

**A FREE BOUNDARY EIGENVALUE PROBLEM**

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1. Introduction

We consider a free boundary problem which was introduced by Tolman [10] and which has already been studied in [1, 2, 3, 4, 5, 6, 7, 8, 9]. It leaves important questions unsolved.

Problem 1. Let  $\Omega$  be a domain bounded with respect to the  $x_1$ -coordinate by  $x_1 = 0$  and  $x_1 = 1$  and having a smooth boundary  $\partial\Omega \in C^2$ . Let  $L$  be a uniformly elliptic operator in  $\Omega$ :

$$Lx = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 x}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial x}{\partial x_i} + c(x)x = 0 \quad (1)$$

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# A Free Boundary Eigenvalue Problem

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## Abstract

The equilibrium equations for a confined axisymmetric plasma lead to a free boundary problem. Here we consider a simple model problem. Related equations are known from other applications (astrophysics, fluid dynamics, atomic physics). If all but one parameter are kept fixed, it is a non-linear elliptic eigenvalue problem which generalizes a related ‘linear’ eigenvalue problem. Given a branch of explicitly known solutions, there exist countably infinitely many bifurcation points. This is shown by a perturbation approach and generalizes results by Sijbrand. The structure of the solutions on the bifurcating branches is investigated numerically: their rapid changing near the bifurcation points as well as their asymptotics. The numerical error caused by the free boundary is well understood (Numer Math 59, 683 – 710 and J. Comp Phys 102, 72 – 77) and was kept small in the investigations reported here.

## 1 Introduction

We consider a simple model problem, which was introduced by Temam [19] and which is already treated in several mathematical monographs [2, 9], but still leaves important questions unanswered.

**Problem 1 :** Let  $\Omega \subset \mathbb{R}^2$  be a domain bounded with respect to the  $x_1$  - coordinate ( $0 < x_* \leq x_1 \leq x^* < \infty$ ) and having a smooth boundary ( $\partial\Omega \in C^4$ ). Let  $\mathcal{L}$  be a uniformly elliptic operator in  $\Omega$ ,

$$\mathcal{L} u := \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( \frac{1}{x_1} \frac{\partial u}{\partial x_i} \right) = \frac{1}{x_1} \Delta^* u, \quad (1)$$

and let  $b \in C^1(\Omega)$  satisfy  $0 < b_1 \leq b(x) \leq b_2$  in  $\bar{\Omega}$ .

Given  $\Omega$  and a constant  $I > 0$ , find  $u$ ,  $\lambda$ ,  $\gamma$ , and  $\Omega_p$  with  $\bar{\Omega}_p \subset \Omega$ ,  $u(x) \neq 0$  in  $\Omega_p$ , such that

$$\mathcal{L} u = \begin{cases} -\lambda b u & \text{in } \Omega_p \\ 0 & \text{in } \Omega_v := \Omega \setminus \bar{\Omega}_p \end{cases} \quad (2)$$

$$u|_{\partial\Omega} = \gamma, \gamma \text{ constant}, \quad u|_{\partial\Omega_p} = 0, \quad \frac{\partial u}{\partial n} \text{ continuous across } \partial\Omega_p, \quad (3)$$

and

$$- \oint_{\partial\Omega_p} \frac{1}{x_1} \frac{\partial u}{\partial n} ds = I. \quad (4)$$

**Remark 1:** Once a solution to this problem is found,  $\lambda$  is the principal eigenvalue of the subproblem

$$\mathcal{L} u + \lambda b u = 0 \quad \text{in } \Omega_p, \quad u|_{\partial\Omega_p} = 0. \quad (5)$$

Owing to the character of elliptic problems, it is in general not possible to find a solution to the full problem from an eigenfunction on some given subdomain  $\Omega_p \subset \Omega$ . Since  $\Omega_p$  has to be found by solving the equations, its boundary  $\partial\Omega_p$  is a *free boundary*.

**Remark 2:** If we know a solution of Problem 1 and multiply  $u$  by an arbitrary positive constant, we obtain another solution with the same  $\lambda$  and  $\Omega_p$ , but with different  $I$  and  $\gamma$ . The integral side condition (4) thus plays the role of a normalization. It singles out one solution from an uninteresting continuum of solutions, same way as this is the custom when dealing with linear eigenvalue problems.

Among the results obtained by Temam [19] are the following: solutions to Problem 1 exist and have Hölder-continuous second derivatives in  $\bar{\Omega}$  (i.e. the solutions are  $C^{2+\alpha}(\bar{\Omega})$ ,  $0 < \alpha < 1$ ); their  $3^{rd}$  derivatives are in  $L^p$ ,  $p \geq 1$  (i.e. the solutions are in  $W^{3,p}$ ,  $p \geq 1$ ).

