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Linear and Nonlinear Stability in Resistive Magnetohydrodynamics *

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Abstract

A sufficient stability condition with respect to purely growing modes is derived for resistive magnetohydrodynamics. Its 'nearness' to necessity is analysed. It is found that for physically reasonable approximations the condition is in some sense necessary and sufficient for stability against all modes. This together with hermiticity makes its analytical and numerical evaluation worthwhile for the optimization of magnetic configurations. Physically motivated test functions are introduced. This leads to simplified versions of the stability functional, which makes its evaluation and minimization more tractable. In the case of special force-free fields the simplified functional reduces to a good approximation of the exact stability functional derived by other means. It turns out that in this case the condition is also sufficient for nonlinear stability.

Nonlinear stability in hydrodynamics and magnetohydrodynamics is discussed especially in connection with 'unconditional' stability and with severe limitations on the Reynolds number. Two examples in magnetohydrodynamics show that the limitations on the Reynolds numbers can be removed but unconditional stability is preserved. Practical stability needs to be treated for limited levels of perturbations or for conditional stability. This implies some knowledge

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of the basin of attraction of the unperturbed solution, which is a very difficult problem. Finally, a special inertia-caused Hopf bifurcation is identified and the nature of the resulting attractors is discussed.

The purpose of this lecture is to report recent progress on resistive or dissipative magnetohydrodynamics (MHD) in general geometry. Though the problem of linear stability has not been completely settled analytically, it has reached the mature level of a general quadratic and Hermitean form ready to be minimized numerically in a way similar to that of the ideal energy principle.

In contrast, nonlinear stability analysis still falls far short of the stage of maturity attained by the linear theory. It was recently possible, however, to prove general statements concerning unconditional stability of three-dimensional force-free fields and unconditional stability with respect to two-dimensinal perturbations.

The paper is organized as follows : Sections 1-3 are devoted to the linear stability condition. The topics are sufficiency and necessity of the condition and simplified versions of it. Nonlinear stability is analyzed in section 4. The topics are force-free fields, two-dimensional perturbations and a special Hopf bifurcation.

1 Sufficient condition for linear stability

In a previous note [1] the author derived a sufficient condition for the stability of purely growing modes,valid for general dissipative systems and general geometries. This condition is applied here for resistive magnetohydrodynamic (MHD) equilibria. These equilibria generally have a flow which, for simplicity, we neglect in the equation of motion, but which we keep in Ohm's law. The equilibrium equations are given by

$$\mathbf{J} \times \mathbf{B} = \nabla P_0, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta_0 \mathbf{J}. \tag{3}$$

As usual $\mathbf{B}$ is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$, $\mathbf{E}$ is the curl-free electric field, $\mathbf{V}$ is the flow velocity due to resistivity η_0 and P_0 is the pressure. The 'existence' of magnetic surfaces is assumed and the resistivity is taken as constant on these surfaces. The equations of the linearized perturbations are

$$\begin{aligned}
\rho\ddot{\xi} + \nabla P_1 - \mathbf{j}\times\mathbf{B} - \mathbf{J}\times\mathbf{b} &= 0, &(4)\\
\mathbf{e} + \dot{\xi}\times\mathbf{B} + \mathbf{V}\times\mathbf{b} - \eta_1\mathbf{J} - \eta_0\mathbf{j} &= 0, &(5)\\
\nabla\times\mathbf{e} &= -\dot{\mathbf{b}}, &(6)\\
\nabla\cdot\mathbf{b} &= 0, &(7)\\
\mathbf{j} &= \nabla\times\mathbf{b}, &(8)\\
\mathbf{B}\cdot\nabla\eta_1 + \mathbf{b}\cdot\nabla\eta_0 &= 0, &(9)\\
P_1 &= -\gamma P_0\nabla\cdot\xi - \xi\cdot\nabla P_0, &(10)
\end{aligned}$$

where ρ is the mass density, P_1, **j**, **b**, **e** and η_1 are the perturbations of, respectively, pressure, current, magnetic field, electric field and resistivity. The boundary conditions are $\mathbf{n}\cdot\mathbf{b} = \mathbf{n}\cdot\xi = \mathbf{0}$, where **n** is the normal to a perfectly conducting wall.

