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FROM A GEOMETRIC POINT OF VIEW**

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¹ Present address:

Universität Bayreuth, Lehrstuhl für Angewandte Mathematik
8580 Bayreuth, Germany

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Abstract

This paper addresses the inverse problem of the existence of surface current MHD equilibria in toroidal geometry with vanishing magnetic field inside. Inverse means that the plasma-vacuum interface rather than the external wall or conductors are given and the latter remain to be determined. This makes a reformulation of the problem possible in geometric terms: what toroidal surfaces with analytic parametrization allow a simple analytic covering by geodesics? If such a covering by geodesics (field lines) exists, their orthogonal trajectories (current lines) also form a simple covering and are described by a function satisfying a nonlinear partial differential equation of the Hamilton-Jacobi type whose coefficients are combinations of the metric elements of the surface. All known equilibria – equilibria with zero and infinite rotational transform and the symmetric ones in the case of finite rotational transform – turn out to be solutions of separable cases of that equation and allow a unified description if the toroidal surface is parametrized in the moving trihedral associated with a closed curve. Analogously to volume current equilibria, the only continuous symmetries compatible with separability are plane, helical and axial symmetry. In the nonseparable case numerical evidence is presented for cases with chaotic behaviour of geodesics, thus restricting possible equilibria for these surfaces. For weak deviation from axisymmetry KAM-type behaviour is observed, i.e. destruction of geodesic coverings with a low rational rotational transform and preservation of those with irrational rotational transform. A previous attempt to establish three-dimensional surface current equilibria on the basis of the KAM theorem is rejected as incomplete, and a complete proof of the existence of equilibria in the weakly nonaxisymmetric case, based on the twist theorem for mappings, is given. Finally, for a certain class of strong deviations from axisymmetry an analytic criterion is formulated which rules out equilibria for these surfaces.

I. Introduction

The problem of the existence of magnetohydrostatic equilibria in toroidal geometry is of crucial importance to magnetic confinement in thermonuclear fusion research. Only a few classes of equilibria, all associated with certain symmetries, are so far known: equilibria with reflection symmetry with respect to a plane, leading to closed magnetic field lines and vanishing rotational transform [1], and equilibria with plane, helical and axial symmetry [2], of which only axial symmetry is compatible with toroidal geometry. It has been proved that the latter three symmetries are the only continuous symmetries which allow the problem to be reduced to a second-order elliptic partial differential equation of the Grad-Shafranov-type [3]. Furthermore, it is conjectured that there are no other equilibria at all [4] and analytical work [5] as well as numerical studies [6] of fully three-dimensional configurations with shear exhibit island formation and ergodic regions at rational surfaces, thus confirming the nonexistence of equilibria for these configurations.

Surface current equilibria are especially simple in that the current is confined to the plasma-vacuum interface and the configuration can consequently support only plasmas with constant pressure but complicated enough to show all the distinctive features which are characteristic of the full problem and which have just been specified. In fact, one of the aims of this paper is to demonstrate that the surface current model allows the same types of equilibria as the full (volume current) model and another two, not (yet) known for volume currents, viz. nonaxisymmetric equilibria with either infinite rotational transform, which have recently been established for surface currents [7], or with finite rotational transform but only weak deviations from axisymmetry, which are established in this paper. So, there is hope that information obtained for surface current equilibria is also beneficial to the general problem.

The main advantage of the surface current model is that its inverse formulation [8] (plasma-vacuum interface given, external wall or conductors to be determined) allows a geometric interpretation which boils the full problem down to investigation of geodesics on a given toroidal surface forming the plasma-vacuum interface. The geometric approach was advocated a long time ago by the author of [9], who applied the method to plane configurations and the axisymmetric circular torus, an equilibrium configuration which had been established shortly before [10]. Arbitrary axisymmetric configurations [11] and a nonaxisymmetric one with vanishing toroidal current [12] have been treated by different methods. It is shown in this paper that all these cases and some others (helical symmetry and infinite rotational transform) can be treated in an elegant and unified manner by using the geometric approach and an appropriate

coordinate system for the surface.

For three-dimensional configurations with finite rotational transform it was not clear till now whether the surface current model shows the same complicated behaviour as the full model, apart from the fact that a similar problem in classical mechanics, viz. force-free motion of a particle constrained to a surface [13], suggests Hamiltonian chaos in nonintegrable cases. This paper demonstrates the occurrence of chaotic behaviour also in the equilibrium problem, clarifies the relation to the mechanical problem and discusses the near-integrable case. The near-integrable case has already been discussed in [14], the authors of which claim to have proved the existence of weakly nonaxisymmetric equilibria with finite (irrational) rotational transform, using KAM theory for weakly perturbed Hamiltonian systems. In our opinion, their treatment of the problem, is, however, unsatisfactory. On the other hand, the existence of a simple geodesic covering of a weakly nonaxisymmetric toroidal surface seems to be well-known to mathematicians [8, 15] but not widely known to MHD theorists, nor does there seem to be an explicit presentation available. This justifies, to our mind, a treatment of that point in some detail. Also discussed are so-called minimal geodesics, which give some hints on the fate of a simple covering for increasing perturbation. Finally, for a certain class of three-dimensional surfaces, viz. surfaces with "big bumps", we are able to prove that no equilibria exist; this, to our knowledge, is the first rigorous nonexistence result in this context.

The paper is organized as follows. Section II contains an exposition of the equilibrium problem and reduces the full problem to a Hamilton-Jacobi-type equation. Section III deals with separable and section IV with nonseparable cases of that equation. Section V formulates the "big bump" criterion and section VI contains some concluding remarks. Two appendices are devoted to discussing the relation of the equilibrium problem to classical mechanics, an attempt to use it as a basis for proving the existence of three-dimensional equilibria, and the area-preserving property of the Poincaré mapping, introduced in section IV.

II. Formulation of the problem

Surface current equilibria are characterized by a current density $\mathbf{j}^*$ confined on the toroidal plasma-vacuum interface S .¹ Thus, inside the surface the pressure p is constant and, for simplicity, a vanishing magnetic field is assumed. Outside the surface the pressure vanishes and the magnetic field $\mathbf{B}$ is a vacuum field with potential λ :

$$\mathbf{B} = \nabla\lambda, \quad (1)$$

$$\Delta\lambda = 0. \quad (2)$$

The equilibrium condition now takes the form of two scalar conditions on S :

$$\mathbf{B}^* \cdot \mathbf{N} = 0, \quad (3)$$

$$\mathbf{B}^{*2} = 2p, \quad (4)$$

where $\mathbf{N}$ is the exterior normal of S . Equation (3) means that S is a magnetic surface and Eq. (4) requires constant magnetic field strength on S . The surface current is given by

$$\mathbf{j}^* = \mathbf{N} \times \mathbf{B}^*, \quad (5)$$

which means, in particular, that magnetic field lines and current lines are orthogonal.

If Eq. (1) is inserted in Eqs. (3), (4), Eq. (3) takes the form of an ordinary von Neumann boundary condition for λ ,

$$\mathbf{N} \cdot \nabla\lambda|_S = 0, \quad (6)$$

but Eq. (4) imposes an additional restriction on the surface derivatives of λ ,

$$\frac{G}{EG - F^2} \left(\frac{\partial\lambda^*}{\partial u} \right)^2 - \frac{2F}{EG - F^2} \frac{\partial\lambda^*}{\partial u} \frac{\partial\lambda^*}{\partial v} + \frac{E}{EG - F^2} \left(\frac{\partial\lambda^*}{\partial v} \right)^2 = 2p, \quad (7)$$

where $E(u, v) = \mathbf{x}_u \cdot \mathbf{x}_u$, $F(u, v) = \mathbf{x}_u \cdot \mathbf{x}_v$ and $G(u, v) = \mathbf{x}_v \cdot \mathbf{x}_v$ are the metric coefficients of the surface S , parametrized by the function $\mathbf{x}(u, v)$. So, the usual von Neumann boundary-value problem of potential theory (with, for example, a von Neumann condition also on an exterior wall) is not appropriate, because the solution will in general not satisfy Eq. (7). Instead, one can try to solve Eq. (7) first and then the initial-value problem posed by Eqs. (2), (6) and the equation

$$\lambda|_S = \lambda^*. \quad (8)$$

¹In this section a quantity with star means that the quantity is defined only on S or denotes the restriction of a quantity to S .

If S allows analytic parametrization and λ^* is an analytic function on S , it is proved in [9] using the existence theorem of Cauchy-Kovalevskaya that a solution of Eqs. (2), (6), (8) exists in a finite region outside S .² This region – the regularity domain – depends on the shape of and the initial values on the surface. Only in simple cases can singularities which bound the regularity domain be calculated explicitly [11, 16]; a rough estimate is that their distance from the surface is of the order of the radius of curvature of the surface.