In [19, 15, 14] Problem 1 is discussed in connection with the **plasma equilibrium problem**. In its derivation the plasma is assumed to be confined in a superconducting vessel. The shaping of the plasma in today's tokamaks, however, is done by field shaping currents, magnetic divertors, and limiters. Problem 1 is thus too simple to be of direct importance to plasma physics. However, it has proved useful as a model problem. Blum et al [4] proved existence of solutions to a more realistic problem, including field shaping coils and a point limiter. Their model is a direct generalization of Problem 1, and their proof makes substantial use of earlier work related to Problem 1 [3, 4].

Some **other applications** lead to model problems with the same or closely related equations, for instance the study of \* self-gravitating axisymmetric rotating fluids as considered in astrophysics; \* steady vortex rings in inviscid incompressible fluid (Squire-Long equation); \* the Thomas-Fermi atomic model (electron clouds moving about fixed ions). For more details see for instance [2, 9].

**Existence proofs and numerical methods** were developed in context with the various applications, see for instance [4, 8, 19] for existence proofs, [5, 11, 12, 13] for numerical methods.

Possible **non-uniqueness of solutions** was noticed quite early for the plasma equilibrium problem with or without free boundary, and has been investigated quite a bit: see, for instance, [3, 6, 10, 13, 16, 17, 18] and the references therein. Nevertheless, those many results do not fit together in an obvious way. It is still not understood how the existence of the free boundary influences the multiplicity of solutions.

## 2 Examples

Several solutions to Problem 1 can be given explicitly [15]. Here we consider only examples to the limiting case, in which the curvature of the torus is neglected. The operator  $\mathcal{L}$  is thus replaced by the Laplacian  $\Delta$  [15].

**Example 1:** The simplest example then possible is probably the 1D problem

$$\begin{aligned} \psi'' + \lambda^2 \psi^+ &= 0 && \text{in } (0, 1), \\ \psi(0) = \psi(1) &= \alpha, \\ \psi'(0) - \psi'(1) &= I, \end{aligned} \quad (6)$$

with

$$\psi^+(x) := \begin{cases} \psi(x) & \text{if } \psi(x) > 0 \\ 0 & \text{if } \psi(x) \leq 0 \end{cases}. \quad (7)$$

Assuming symmetry of the solution with respect to  $x = 1/2$ , we make the ansatz

$$\psi(x) = \begin{cases} a + b x & \text{if } 0 \leq x \leq x_f \\ c \sin \lambda(x - x_f) & \text{if } x_f \leq x \leq 1 - x_f \\ a + b(1 - x) & \text{if } 1 - x_f \leq x \leq 1 \end{cases}. \quad (8)$$

The continuity of  $\psi$  and  $\psi'$  in the free-boundary point  $x_f$ , the fact that  $\lambda^2$  is the eigenvalue of  $\frac{\partial^2}{\partial x^2}$  in the interval  $(x_f, 1 - x_f)$ , and the boundary conditions provide five equations for the six unknowns  $\lambda$ ,  $\alpha$ ,  $x_f$ ,  $a$ ,  $b$ , and  $c$ . Thus we can choose one of these parameters arbitrarily. Prescribing the inner parameter  $x_f$  gives good insights and explains why there exist no solutions for certain values of  $(\alpha, I, \lambda, \psi_{\max})$ . We find  $I = 2b$ ,  $\psi_{\max} = c$  and

$$0 \leq x_f = -\frac{\alpha}{b} < \frac{1}{2} \Rightarrow \pi \leq \lambda = \frac{\pi}{1 - 2x_f} < \infty \quad \text{and} \quad 0 \leq c = \frac{b}{\lambda} \leq \frac{b}{\pi}. \quad (9)$$

Fig. 1 shows solutions to eqs. (6).

**Example 2:** Under the assumption that

$$\Delta \psi + \lambda^2 \psi^+ = 0 \quad (10)$$

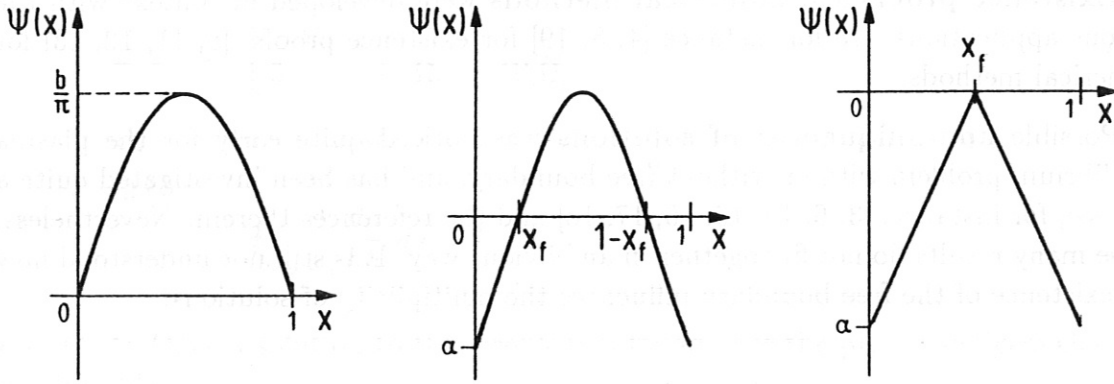


Figure 1: Sketch of explicitly known 1D solutions: a) limiting case with coinciding fixed and free boundary,  $2|\alpha|/I = 0$ ; b) typical case  $0 < 2|\alpha|/I < 1/2$ ; c) limiting case in which the inner domain  $\Omega_p$  is degenerated to the empty set,  $2|\alpha|/I = 1/2$ .