Let us express **e** and **b** in terms of the vector potential **A** and take the gauge of zero scalar potential :

$$\begin{aligned}
\mathbf{e} &= -\dot{\mathbf{A}},\\
\mathbf{b} &= \nabla\times\mathbf{A},
\end{aligned}$$

with the boundary condition $\mathbf{n}\times\mathbf{A} = \mathbf{0}$. We insert **j** from eq.(5) into eq.(4) to obtain a system written in terms of $\Psi = \begin{pmatrix}\xi\\ \mathbf{A}\end{pmatrix}$:

$$N\ddot{\Psi} + P\dot{\Psi} + Q\Psi = 0, \qquad (11)$$

where N, P and Q are given by, respectively,

$$N = \begin{pmatrix}\rho & 0\\ 0 & 0\end{pmatrix},$$

$$P = \begin{pmatrix}\mathbf{B}/\eta_0\times(\cdots\times\mathbf{B}) & (\cdots\times\mathbf{B}/\eta_0)\\ -(\cdots\times\mathbf{B}/\eta_0 & 1/\eta_0\end{pmatrix},$$

and

$$Q = \begin{pmatrix} \nabla(-\gamma P_0(\nabla\cdot\ldots)) & -\mathbf{J}\times(\nabla\times\cdots) \\ -\nabla(\ldots\cdot\nabla P_0) & -1/\eta_0\nabla P_0(\mathbf{B}\cdot\nabla)^{-1}(\nabla\times\ldots\cdot\nabla\eta_0) \\ & +\mathbf{B}/\eta_0\times(\mathbf{V}\times\nabla\times\cdots) \\ & \\ 0 & \nabla\times\nabla\cdots \\ & +\mathbf{J}/\eta_0(\mathbf{B}\cdot\nabla)^{-1}(\nabla\times\ldots\cdot\nabla\eta_0) \\ & -\mathbf{V}/\eta_0\times\nabla\times\cdots \end{pmatrix}.$$

The first two matrix operators are symmetric and positive. The last operator Q is obviously not selfadjoint. For this reason we cannot find a Lyapunov functional which would lead to a necessary and sufficient condition for stability as in, for example, [2] or [3].

As shown in [1], one can, however, write a sufficient condition for stability against purely growing modes in the form

$$\delta W = (\Psi, Q\Psi) \geq 0, \tag{12}$$

where the scalar product is defined with purely real quantities. Only the symmetric part Q_S of Q survives in eq.(12), but if a symmetrized form for eq.(12) is wanted, it is easy to construct Q^+, the adjoint of Q, by integration by parts, and use $Q_S = (Q + Q^+)/2$ instead of Q in eq.(12).

Criterion (12) implies volume integrations which can be reduced to integrations on the magnetic surfaces and integrations across them. The operator $(\mathbf{B}\cdot\nabla)^{-1}$ in eq.(12), which comes from integration of eq.(9), is singular across the rational surfaces ($1/x$ singularity). This singularity is physically prohibited by the breakdown of eq.(9) due to a finite heat conduction κ ($\kappa_{\parallel}$ is assumed to be infinite and $\kappa_{\perp} = 0$ for eq.(9)). In fact, η_1 should not become infinite on the rational magnetic surfaces, but small. It is then natural to define the integrations across the surfaces in the sense of Cauchy principal parts (no delta functions) as in [3]. Note here that these singularities are not aggravated by the above-mentioned symmetrizing integrations by parts, because they occur on the surfaces.

Let us now write δW explicitly:

$$\delta W \;=\; \int d\tau(\gamma P_0(\nabla\cdot\xi)^2 + (\xi\cdot\nabla P_0)\nabla\cdot\xi)$$

$$
\begin{aligned}
&+\int d\tau(\nabla\times\mathbf{A})^2-\int d\tau\xi\times\mathbf{J}\cdot\nabla\times\mathbf{A}+ \\
&+p\int d\tau\mathbf{J}\cdot(\mathbf{A}-\xi\times\mathbf{B})(\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A}) \\
&-\int d\tau(\mathbf{A}-\xi\times\mathbf{B})\cdot\mathbf{V}\times(\nabla\times\mathbf{A})1/\eta_0.
\end{aligned}
\tag{13}
$$

If we choose in δW the MHD test function $\mathbf{A}=\xi\times\mathbf{B}$, then δW reduces to δW_{MHD}. In the tokamak scaling (large axial wavelength and magnetic fields) and for $\mathbf{J}=\mathbf{e_z}J$, $\eta_0 J=ct\cdot$, $\xi=\mathbf{e_z}\times\nabla U$, $\mathbf{V}=0$, δW reduces to the necessary and sufficient condition found in [3].

It is more convenient to treat δW in Hamada-like coordinates especially for the term $(\mathbf{B}\cdot\nabla)^{-1}$, which also appears in [3]. The symmetrization of Q, if desired, can be done either analytically in the same coordinates by integration by parts or after discretization in the case of numerical evaluation by computing the adjoint matrix.

The equilibrium quantities in eq.(13) should satisfy equations (1)-(3). To determine the contribution of the last integral in eq.(13), one requires a knowledge of unavoidable [4] Pfirsch-Schlüter- like flows, which are important especially for stellarators. The flow in a tokamak can probably be neglected if the aspect ratio is large enough and the poloidal currents are weak. One can then take $\nabla\times\eta_0\mathbf{J}\approx\mathbf{0}$ as in [3].