In conclusion, if we restrict ourselves to the analytic case, the inverse problem reduces to a search for toroidal surfaces with analytic parametrization which allow a nontrivial analytic solution λ^* of Eq. (7). It should be noted that λ^* may be multi-valued but $\nabla\lambda^* = \mathbf{B}^*$ has to be single-valued on S .

To get a more geometric formulation of the problem, consider the magnetic field lines on S . From Eqs. (1), (3), (4) and the identity

$$\nabla\left(\frac{1}{2}\mathbf{B}^2\right) = \mathbf{B} \times (\nabla \times \mathbf{B}) + \frac{1}{B}\mathbf{B}(\mathbf{B} \cdot \nabla B) + B^2\boldsymbol{\kappa}, \quad (9)$$

with $\boldsymbol{\kappa} = (\mathbf{B}/B) \cdot \nabla(\mathbf{B}/B)$ being the curvature vector of the field line, it is concluded that $\boldsymbol{\kappa} \parallel \mathbf{N}$, the condition for magnetic field lines to be geodesics. Their orthogonal lines, the current lines, are equidistant [17], p. 41, and are just the lines $\lambda^* = \text{const}$ (see Eq. (1)). As the magnetic field must not vanish on S and field lines cannot intersect, one obtains the equivalent characterizations of the above problem: Find analytic toroidal surfaces which allow a simple³ analytic covering by geodesics or by equidistant lines (a more formal definition of a geodesic covering is given in section V).

For $\mathbf{x}(u, v)$ to be a proper parametrization of S it is required, that the lines $u = \text{const}$ close after one toroidal (the long way) and no poloidal (the short way) circuit and vice versa for lines $v = \text{const}$; furthermore, these lines should be nowhere tangential to one another. If the parametrization is orthogonal ($F \equiv 0$), Eq. (7) simplifies to

$$\frac{1}{E} \left(\frac{\partial \lambda^*}{\partial u} \right)^2 + \frac{1}{G} \left(\frac{\partial \lambda^*}{\partial v} \right)^2 = 2p. \quad (7a)$$

The contravariant components of the magnetic field,

$$\mathbf{B}^* = B^{*u} \mathbf{x}_u + B^{*v} \mathbf{x}_v, \quad (10)$$

²In fact, the proof in [9] refers only to orthogonal parametrizations, which does not suffice for our purposes, because it is not definite that any toroidal surface allows a global, orthogonal parametrization. But generalization of the proof to nonorthogonal parametrizations is straightforward.

³“Simple” means that one and only one line of the covering passes through every point of S .

take the form

$$\begin{aligned} B^{*u} &= \frac{G}{EG - F^2} \frac{\partial \lambda^*}{\partial u} - \frac{F}{EG - F^2} \frac{\partial \lambda^*}{\partial v}, \\ B^{*v} &= -\frac{F}{EG - F^2} \frac{\partial \lambda^*}{\partial u} + \frac{E}{EG - F^2} \frac{\partial \lambda^*}{\partial v}, \end{aligned} \quad (11)$$

or in the orthogonal case

$$B^{*u} = \frac{1}{E} \frac{\partial \lambda^*}{\partial u}, \quad B^{*v} = \frac{1}{G} \frac{\partial \lambda^*}{\partial v}. \quad (11a)$$

Integration of the ordinary differential equation

$$\frac{dv}{du} = \frac{B^{*v}}{B^{*u}} \quad (12)$$

finally leads to the field lines $v(u)$ and allows definition of the rotational transform ι :

$$\frac{1}{\iota} = \lim_{n \rightarrow \infty} \frac{1}{nl} \int_0^{2\pi n} \frac{B^{*v}}{B^{*u}}(u) du, \quad (13)$$

with u having the period 2π and v the period l .

This section concludes with three remarks.

1) The reason for concentrating on analytic surfaces and analytic functions λ^* is obviously the fact that in this case a powerful existence result is available, but the assumption is not as restrictive as it looks at first glance. Almost all configurations of physical interest can be represented by analytic surfaces and, concerning the initial-value problem, only an analytic λ^* is compatible with the condition (6) on an analytic initial surface, i. e. a nonanalytic solution λ^* of Eq. (7) cannot be continued into the exterior region of S [18]. It should also be noted that the initial-value problem for elliptic operators is ill-posed in the sense that the solution does not necessarily depend continuously on the initial data (see Hadamard's counter-example [19], p. 229). The mere existence problem, however, is not spoiled by this point.

2) If a nonvanishing magnetic field $\mathbf{B}_i$ is allowed inside the toroidal surface, a von Neumann problem for the interior of S has to be solved first. The induced boundary values $\mathbf{B}_i^*$ then appear in Eqs. (4), (5): B^{*2} and $\mathbf{B}^*$ have to be replaced by $B^{*2} - B_i^{*2}$ and $\mathbf{B}^* - \mathbf{B}_i^*$, respectively. So, the problem can still be reduced to solving an equation of the type Eq. (7) (now, with a nonconstant right-hand side), but characterization of the problem by a geodesic covering of the surface is no longer possible.

3) In an equilibrium configuration with magnetic surface S the fact that magnetic lines are geodesics does not necessarily mean that B is constant on S . In fact, with $\kappa \parallel \mathbf{N}$ it only follows from Eq. (9) that $\nabla_{\perp} B \parallel \mathbf{N}$, or, using Eq. (5),

$$\mathbf{j}^* \cdot \nabla B = 0. \quad (14)$$

Examples of configurations with geodesic magnetic field lines where B varies along the field lines are provided by axisymmetric mirrors, as mentioned in, for example, [20]. The statement after Eq. (9) asserts only that, given a simple geodesic covering of S , a vacuum magnetic field whose modulus is constant all over S can be found: The potential of that field, in fact, is given by a function $\tilde{\lambda}^*$ whose level lines are just the (equidistant) orthogonal trajectories of the geodesics. The condition of equidistantness reads

$$(\nabla \tilde{\lambda}^*)^2 = f(\tilde{\lambda}^*), \quad (15)$$

with $f(\tilde{\lambda}^*)$ reflecting the above-mentioned freedom. The redefined potential

$$\lambda^* = \int^{\tilde{\lambda}^*} \sqrt{\frac{2p}{f(\lambda)}} d\lambda \quad (16)$$

then satisfies Eq. (7), and its magnetic field is therefore constant on S .

III. Separable cases

All known surface current equilibria correspond to separable cases of Eq. (7) in the sense that all metric coefficients of S are independent of one coordinate, allowing a complete integration of Eq. (7), or at least that one coefficient is independent, allowing a special solution of Eq. (7).

An appropriate parametrization of an analytic toroidal surface is given by

$$\mathbf{x}(u, v) = \mathbf{C}(v) + \rho(u, v) [-\mathbf{n}(v) \cos u + \mathbf{b}(v) \sin u], \quad (17)$$

with $\mathbf{C}(v)$ being a closed curve with curvature κ and torsion τ , analytically parametrized by its arc length v , $\rho(u, v)$ being an analytic positive function of the variable u (period 2π) and v (period $l(\mathbf{C})$) and $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is the moving trihedral associated with $\mathbf{C}$ (see Fig. 1). (Note that for a plane curve $\mathbf{n}$ always points in the convex direction and, therefore, in the ordinary torus the line $u = 0$ describes the outer equator.) It is further assumed that the generating curve $\mathbf{C}$ and the radius ρ are chosen in such a way that S does not intersect itself; this means especially that

$$\kappa \rho \cos u > -1, \quad \forall u, v. \quad (18)$$

With the *Serret-Frenet* formulas

$$\mathbf{t}' = \kappa \mathbf{n}, \quad \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \mathbf{b}' = -\tau \mathbf{n}, \quad (19)$$

where the prime denotes derivation with respect to the arc length v , the metric coefficients become

$$E = \rho^2 + \dot{\rho}^2, \quad F = -\tau \rho^2 + \dot{\rho} \rho', \quad G = \Delta^2 + \tau^2 \rho^2 + \rho'^2, \quad (20)$$

with

$$\Delta = 1 + \kappa \rho \cos u, \quad (21)$$

and the dot denotes derivation with respect to the angle u .

