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ABSTRACT

Wave absorption in a torus is found to differ significantly from the classical Landau result. Instead of the isolated resonance at $v = \omega/k$, the parametric beating between the particles' periodicity and the wave periodicity gives rise to an infinite, discrete spectrum of resonances. The spread of wave absorption over the velocity space leads to an enhancement in damping, particularly for the high-phase-velocity waves. Physical interpretation of Landau damping as a diffusion-like process ocurring at the parametric resonances is invoked to explain the phenomenon of toroidal Landau damping.

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Landau damping forms one of the central concepts in plasma waves. It is pivotal to the understanding of fundamental phenomena such as damping of electrostatic waves, instabilities both in astrophysical and fusion plasmas, as well as to vital applications like steady-state current drive in tokamaks. Particles streaming with the phase velocity $v = \omega/k$ of a longitudinal plasma wave strongly interact with the wave. In the linear limit, depending on the phase relative to the wave, they gain or lose energy monotonically. Averaging over a Maxwellian distribution with random initial phases, Landau[1] showed that there is net absorption of the wave energy by the group of particles moving at the wave phase velocity. Landau's result both fascinated and perplexed physicists until it was unequivocally verified in the experiments of Malmberg and Wharton[2] and Derfler and Simonen[3], some 20 years later. Alternative derivations of Landau's result via direct evaluation of the particles' kinetic energy integrated over the distribution function are given by Stix[4] and Swanson[5].

A clear physical picture of Landau damping remains elusive to this day. The best-known physical interpretation is due to Bohm and Gross[6] according to which particles moving slightly slower than the wave gain energy, while those moving faster lose energy. Depending upon the slope of the distribution function, there is net acceleration or deceleration of the particles moving at the wave phase velocity. Although this explanation is fortuitously correct in straight geometry, it has inadvertently contributed to erroneous reasoning when applied to Landau damping in toroidal geometry. For example, the existing current-drive theory[7] for the realization of a steady-state tokamak assumes that the wave imparts its momentum to particles moving at its phase velocity averaged over the flux surface. This will be shown to be qualitatively misleading and quantitatively incorrect. Before proceeding further, it would be instructive to offer a more uniformly valid interpretation of Landau damping as a diffusion-like process in velocity space. The change in the particle's velocity may be expressed as

$$\Delta v = \int_0^t \frac{eE}{m} \cos\left[(kv-\omega)t + \varphi_0\right] dt = \frac{eE}{m} \frac{\sin\left[(kv-\omega)t+\varphi_0\right] - \sin\varphi_0}{kv-\omega} \, .$$

After averaging $(\Delta v)^2$ over φ_0, diffusivity may be expressed as

$$D(v) = \lim_{t\to\infty} \frac{<(\Delta v)^2>}{t} = \frac{\pi}{2}\left(\frac{eE}{m}\right)^2 t\left[\frac{\sin^2(kv-\omega)t}{\pi(kv-\omega)^2t^2} + \frac{\{\cos(kv-\omega)t-1\}^2}{\pi(kv-\omega)^2t^2}\right] .$$

As $t \to \infty$, the first term in the square brackets may be approximated by the Dirac delta function $\delta(kvt - \omega t)$, while the second term becomes vanishingly small. The diffusion flux is given by

$$\phi(v) = -n_0 D(v)\frac{\partial F}{\partial v} = -n_0\frac{\pi}{2}\left(\frac{eE}{m}\right)^2 \frac{\partial F}{\partial v}\, t\,\delta(kvt-\omega t) ,$$

where $F(v)$ is the velocity distribution function. The energy absorption rate becomes

$$\frac{\partial W}{\partial t} = \int mv\phi(v)\,dv = -\int \pi\omega_{pe}^2\frac{v}{k}\left(\frac{1}{2}\epsilon_0 E^2\right)\frac{\partial F}{\partial v}\,\delta(kvt-\omega t)\,d(kvt) ,$$

where ϵ_0 is the dielectric permittivity of free space and ω_{pe} is the plasma frequency. After performing the integration, one obtains the classical Landau result

$$\frac{\partial W}{\partial t} = -\frac{\pi\omega_{pe}^2}{\omega}v_p^2\left.\frac{\partial F}{\partial v}\right|_{v_p} W , \tag{1}$$

where $v_p = \omega/k$ is the wave phase velocity and W is the energy density. In deriving Eq. (1), it was tacitly assumed that the step sizes $\pm\Delta v$ are equally probable; also, while averaging over the initial phases, the past history of the particle was ignored. Both of these conditions are satisfied in the linear approximation.