in a circle has radially symmetric solutions, it can be reduced to the ODE

$$\psi'' + \frac{1}{r}\psi' + \lambda^2\psi = 0 \quad (11)$$

and then solved explicitly. If we assume that the free boundary lies at radius  $r_f$ , we get

$$\psi(r) = \begin{cases} J_0(\lambda r) & \text{if } 0 \leq r \leq r_f \\ A \ln \frac{r}{r_f} & \text{if } r \geq r_f \end{cases} \quad (12)$$

with

$$\begin{aligned} \lambda r_f &= j_{0,1} \quad (\text{first zero of } J_0), \\ A &= \lambda r_f J_0'(\lambda r_f) = j_{0,1} J_0'(j_{0,1}), \end{aligned} \quad (13)$$

$$\oint_{r=r_f} \frac{\partial \psi}{\partial n} dl = 2 \pi A.$$

This solution was already given by Bandle [1] in a slightly different formulation. Counting conditions (equations) and unknown coefficients, we find that we again can choose one parameter arbitrarily. The free boundary radius  $r_f$  seems to be the best choice. In Fig. 2 cross sections of solutions of (10) are shown, with the origin  $r = 0$  situated in the center  $(1/2, 1/2)$  of the unit square  $D = (0, 1) \times (0, 1)$  and

$$\oint_{\partial D} \frac{\partial \psi}{\partial n} dl = 2 \pi j_{0,1} J_0'(j_{0,1}). \quad (14)$$

For  $r_f \geq 1/\sqrt{2}$  we get in  $D$  simply the Bessel function. For  $1/\sqrt{2} > r_f > 0$  we get in  $D$  solutions of type (12). Their curvatures (2nd derivatives) strongly increase for  $r_f \rightarrow 0$ . In the limiting case  $r_f = 0$  the solution forms a rotationally symmetric cusp, and is not differentiable in its maximum.

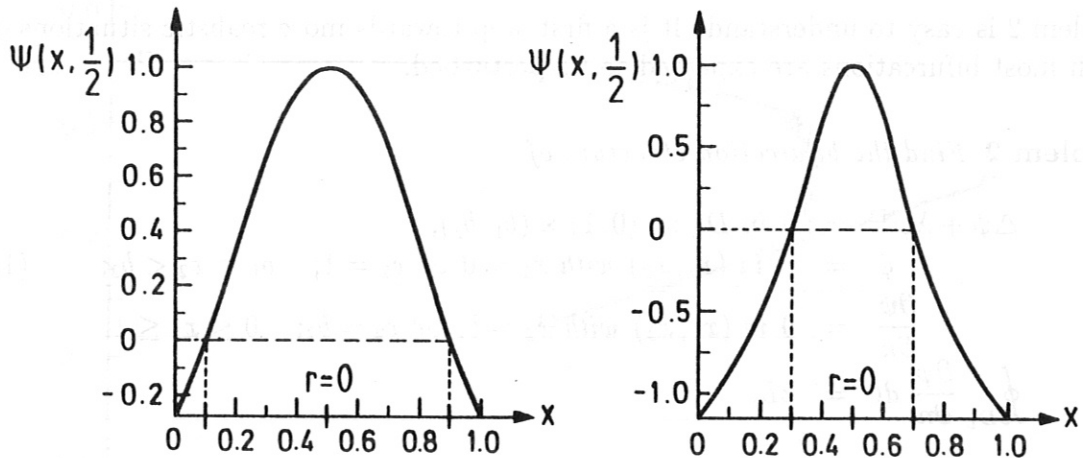


Figure 2: Radially symmetric free boundary solutions in  $D = (0, 1) \times (0, 1)$  (see text for details). a) Solution with  $r_f = 0.4$ , b) solution with  $r_f = 0.2$ .

### 3 Numerical Model & Formulation of the Problem

In [15] and [7], the author and B. Fornberg studied **the numerical error of finite-difference schemes at free boundaries**. They used a numerical model problem related to Problem 1, and compared numerically obtained solutions with explicitly known solutions, in 1D and in 2D. The code used discretization by standard centered finite differences of 2nd order; prescribed boundary values on fixed boundary, unknown eigenvalue  $\lambda$ ; bordering algorithm and Gauss-Newton iterations.

First approach: no special treatment of the free boundary. This approach is used in many free boundary computations [3, 5, 12, 13, 16]. It was found that the actual size of the maximum error depends on the position of the grid points with respect to the free boundary. The error is not asymptotically equal to some constant times  $h^2$ . But it can be bounded by some (larger) constant times  $h^2$ . Extrapolation  $h \rightarrow 0$  and multigrid are of doubtful value. Higher order methods reduce to  $\mathcal{O}(h^2)$  methods unless there is special treatment of the free boundary.