A. Symmetric configurations

Symmetric surfaces are mathematically described as surfaces allowing one or several one-parameter groups of isometries. Isometries are one-to-one mappings of the surface which leave the metric invariant. Locally, it is always possible to choose a parametrization such that the lines $u = \text{const}$ are the trajectories of the isometry group and the lines $v = \text{const}$ are their orthogonals; the metric coefficients then take the form

$$E = E(u), \quad F = 0, \quad G = G(u). \quad (22)$$

These are, however, the metric coefficients of a surface of revolution (in appropriate parametrization) and we have the result [21]: A surface admits a one-parameter group of isometries if and only if it is isometric to a surface of revolution. This class includes surfaces with plane symmetry ($G = \text{const}$) and helical symmetry as well. To realize the latter statement, we parametrize the helical surface by

$$\mathbf{x}(\bar{u}, \bar{v}) = \rho(\bar{u}) \cos \bar{v} \mathbf{e}_x + \rho(\bar{u}) \sin \bar{v} \mathbf{e}_y + [\rho(\bar{u}) + h \bar{v}] \mathbf{e}_z, \quad (23)$$

with $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ being a Cartesian frame. The metric coefficients are

$$\bar{E} = \rho^2 + \dot{\rho}^2, \quad \bar{F} = h \dot{\rho}, \quad \bar{G} = \rho^2 + h^2, \quad (24)$$

which, after a change of parameters

$$u = \bar{u}, \quad v = \bar{v} + \int^{\bar{u}} \frac{h \dot{\rho}}{\rho^2 + h^2}(\tilde{u}) d\tilde{u}, \quad (25)$$

take the form of Eq. (22).

The main advantage of surfaces with a metric of form (22) – and, as far as is known, they are the only ones – is the possibility of a complete integration of Eq. (7a). In fact, the ansatz [9]⁴

$$\lambda = C v + f(u), \quad (26)$$

with C being a real constant, reduces Eq. (7a) to a quadrature:

$$\lambda(u, v) = C v + \int^u \sqrt{E(2p - C^2/G)}(\tilde{u}) d\tilde{u}. \quad (27)$$

This result is the equivalent of a result derived in [3] for volume current equilibria, stating that the plane, helical and axial symmetries are the only continuous symmetries allowing a reduction of the problem to a second-order elliptic partial differential equation which has solutions under quite general assumptions.

The only surface in this class, which is toroidally closed, is of course the axisymmetric torus. A suitable parametrization is obtained if in Eq. (17) $C(v)$ is chosen as a circle ($\kappa = \text{const}$, $\tau = 0$) and ρ depends solely on u . The metric coefficients in Eq. (20) are then of the form of Eq. (22), and λ (and by this the current lines) is the result of a quadrature:

$$\lambda(u, v) = C v + \int^u \sqrt{(\rho^2 + \dot{\rho}^2)(2p - C^2/\Delta^2)}(\tilde{u}) d\tilde{u}, \quad (28)$$

⁴In this and the following sections we again omit the star because all quantities occurring henceforth are defined on the surface only.

with Δ given in Eq. (21). The magnetic field (Eq. (11a)) is given explicitly by

$$B^u = \sqrt{\frac{2p - C^2/\Delta^2}{\rho^2 + \dot{\rho}^2}}, \quad B^v = \frac{C}{\Delta^2}, \quad (29)$$

and the magnetic field lines (or geodesics) again by a quadrature:

$$v(u) = \int^u \frac{1}{\Delta} \sqrt{\frac{\rho^2 + \dot{\rho}^2}{2p\Delta^2/C^2 - 1}}(\tilde{u}) d\tilde{u}. \quad (30)$$

The problem of a geodesic covering is most judiciously solved by a discussion of all possible geodesics on the surface. This was done in [22] for the case of a circular poloidal cross-section ($\rho = \text{const}$)⁵ by an integration of Eq. (30) in terms of elliptic functions, and in [23] for arbitrary convex cross-sections.

The qualitative behaviour of geodesics on a surface of revolution can be read from *Clairaut's* formula, connecting the angle β of the geodesic against an azimuthal line and its distance d from the axis of symmetry:

$$d \cos \beta = \text{const}. \quad (31)$$

As a result, five different types of geodesics can be distinguished: the meridians, the inner and the outer equator, geodesics crossing the outer equator at an angle β given by $\cos \beta = \cos \beta_c := d_i/d_o$ (d_i and d_o being the distances of the inner and the outer equator, respectively, from the axis of symmetry) and approaching the inner equator asymptotically, geodesics winding around the torus ($\pi/2 > |\beta| > \beta_c$) and geodesics wriggling around the outer equator ($\beta_c > |\beta| > 0$). From these only the geodesics of the first and fourth types allow a geodesic covering of the surface. They correspond to solutions λ of Eq. (28) with $C = 0$ and $0 < C^2 < 2p\kappa^2 d_i^2$, respectively. Note that these solutions are analytic, as far as $\rho(u)$ is a proper analytic parametrization and Eq. (18) is satisfied. The rotational transform (Eq. (13)) is infinite in the former case and varying in the interval $(0, \infty)$ in the latter.

Two remarks conclude this subsection.

- 1) If a surface admits more than one one-parameter group of isometries, it is of constant curvature [21]. For toroidal surfaces this constant has to be zero because of the *Gauss-Bonnet* theorem. The only surfaces of everywhere vanishing curvature are cylinders, cones and tangent surfaces [17], p. 31, 57. Thus, any symmetry further to axisymmetry is not possible for toroidal surfaces.
- 2) If the convexity constraint on the cross-section of an axisymmetric configuration is

⁵This case was also treated in [9], on the basis of a discussion of the magnetic field (Eq. (29)) and corresponding current field.

dropped, more complex behaviour of the geodesics is possible. For example, several local inner equators with geodesics approaching them asymptotically and local outer equators with geodesics wriggling around them may occur. Only geodesics winding around the torus, however, allow geodesic coverings in this case as well.

B. Configurations with infinite rotational transform

Configurations of this type have poloidally closed magnetic field lines. They have only recently been discovered [7] and are important in that they constitute toroidally closed configurations that are nonaxisymmetric and do not use any discrete symmetry, either.

For two classes of surfaces these purely poloidal equilibria can easily be constructed. If the generating curve $C(v)$ is an arbitrary analytic space curve ($\tau \neq 0$) and the radius ρ is constant, Eq. (17) describes a so-called canal surface. Equation (7) then takes the form

$$\frac{\Delta^2 + \rho^2 \tau^2}{\rho^2 \Delta^2} \left(\frac{\partial \lambda}{\partial u} \right)^2 - \frac{2\tau}{\Delta^2} \frac{\partial \lambda}{\partial u} \frac{\partial \lambda}{\partial v} + \frac{1}{\Delta^2} \left(\frac{\partial \lambda}{\partial v} \right)^2 = 2p \quad (32)$$

with the special analytic solution

$$\lambda(u, v) = \sqrt{2p} \rho \left[u + \int^v \tau(\tilde{v}) d\tilde{v} \right]. \quad (33)$$

The magnetic field (Eq. (11)) is, as expected, purely poloidal:

$$B^u = \frac{\sqrt{2p}}{\rho}, \quad B^v = 0. \quad (34)$$

If the generating curve is plane ($\tau \equiv 0$), an arbitrary poloidal cross-section is allowed ($\rho = \rho(u)$). The parametrization equation (17) is now orthogonal and an analytic solution of Eq. (7a) is

$$\lambda(u, v) = \sqrt{2p} \int^u \sqrt{\rho^2 + \dot{\rho}^2}(\tilde{u}) d\tilde{u}. \quad (35)$$

Note that in both cases the magnetic field lines are plane curves. If nonvanishing torsion is combined with arbitrary cross-section, it is readily shown that plane poloidal sections are no longer geodesics (compare the surface normal with the normal of the curve!). Thus, if purely poloidal equilibria exist in this case, the field lines are necessarily distorted. The existence of such equilibria is expected but not yet proved.

Remark: In the case of non-vanishing torsion the system of coordinates u, v can be locally orthogonalized by the following transformation:

$$\bar{u} = u + \int_0^v \tau(\tilde{v}) d\tilde{v}, \quad \bar{v} = v. \quad (36)$$

To make the system $\bar{u}, \bar{v}$ global, one has to ensure that the curves $\bar{u} = \text{const}$ close after one toroidal and no poloidal circuit. This puts the following condition on the torsion:

$$\oint \tau d\tilde{v} = 0, \quad (37)$$

which is trivially satisfied by plane curves, but there are also space curves satisfying this condition, e.g. curves on a sphere [17], p. 160. Condition (37) characterizes at the same time surfaces which combine poloidally closed field lines and toroidally closed current lines.

C. Configurations with vanishing rotational transform

Configurations of this type were constructed a long time ago in an attempt to find equilibria without toroidal current [12]. Only two aspects are to be considered here, viz. how the equilibrium fits into our framework, and the importance of a discrete symmetry for the existence of a solution. A more physical discussion of the equilibrium may be found in [12].