The main advantage of (13) is that it can be numerically evaluated by spectral methods well known in ideal MHD stability and recently extended to MHD stability of stellarator equilibria. A second positive aspect is that this approach to resistive MHD stability is the only one which takes real geometry into account together with the complex flows it generates, and in an exact way at that.

2 Necessity of the condition

As already known (see [6]), conditions (12)-(13) become necessary and sufficient for all modes if $Q_a=0$, Q_a being the antisymmetric part of Q. In the incompressible case with tokamak ordering, Tasso and Virtamo derived some time ago a necessary and sufficient condition (see [3]) which has been evaluated numerically (see [7]). In the general case it does not seem possible to find a system of dynamic variables for which $Q_a=0$. One can, however, 'upgrade' conditions (12)-(13) for two interesting situations : 1) for $Q_a \approx \epsilon$ small, which relates to the tokamak scaling and 2) $N = 0$, or neglecting inertia, which is valid for time scales much larger than the Alfven or acoustic time scales.

2.1 $Q_a \approx \epsilon$

Let us first show that for $Q_a = 0$ any unstable mode must be purely growing. For that purpose let us assume that

$$\Psi = e^{\omega t}\Psi_0(\mathbf{r}), \tag{14}$$

$$\omega = i\omega_0 + \gamma_0 \tag{15}$$

with ω_0 and γ_0 real. Inserting (14) and (15) in (11) yields

$$(i\omega_0 + \gamma_0)^2 N\Psi_0 + (i\omega_0 + \gamma_0)P\Psi_0 + (Q_s + Q_a)\Psi_0 = 0. \tag{16}$$

Taking the scalar product of (16) with Ψ_0^*, integrating over the plasma volume and using the usual notation for the scalar product reduces (16) to

$$[(\gamma_0^2 - \omega_0^2) + 2i\gamma_0\omega_0](\Psi_0, N\Psi_0) + (\gamma_0 + i\omega_0)(\Psi_0, P\Psi_0) + (\Psi_0, (Q_s + Q_a)\Psi_0) = 0. \tag{17}$$

Since N, P and Q_s are Hermitean, the imaginary part of (17) is

$$2\gamma_0\omega_0(\Psi_0, N\Psi_0) + \omega_0(\Psi_0, P\Psi_0) - (\Psi_0, Q_a\Psi_0) = 0. \tag{18}$$

Since N and P are positive and if we assume $Q_a = 0$ and $\gamma_0 \geq 0$, it follows from (18) that $\omega_0 = 0$. This proves that for $Q_a = 0$ exponentially unstable modes must be purely growing.

If it is assumed that condition (12) is violated for some test function, it follows that for $\epsilon = 0$ a purely growing mode with $\omega_0 = 0$ exists and satisfies (16) for $Q_a = 0$. Now supposing that Q_a is small and of order ϵ, we expand (11) up to first order in ϵ

$$\Psi = \Psi_0 + \epsilon\Psi_1, \tag{19}$$

$$\omega = \gamma_0 + \epsilon\omega_1, \tag{20}$$

$$\gamma_0^2 N\Psi_0 + \gamma_0 P\Psi_0 + Q_s\Psi_0 = 0, \tag{21}$$

$$2\gamma_0\omega_1 N\Psi_0 + \gamma_0^2 N\Psi_1 + \omega_1 P\Psi_0 + \gamma_0 P\Psi_1 + Q_s\Psi_1 + Q_a\Psi_0 = 0. \tag{22}$$

Taking the scalar products of (21) and (22) with $\Psi_0^\star$ and integrating over the plasma volume, we obtain

$$\gamma_0^2(\Psi_0, N\Psi_0) + \gamma_0(\Psi_0, P\Psi_0) + (\Psi_0, Q_s\Psi_0) = 0, \tag{23}$$

$$2\gamma_0\omega_1(\Psi_0, N\Psi_0) + \omega_1(\Psi_0, P\Psi_0) + (\Psi_0, Q_a\Psi_0) + \\ \gamma_0^2(\Psi_0, N\Psi_1) + \gamma_0(\Psi_0, P\Psi_1) + (\Psi_0, Q_s\Psi_1) = 0. \tag{24}$$

Using (21) and the fact that N, P and Q_s are Hermitean reduces (24) to

$$\omega_1[2\gamma_0(\Psi_0, N\Psi_0) + (\Psi_0, P\Psi_0)] + (\Psi_0, Q_a\Psi_0) = 0. \tag{25}$$

Since the original system of equations is real and the mode is purely growing ($\omega_0 = 0$), Ψ_0 can be chosen real without loss of generality. It then follows from (25) together with the positivity of γ_0, N and P and the antisymmetry of Q_a that

$$\omega_1 = 0. \tag{26}$$

This means that if the purely growing eigenmode were affected by $Q_a \approx \epsilon$, the effect would be of order ϵ^2 or higher. A small Q_a affects the unstable spectrum very weakly.