The above derivation holds important clues to the essential nature of linear Landau damping. If a given geometrical configuration displays a wave-particle resonance such that particles with velocity v repetitively gain/lose velocity with step size Δv during each periodic orbit of duration Δt, linear Landau damping with an effective diffusivity $D(v) =< (\Delta v)^2 > /\Delta t$ would be present. Note that in the linear approximation, the random initial phases would cause $\pm\Delta v$ to be invariably equally probable. Linear Landau damping, therefore, may be seen as a resonant diffusion process with a solitary resonance at $v = \omega/k$ in straight geometry. For finite wave amplitudes and in the absence of randomizing collisions, particles retain their phase memory and a nonlinear treatment would be required.

Unlike the straight geometry case, the toroidal geometry is capable of supporting an infinite number of discrete resonances through the parametric beating between the particles' periodicity and the wave periodicity. Non-relativistic wave damping in a large aspect-ratio tokamak has been studied by Grishanov and Nekrasov[8]. A simplified derivation retaining Landau damping but excluding cyclotron damping is given in Ref.9. Starting with the linearized drift-kinetic equation, the expression for the parallel current (to the local magnetic field direction) in a Maxwellian plasma for an electric field excitation $E_{\parallel} = E_m \exp(im\theta)$ is given as (from Eqs.(20-22) of Ref. 9)

$$j_{\parallel} = \frac{\omega_{pe}^2 \epsilon_0}{i\omega} \sum_s E_m \left[\Psi_{u,m}^{(s)} + \Psi_{t,m}^{(s)} \right] , \tag{2}$$

where $s = \pm 1$ for the parallel velocity component $v_{\parallel} \gtrless 0$,

$$\Psi_{u,m}^{(s)}(\theta) = \frac{2}{\sqrt{\pi}} \frac{iqnU_p}{h_\phi(1 + \varepsilon \cos\theta)} \int_0^\infty U^3 \exp(-U^2)\, dU \int_0^{1-\varepsilon} d\Lambda$$

$$\left[\int_{-\pi}^{\theta} \frac{\exp\left(imy + i\left\{ X_u^{(s)}(y) - X_u^{(s)}(\theta) \right\} \right)}{1 - \exp\{-2iX_u^{(s)}(\pi)\}} dy \right.$$

$$\left. + \int_{\theta}^{\pi} \frac{\exp\left(imy + i\left\{ X_u^{(s)}(y) - X_u^{(s)}(\theta) - 2X_u^{(s)}(\pi) \right\} \right)}{1 - \exp\{-2iX_u^{(s)}(\pi)\}} dy \right] , \tag{3}$$

$$\Psi_{t,m}^{(s)}(\theta) = \frac{2}{\sqrt{\pi}} \frac{iqnU_p}{h_\phi(1 + \varepsilon \cos\theta)} \int_0^\infty U^3 \exp(-U^2)\, dU \int_{1-\varepsilon}^{1+\varepsilon\cos\theta} d\Lambda$$

$$\left[\int_{-\theta_m}^{\theta_m} \frac{\exp\left(imy + i\left\{ X_t^{(-s)}(y) - X_t^{(s)}(\theta) - X_t^{(s)}(\theta_m) - X_t^{(-s)}(\theta_m) \right\} \right)}{\exp\left\{ -2iX_t^{(s)}(\theta_m) \right\} - \exp\left\{ -2iX_t^{(-s)}(\theta_m) \right\}} dy \right.$$

$$- \int_{-\theta_m}^{\theta} \frac{\exp\left(imy + i\left\{ X_t^{(s)}(y) - X_t^{(s)}(\theta) - 2X_t^{(-s)}(\theta_m) \right\} \right)}{\exp\left\{ -2iX_t^{(s)}(\theta_m) \right\} - \exp\left\{ -2iX_t^{(-s)}(\theta_m) \right\}} dy$$