If in Eq. (17) the lines $v = \text{const}$ are chosen as (poloidally closed) current lines, the condition of equidistantness reads

$$G - F^2 / E = f(v), \quad (38)$$

with a function f depending on v only; a special solution of Eq. (7) is then

$$\lambda(u, v) = \sqrt{2p} \int^v \sqrt{f(\tilde{v})} d\tilde{v}. \quad (39)$$

For a plane generating curve ($\tau \equiv 0$) Eq. (38) is equivalent to a nonlinear partial differential equation for the unknown function $\rho(u, v)$:

$$\frac{\rho'^2}{f - \Delta^2} - \frac{\dot{\rho}^2}{\rho^2} = 1. \quad (40)$$

By introducing new variables x, α (see Fig. 2) of the form

$$x = \rho \cos u, \quad \tan(u - \alpha) = \frac{\dot{\rho}}{\rho}, \quad (41)$$

Eq. (40) can be transformed into a linear differential equation for the unknown function $x(v, \alpha)$:

$$x_v + g_x \sin \alpha x_\alpha = g \cos \alpha, \quad (42)$$

with

$$g(v, x) := \pm \sqrt{f(v) - [1 + \kappa(v)x]^2}. \quad (43)$$

The characteristic curves of Eq. (42) are just the magnetic field lines [12] and are described by

$$v' = 1, \quad (44)$$

$$x' = g \cos \alpha, \quad \alpha' = g_x \sin \alpha. \quad (45)$$

Equation (44) states that v serves as curve parameter. One may introduce for convenience the variable $y = \rho \sin u$, and once a solution $x(v)$, $\alpha(v)$ of the system Eq. (45) is determined, $y(v)$ is given by quadrature of the equation

$$y' = g \sin \alpha. \quad (46)$$

The essential point now is to ensure that the magnetic lines close after one toroidal and no poloidal circuit. This can be achieved by choosing a generating curve with a $2n$ -fold symmetry, such as in [12],

$$R(\varphi) = 1 - \frac{\cos(2n\varphi)}{1 + 4n^2}, \quad (47)$$

with R , φ here being polar coordinates in the equatorial plane (see Fig. 3). Note that the curvature is symmetric with respect to all symmetry points:

$$\varphi_\nu = \nu\pi/n, \quad \nu = 0, 1, \dots, 2n - 1, \quad (48)$$

and vanishes especially there. If in Eq. (43) $f(v)$ is chosen as $f \equiv 1$, x' and α' also vanish in the symmetry points, which means that the solutions of Eq. (45), constituting the toroidal surface, show an n -fold symmetry and therefore close. It remains to show that the surface does not self-intersect. In [12] only the innermost and outermost magnetic lines were discussed (see Fig. 3) and it was concluded that only for $n \geq 3$ can self-intersection be avoided. In general, the system of equations (45) has to be integrated with an arbitrary poloidal section as initial values. Two examples of such an integration performed between two consecutive symmetry planes are shown in Figs. 4a, 4b. They demonstrate that arbitrary initial sections lead to self-intersections, but that they can be avoided if the initial section is chosen sufficiently convex and smooth. Finally, it should be noted, that an analytic initial section and analytic solutions of Eq. (45) allow analytic parametrization of the surface.

Remark: If the restriction of plane current lines is dropped, more general equilibria are expected to exist [24]. The bumpiness of the configuration, however, is unavoidable and a direct consequence of the equidistantness of the current lines.

IV. The nonseparable case

In this section initially some simple nonseparable cases are investigated by means of a numerical integration of the geodesic equation. It is found that in the cases considered the geodesics show – depending on the rotational transform and the degree of nonaxisymmetry – regular behaviour (simple covering), quasiregular behaviour (covering with self-intersection and possibly gaps) or chaotic behaviour (no covering at all). For the case of weak deviation from axisymmetry the existence of a simple covering is proved by means of the twist theorem. Finally, a special class of geodesics (“minimal geodesics”) is discussed which allows a qualitative understanding of the fate of a simple covering for increasing deviation from axisymmetry.

Functions $u(t)$, $v(t)$ with arbitrary parameter t , describing a geodesic with finite rotational transform, have to obey the nonlinear ordinary differential equation [17], p. 41,

$$\frac{d^2 u^1}{dt^2} \frac{du^2}{dt} - \frac{d^2 u^2}{dt^2} \frac{du^1}{dt} + \left(\frac{du^2}{dt} \Gamma_{ik}^1 - \frac{du^1}{dt} \Gamma_{ik}^2 \right) \frac{du^i}{dt} \frac{du^k}{dt} = 0, \quad (49)$$

with Γ_{ik}^r being the Christoffel symbols

$$\Gamma_{ik}^r = \frac{g^{lr}}{2} \left(\frac{\partial g_{il}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^l} + \frac{\partial g_{lk}}{\partial u^i} \right), \quad (50)$$

g_{ik} the metric coefficients

$$g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G, \quad (51)$$

g^{lr} its inverse, and $u^1 = u$, $u^2 = v$ the surface parameters. If v is taken as curve parameter, Eq. (49) takes the form

$$u'' - \Gamma_{11}^2 u'^3 + (\Gamma_{11}^1 - 2\Gamma_{12}^2) u'^2 - (\Gamma_{22}^2 - 2\Gamma_{12}^1) u' + \Gamma_{22}^1 = 0. \quad (52)$$

In the axisymmetric case (see Eqs. (20) - (22)) Eq. (52) further simplifies to ⁶

$$u'' + \left(\frac{\dot{E}}{2E} - \frac{2\dot{\Delta}}{\Delta} \right) u'^2 - \frac{\Delta \dot{\Delta}}{E} = 0, \quad (53)$$

which is equivalent to

$$u'' u' + \left(\frac{E'}{2E} - \frac{2\Delta'}{\Delta} \right) u'^2 - \frac{\Delta \Delta'}{E} = 0, \quad (54)$$

⁶As already noted in section III, the dot and prime mean derivation with respect to u and v , respectively.

and allows a first integration (compare Eq. (30)):

$$\frac{E}{\Delta^4} u'^2 + \frac{1}{\Delta^2} = \bar{C}. \quad (55)$$

In the nonaxisymmetric case at least one of the metric coefficients also depends on v and no further integrable case of Eq. (49) is known.

A. Numerical results

Equation (52) for the geodesic lines is solved numerically for two classes of nonaxisymmetric surfaces and for a large number of initial values. The surfaces are specified by the parametrization $\mathbf{x}(u, v)$ as given by Eq. (17), based on a generating curve $\mathbf{C}(v)$. The distance from $\mathbf{C}$, assumed to be plane, to the center of a polar coordinate system defines the major radius $R(v)$. In the first class this major radius varies toroidally according to

$$R(v) = R_0 [1 + R_1 \sin(m_R v)], \quad (56)$$

with integer m_R , while the minor radius $\rho(u, v) = \rho_0$ stays constant so that the poloidal cross-section is a fixed circle. The second class has a constant major radius $R(v) = R_0$, but the poloidal cross-section is a function of the toroidal angle v :

$$\rho(u, v) = \rho_0 + \rho_1 \sin(m_r v), \quad (57)$$

with integer m_r . We shall also briefly mention results for a circular torus ($R, \rho = \text{const}$) with a single bump on the surface.

The periodicity length l of the variable v around the full torus is fixed at 2π in Eqs. (56), (57). In order to reconcile this with the arc length property of v , inherent in Eqs. (19), (20), the average R_0 of the major radius has to be adapted accordingly. For a non-constant major radius, $R_1 \neq 0$, this leads to a deviation from $R_0 = 1$, the value valid for $R = \text{const}$.

For the integration a fourth-order modified predictor-corrector method with automatic adjustment of the increment during the computation is used.

Apart from observation of geodesics $u(v)$ as such, it is advantageous to monitor their behaviour on a fixed poloidal or toroidal cut. In this subsection a poloidal cut is used throughout. If a geodesic line is to produce a simple covering of the surface, the slope γ with which the cut is crossed must be a smooth function of the position on the cut. In other words, the repeated mapping

$$(\gamma, u)_n \rightarrow (\gamma, u)_{n+1}, \quad (58)$$

$n = 0, 1, \dots, N$, from one intersection to the next must give a smooth invariant curve $\gamma = \gamma(u)$, in the limit $N \rightarrow \infty$. Several such mappings which are obtained from different starting values γ_0, u_0 can be combined into a single picture. This facilitates the discussion of the results.

Figure 5, as a basis for comparison, shows the iterated mapping (58) obtained for an axisymmetric torus, with major radius $R = R_0 = 1$ and circular poloidal cross-section with radius $\rho = \rho_0 = 0.2$. Eight geodesic lines are started at the same position $u_0 = 0$ (crossing with the outer equator) on a poloidal cut $v = v_0 = \text{const}$ with slopes γ_0 , measured with respect to lines $u = \text{const}$, and ranging equidistantly from 51.5° to 72.5° . (The angle γ , as employed here, is defined in the u, v plane by $\tan \gamma = u'(v_0)$. The angle β , in contrast, see section III and appendix B, refers to the angle on the toroidal surface.) Smooth curves are obtained (in the limit $N \rightarrow \infty$), in conformity with Eq. (30). For the lowest γ_0 the curve $\gamma(u)$ does not cover the whole periodicity interval $0 \leq u \leq 2\pi$, since for this small starting angle the geodesic belongs to those that wriggle around the equator, see section III.