2.2 $N = 0$, or neglecting inertia

Equation (11) becomes

$$P\dot{\Psi} + (Q_s + Q_a)\Psi = 0. \tag{27}$$

Taking the scalar product of (27) with Ψ real and integrating over the volume, we obtain

$$\frac{\partial(\Psi, P\Psi)}{\partial t} = -2(\Psi, Q_s\Psi). \tag{28}$$

We see that the positive form $(\Psi, P\Psi)$ is a Liapunov functional if condition (12) or (13) is verified. Now these conditions are sufficient for stability against all modes, not only the purely growing ones.

The analysis presented in this section cannot make condition (12) necessary and sufficient for all modes but does give more weight to it. One could say that the condition is 'nearly' necessary so that its analytical and numerical evaluations may be worth doing. As mentioned in [5], condition

(12) reduces to the ideal MHD energy principle and to the resistive energy principle of Tasso and Virtamo (see [3]) in the appropriate limits. Extensive numerical calculations in the particular limits for ideal MHD (see [8]) and for resistive MHD (see [7]) show that both ideal and resistive modes can be stabilized if β is small enough, the safety factor large enough and the current distribution well chosen.

Condition (12), however, may be violated in general by test functions reminiscent of resistive ballooning or resistive drift modes or other residual modes. Nevertheless, its degree of violation can be taken as a 'measure' of the optimization of magnetic configurations. As in previous work (see [7]), numerical evaluation of the condition is made possible by its Hermitean form.

3 Simplified versions of the condition

The test functions ξ and $\mathbf{A}$ in (13) are general and together constitute a six-dimensional test function space. As mentioned above and in [5], ideal MHD can be recovered by restricting to $\mathbf{A} = \xi \times \mathbf{B}$. In the tokamak scaling and for $\nabla \cdot \xi = 0$ one recovers the resistive principle derived in [3] .

In this section we introduce a physical restriction by the following arguments. Perpendicularly to the magnetic field a weakly dissipative plasma behaves like in ideal MHD but parallel to $\mathbf{B}$ it may behave quite differently, essentially because of resistivity. This suggests the following restriction in test function space :

$$\mathbf{A} = \xi \times \mathbf{B} + \mathbf{A_{par}}, \tag{29}$$

where $\mathbf{A_{par}}$ is the part of $\mathbf{A}$ parallel to $\mathbf{B}$. Crossing relation (29) with $\mathbf{B}$, we find for ξ

$$\xi = \frac{\mathbf{B} \times \mathbf{A}}{B^2} + \xi_{par}, \tag{30}$$

where ξ_{par} is the part of ξ parallel to $\mathbf{B}$.

If we insert (29) and (30) in (13), we obtain a first simplified version of (13) :

$$\delta W \;=\; \int d\tau (\gamma P_0 (\nabla \cdot (\frac{\mathbf{B} \times \mathbf{A}}{B^2} + \xi_{par}))^2 + (\frac{\mathbf{B} \times \mathbf{A}}{B^2} \cdot \nabla P_0) \nabla \cdot (\frac{\mathbf{B} \times \mathbf{A}}{B^2} + \xi_{par}))$$

$$+\int d\tau(\nabla\times\mathbf{A})^2-\int d\tau(\frac{\mathbf{B}\times\mathbf{A}}{B^2}+\xi_{par})\times\mathbf{J}\cdot\nabla\times\mathbf{A}+$$
$$+p\int d\tau(\mathbf{J}\cdot\mathbf{A}_{par})\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A})$$
$$-\int d\tau\mathbf{A}_{par}\cdot\mathbf{V}\times(\nabla\times\mathbf{A})1/\eta_0. \quad (31)$$

Instead of a six-dimensional test function space we now have a four-dimensional one. One is tempted to minimize (31) with respect to ξ_{par} as in ideal MHD. This does not lead to $\nabla\cdot\xi=0$ but to a rather complicated expression together with a difficult equation for ξ_{par}. Despite this fact, expression (31) is already simple enough to minimize either numerically or analytically (e.g. for perturbations localized about magnetic surfaces or magnetic lines).