A **modification of the difference scheme near the free boundary** was developed such that the discretization error becomes independent of the location of the grid points w.r.t. the free boundary. For this modified scheme, error extrapolation (from 2<sup>nd</sup> to 3<sup>rd</sup> order of accuracy) is possible. In [7] this scheme was generalised to more general elliptic equations in three and more dimensions, and also to equations with discontinuous r.h.s.

An extension of the code developed for these investigations was **used in the computations reported here**. Since we found perturbed bifurcations in our numerical model of Problem 1, it seems adequate to investigate bifurcation questions first on a problem even simpler than Problem 1. As we shall see, the bifurcation structure of

Problem 2 is easy to understand. It is a first step towards more realistic situations, in which most bifurcations are expected to be perturbed.

**Problem 2** Find the bifurcation structure of

$$\begin{aligned} \Delta\psi + \lambda\psi^+ &= 0 \text{ in } D_L := (0, 1) \times (b_1, b_2), \\ \psi &= \alpha \text{ in } (x_1, x_2) \text{ with } x_1 = 0 \text{ or } x_1 = 1; \quad b_1 < x_2 < b_2; \\ \frac{\partial\psi}{\partial n} &= 0 \text{ in } (x_1, x_2) \text{ with } x_2 = b_1 \text{ or } x_2 = b_2; \quad 0 \leq x_1 \leq 1; \\ \oint_{\partial D_L} \frac{\partial\psi}{\partial n} dl &= 2L, \end{aligned} \quad (15)$$

where  $\psi$  and  $\lambda$  are both unknown,  $b_2 - b_1 =: L < \infty$ , and  $\psi^+$  is defined as in (7).

## 4 A Bifurcation Diagram

The 1D solutions given in eq (8) form a branch of solutions of eqs (15) in  $D_1 = (0, 1) \times (0, 1)$ . For this solution branch, the variation of the boundary value  $\alpha$  is equivalent to variation of  $x_f$ , the  $x_1$ -location of the free boundary. The branch only exists for  $0 \geq \alpha > -0.5$ . The solutions of the limiting cases  $\alpha = 0$  and  $\alpha = -0.5$  are shown in Fig. 1. Figure 3 a and b show  $\psi_{\max}$  and  $\lambda$  as functions of  $\alpha$ , respectively. We get

$$\lim_{\alpha \rightarrow -0.5} \psi_{\max} = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow -0.5} \lambda = \infty.$$

As we can see from Fig. 3 as well, there is a second branch of solutions which bifurcates from the trivial branch of 1D solutions. The bifurcated solutions are 2-dimensional. Their structure is illustrated in Figs 4 and 6. Fig 4 shows a sequence of free boundaries for  $\alpha$ -values in the range

$$\alpha_o \approx -0.15312 \geq \alpha \geq -0.8. \quad (16)$$

Fig. 6 shows level lines of several of the solutions which contributed to Fig. 4.

After one solution on the bifurcated branch was found ( for  $\alpha_1 := -0.17$ , solid square in Fig. 3), the whole branch was computed by simple continuation with growing  $\alpha$  in the interval  $[\alpha_1, \alpha_o]$  and with decreasing  $\alpha$  in the interval  $[-0.8, \alpha_1]$ . Every 10th computed solution is marked by a sign ('□' for  $\psi_{\max}$ -values and 'o' for  $\lambda$ -values). The bifurcating branch seems to exist for all  $\alpha \in [\alpha_o, -\infty)$ . For the limit  $\alpha \rightarrow -\infty$  we expect  $\lambda(\alpha) \rightarrow \infty$ . The solutions become more and more radially symmetric. This is demonstrated in Fig. 5 which shows the quantity

$$\psi_{diff}(\alpha) := \frac{\psi_{\max}(\alpha) - \psi_{mid}(\alpha)}{\psi_{\max}(\alpha) - \psi_{11}(\alpha)} \quad (17)$$



