y}$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}((\dot{Y}, \dot{Y}) + (Y, CY)) &= -((Y, A_s Y)(Y, M_s Y) + \\ &\quad (\dot{Y}, B_s \dot{Y})(\dot{Y}, N_s \dot{Y}) - (\dot{Y}, P_s \dot{Y})).\end{aligned} \tag{5}$$

Two inequalities can be extracted from (5)

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}((\dot{Y}, \dot{Y}) + (Y, CY)) &\leq -\beta_0\nu_0((\dot{Y}, \dot{Y}) + (Y, CY) + \\ (Y, (\frac{\mu_0}{\beta_0\nu_0}A_s - C)Y) - \frac{\pi_1}{\beta_0\nu_0})(\dot{Y}, \dot{Y}),\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}((\dot{Y}, \dot{Y}) + (Y, CY)) &\geq -\beta_1\nu_1((\dot{Y}, \dot{Y}) + (Y, CY) + \\ (Y, (\frac{\mu_1}{\beta_1\nu_1}A_s - C)Y) - \frac{\pi_0}{\beta_1\nu_1})(\dot{Y}, \dot{Y}).\end{aligned} \tag{7}$$

From inequality (7) we have instability around the origin, and in case

$$\frac{\mu_1}{\beta_1\nu_1}A_s - C \leq 0 \tag{8}$$

the instability persists if

$$(\dot{Y},\dot{Y}) + (Y, CY) \leq \frac{\pi_0}{\beta_1\nu_1}. \tag{9}$$

From inequality (6) and

$$\frac{\mu_0}{\beta_0\nu_0}A_s - C \geq 0 \tag{10}$$

it can be seen that the system is stable if

$$(\dot{Y},\dot{Y}) + (Y, CY) \geq \frac{\pi_1}{\beta_0\nu_0}. \tag{11}$$

This leads to the definition of an attractive region in the $(\dot{Y},\dot{Y}),(Y,CY)$ plane (see Fig. 2)

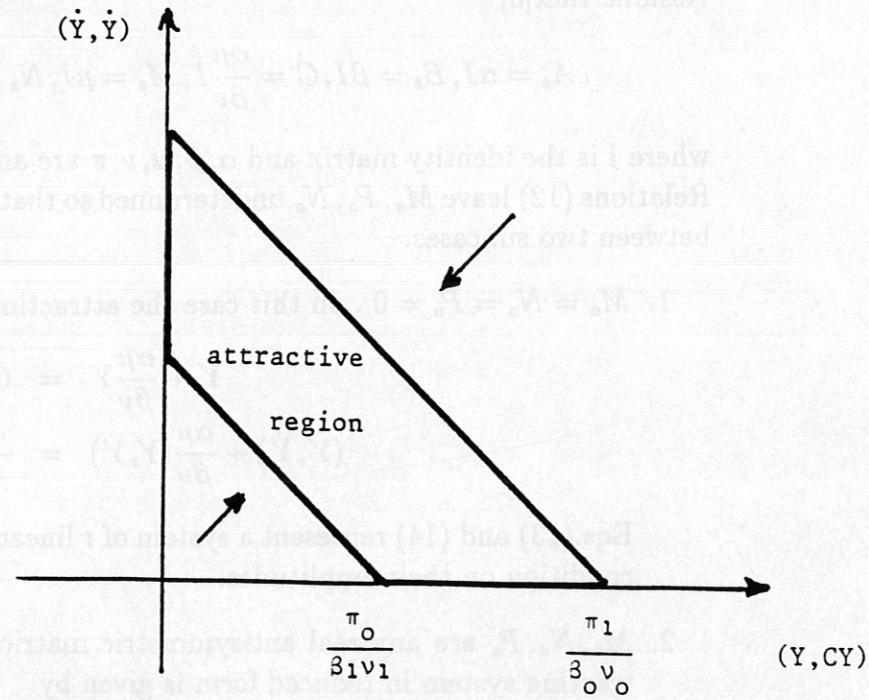

Fig. 2

Under Conditions (8),(10) any solution of system (4) will be 'trapped' after some time in the attractive region defined by the bounds on the righthand sides of (9) and (11). The detailed asymptotic behaviour is very difficult to study, and one should expect, in general, a kind of high dimensional 'strange attractor'.