$$\left. - \int_{\theta}^{\theta_m} \frac{\exp\left(imy + i\left\{ X_t^{(s)}(y) - X_t^{(s)}(\theta) - 2X_t^{(s)}(\theta_m) \right\} \right)}{\exp\left\{ -2iX_t^{(s)}(\theta_m) \right\} - \exp\left\{ -2iX_t^{(-s)}(\theta_m) \right\}} dy \right] , \tag{4}$$

$$X^{(s)}_{u,t}(\eta) = -X^{(s)}_{u,t}(-\eta) = \int_0^\eta \chi^{(s)}\, d\eta = A(\eta) - s\,\frac{B_{u,t}(\eta)}{U}\ , \tag{5}$$

$$\chi^{(s)}(\eta) = \frac{nq}{1+\varepsilon\cos\eta} - \frac{snq}{h_\phi\sqrt{1-\frac{\Lambda}{1+\varepsilon\cos\eta}}}\frac{U_p}{U},$$

$$A(\eta) = nq\int_0^\eta \frac{d\eta}{1+\varepsilon\cos\eta} = \frac{2nq}{\sqrt{1-\varepsilon^2}}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\eta}{2}\right),$$

$$B_u(\eta) = \frac{nqU_p}{h_\phi}\int_0^\eta \frac{d\eta}{\sqrt{1-\frac{\Lambda}{1+\varepsilon\cos\eta}}}$$

$$= \frac{nqU_p}{h_\phi}\frac{2(1+\varepsilon)}{\sqrt{(1-\varepsilon)(1+\varepsilon-\Lambda)}}\prod\left(\frac{2\varepsilon}{1-\varepsilon};\phi_u\middle|\frac{2\varepsilon\Lambda}{(1-\varepsilon)(1+\varepsilon-\Lambda)}\right)\ ,$$

$$B_t(\eta) = \frac{nqU_p}{h_\phi}\int_0^\eta \frac{d\eta}{\sqrt{1-\frac{\Lambda}{1+\varepsilon\cos\eta}}}$$

$$= \frac{nqU_p}{h_\phi}(1+\varepsilon)\sqrt{\frac{2}{\varepsilon\Lambda}}\prod\left(\frac{1+\varepsilon-\Lambda}{\Lambda};\phi_t\middle|\frac{(1-\varepsilon)(1+\varepsilon-\Lambda)}{2\varepsilon\Lambda}\right)\ ,$$

$$\phi_u = \sin^{-1}\sqrt{\frac{1-\varepsilon}{2}}\sqrt{\frac{1-\cos\eta}{1+\varepsilon\cos\eta}}\ ,$$

$$\phi_t = \sin^{-1}\sqrt{\frac{\varepsilon\Lambda}{1+\varepsilon-\Lambda}}\sqrt{\frac{1-\cos\eta}{1+\varepsilon\cos\eta}}\ ,$$

subscripts u and t denote untrapped and trapped particles, respectively, $\varepsilon = r/R_0$, R_0 is the torus major radius, θ and ϕ are the poloidal and toroidal angles, m and n are the respective wave numbers, q is the safety factor, $h_\phi = B_\phi/B$, $U = v/\sqrt{2}v_{te}$, $U_\parallel = v_\parallel/\sqrt{2}v_{te}$, $U_p = v_p/\sqrt{2}v_{te}$, $\Lambda = 2\mu B_0/v^2$, $\mu = (1+\varepsilon\cos\theta)v_\perp^2/2B_0$, B_0 is the magnetic field at the axis, $\theta_m = \cos^{-1}[(\Lambda-1)/\varepsilon]$ is the maximum azimuthal excursion for the trapped particles, and $\prod$ is the elliptic integral of the third kind (its origin is discussed in Eq.(24) of Ref. 9).

Of the three integrations in Eqs.(3) and (4), the one over velocity U is by far the most problematic. The numerical evaluation carried out in Ref. (9) is time consuming and lacks accuracy. A straightforward approach for obtaining only the imaginary part of Ψ would consist in contour integration around the singularities ocurring at the zeros of the denominators in Eqs.(3) and (4). The poles in Eq.(3) correspond to

$$X_u^{(s)}(\pi) = N\pi \ ,$$

which using Eq.(5) gives

$$U_N = \frac{B_u(\pi)}{A(\pi) - N\pi} \ , \tag{6}$$

where $U_N \gtrless 0$ corresponds to $s = \pm 1$. Using

$$\lim_{U \to U_N} \frac{U - U_N}{1 - \exp\left\{-2iX_u^{(s)}(\pi)\right\}} = \frac{U_N^2}{2iB_u(\pi)}$$