For interpretation of the results in the nonaxisymmetric cases below it is helpful to know how the rotational transform ι of the geodesics, Eq. (13), depends on the starting angle γ_0 , in the axisymmetric case. In Fig. 6 this is shown for $R = 1$ and two values of ρ_0 , and again with $u_0 = 0$. It is evident that ι drops to zero extremely steeply ($\sim 1/\ln(\gamma_0 - \gamma_c)$) as γ approaches its critical value γ_c below which the orbits cease to make full poloidal turns.

If the axisymmetry of the toroidal surface is perturbed by a modulation of the major radius according to Eq. (56), with, for example, $R_1 = 0.05$ and $m_R = 2$, corresponding to an elliptic deformation, Fig. 5 changes into Fig. 7: Some of the curves survive in slightly modified form, while others are destroyed and replaced by a broad chaotic region, to which we come back below. A more detailed analysis using more closely spaced initial values γ_0 shows that the curves with rational $\iota = m/n$ are destroyed. In their place nested islands, again with $\iota = m/n$, appear, see, for example, Fig. 8a, showing one of the $\iota = 4/1$ islands which are nested inside each other, and two surviving curves with $\iota \approx 3.7447$ and 4.2959 which confine the $\iota = 4$ island family. (It was only by chance, that no islands were encountered at the parameters used in Fig. 7.)

Further analysis reveals that the outer region of the island families themselves are adorned by chaotic regions and a subisland structure. Figure 8b shows these effects, in a magnification of the end region of the $4/1$ islands. On the inner equator, $u = \pi$, eleven equidistant starting slopes γ_0 were used.

The broad chaotic region in Fig. 7 fits into the picture if one remembers the ex-

tremely large shear $d\iota/d\gamma_0$ in the neighbourhood of the separatrix in the axisymmetric case (see Fig. 6). In the perturbed case the resulting islands overlap and thus create an extended chaotic region.

The qualitative picture obtained does not change if the poloidal cut $v_0 = 0$, used in Figs. 7 - 8, is changed to another position.

Although the mapping considered so far is in general not area-preserving, its features are reminiscent of the KAM-type behaviour of area-preserving maps [25]. That KAM theory is indeed applicable is discussed in the next subsection and appendix B.

After having discussed the properties of the mapping induced by the geodesics as they cross a poloidal cut, we return to the global behaviour of the geodesics on the toroidal surface S as a whole. As expected and more closely considered in the next subsection, geodesic curves whose iterated mapping gives a 2π -periodic, smooth curve also give a simple covering of S , i.e. a covering without self-intersection and without gaps, in the limit of infinite continuation. Figure 9, for example, corresponds to such a case. It shows twenty toroidal revolutions of a geodesic line projected onto the u, v plane and corresponds to the upper curve of the iterated mapping in Fig. 8a. Infinitely many coverings of this type, and hence infinitely many MHD equilibria, characterized by their irrational value of ι thus exist for the nonaxisymmetric surface considered.

Figure 10a, in contrast, shows a geodesic orbit which neither covers the whole surface nor is free of self-intersections. Its mapping on the poloidal cut gives the island shown in Fig. 8a. An orbit which also intersects itself but seems to cover the whole surface in the long run is presented in Fig. 10b. Its mapping yields the chaotic island boundary in Fig. 8b. A further variant of self-intersecting geodesic orbits is shown in Fig. 11. The orbit ergodically switches between wriggling around the outer equator and full poloidal turns around the torus. Its starting values $u_0 = 0$ and $\gamma_0 = 54.5^\circ$ are in the large chaotic domain in Fig. 7.

In order to corroborate our numerical results, in particular the finding that simple coverings exist for nonaxisymmetric surfaces in a KAM-like fashion, we also investigate another type of nonaxisymmetry, as described by Eq. (57). We take $m_r = 1$. This implies a fat torus around $v = \pi/2$ and a thin torus at the opposite side, around $v = 3\pi/2$. Our figures refer to $R_0 = 1$, $\rho_0 = 0.1$ and $\rho_1 = 0.02$. Figure 12, analogously to Fig. 7, shows the mapping at a poloidal cut, $v_0 = 3\pi/2$, for eight starting angles, ranging equidistantly from 37° to 58° . The starting point is again the outer equator. The nonaxisymmetry in this case is quite strong so that only one of the initial slopes leads to an invariant curve with full poloidal interval $0 \leq u \leq 2\pi$. Another one produces an island, while the others end up in chaos. A magnification of this figure

with fifteen equidistant starting slopes, this time starting at the inner equator, is shown in Fig. 13. Invariant curves, islands, subislands and chaotic island boundaries are evident, qualitatively just as in the case considered before. A new type of geodesic can be observed which does not make it toroidally around the torus. Instead, the orbit asymptotically approaches a poloidally closed curve. Since the aim here, however, is not to categorize all classes of geodesics as such we do not further dwell on it.

If the amount of nonaxisymmetry is increased in both cases considered above, the percentage of self-intersecting geodesics increases, while the surface-covering geodesics are harder and harder to find. In addition, gaps would be expected to appear in otherwise smooth coverings (see subsection C below) if the nonaxisymmetry becomes too strong. As an extreme example we also investigated an axisymmetric torus with a single big bump. We did not observe such covering, however. Either self-intersecting orbits similar to Figs. 10b and 11 are obtained or orbits which asymptotically approach a closed curve. Nonintersecting coverings with gaps, therefore, are either of small measure compared with other types or they are destroyed by unavoidable numerical errors.

B. The near-integrable case

In the last section numerical evidence was presented for the preservation of invariant curves in the Poincaré mapping generated by the intersection of the circulating geodesics with a fixed meridian. A rigorous proof of this and hence of the existence of a simple covering of a weakly nonaxisymmetric toroidal surface can be given with the help of the twist theorem [26, 27]. The twist theorem guarantees the existence of invariant curves with “sufficiently irrational” rotational transform in area-preserving mappings of the annulus which are close to a monotonic twist mapping. The purpose of this section is to show in some detail how to apply this theorem and that all prerequisites can be met.⁷

An appropriate mapping is now generated by the intersection of the geodesics with the outer equator $u = 0$. As coordinates having the area-preserving property, one may use a variable proportional to the arc length along the outer equator between consecutive intersection points and the cosine of the angle between the equator and geodesic (see [28] or, for a derivation in the Hamiltonian framework, appendix B). In the unperturbed (axisymmetric) case an angle variable θ and a radial variable r may

⁷A previous attempt to prove the existence of equilibria for weakly nonaxisymmetric configurations in the framework of KAM theory is discussed in appendix A.

be defined by

$$\theta := 2\pi v/l, \quad (59)$$

$$r := \cos \beta = \left(1 + \frac{E}{G} \left/ \left(\frac{dv}{du} \right)^2 \right)^{-1/2} \quad (60)$$

with $v(u)$ given in Eq. (30). In these coordinates the Poincaré mapping takes the form of a monotonic twist mapping:

$$\theta_1 = \theta_0 + 2\pi q(r_0), \quad (61)$$

$$r_1 = r_0 \quad (62)$$

of the annulus $A : r' \leq r \leq r''$ (corresponding to $\beta_c < \beta'' \leq \beta \leq \beta' < \pi/2$ with β_c defined in section III.A). $q(r_0)$ is given by

$$q(r_0) = \frac{1}{l} \int_0^{2\pi} \frac{1}{\Delta} \sqrt{\frac{\rho^2 + \dot{\rho}^2}{2p\Delta^2/C^2 - 1}} du, \quad (63)$$

where the right-hand side depends via the parameter $2p/C^2$ on the initial gradient $(dv/du)_0$ and thereby on r_0 . This dependence is monotonic,

$$\frac{dq}{dr_0} > 0, \quad (64)$$

and analytic, as can easily be verified. Equation (62) is a direct consequence of *Clairaut's* formula (31).