If, however, the attractive region is zero in the $(\dot{Y},\dot{Y})$,(Y,CY) plane, then the attractor of system (4) obeys itself a system of conservative nonlinear oscillators [6] together with a 'shell' condition as will be shown below.

3 Special cases of attracting systems

Under special choices of the matrices A_s, B_s, M_s, N_s, P_s and C the attracting region of Fig. 2 can be made to shrink to zero.

3.1 First choice

Assume that[3]

$$A_s = \alpha I, B_s = \beta I, C = \frac{\alpha\mu}{\beta\nu} I, M_s = \mu I, N_s = \nu I, P_s = \pi I, \tag{12}$$

where I is the identity matrix and $\alpha, \beta, \mu, \nu, \pi$ are any real positive numbers. Relations (12) leave M_a, P_a, N_a undetermined so that we are led to distinguish between two subcases.

1. $M_a = N_a = P_a = 0$. In this case the attracting system is given by

$$\ddot{Y} + \frac{\alpha\mu}{\beta\nu} Y = 0, \tag{13}$$

$$(\dot{Y},\dot{Y}) + \frac{\alpha\mu}{\beta\nu}(Y,Y) = \frac{\pi}{\beta\nu}. \tag{14}$$

   Eqs.(13) and (14) represent a system of r linear oscillators with a 'shell' condition on their amplitudes.

2. M_a, N_a, P_a are any real antisymmetric matrices. In this case the attracting system in reduced form is given by

$$\ddot{Y} + ((Y,Y)M_a + (\dot{Y},\dot{Y})N_a - P_a)\dot{Y} + Y = 0 \ , \tag{15}$$

$$(\dot{Y},\dot{Y}) + (Y,Y) = \epsilon = \frac{\pi}{\beta\nu}. \tag{16}$$

Apart from the case of $r = 2$ oscillators which is completely integrable [7], system (15) is not expected to be, in general, integrable [8]. It is shown [6] that system (15) does not possess, in general, a Lagrangean formulation in terms of Y and that (16), the only known constant of motion, cannot play the role of a noncanonical Hamiltonian.

3.2 Second choice

Assume that[9]

$$B_s = \beta I, M_s = \mu I, N_s = \nu I, P_s = \pi I, C = \frac{\mu}{\beta\nu} A_s. \tag{17}$$

In contrast to (16), the new 'shell' condition is given by

$$(\dot{Y}, \dot{Y}) + (Y, CY) = const. \tag{18}$$

The attracting system is now

$$\ddot{Y} + ((Y,Y)M_a + (\dot{Y},\dot{Y})N_a - P_a)\dot{Y} + CY = 0. \tag{19}$$

4 Statistics of the attracting systems (15) and (19)

Introduce

$$X = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix},$$

the 'shell' conditions (16) and (18) become

$$\sum_{i=1}^{i=2r} x_i^2 \;=\; \epsilon, \tag{20}$$

$$\sum_{i=1}^{i=r} x_i^2 + \sum_{i,j=r+1}^{i=2r} x_i c_{ij} x_j \;=\; const. \tag{21}$$

It can be shown [6] that systems (15) and (19) represent incompressible flows in the phase space X or

$$\sum_{i=1}^{i=2r} \frac{\partial \dot{x}_i}{\partial x_i} = 2 \sum_{i,j=r+1}^{2r} x_i n_{ij} x_j = 0. \tag{22}$$

Liouville's theorem (22) and the shell conditions (20) and (21) allow us to define microcanonical distributions if an assumption of ergodicity is introduced. Canonical distributions can also be derived if the systems were in contact with an 'amplitude bath'.