one obtains from Eq.(3) after summing over $s = \pm 1$,

$$\mathrm{Im}\left[\Psi_{u,m}(\theta)\right] = \frac{\sqrt{\pi} q n U_p}{h_\phi(1+\varepsilon\cos\theta)} \int_0^{1-\varepsilon} \frac{d\Lambda}{B_u(\pi)} \sum_N |U_N|^5 \exp(-U_N^2)$$

$$\int_{-\pi}^{\pi} \exp\left[imy + i\left\{X_{uN}(y) - X_{uN}(\theta)\right\}\right] dy \ ,$$

where only one-half of the residue contributes to the integral, and

$$X_{uN}(\eta) = A(\eta) - \frac{B_u(\eta)}{U_N} \ .$$

Averaging over θ gives

$$\mathrm{Im}\left[\Psi_{u,m}\right] = \frac{2qnU_p}{\sqrt{\pi} h_\phi} \int_0^{1-\varepsilon} \frac{d\Lambda}{B_u(\pi)} \sum_N |U_N|^5 \exp(-U_N^2) \left[\int_0^{\pi} \cos\left[m\theta + X_{uN}(\theta)\right] d\theta\right]^2 . \tag{7}$$

Equation (7) exhibits the effect of parametric resonances on toroidal Landau damping ocurring at the velocities given by Eq.(6). U_N is related to the parallel velocity by

$$U_{\parallel N} = U_N \sqrt{1 - \frac{\Lambda}{1+\varepsilon\cos\theta}} \ .$$

The primary resonance with $N = 0$ occurs for the group of particles moving on the average at the parallel phase velocity of the wave. For $\varepsilon \to 0$, $X_{u0}(\theta) \to 0$ and for the case $m = 0$, the integrand over θ in Eq.(7) becomes $\cos(\theta) \to 1$. It can be shown that in the limit $\varepsilon \to 0$, Eq.(7) reproduces the classical Landau damping result for the straight geometry. Secondary resonances with $N \neq 0$, corresponding to the case when the particle either gains or loses N complete wavelengths during one period of its orbit, contribute very little to the damping process for the case $\varepsilon \to 0$. However, for larger ε and most particularly for large toroidal wave numbers n, even a small slippage in phase destroys any semblance of cohesion between the wave and the particle, and the step size Δv may be drastically reduced causing decreased Landau damping for the $N = 0$ primary resonance. However, secondary resonances for $N \neq 0$ begin to gain prominence, both enhancing absorption and enlarging the velocity spread over which Landau damping extends. The enhancement in damping is most dramatic for $U_p \gg 1$, since the secondary resonances ocurring at lower velocities involve a much larger population of particles. This may have direct application to the lower-hybrid-current-drive spectral-gap problem and will be dealt with in a separate communication. Each of the resonances corresponding to a specific value of N is broadened by the Λ integration in Eq.(7). For large n, this broadening in combination with the decreased separation between the individual resonances would lead to a continuous absorption spectrum for Landau damping.

From Eq.(4) the Landau poles for the trapped particles occur at

$$X_t^{(-1)}(\theta_m) - X_t^{(+1)}(\theta_m) = N\pi \ ,$$

or

$$U_N = \frac{2B_t(\theta_m)}{N\pi} \ . \tag{8}$$

Proceeding as before, one obtains

$$\mathrm{Im}\left[\Psi_{t,m}\right] = \frac{qnU_p}{\sqrt{\pi}h_\phi} \int_0^\pi d\theta \int_{1-\varepsilon}^{1+\varepsilon\cos\theta} \frac{d\Lambda}{B_t(\theta_m)} \sum_N |U_N|^5 \exp(-U_N^2)$$

$$\cos\left[m\theta + X_{tN}(\theta)\right] \int_0^{\theta_m} \left[\cos\left\{my + X_{tN}(y)\right\} - (-1)^N \cos\left\{my + X_{tN}(y)\right\}\right] dy \ , \tag{9}$$

where

$$X_{tN}(\eta) = A(\eta) - \frac{B_t(\eta)}{U_N} \ .$$

From Eq.(8) one observes that for the trapped particles there is no $N = 0$ resonance; hence, the energy absorption is lowered in comparison with that of the untrapped particles. This would enhance low-phase-velocity current drive as was shown in Ref. 9.