For small deviations from axisymmetry,

$$\max_{u \in [0, 2\pi], v \in [0, l]} \left(|g_{ik} - g_{ik}^0| + \left| \frac{\partial g_{ik}}{\partial u_r} - \frac{\partial g_{ik}^0}{\partial u_r} \right| \right) < \delta, \quad (65)$$

with g_{ik}^0 describing the axisymmetric case, circulating geodesics remain circulating and are described by Eq. (49) or, using θ instead of v and u as curve parameter, by an equation of the form

$$\ddot{\theta} + \sum_{i=0}^3 C_i \dot{\theta}^i = 0, \quad (66)$$

with $C_i(\theta, u)$ depending analytically on θ and u . The solution

$$\theta = T(u; \theta_0, r_0) \quad (67)$$

of Eq. (66) then also depends analytically on u and the initial values θ_0 and r_0 (this is a consequence of the unique solvability of Eq. (66) and the *Cauchy-Kovalevskaya* theorem). The perturbed twist mapping now takes the form

$$\begin{aligned} \theta_1 &= \theta_0 + 2\pi q(r_0) + f(\theta_0, r_0), \\ r_1 &= r_0 + g(\theta_0, r_0), \end{aligned} \quad (68)$$

with

$$\begin{aligned} f(\theta_0, r_0) &= T(2\pi; \theta_0, r_0) - 2\pi q(r_0), \\ g(\theta_0, r_0) &= R(2\pi; \theta_0, r_0) - r_0 \end{aligned} \quad (69)$$

and ⁸

$$R(u; \theta_0, r_0) := \left(\frac{\tilde{F}}{\tilde{G}} + \frac{dT}{du} \right) \left(\frac{\tilde{E}}{\tilde{G}} + 2 \frac{\tilde{F}}{\tilde{G}} \frac{dT}{du} + \left(\frac{dT}{du} \right)^2 \right)^{-1/2}. \quad (70)$$

In the perturbed case the coordinate θ is in general no longer proportional to the arc length along the outer equator $u = 0$. In order to restore a variable proportional to the arc length as curve parameter, another coordinate transformation is necessary:

$$\theta \rightarrow \bar{\theta}(\theta), \quad (71)$$

where $\bar{\theta}$ is an analytic invertible function of θ with the periodicity property $\bar{\theta}(\theta + 2\pi) = \bar{\theta} + 2\pi$. Moreover, $\bar{\theta}$ depends continuously on the unperturbed case:

$$|\bar{\theta}(\theta) - \theta| \rightarrow 0 \quad \text{for } \delta \rightarrow 0. \quad (72)$$

Written in the variables $\bar{\theta}, r$ the perturbed twist mapping is area preserving and has the same form as Eq. (68)

$$\begin{aligned} \bar{\theta}_1 &= \bar{\theta}_0 + 2\pi q(r_0) + \bar{f}(\bar{\theta}_0, r_0), \\ r_1 &= r_0 + \bar{g}(\bar{\theta}_0, r_0), \end{aligned} \quad (73)$$

with modified perturbations $\bar{f}, \bar{g}$. These are analytic in $\bar{\theta}_0$ and r_0 and satisfy for a given $\bar{\delta} > 0$

$$\max_{r_0, \bar{\theta}_0 \in A} (|\bar{f}(r_0, \bar{\theta}_0)| + |\bar{g}(r_0, \bar{\theta}_0)|) < \bar{\delta} \quad (74)$$

if δ is sufficiently small. Equation (74) follows from the fact that solutions of Eq. (66) depend continuously on the coefficients C_i , and from Eq. (72).

All prerequisites of the twist theorem are thus satisfied and its analytic version [27] guarantees the existence of an invariant curve

$$\bar{\theta}_1 = \theta' + h(\theta'), \quad (75)$$

$$r = r_0 + k(\theta'), \quad (76)$$

with h, k analytic, satisfying

$$\max_{\theta' \in [0, 2\pi]} (|h(\theta')| + |k(\theta')|) < \epsilon \quad (77)$$

⁸ $\tilde{E}, \tilde{F}, \tilde{G}$ are the metric coefficients with respect to the coordinates u, θ .

for a given $\epsilon > 0$ ($\bar{\delta}$ depending on ϵ) and r_0 so that $q(r_0)$ satisfies

$$\left| q(r_0) - \frac{m}{n} \right| \geq \frac{c}{n^\mu} \quad (78)$$

for some $c, \mu > 0$ and all integer $n, m > 0$.

Equation (75) may be inverted and defines, together with Eqs. (76) and (71), an analytic function

$$r = \tilde{r}(\theta), \quad (79)$$

which defines, if inserted in Eq. (67), a one-parameter field of geodesics:

$$\theta = T(u; \theta_0, \tilde{r}(\theta_0)) =: \tilde{T}(u; \theta_0). \quad (80)$$

Equation (80) provides the sought-after geodesic covering of the weakly nonaxisymmetric toroidal surface S .

For completeness, one should still show that different geodesics of Eq. (80) do not intersect and every point of S is reached by some geodesic. To provide an answer to the former point, assume that the geodesic $\tilde{T}_0(u) := \tilde{T}(u; \theta_0)$ is intersected by $\tilde{T}_1(u)$ with $\theta_1 < \theta_0$, and consider $\tilde{T}_S$ with

$$\theta_S := \sup\{\theta : \tilde{T}(u; \theta) \leq \tilde{T}_0(u), \quad u \in [0, 2\pi]\}. \quad (81)$$

Then, either $\tilde{T}_S(u) < \tilde{T}_0(u)$ for all u , which means, because of compactness, that

$$\min_{u \in [0, 2\pi]} |\tilde{T}_S(u) - \tilde{T}_0(u)| > 0, \quad (82)$$

or there is a u_0 with $\tilde{T}_S(u_0) > \tilde{T}_0(u_0)$ or there is a u_0 with $\tilde{T}_S(u_0) = \tilde{T}_0(u_0)$ and $\frac{\partial \tilde{T}_S}{\partial u}(u_0) \neq \frac{\partial \tilde{T}_0}{\partial u}(u_0)$. The first and second cases contradict the continuous dependence of $\tilde{T}$ on its initial value θ and the third case is in contradiction to the uniqueness of the solution of the geodesic equation (66). A similar argument shows that a geodesic line passes through every point of S and, therefore, Eq. (80) is in fact a simple covering of S .

Finally, one should note that the solution of Eq. (80) for θ_0 yields the magnetic lines as level sets of an analytic function $T_0(u, \theta)$ or $T(u, v)$ (if θ is again replaced by v):

$$\theta_0 = T_0(u, \theta) =: T(u, v), \quad (83)$$

and integration of

$$\frac{dv}{du} = \frac{ET_v - FT_u}{GT_u - FT_v} \quad (84)$$

and subsequent solving for v_0 yields the current lines as level sets of an analytic function $\lambda(u, v)$:

$$v_0 = \lambda(u, v). \quad (85)$$

The function $\lambda(u, v)$ is, of course, a proper solution of Eq. (7).

Remark:

The constant C in Eq. (78) actually depends on the perturbation in the sense of vanishing C for vanishing perturbation. This means that the measure of non-preserved invariant curves goes to zero for vanishing perturbation. The invariant curves with rational rotational transform are, in fact, destroyed according to the *Poincaré-Birkhoff* theorem and are the germ of the chaotic regions, which were discussed in the preceding subsection and were also extensively studied in numerous publications (see, for example, [25]).

C. Minimal geodesics

Minimal geodesics are a special class of geodesics which have been thoroughly studied on toroidal surfaces [29] and which provide some insight into the process of disintegration of a simple covering [30], [31].

Geodesics on a toroidal surface S are called minimal (or of class A in [29]) if they are minimal with respect to all curves having the same end points and having the same number of poloidal and toroidal circuits; especially closed minimal geodesics are minimal in their homotopy class. It is well known that, given two arbitrary points on S , there is at least one minimal geodesic connecting these points. In [29] the following facts about geodesics, among others, are proved: For every real number ι there is a minimal geodesic having the rotational transform ι , and a minimal geodesic does not intersect itself. Especially the last property makes minimal geodesics prime candidates for simple coverings. In fact, in the axisymmetric case all geodesics with the exception of the outer equator and the geodesics wriggling around it are minimal (see section III.A). A complete classification of minimal geodesics on a toroidal surface is contained in [30]. For irrational ι there are three cases: Either there is a minimal geodesic passing through every point of S , or there is a minimal geodesic intersecting every closed minimal geodesic in a Cantor set. In both of these cases the minimal geodesic γ is recurrent, which means that γ returns to every neighbourhood of each of its points. In the third, nonrecurrent, case γ is enclosed between two minimal geodesics which are asymptotic to γ . So, the following picture concerning the fate of a simple covering of the axisymmetric torus for increasing perturbation emerges: For the axisymmetric

torus every simple covering with finite ι is built from recurrent minimal geodesics. For rational ι and increasing perturbation the covering is immediately destroyed; for irrational ι the covering persists for a while, depending on the "irrationality" of ι , and then gaps develop. Depending on the perturbation, these gaps grow and, ultimately, the notion of a covering becomes meaningless.