These distributions lead for the first choice associated with (16) to an equipartition in the amplitude expectations (see discussion in [10]). The second choice leads to a richer class of attracting systems whose statistics generate a larger class of fluctuation spectra [9]. In particular, a $\frac{1}{f^2}$ fluctuation spectrum can be produced, which compares very well with experimental observations of magnetic fluctuations [11].

5 Connection with turbulence and outlook

In the previous section we were able to apply a slightly extended Gibbs procedure because we were in possession of constants of motion and of a Liouville theorem. Both properties are related to the fact that the attractive region of Fig. 2 has zero thickness. This corresponds to exact compensation of driving and damping at each time for each oscillator. Real situations of turbulence correspond to a huge region of attraction usually estimated from above only [12]. There is no way to squeeze it to zero by making some choice of interaction matrices. In other words we cannot use equilibrium statistics or some simple extension of it as in section 4.

Nonequilibrium statistics can, in principle, be formulated using the 'maximum entropy' approach advocated by Jaynes [13]. It consists in maximizing the entropy

$$S = -\int f \ln f dX, \tag{23}$$

subject to appropriate constraints which represent our knowledge about the system

$$\int f g_i dX = C_i. \tag{24}$$

It leads to

$$f = \frac{e^{-\sum \lambda_i g_i}}{Z}, \tag{25}$$

where the λ_i are the Lagrange multipliers and Z the normalizing extended partition function. The determination of $\lambda_i$in terms of the known C_i is made by inserting (25) into (24).

Unfortunately, the main problem in this procedure is in the choice of the constraints (see also [14, 15]) which have to be consistent with an extended Liouville equation for f

$$\frac{\partial f}{\partial t} + \frac{\partial(\dot{x}_i f)}{\partial x_i} = 0, \tag{26}$$

in particular for the time-independent case. This is very hard to check. The constraints could come from the experiment, which means that the phenomenology has to be done first. This is a matter of trial and error and one is never sure of the consistency with the Liouville equation. Even in equilibrium statistics the choice of constants of motion may be a problem but one is at least sure to be consistent with Liouville's equation.

Independently of the kind of statistics, the existence of an attractive region as demonstrated in section 2, is certainly useful especially if the thickness of the attractive region is not too large. Better estimates of this region would be desirable. Since dissipative fluid systems are expected to have asymptotically a large but finite number of determining modes [16], the existence of attractive regions and the refinement of their bounds may become a powerful tool in fluid and Plasma turbulence.

References

[1] B. van der Pol. *Phil. Mag. 43, 177 (1922)*

[2] P. Hagedorn. *'Nichtlineare Schwingungen' Akademie Verlagsgesellschaft Wiesbaden, 1978*

[3] H. Tasso. *Z. Naturforsch. 41a, 987 (1986)*

[4] S. Smale. *'Differentiable Dynamical Systems' Bull. Amer. Math. Soc. 73, 747 (1967)*

[5] D. Ruelle, F. Takens. *Comm. Math. Phys. 20, 167 (1971)*

[6] H. Tasso. *Z. Naturforsch. 42a, 1377 (1987)*

[7] H. Tasso. *ZAMP J. Appl. Math. Phys. 38, 143 (1987)*

[8] H. Tasso. *Z. Naturforsch. 42a, 1237 (1987)*

[9] H. Tasso. *Z. Naturforsch. 44a, 325 (1989)*

[10] H. Tasso. *IPP report 6/277 (Oct. 1988)*

[11] P. A. Duperrex et al. *Phys. Lett. 106 A, 133 (1984)*

[12] R. Temam. *'Infinite-Dimensional Dynamical Systems in Mechanics and Physics', Springer Verlag, New York, 1988*

[13] E. T. Jaynes. *Phys. Rev. 106, 620 (1957)*

[14] R. B. Evans. Proc. Maximum Entropy Formalism Conf. MIT 1978, The MIT Press, Cambridge

[15] J. Rissanen. *in Proc. 'Complexity, Entropy and the Physics of Information', Santa Fe Conf. Ed. W. H. Zurek, Addison-Wesley, California,1990*

[16] B. Nicolaenko, B. Scheurer, R. Temam. *Physica 16 D, 155 (1985)*