From Eqs.(2), (7) and (9), one obtains the energy absorption rate

$$\frac{\partial W}{\partial t} = \frac{1}{2}\mathrm{Re}(j_{\parallel} E_{\parallel}^{*}) = \frac{\omega_{pe}^2}{\omega}\Psi^i \ ,$$

where $\Psi^i = \mathrm{Im}\,[\Psi_{u,m} + \Psi_{t,m}]$. Comparing with the derivation of energy absorption rate using the diffusion model leading to Eq.(1), one may formally relate diffusivity to Ψ^i as

$$D(U_{\parallel}) = -\frac{\omega_{pe}^2}{\omega}\frac{W}{\Theta}\frac{1}{U_{\parallel}(\partial F/\partial U_{\parallel})}\Psi^i(U_{\parallel}) \ , \qquad (10)$$

where $\Theta = mn_0 v_{te}^2$ and $\Psi(U_{\parallel})$ can be obtained from $\Psi(U)$.

The remaining integrations over Λ and θ in Eqs.(7) and (9) are performed using Gauss quadrature. Figure 1 displays the velocity spread in diffusivity (drawn to relative scale) for increasing ε for the case $U_p = 5$, $m = 0$ and $n = 50$. For lower n, the individual resonances no longer overlap and become distinct as seen in Fig.2. Figure 3 shows the enhancement in Landau damping with increasing ε, especially for large U_p.

To summarize the findings of this paper: (1) Linear Landau damping is the physical consequence of resonant diffusion in velocity space. (2) The multiplicity of resonances due to parametric beating between the particle and wave periodicities in a tokamak leads to a broadening of the Landau damping spectrum in velocity space. (3) For the low-phase-velocity waves, the dominant toroidal effect is to redistribute Landau absorption in favor of circulating particles at the expense of trapped particles[9]. (4) For $U_p \gg 1$, the secondary resonances enhance the net energy absorption, in additon to broadening the velocity spectrum of Landau damping.

These findings indicate a need for a complete reevaluation of the existing current-drive theories in tokamak plasmas.

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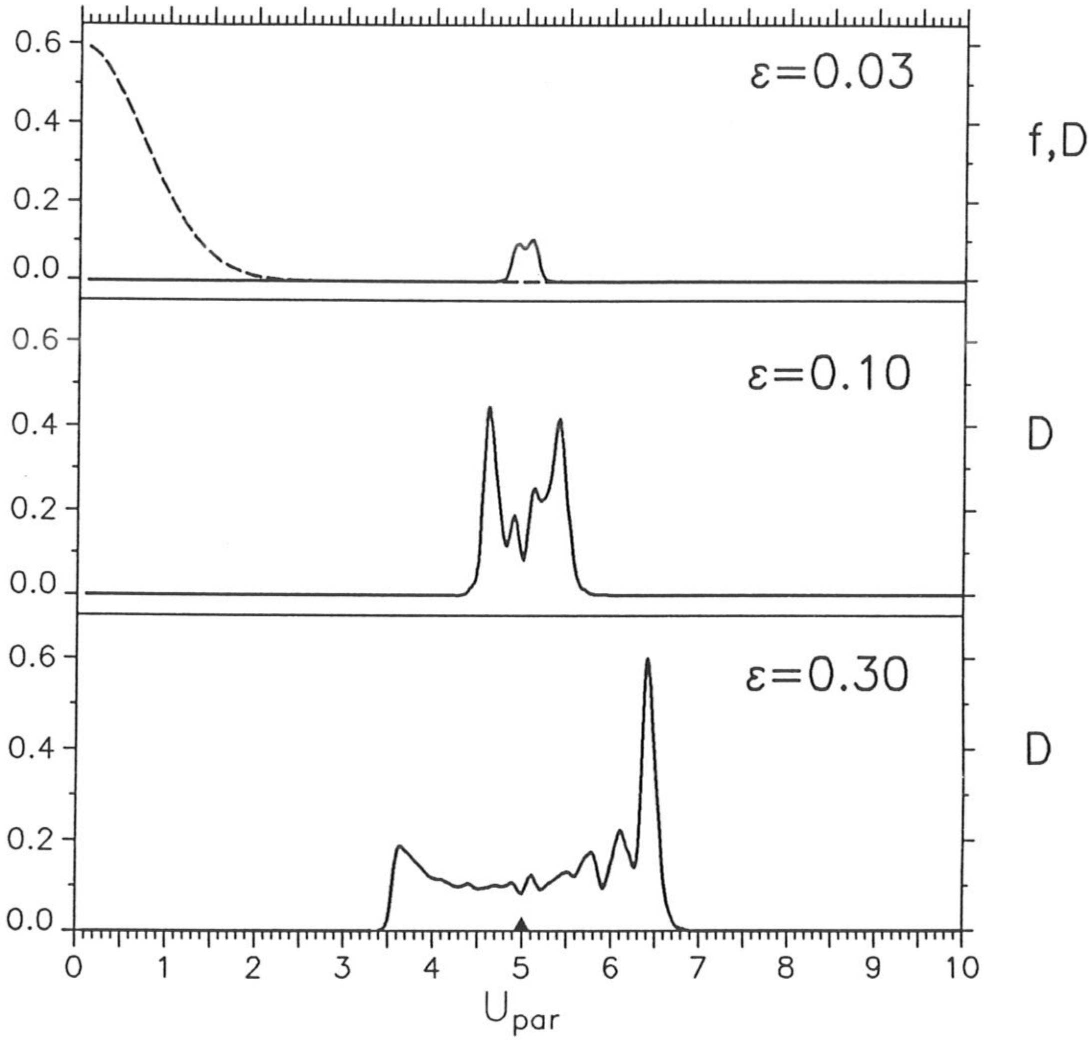