A further simplification consists in setting $\nabla\cdot\xi=0$ from the outset. In this case one can solve for ξ_{par} by setting

$$\xi_{par}=\alpha\mathbf{B} \quad (32)$$

and using (30) to obtain

$$\nabla\cdot(\frac{\mathbf{B}\times\mathbf{A}}{B^2})+\mathbf{B}\cdot\nabla\alpha=0, \quad (33)$$

whose solution is

$$\alpha=-(\mathbf{B}\cdot\nabla)^{-1}\nabla\cdot\frac{\mathbf{B}\times\mathbf{A}}{B^2}. \quad (34)$$

Inserting (32) and (34) in (31), we obtain a further simplified version of (13):

$$\begin{aligned}\delta W &= \int d\tau((\nabla\times\mathbf{A})^2-(\frac{\mathbf{B}\times\mathbf{A}}{B^2})\times\mathbf{J}\cdot\nabla\times\mathbf{A})\\ &-p\int d\tau((\mathbf{B}\cdot\nabla)^{-1}\nabla\cdot\frac{\mathbf{B}\times\mathbf{A}}{B^2})\nabla P_0\cdot\nabla\times\mathbf{A}+\\ &p\int d\tau(\mathbf{J}\cdot\mathbf{A}_{par})(\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A})\\ &-\int d\tau\mathbf{A}_{par}\cdot\mathbf{V}\times(\nabla\times\mathbf{A})1/\eta_0. \quad (35)\end{aligned}$$

3.1 Application to force-free fields

In the case of a resistive field obeying

$$\mathbf{J}=\lambda\mathbf{B} \quad (36)$$

with $\lambda = ct.$, one knows (see [10]) that η_0 also has to be constant and $\mathbf{V} = 0$. The field satisfies

$$\dot{\mathbf{B}} = -\eta_0 \lambda^2 \mathbf{B}. \tag{37}$$

Though expression (13) is derived in [9] for time-independent equilibria, it should hold in the limit $\eta_0 \to 0$. Therefore, inserting (36) and (37) in (35) as well as $\mathbf{V} = \nabla P_0 = \eta_0 = 0$, then reduces δW from (35) to

$$\delta W = \int d\tau ((\nabla \times \mathbf{A})^2 - \lambda \mathbf{A}_{perp} \cdot \nabla \times \mathbf{A}), \tag{38}$$

where $\mathbf{A}_{perp}$ is the part of $\mathbf{A}$ perpendicular to $\mathbf{B}$. Expression (38) compares very well with the exact δW derived in [11] for the field (36)-(37),

$$\delta W = \int d\tau ((\nabla \times \mathbf{A})^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) \geq 0, \tag{39}$$

which is sufficient for stability. The difference between (38) and (39) is in an $\mathbf{A}_{par}$ term not containing the singularity $(\mathbf{B} \cdot \nabla)^{-1}$, which means that this term vanishes smoothly for $\eta_0 \to 0$.

In view of the physical (but formally not exact) restrictions in the test function space this is a remarkable result and gives us hope that expressions (31) and (35) for the simplified δW are good even for equilibria with pressure and $\lambda \neq ct$. In these cases, however, inaccuracies of the kind above can be amplified by the singularity $(\mathbf{B} \cdot \nabla)^{-1}$ despite the 'principal part' before the integral.

4 Nonlinear stability

The stability of complex systems such as fluids or plasmas is usually investigated in the linearized case. Obviously, the linearization is done in order to simplify the analysis and obtain a first insight into the problem. This is, however, by no means sufficient for practical stability for the following reasons : If a system is linearly stable, it implies stability only for infinitesimal perturbations. If it is linearly unstable, it may saturate at a low or high level in the nonlinear regime. Since for practical situations the perturbations are finite and the saturation levels critical, the study of nonlinear stability, especially for fluids and plasmas, becomes an important and sometimes crucial issue.

In hydrodynamics (HD) the planar Couette flow and the Poiseuille flow in a circular pipe are both linearly stable for all Reynolds numbers (see [12] and [13]). In practical situations turbulence occurs at Reynolds numbers larger than roughly one thousand. It is attributed to nonlinear instabilities or instabilities due to finite perturbations. This view was lent support by simple amplitude expansions [13] and numerical calculations [14] .

In HD and magnetohydrodynamics (MHD) exact sufficient criteria for nonlinear stability exist (see [15], [16] and [17]). Such criteria are powerful and robust, and provide nonlinear stability for arbitrary perturbation levels. In other words, they ensure so-called unconditional stability, which in a certain sense is too good and is not needed for practical stability, since the perturbations can be assumed to be limited in an experiment, especially if one wants to avoid strong vibrations etc. Accordingly, the critical Reynolds numbers delivered by these criteria are too low, of the order of 5 (see [15]) and 20 (see [17]).

Unfortunately, no rigorous criteria are available in HD in the range of Reynolds numbers larger than roughly 20 . This lack of knowledge is precisely in the range where the nonlinear stability margin will probably depend upon the perturbation level. This is equivalent to saying that what is missing is a knowledge of the basin of attraction of the unperturbed solution in functional space. For very low Reynolds numbers 5 to 20 the basin of attraction is infinite and for very large Reynolds numbers it is probably infinitesimal or very small.

Fortunately, the situation is not as bad in MHD. It is possible there to find unperturbed equilibria with zero flow which are unconditionally stable for all magnetic Reynolds numbers. A first example is the case of so-called force-free fields, whose nonlinear stability was recently analyzed by the author [18].

