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Simplified Versions of a Stability Condition in Resistive MHD

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Abstract

In a general stability condition obtained by the author in a previous work physically motivated test functions are introduced. This leads to simplfied versions of the stability functional, which makes its evaluation and minimization more tractable. In the case of special force-free-fields the simplified functional reduces to a good approximation of the exact stability functional derived by other means.It turns out that in this case the condition is sufficient for nonlinear stability also.

The purpose of this note is to simplify the general resistive stability condition presented in [1] by restricting the space of test functions to a "physically desirable" space. The condition proposed in [1] has the form of an "energy principle" whose sufficiency with respect to purely growing modes was proved in [2]. The analysis in [1] suggested that this condition can be considered as "nearly" necessary and sufficient for all modes under reasonable physical approximations. This condition or "energy principle" is given by(see [1] and [2])

$$\delta W = \int d\tau(\gamma P_0(\nabla\cdot\xi)^2 + (\xi\cdot\nabla P_0)\nabla\cdot\xi)$$

$$+\int d\tau(\nabla\times\mathbf{A})^2-\int d\tau\xi\times\mathbf{J}\cdot\nabla\times\mathbf{A}+$$
$$+p\cdot p\cdot\int d\tau\mathbf{J}\cdot(\mathbf{A}-\xi\times\mathbf{B})(\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A})$$
$$-\int d\tau(\mathbf{A}-\xi\times\mathbf{B})\cdot\mathbf{V}\times(\nabla\times\mathbf{A})1/\eta_0 \qquad (1)$$

where P_0, $\mathbf{J}$ and $\mathbf{B}$ are, as usual, the equilibrium pressure, the current density and the magnetic field respectively. $\mathbf{V}$ is the Pfirsch-Schlüter flow velocity and η_0 the unperturbed resistivity.ξ is the fluid displacement and $\mathbf{A}$ the perturbed vector potential. p.p. denotes the "principal part" and the integration is over the plasma volume with perfectly conducting boundary conditions, all quantities in the integrand being taken real. For ξ and $\mathbf{A}$ complex it is recommended to symmetrize (1) by integration by parts (see [2]).

The test functions ξ and $\mathbf{A}$ in (1) are general and constitute together a six-dimensional test-function-space. As mentioned in [2] ideal MHD can be recovered by restricting to $\mathbf{A}=\xi\times\mathbf{B}$. In the tokamak scaling and for $\nabla\cdot\xi=0$ one recovers the resistive principle derived in [3] .

In this note we introduce a physical restriction by the following arguments. Perpendicular to the magnetic field a weakly dissipative plasma behaves like in ideal MHD but parallel to $\mathbf{B}$ it may behave quite differently essentially because of resistivity. This suggests the following restriction in test-function-space

$$\mathbf{A}=\xi\times\mathbf{B}+\mathbf{A}_{\mathbf{par}} \qquad (2)$$

where $\mathbf{A}_{\mathbf{par}}$ is the part of $\mathbf{A}$ parallel to $\mathbf{B}$. Crossing relation (2) with $\mathbf{B}$ we find for ξ

$$\xi=\frac{\mathbf{B}\times\mathbf{A}}{B^2}+\xi_{par} \qquad (3)$$

where ξ_{par} is the part of ξ parallel to $\mathbf{B}$.

If we insert (2) and (3) in (1) we obtain a first simplified version of (1)

$$\delta W = \int d\tau(\gamma P_0(\nabla\cdot(\frac{\mathbf{B}\times\mathbf{A}}{B^2}+\xi_{par}))^2+(\frac{\mathbf{B}\times\mathbf{A}}{B^2}\cdot\nabla P_0)\nabla\cdot(\frac{\mathbf{B}\times\mathbf{A}}{B^2}+\xi_{par}))$$
$$+\int d\tau(\nabla\times\mathbf{A})^2-\int d\tau(\frac{\mathbf{B}\times\mathbf{A}}{B^2}+\xi_{par})\times\mathbf{J}\cdot\nabla\times\mathbf{A}+$$
$$+p\cdot p\cdot\int d\tau(\mathbf{J}\cdot\mathbf{A}_{par})\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A})$$

$$- \int d\tau \mathbf{A}_{par} \cdot \mathbf{V} \times (\nabla \times \mathbf{A}) 1/\eta_0. \tag{4}$$

Instead of a six-dimensional test-function-space we have now a four-dimensional one. One is tempted to minimize (4) with respect to ξ_{par} as in ideal MHD. This does not lead to $\nabla \cdot \xi = 0$ but to a rather complicated expression together with a difficult equation for ξ_{par}. Despite this fact expression (4) is already simple enough to minimize either numerically or analytically(e.g. for perturbations localized about magnetic surfaces or magnetic lines).