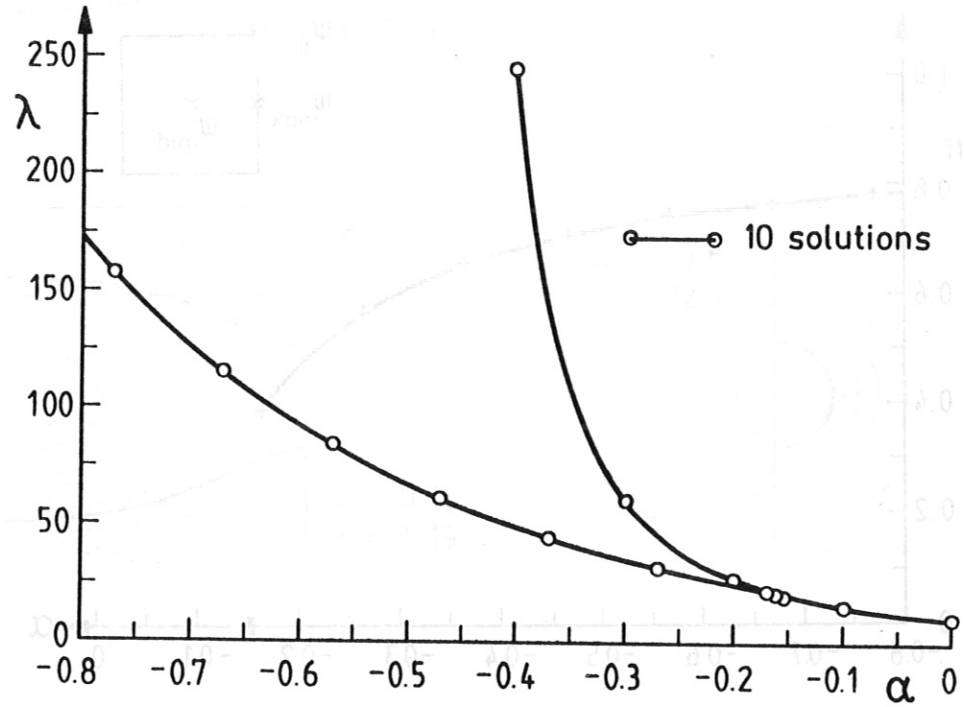
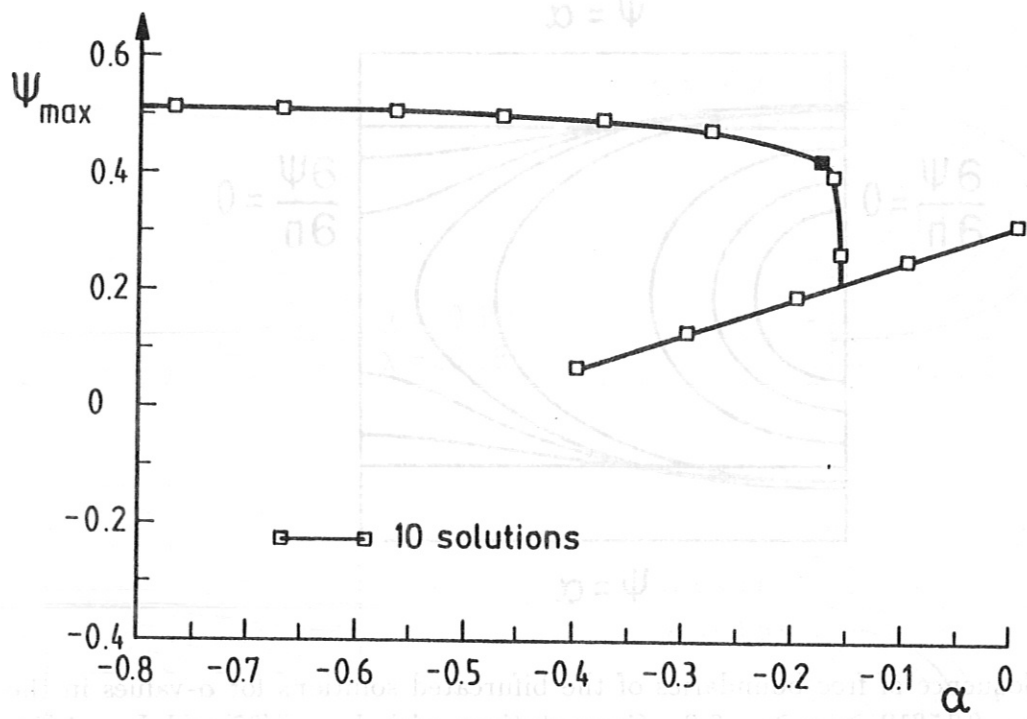


Figure 3: Bifurcation behavior obtained for varying boundary value  $\alpha$ , in two different projections: a)  $\psi_{\max}$  versus  $\alpha$ ; and b) eigenvalue  $\lambda$  versus boundary value  $\alpha$ . For the 1D solution we get  $\lim_{\alpha \rightarrow -0.5} \psi_{\max}(\alpha) = 0$  and  $\lim_{\alpha \rightarrow -0.5} \lambda(\alpha) = \infty$ . The bifurcated solutions seem to exist for all  $\alpha \leq \alpha_o$ , where  $\alpha_o$  is the value at which bifurcation happens.