4.1 Nonlinear stability of force-free fields

Let us assume as unperturbed solution

$$\mathbf{j} = \lambda \mathbf{B} \tag{40}$$

with $\lambda = ct.$ bounded by a perfectly conducting wall.

The equations of motion are those of incompressible MHD with a material resistivity η constant in space and time. For any finite perturbations $\mathbf{v}$ of the velocity field with $\mathbf{n}\cdot\mathbf{v}=0$ at the boundary and $\mathbf{A}$ of the vector potential with $\mathbf{n}\times\mathbf{A}=0$ at the boundary the equations of motion are [18]

$$\dot{\mathbf{v}}+\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{J}_0\times\mathbf{B}_1+\mathbf{j}_1\times\mathbf{B}_0+\mathbf{j}_1\times\mathbf{B}_1, \tag{41}$$

with $\nabla\cdot\mathbf{v}=0$,

$$\dot{\mathbf{A}}=\mathbf{v}\times(\mathbf{B}_0+\mathbf{B}_1)-\eta\mathbf{j}_1, \tag{42}$$

$$\dot{\mathbf{B}}_1=\nabla\times(\mathbf{v}\times(\mathbf{B}_0+\mathbf{B}_1)-\eta\mathbf{j}_1). \tag{43}$$

Taking the scalar product of (41) with $\mathbf{v}$ and that of (43) with $\mathbf{B}_1$, adding and integrating over the volume, we obtain

$$\frac{\partial}{\partial t}\int\frac{d\tau}{2}(v^2+\mathbf{B}_1^2)=\lambda\int d\tau\mathbf{v}\times\mathbf{B}_0\cdot\mathbf{B}_1-\int d\tau\eta\mathbf{j}_1^2. \tag{44}$$

Many quadratic and cubic terms integrate to zero because of the boundary condition being taken as perfectly conducting. Taking the scalar product of (42) with $\mathbf{B}_1$, we can solve for $\mathbf{v}\times\mathbf{B}_0\cdot\mathbf{B}_1$, and, inserting into (44), we obtain

$$\frac{\partial}{\partial t}\int\frac{d\tau}{2}(v^2+\mathbf{B}_1^2-\lambda\mathbf{A}\cdot\nabla\times\mathbf{A})=-\eta\int d\tau(\mathbf{j}_1^2-\lambda\mathbf{B}_1\cdot\mathbf{j}_1) \tag{45}$$

or

$$\frac{d}{dt}\int d\tau\frac{1}{2}(\mathbf{v}^2+(\nabla\times\mathbf{A})^2-\lambda\mathbf{A}\cdot\nabla\times\mathbf{A}) \;=\; -\eta\int d\tau((\nabla\times\nabla\times\mathbf{A})^2-\lambda\nabla\times\mathbf{A}\cdot\nabla\times\nabla\times\mathbf{A}). \tag{46}$$

Since (46) also holds for the linearized case, which was discussed a long time ago in [11], we reproduce the proof given there for the sufficiency of

$$\int d\tau\frac{1}{2}((\nabla\times\mathbf{A})^2-\lambda\mathbf{A}\cdot\nabla\times\mathbf{A})\geq 0 \tag{47}$$

for nonlinear stability. Note first that $\mathbf{n}\times\mathbf{A}=0$ implies $\mathbf{n}\cdot\nabla\times\mathbf{A}=0$ at the boundary, so that if (47) is satisfied for $\mathbf{n}\times\mathbf{A}=0$, then the right-hand side of (46) will be satisfied for $\mathbf{n}\cdot\nabla\times\mathbf{A}=0$. By means of the Lyapunov theorems the expression under the time derivative of the left-hand side of (46) is a Lyapunov function if (47) is verified. Condition (47) is sufficient for stability independently of the values of the resistivity and viscosity. As mentioned above, there is nothing like this in HD .

4.2 Two-dimensional perturbations

A less spectacular example in MHD is the nonlinear stability of a straight z-pinch or tokamak surrounded by perfectly conducting walls. Here it is possible to prove nonlinear stability with respect to 2-dimensional perturbations if the current density is homogeneous, the velocity of the unperturbed fluid $\mathbf{v}_0$ being zero. The equilibrium is given by

$$\Delta\Psi = J_0 = -P'(\Psi), \qquad (48)$$

$$\mathbf{j}_0 = \mathbf{e}_z J_0, \qquad (49)$$

$$\mathbf{v}_0 = 0. \qquad (50)$$

Ψ denotes the flux of the poloidal magnetic field, J_0 is the current density in the z direction and $P(\Psi)$ is the pressure as a function of Ψ. A constant magnetic field B_z in the z direction could be added without changing the shape of Ψ, which is determined by (48) for any given boundary condition on Ψ.