Remarks:

1) Concerning the interpretation of our findings in subsection A in terms of minimal geodesics one should be cautious: In the axisymmetric case all circulating geodesics are minimal; so any simple geodesic covering consists of minimal geodesics. But this need not be so in the nonaxisymmetric case and non-minimal simple coverings are not excluded. On the other hand, we fail to find a minimal one under the coverings with gaps, the reason for this being not quite clear.

2) If one considers Poincaré mappings instead of flows, the corresponding generalization of an invariant curve is the so-called *Mather set*, a subset of a closed, Lipschitz continuous curve (see, for example, [31]). One should note that the disintegration of an invariant curve happens through loss of smoothness. While the invariant curve is described by a smooth function

$$\theta = f(t), \quad r = g(t), \quad (86)$$

the *Mather set* is described by merely monotonic functions with, in general, countably many discontinuities. These discontinuities are responsible for the Cantor-like structure of the *Mather set*.

V. The “big bump” criterion

There are surfaces which obviously do not allow a simple geodesic covering. On a hemisphere, for example, all geodesics are part of great circles. Great circles, however, intersect one another on a hemisphere (with boundary) at least once. So, one can expect that for surfaces with big enough bumps equilibria do not exist. In the following, this idea is stated more precisely.

Def. 1 A “big bump”⁹ is an open subset $B \subset S$ of the surface S with the properties:

- a) ∂B is a geodesic circle with center $\mathbf{p}_0$ and radius r ,
- b) the Gaussian curvature K satisfies the condition $K(\mathbf{x}) \geq K_0$ for all $\mathbf{x} \in B$ and some $K_0 > 0$ and the radius r satisfies the inequality $r > \pi/(2\sqrt{K_0})$.

Def. 2 A domain $D \subset S$ is called simply covered by geodesics if there is a function

$$\begin{cases} D_0 := [0, l] \times [-a, a] \rightarrow S; & a, l > 0 \\ s, t \mapsto \mathbf{x}(s, t) \end{cases}$$

with the properties:

- a) $\mathbf{x}$ is one-to-one on D_0 and $D \subset \mathbf{x}(D_0)$,
- b) $\mathbf{x}$ is sufficiently often differentiable¹⁰,
- c) $\mathbf{x}_t(s) := (\mathbf{x}(s, t)$ with t fixed) is a geodesic for all $t \in [-a, a]$ ¹¹,
- d) the associated Jacobi field $\mathbf{J}(s, t) := \frac{\partial \mathbf{x}}{\partial t}(s, t) \neq 0$ for all $s, t \in D_0$.

The second definition merits two remarks:

- 1) If $\gamma(s) := \mathbf{x}_0(s)$ is parametrized by the arc length s , the associated Jacobi field $\mathbf{J}_\gamma(s) := \mathbf{J}(s, 0)$ satisfies the condition

$$\mathcal{D}\mathcal{D} \mathbf{J}_\gamma(s) + K(s) [[\gamma'(s) \times \mathbf{J}_\gamma(s)] \times \gamma'(s)] = 0, \quad (87)$$

with $\mathcal{D}$ denoting the covariant derivation, K the Gaussian curvature, and prime derivation with respect to the arc length. Given γ , the solution $\mathbf{J}_\gamma$ of Eq. (87) is unique if $\mathbf{J}_\gamma$ additionally satisfies

$$\mathbf{J}_\gamma \cdot \gamma' = 0 \quad \forall s \in [0, l]. \quad (88)$$

- 2) For Jacobi fields not identically vanishing along some geodesic γ condition d) is in fact a consequence of condition a). The condition $\mathbf{J}_\gamma(s_0) = 0$ marks points $\mathbf{x}_0(s_0)$ of intersection of “infinitely close” geodesics.

⁹For a similar definition appropriate to minimal geodesics see [30].

¹⁰In this context three times is enough.

¹¹ s is not necessarily the arc length.

Theorem: A surface with a big bump has no simple geodesic covering.

Proof: Assume $B \subset S$ has a simple geodesics covering. Without loss of generality, let $\gamma(s)$ be the geodesic through the center $\mathbf{p}_0$ of B , parametrized with arc length s . According to a theorem about conjugate points [32], for every geodesic γ on a surface B with Gaussian curvature $K(\mathbf{x}) \geq K_0 > 0$ the distance between any consecutive points $\mathbf{p} = \gamma(s_1)$ and $\mathbf{q} = \gamma(s_2)$ of γ with $\mathbf{J}_\gamma(s_1) = \mathbf{J}_\gamma(s_2) = 0$ (conjugate points) is at most $\pi/\sqrt{K_0}$. Because of the length of γ in B $l_\gamma \geq 2r > \pi/\sqrt{K_0}$, there is at least one point $\gamma(s_0)$ contained in B with $\mathbf{J}_\gamma(s_0) = 0$. This is in contradiction to condition d) in the definition of a geodesic covering.

The criterion formulated in this section is, contrary to the case of small deviations from axisymmetry considered in the last section, local in character and makes a negative statement for toroidal surfaces with some extreme deviation from axisymmetry. What is desirable but seemingly not yet available, are negative or positive (i.e. existence-) results, for certain classes of equilibria, which exhibit strong deviations from axisymmetry. Candidates are stellarator and other fully three-dimensional configurations, which are important in applications and have so far been treated only numerically [33].

(87)

(88)

VI. Summary

The aim of this paper was to give a rather complete treatment of the inverse existence problem of surface current equilibria for toroidal surfaces. An important tool here is a geometric characterization of surface current equilibria which made it possible to describe known equilibria in a unified and lucid manner and discover new ones, which would otherwise have hardly been found. It serves to demonstrate the existence of equilibria with infinite rotational transform and plane magnetic field lines as well as with vanishing rotational transform and plane current lines, and the existence of axisymmetric equilibria and of equilibria close to axisymmetry. The axisymmetric case is the only symmetric equilibrium which is compatible with toroidal geometry and the only case which is integrable in the Hamiltonian sense. The first two types of equilibria are still separable cases of Eq. (7), whereas the fourth type is nonseparable and its existence crucially depends on the degree of nonaxisymmetry and "irrationality" of its rotational transform.

The question now naturally arises whether this list is complete or not. Clearly, for the cases with vanishing and infinite rotational transform we expect more general equilibria than those whose current lines and magnetic field lines, respectively, are confined to planes. Equilibria with, for example, infinite rotational transform, nonplanar generating curve and noncircular poloidal cross-section are likely to exist but require necessarily nonplanar field lines. In any case, such more general equilibria (if they exist) fit into our list and do not add anything essentially new. More interesting is the case with finite rotational transform. The numerical results as well as the analytic considerations suggest that no equilibria exist beyond a "neighbourhood" of the axisymmetric case; but this is of course no proof. Only for a certain kind of strong deviation from axisymmetry, "big bumps", could a negative result be rigorously proved. So, further special solutions of Eq. (7), although unlikely, cannot yet be excluded. Nevertheless, bearing these caveats in mind, we conjecture that the list is complete.

Another point, deserving comment, is the analyticity requirement throughout this paper. If this requirement is dispensed with, more general surfaces, including interesting ones such as a race track-type surface consisting of cylinders and halves of axisymmetric tori, are allowed. But we do not expect new types of equilibria. The reason for this is that the essential obstacle to the existence of equilibria is the combination of toroidal geometry and lack of symmetry and not questions of smoothness or expansions. So, we consider the restriction to analytic surfaces to be not essential.

Finally, possible implications for volume current equilibria should be mentioned. First, three-dimensional equilibria with infinite rotational transform should also exist

for volume currents and are discussed in a forthcoming publication. Second, the case of weakly nonaxisymmetric configurations with finite rotational transform probably represents the only true difference in surface and volume current models, in the sense that classical solutions of the steady MHD equations in this case most likely do not exist. But this point is not yet rigorously proved, and it does not mean that generalized solutions (e.g. minimal solutions of an appropriate variational problem) are excluded.

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Appendix A

This appendix is intended to clarify the relation between the equilibrium problem and classical mechanics and discusses an attempt to use this relation for proving the existence of three-dimensional equilibria.

Free motion of a particle on a surface S with metric g_{kl} may be described by a Hamiltonian

$$H = \frac{1}{2} g^{kl}(u^1, u^2) p_k p_l, \quad k, l = 1, 2 \quad (\text{A.1})$$

and the four canonical equations

$$\frac{du^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp^j}{dt} = -\frac{\partial H}{\partial u_j}, \quad (\text{A.2})$$

with g^{kl} being the inverse matrix of g_{kl} , p_k the conjugate momenta and $u^1 = u$, $u^2 = v$ the surface parameters. The corresponding Hamilton-Jacobi equation reads (in its time-separated form)

$$g^{kl} \frac{\partial \lambda}{\partial u_k} \frac{\partial \lambda}{\partial u_l} = b, \quad k, l = 1, 2, \quad (\text{A.3})$$

with b being an integration constant. By means of Eq. (51) this equation can be identified with Eq. (7) of the equilibrium problem.