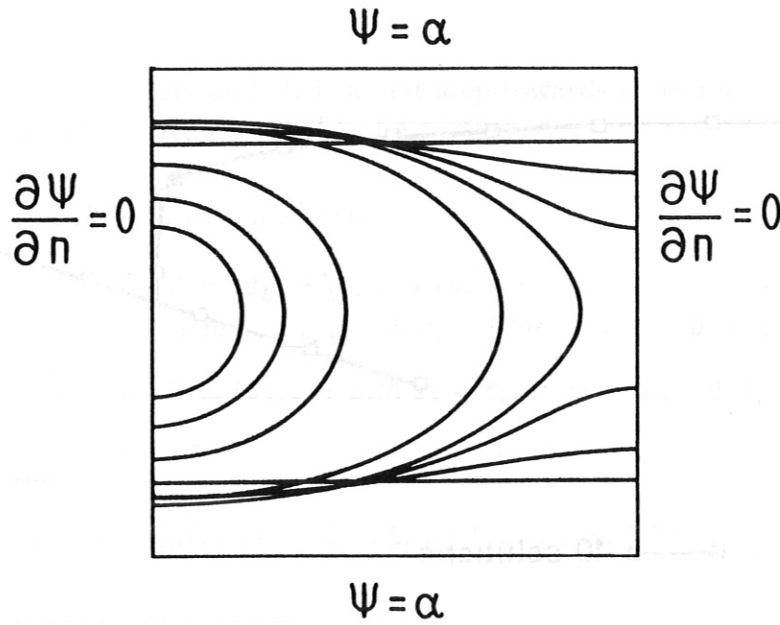


Figure 4: Sequence of free boundaries of the bifurcated solutions for  $\alpha$ -values in the range  $\alpha_0 \approx -0.15312 \geq \alpha \geq -0.8$ . Computations with  $h = 1/32$  and  $h = 1/64$ .  $\alpha$ -values used:  $-0.15312$ ;  $-0.155$ ;  $-0.160$ ;  $-0.17$ ;  $-0.20$ ;  $-0.4$ ;  $-0.6$ ;  $-0.8$ .

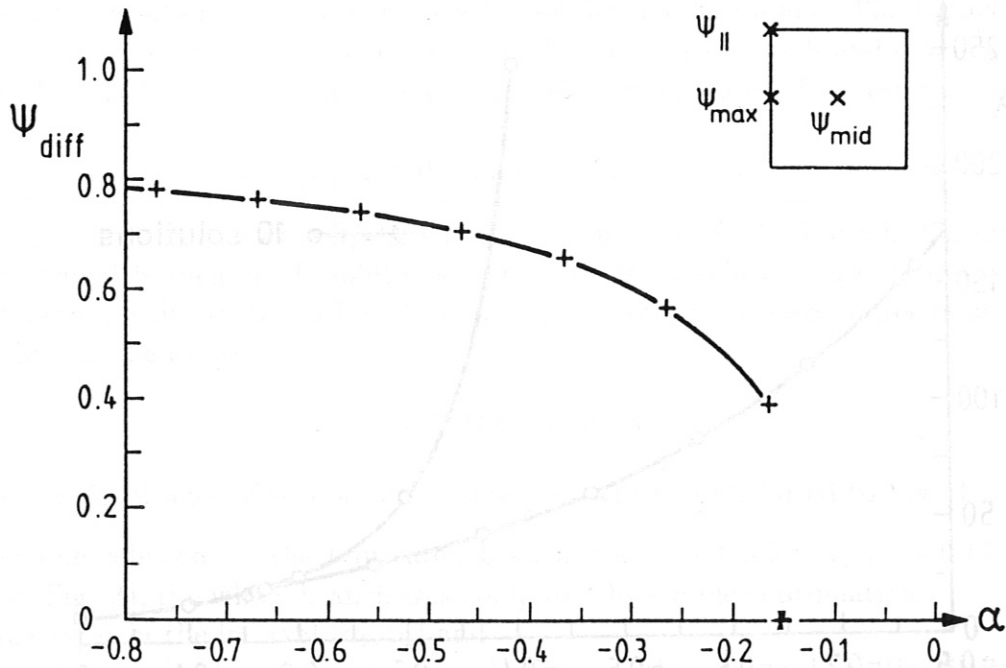


Figure 5:  $\psi_{diff} = \frac{\psi_{max} - \psi_{mid}}{\psi_{max} - \psi_{11}}$  versus  $\alpha$ . In this picture, the trivial branch corresponds to  $\psi_{diff} \equiv 0$ . The bifurcation point in  $\alpha_0$  is marked by a +. The bifurcating branch is only shown for  $\alpha \leq \alpha_1 = -0.17$  (see text for details).