The MHD equations of motion for an incompressible fluid with mass density equal to unity are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{j}_1\times\mathbf{B}_0 + \mathbf{j}_0\times\mathbf{B}_1 + \mathbf{j}_1\times\mathbf{B}_1 - \nabla P_1 + \mu\Delta\mathbf{v}, \qquad (51)$$

$$-\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla\times(\mathbf{v}\times(\mathbf{B}_0+\mathbf{B}_1)) + \eta\nabla\times\mathbf{j}_1, \qquad (52)$$

where $\mathbf{v}$ and $\mathbf{B}_1$ are finite perturbations of the velocity and the magnetic fields having $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{B}_1 = 0$ at the boundary. Taking the scalar product of (51) with $\mathbf{v}$ and that of (52) with $\mathbf{B}_1$, adding and integrating over the volume, we obtain

$$\frac{d}{dt}\frac{1}{2}\int(\mathbf{v}^2+\mathbf{B}_1^2)d\tau = \int \mathbf{v}\cdot\mathbf{j}_0\times\mathbf{B}_1 d\tau - \mu\int(\nabla\times\mathbf{v}^2)d\tau - \eta\int(\nabla\times\mathbf{B}_1^2). \quad (53)$$

Many quadratic and cubic terms integrate to zero because of the boundary conditions. The right-hand side of (53) would be negative if the first integral on the right-hand side of (53) vanished. We now prove that this is the case if $\mathbf{j}_0 = ct$. Introducing the vector potential, we have

$$\int \mathbf{v}\times\mathbf{j}_0\cdot\nabla\times\mathbf{A}d\tau = \int \mathbf{A}\cdot(\mathbf{j}_0\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{j}_0)d\tau = 0. \qquad (54)$$

The last equality is due to the two-dimensionality of $\mathbf{v}$ and the assumption of a constant vector $\mathbf{j}_0$. This means that the expression under the time derivative on the left-hand side of (53) is a Lyapunov function, from which nonlinear stability follows. Again this condition is independent of the Reynolds and magnetic Reynolds numbers, in contrast to, for example, [17], but the stability is unconditional.

For the above two examples we were able to obtain sufficient and unconditional stability conditions without limitations on the Reynolds numbers. Our examples are of course special and contain an important ingredient $\mathbf{v}_0 = 0$, i.e. no flow in the unperturbed state. This is nontrivial only in the MHD cases. In contrast, the criteria [15], [16] and [17] are very general but imply severe limitations on the Reynolds numbers. The examples given in this note and the general criteria [15], [16] and [17] have one thing in common, viz-they all deal with unconditional stability.

As mentioned at the beginning, this is not necessary for practical stability. If we want to get rid of unconditional stability in our proofs, we have to deal with finite basins of attraction in functional spaces. Practical stability is tied to this very difficult problem.

4.3 A manifest Hopf bifurcation

In this section we consider the case for which condition (12) is satisfied and prove that, if the inertial term can cause some additional overstability, the modes appearing in this way meet the requirements of the centre manifold theorem [19]. This means that they can be stabilized nonlinearly through a Hopf bifurcation, resulting in a limit cycle or nonlinear periodic oscillation. If $N \neq 0$ in equation (11), inertia-caused overstable modes can occur : In a special example [20], the overstability occurs only in the compressible case, primarily at the magnetoacoustic resonance.

Let us now consider the case for which (12) is satisfied but (11) is overstable for $N \neq 0$. Any overstable mode of (11) is given by

$$\xi = \Psi e^{(i\omega+\gamma)t}, \tag{55}$$

where ω and γ are real and satisfy

$$\begin{aligned}(\gamma^2-\omega^2)(\Psi, N\Psi)+\gamma(\Psi, P\Psi)+(\Psi, Q_s\Psi) &= 0, &(56)\\ 2\gamma\omega(\Psi, N\Psi)+\omega(\Psi, P\Psi)+(\Psi, Q_a\Psi) &= 0. &(57)\end{aligned}$$

We see from (56) and (57) and, generally, from the reality of the operators in (11) that $\xi^\star = \Psi^\star e^{(-i\omega+\gamma)t}$ is also an eigenmode of (11). It follows that the modes due to the inertia operator N always come in pairs with opposite sign of the real frequencies but the same growth rate, all other modes being damped because of (12). These features are precisely the principal ingredients of the centre manifold theorem [19]. In summary, if (12) is satisfied, inertia-caused overstability can lead to a Hopf bifurcation resulting in a periodic nonlinear oscillation.

It may be instructive to look at the following example, consisting of two ordinary differential equations :

$$\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} q_s & q_a \\ -q_a & q_s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0, \qquad (58)$$

where n and p are positive. The eigenvalues are given by

$$(n\Omega^2 + p\Omega + q_s)^2 + q_a^2 = 0. \qquad (59)$$

The solutions of (59) are

$$\Omega = \frac{-p \pm \sqrt{p^2 - 4n(q_s \mp iq_a)}}{2n}. \qquad (60)$$

If we choose $q_s = \frac{p^2}{4n}$, which satisfies condition (12), the threshold for instability is $n|q_a| = \frac{p^2}{2} = 2nq_s$.