A further simplification consist in setting $\nabla \cdot \xi = 0$ from the outset. In this case one can solve for ξ_{par} setting

$$\xi_{par} = \alpha \mathbf{B} \tag{5}$$

and using (3) to obtain

$$\nabla \cdot (\frac{\mathbf{B} \times \mathbf{A}}{B^2}) + \mathbf{B} \cdot \nabla \alpha = 0 \tag{6}$$

whose solution is

$$\alpha = -(\mathbf{B} \cdot \nabla)^{-1} \nabla \cdot \frac{\mathbf{B} \times \mathbf{A}}{B^2}. \tag{7}$$

Inserting (5) and (7) in (4) we obtain as a further simplified version of (1)

$$\begin{aligned} \delta W &= \int d\tau ((\nabla \times \mathbf{A})^2 - (\frac{\mathbf{B} \times \mathbf{A}}{B^2}) \times \mathbf{J} \cdot \nabla \times \mathbf{A}) \\ &- p \cdot p \cdot \int d\tau ((\mathbf{B} \cdot \nabla)^{-1} \nabla \cdot \frac{\mathbf{B} \times \mathbf{A}}{B^2}) \nabla P_0 \cdot \nabla \times \mathbf{A} + \\ & p \cdot p \cdot \int d\tau (\mathbf{J} \cdot \mathbf{A}_{par})(\mathbf{B} \cdot \nabla)^{-1} (1/\eta_0)(\nabla \eta_0 \cdot \nabla \times \mathbf{A}) \\ &- \int d\tau \mathbf{A}_{par} \cdot \mathbf{V} \times (\nabla \times \mathbf{A}) 1/\eta_0. \end{aligned} \tag{8}$$

Application to force-free-fields

In the case of a resistive field obeying

$$\mathbf{J} = \lambda \mathbf{B} \tag{9}$$

with $\lambda = ct.$ one knows (see [4]) that also η_0 has to be constant and $\mathbf{V} = 0$. The field satisfies

$$\dot{\mathbf{B}} = -\eta_0 \lambda^2 \mathbf{B}. \tag{10}$$

Though expression (1) is derived in [1] for time-independent equilibria it should hold in the limit $\eta_0 \to 0$. Therefore inserting (9) and (10) in (8) as well as $\mathbf{V} = \nabla P_0 = \eta_0 = 0$, then δW from (8) reduces to

$$\delta W = \int d\tau((\nabla \times \mathbf{A})^2 - \lambda \mathbf{A}_{perp} \cdot \nabla \times \mathbf{A}) \tag{11}$$

where $\mathbf{A}_{perp}$ is the part of $\mathbf{A}$ perpendicular to $\mathbf{B}$. Expression (11) compares very well with the exact δW derived in [5] for the field (9)-(10), which is

$$\delta W = \int d\tau((\nabla \times \mathbf{A})^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) \geq 0, \tag{12}$$

sufficient for stability. The difference between (11) and (12) is in an $\mathbf{A}_{par}$ term not containing the singularity $(\mathbf{B} \cdot \nabla)^{-1}$ which means that this term vanishes smoothly for $\eta_0 \to 0$.

In view of the physical (but formally not exact) restrictions in the test-function-space this is a remarkable result and gives us hope that expressions (4) and (8) for the simplified δW are good even for equilibria with pressure and $\lambda \neq ct.$ .In those cases, however, inaccuracies of the kind above can be amplified by the singularity $(\mathbf{B} \cdot \nabla)^{-1}$ despite the "principal part" before the integral.

Finally let us prove that condition (12) derived in [5] for the linear case still holds for nonlinear resistive incompressible fluids. The finitely perturbed equations are

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_1 \times \mathbf{B}_1 \tag{13}$$

with $\nabla \cdot \mathbf{v} = 0, \mathbf{J}_0 = \lambda \mathbf{B}_0, \lambda = ct.$

$$\dot{\mathbf{A}} = \mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1) - \eta \mathbf{j}_1, \tag{14}$$

$$\dot{\mathbf{B}_1} = \nabla \times (\mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1) - \eta \mathbf{j}_1). \tag{15}$$

Multiply scalarly (13) by $\mathbf{v}$ and (15) by $\mathbf{B}_1$ add and integrate over the volume to obtain

$$\frac{\partial}{\partial t} \int \frac{d\tau}{2} (v^2 + \mathbf{B}_1^2) = \lambda \int d\tau \mathbf{v} \times \mathbf{B}_0 \cdot \mathbf{B}_1 - \int d\tau \eta \mathbf{j}_1^2. \tag{16}$$

Many quadratic and cubic terms integrate to zero because of the boundary condition taken as perfectly conducting. Multiplying (14) scalarly by $\mathbf{B}_1$ we can solve for $\mathbf{v} \times \mathbf{B}_0 \cdot \mathbf{B}_1$ and insert into (16) to obtain

$$\frac{\partial}{\partial t} \int \frac{d\tau}{2} (v^2 + \mathbf{B}_1^2 - \lambda \mathbf{A} \cdot \nabla \times \mathbf{A}) = -\eta \int d\tau (\mathbf{j}_1^2 - \lambda \mathbf{B}_1 \cdot \mathbf{j}_1). \tag{17}$$

Equation (17) is identical with the stability equation derived in [5] for linear perturbations. This proves that condition (12) is sufficient for linear as well as for nonlinear stability.

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