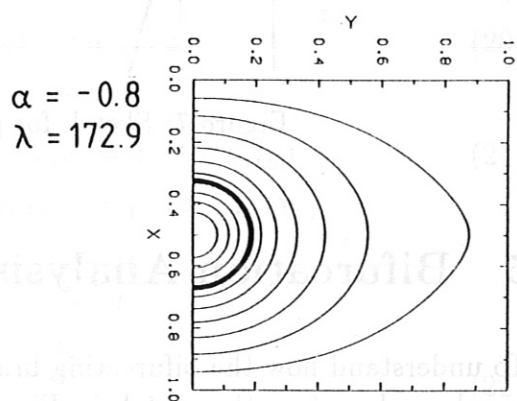
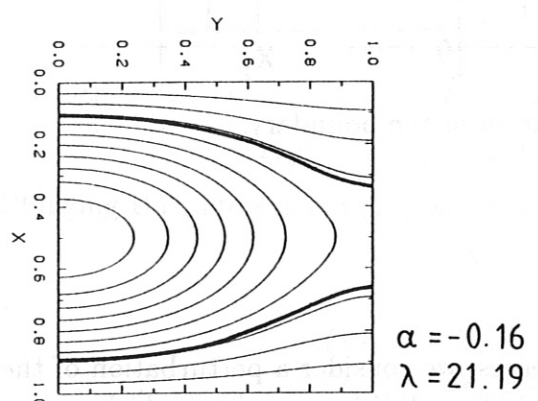
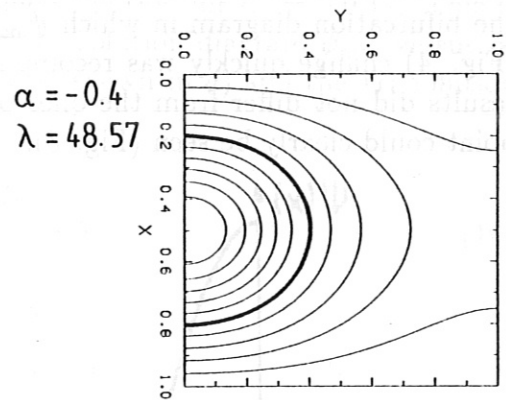
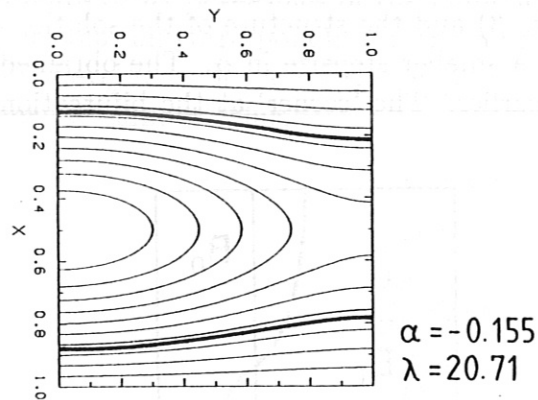
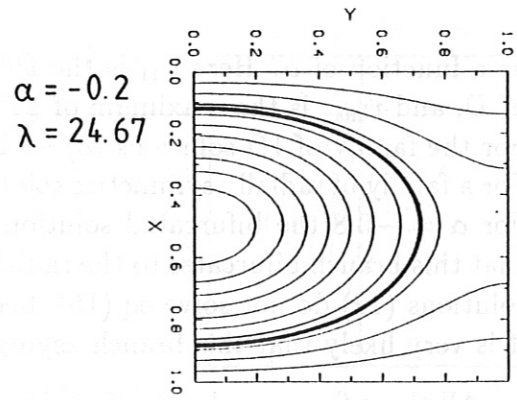
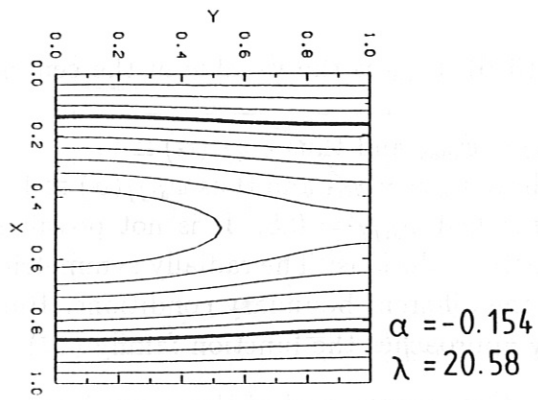


Figure 6: Level lines of several of the solutions on the bifurcated branch: 10 equidistant levels plus the line  $\psi \equiv 0$ .

as a function of  $\alpha$ . Here  $\psi_{11}$  is the  $\psi$ -value in  $(0, 0)$ ,  $\psi_{mid}$  is the  $\psi$ -value in the center of  $D$ , and  $\psi_{max}$  is the maximum of  $\psi$ .

For the family of 1D solutions (8) we have  $\psi_{mid} = \psi_{max}$  and thus  $\psi_{diff}(\alpha) \equiv 0$ .

For a family of radially symmetric solutions we have  $\psi_{mid} = \psi_{11}$  and thus  $\psi_{diff}(\alpha) \equiv 1$ .

For  $\alpha = -0.8$  the bifurcated solution nearly reached  $\psi_{diff} = 0.8$ . It is not possible that this branch bifurcates to the radially symmetric solutions: The radially symmetric solutions (12) do not solve eq (15), because of the different boundary conditions. But it is very likely that this branch asymptotically approaches the function family (12).