It is conceivable that the Mirnov or 'precursor' oscillations, seen in tokamaks prior to 'disruptions', are of that kind, though they need not be related to that special instability. Pursuing this speculation, we could say that a further increase of density (or inertia) may lead to a 'disruption' because the limit cycle can no longer be maintained.

One of the puzzling questions is how to describe and follow such highly nonlinear systems for long times. In hydrodynamics, it is believed that such systems become turbulent or will tend to a fractal attractor with a large Hausdorff dimension increasing with the Reynolds number [21]. Estimates of the Hausdorff dimension of attractors are made, however, from above [21, 22] and do not necessarily reflect the real situation. Especially in MHD, this picture could be completely wrong: Nontrivial examples (see sections 4.1 and

4.2) are known to be nonlinearly and unconditionally stable for all magnetic Reynolds numbers [18, 23] , which means that the attractors have zero Hausdorff dimension, though the estimates [21] would deliver huge upper bounds such as 10^{10} or even higher. In other words, nonlinear behaviour and turbulence in MHD is much more configuration-dependent than in hydrodynamics. The reason is that the dimension of attractors or the number of determining modes is, to an extreme degree, configuration-dependent.

Let us finally note that, if condition (12) is not satisfied, other unstable eigenmodes can occur, but since the spectrum is not known, it will be very hard, in general, to check whether the eigenvalues satisfy the requirements of the centre manifold theorem.

5 Outlook

Two main future lines of research can be foreseen. Sections 1-3 show that the linear stability of resistive plasmas, in general geometry, can now be handled by a Hermitean form, similar to the energy principle of ideal MHD. The first line of activity will be the analytical and numerical minimization of that form. Though not straightforward, this task could be done in the near future.

Concerning the second line on nonlinear stability, the unsolved problems are those of general equilibria with pressure gradients. The successful use of Lyapunov functionals in section 4, in particular for force-free fields, gives an indication of the approach to be used, but progress in this area is unpredictable.

References

[1] H. Tasso. *Phys. Let. 94A, 217 (1983)*

[2] H. Tasso. *Plasma Phys. 17, 1131 (1975)*

[3] H. Tasso, J. T. Virtamo. *Plasma Phys. 22, 1003 (1980)*

[4] H. Tasso. *Lectures on Plasma Physics IFUSP/P-181- Sao Paulo (1979)*

[5] H. Tasso. *Phys. Lett. A147, 28 (1990)*

[6] E. M. Barston. *Phys. Fluids 12, 2162 (1969)*

[7] W. Kerner, H. Tasso. *Phys. Rev. Let. 49, 654 (1982)*

[8] R. Gruber, J. Rappaz. *'Finite Element Methods in Linear Ideal Magnetohydrodynamics', Springer Verlag, Berlin (1985)*

[9] H. Tasso. *Phys. Lett. A161, 289 (1991)*

[10] A. D. Jette. *J. Math. Anal. and Appl. 29, 109 (1970)*

[11] H. Tasso. *Lecture at 'Theoretical and Computational Plasma Physics' at Trieste. p.321, IAEA, Vienna 1978. See also IPP 6/151, Dec. 1976*

[12] P. G. Drazin, W. H. Reid. *'Hydrodynamic Stability', Cambridge University Press, 1981*

[13] J. T. Stuart. *in 'Transition and Turbulence', Editor R. E. Meyer, Academic Press, 1981*

[14] S. A. Orszag, A. T. Patera. *in 'Transition and Turbulence', Editor R. E. Meyer, Academic Press, 1981*

[15] J. Serrin. *'Handbuch der Physik', Springer Verlag, 1959*

[16] G. P. Galdi, S. Rionero. *'Weighted Energy Methods in Fluid Dynamics and Elasticity', Lecture Notes in Math.,Springer Verlag, 1985*

[17] H. Tasso, S. J. Camargo. *Nuovo Cimento 107 B, 733 (1992)*

[18] H. Tasso. *Phys. Lett. A 169, 396 (1992)*

[19] J. E. Marsden, M. Mc Cracken. *'The Hopf Bifurcation and its Applications', Springer Verlag, New York, 1976*

[20] H. Tasso, M. Cotsaftis. *Plasma Phys. (part C) 7, 29 (1965)*

[21] R. Temam. *'Infinite-Dimensional Dynamical Systems in Mechanics and Physics', Springer Verlag, New York, 1988*

[22] P. Constantin. *Commun. Math. Phys. 129, 241 (1990)*

[23] H. Tasso. *submitted to Phys. Lett. A*