Fig.1 Velocity spread of Landau damping for increasing ε.

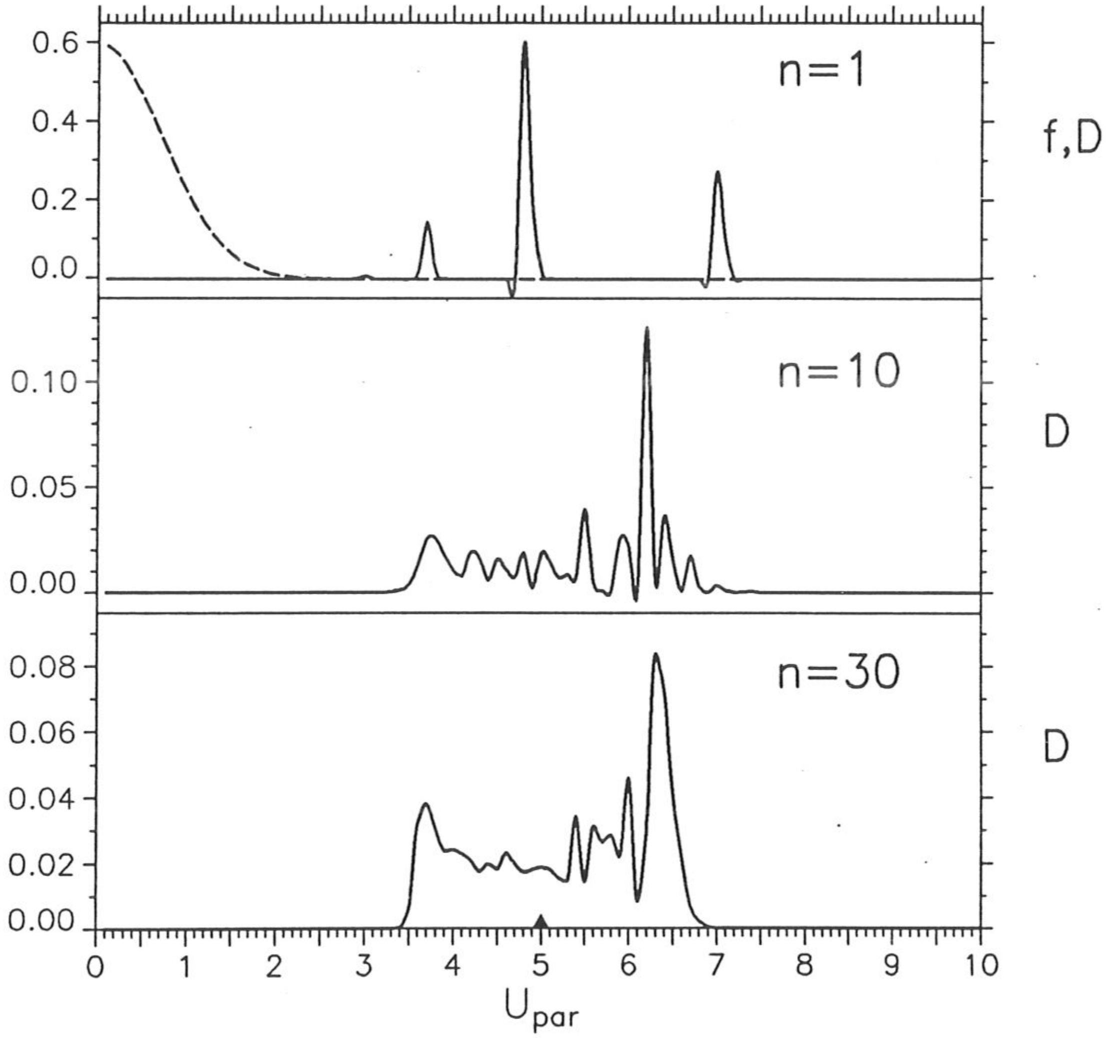

Fig.2 Damping spectrum