Equation (A.3) is a nonlinear partial differential equation for the unknown function $\lambda(u, v)$, whose characteristic equations are just the canonical equations (A.2). Any solution

$$\lambda = \lambda(u, v; a, b) + c \quad (\text{A.4})$$

of Eq. (A.3) which depends on – besides b and a trivial additive constant c – another parameter a and everywhere obeys the condition

$$\text{rank} \left\{ \frac{\partial(\lambda, \lambda_u, \lambda_v)}{\partial(a, b)} \right\} = 2 \quad (\text{A.5})$$

is called a complete integral of Eq. (A.3) [19], p. 24 and allows a complete integration of Eq. (A.2) [13], p. 314. Integrable (in this Hamiltonian sense) is the motion on surfaces of revolution, as shown in section III.A, or, as a famous long-known example, the motion on an ellipsoid [34], p. 95. In the former case the constants p and C in Eq. (30) combine into a single parameter which is related to the rotational transform Eq. (13). In contrast to this, the cases with zero and infinite rotational transform, described in section III.B, III.C, correspond to special solutions of Eq. (A.3) and are not integrable. So, integrability is not a prerequisite for the existence of surface current equilibria.

On the other hand, it is asserted in [14] that the existence of a KAM surface is sufficient for the existence of an equilibrium. Besides the fact that reference [14] does not discuss the precise conditions for the application of the KAM theorem, this is not true. It is in fact true, as the authors in [14] state, that the KAM theorem – if applicable – guarantees single-valued solutions of Eq. (A.2), viz. quasiperiodic motions in *phase space*, simply covering phase space tori there. But the projection of the motion to *configuration space* (a given toroidal surface close to axisymmetry) is neither single-valued in general nor does it cover the surface completely. As long as the relation between phase space and configuration space is not clarified, the argument in [14] is at best incomplete.

We conclude again with two remarks:

- 1) Solutions λ of Eq. (A.3) with periodic metric do not automatically have periodic derivatives λ_u , λ_v . This is an additional requirement due to the toroidal configuration space.
- 2) If one neglects the global connection, motion on a surface may even be described as motion on a plane in the presence of a potential. If one introduces so-called isothermal coordinates x , y , which is locally always possible, the metric takes the form

$$g_{kl} = g(x, y) \delta_{kl} \quad (\text{A.6})$$

and the Hamiltonian is

$$H = \frac{1}{2g} (p_x^2 + p_y^2). \quad (\text{A.7})$$

The trajectories, generated by Eq. (A.7), with energy h_0 , are, however, the same as those of the Hamiltonian

$$\tilde{H} = \frac{1}{2} (p_x^2 + p_y^2) - h_0 g(x, y) \quad (\text{A.8})$$

with vanishing energy. Integrable cases of Eq. (A.8) for periodic $g(x, y)$ (flat torus) are discussed in [34], p. 107.

Appendix B

In order to obtain an area-preserving mapping generated by the intersection of circulating geodesics with a fixed, closed curve, one can make use of the Hamiltonian description of geodesics. Their Hamiltonian is again given by Eq. (A.1). This follows by considering, for example, the characteristics of Eq. (A.3), which leads back to Eq. (A.1). A Poincaré cut at, say, $u^1 = \text{const}$ generates a mapping of the canonical pair (p_2, u^2) ,

$$(p_2, u^2)_0 \rightarrow (p_2, u^2)_1, \quad (\text{B.1})$$

which is area-preserving; see, for example, [25]. The invariant area element $dp_2 du^2$ can then easily be expressed in terms of geometric quantities:

From the definition of the g^{ik} we have

$$g^{11} = G/D, \quad g^{12} = -F/D, \quad g^{22} = E/D, \quad (\text{B.2})$$

where $D = EG - F^2$. The canonical equations (A.2) give

$$\frac{du^2}{dt} \bigg/ \frac{du^1}{dt} = \frac{du^2}{du^1} = \frac{-Fp_1 + Ep_2}{Gp_1 - Fp_2}. \quad (\text{B.3})$$

If p_1 is expressed in terms of p_2 , u^1 , u^2 and H , Eq. (B.3) yields

$$p_2 = \sqrt{2H} \frac{Fdu^1 + Gdu^2}{ds}, \quad (\text{B.4})$$

where $ds^2 = E(du^1)^2 + 2Fdu^1 du^2 + G(du^2)^2$. With

$$\begin{aligned} d\mathbf{x} \cdot \mathbf{x}_{u^2} &= (\mathbf{x}_{u^1} du^1 + \mathbf{x}_{u^2} du^2) \cdot \mathbf{x}_{u^2} = Fdu^1 + Gdu^2 \\ &= ds |\mathbf{x}_{u^2}| \cos \beta, \end{aligned} \quad (\text{B.5})$$

and the length element ds_2 along the line $u^1 = \text{const}$,

$$ds_2 = \sqrt{G} du^2, \quad (\text{B.6})$$

one obtains

$$p_2 du^2 = \sqrt{2H} ds_2 \cos \beta, \quad (\text{B.7})$$

where β is the angle between the geodesic line and the line $u^1 = \text{const}$. From Eq. (B.7) it immediately follows that a variable proportional to the arc length s_2 and $\cos \beta$ form a pair of area-preserving variables.

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Fig. 1 Section of the toroidal surface and its proper parametrization.

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Fig. 2 New coordinates x , α .

Fig. 3 Equatorial cut through the torus for $n = 4$. $\alpha = 0$ corresponds to the outermost ($\equiv$ generating curve) and $\alpha = \pi$ to the innermost magnetic field line of the configuration.

Fig. 4a Poloidal section of two consecutive symmetry planes, projected into the same plane. Only the upper half-plane of the configuration, symmetric with respect to the x axis, is shown; also shown are magnetic field lines (dotted). An elliptic initial section generates a well-defined surface. The calculation was made for the generating curve of Fig. 3.

Fig. 4b Same as Fig. 4a, but with a nonconvex initial section. This leads to self-intersection.

Fig. 5 Mapping from one crossing of a poloidal cut to the next when a geodesic line is followed around a toroidal surface. u = position on poloidal cut, γ = angle to the line $u = \text{const}$. Axisymmetric case. $R_0 = 1$, $\rho_0 = 0.2$. Eight equidistant starting angles $\gamma_0 = 51.5^\circ - 72.5^\circ$ at fixed starting position $u_0 = 0$. $N = 1300$ toroidal revolutions.

Fig. 6 Rotational transform ι versus initial angle γ_0 at fixed $u_0 = 0$ for two values of the minor radius ρ_0 . Axisymmetric case. $R_0 = 1$.

Fig. 7 Mapping in nonaxisymmetric case: $R(v) = R_0 (1 + R_1 \sin 2v)$. $R_0 = 0.9975$, $R_1 = 0.05$, $\rho_0 = 0.2$. Initial values as in Fig. 5. Poloidal cut at $v_0 = 0$.

Fig. 8a Same as Fig. 7 with initial values $\gamma_0 = 57.5^\circ, 59.5^\circ$ and 59.9° .

Fig. 8b Magnification of boundary region of Fig. 8a. Eleven equidistant starting angles $\gamma_0 = 20^\circ - 24^\circ$ at opposite starting point $u_0 = \pi$.

Fig. 9 Geodesic orbit in u, v plane. $R_0 = 0.9975$, $R_1 = 0.05$. Simple covering. Starting values $u_0 = 0$, $\gamma_0 = 59.9^\circ$, $N = 20$.

Fig. 10a Same as Fig. 9. Covering with self-intersection and gaps. $u_0 = 0$, $\gamma_0 = 59.5^\circ$, $N = 25$.

Fig. 10b Same as Fig. 9. Covering with self-intersection but possibly no gaps for $N \rightarrow \infty$. $u_0 = \pi$, $\gamma_0 = 22^\circ$, $N = 30$.

Fig. 11 Same as Fig. 9. Covering with self-intersection and ergodic switching of orbit type. $u_0 = 0$, $\gamma_0 = 54.5^\circ$, $N = 30$.

Fig. 12 Mapping in nonaxisymmetric case: $r(v) = \rho_0 + \rho_1 \sin v$. $\rho_0 = 0.1$, $\rho_1 = 0.02$, $R_0 = 1$. Eight equidistant starting angles $\gamma_0 = 37^\circ - 58^\circ$ at $u_0 = 0$. Poloidal cut at $v_0 = 3\pi/2$. $N = 1300$.

Fig. 13 Magnification of Fig. 12. Fifteen equidistant starting angles $\gamma_0 = 36^\circ - 43^\circ$, at $u_0 = \pi$. $N = 800$.