versus toroidal wave number n.

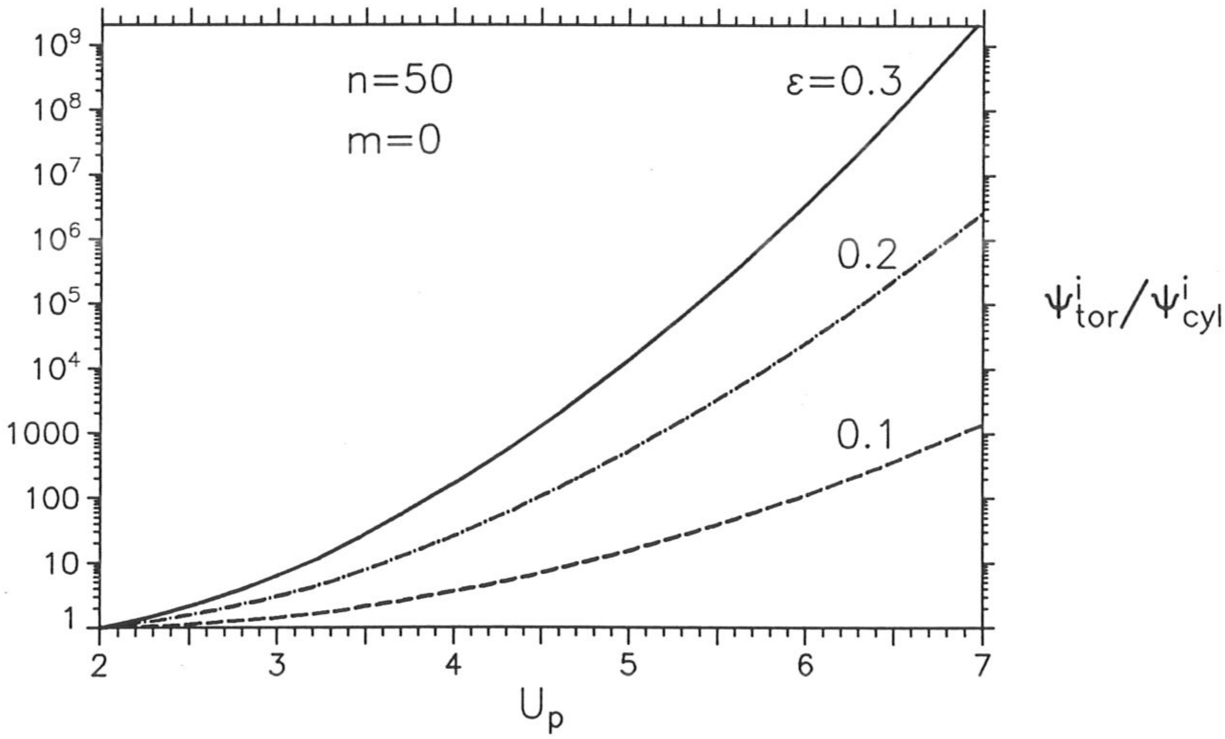

Fig.3 Enhancement of Landau damping in a torus.