All these figures are *not affected by discretization errors*: most of the computations were done twice, with  $h = 1/32$  and  $h = 1/64$ , and the differences between the obtained values for both cases are so small that they cannot be seen in the figures. That part of the bifurcation diagram in which  $\psi_{max}(\alpha)$  (Fig. 3) and the structure of the solutions (Fig. 4) change quickly was recomputed with a smaller stepsize in  $\alpha$ . The obtained results did not differ from the ones obtained earlier. The 'corner' at the bifurcation point could clearly be seen (Fig. 3).

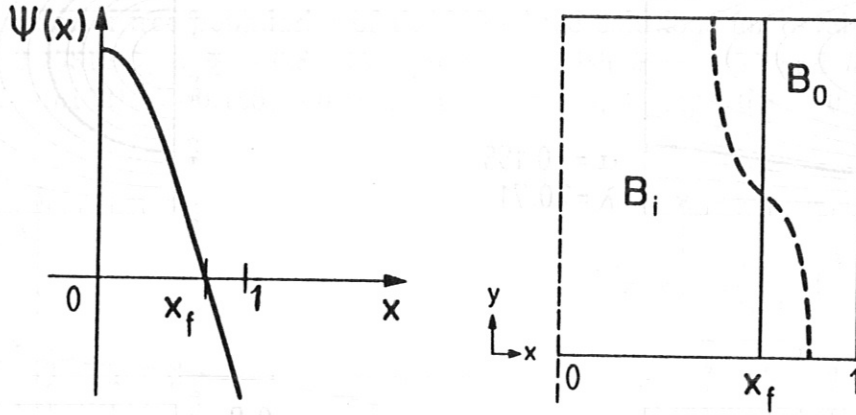


Figure 7: Sketch for perturbation of the boundary.

## 5 Bifurcation Analysis

To understand how the bifurcating branch emerges, we consider a perturbation of the free boundary (see the sketch in Fig. 7). To get analytic expressions which are as simple as possible, we look at the transformed equation

$$\begin{aligned} \Delta\psi + \nu^2\psi &= 0 \text{ in } B_L := (-1, 1) \times (0, L), \\ \psi &= \alpha \text{ in } (x, y) \text{ with } x = -1 \text{ or } x = 1; \quad 0 < y < L; \\ \frac{\partial\psi}{\partial n} &= 0 \text{ in } (x, y) \text{ with } y = 0 \text{ or } y = L; \quad -1 \leq x \leq 1; \end{aligned} \quad (18)$$

Any solution  $(\psi, \hat{\psi})$  of (18) has to satisfy:

$$\Delta\psi + \nu^2\psi = 0 \text{ in } B_i$$

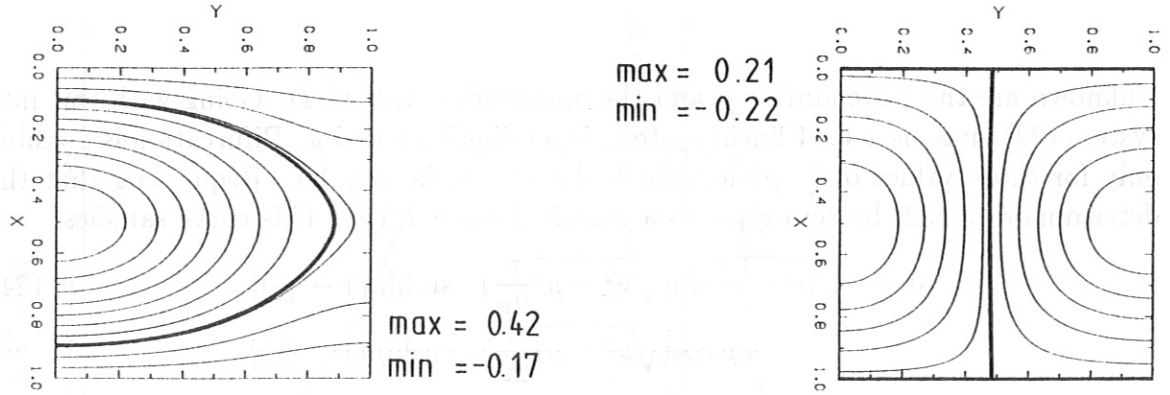


Figure 8: a) Level lines of the numerically obtained solution for  $\alpha_1 = -0.17$ . This is the solution corresponding to the solid square in the solution diagram  $\psi_{\max}$  versus  $\alpha$ . b): Level lines of the difference between the solution shown in a) and the 1D solution for the same  $\alpha$  value.

$$\begin{aligned}
 \Delta \hat{\psi} &= 0 \text{ in } B_o \\
 \psi(x_f, y) &= \hat{\psi}(x_f, y) = 0 \\
 \frac{\partial \psi}{\partial x}(x_f, y) &= \frac{\partial \hat{\psi}}{\partial x}(x_f, y); \\
 \frac{\partial \psi}{\partial n_x}(0, y) &= \frac{\partial \hat{\psi}}{\partial x}(0, y) = 0 \\
 \hat{\psi}(1, y) &= 0
 \end{aligned} \tag{19}$$

System (19) is solved by the 1D function

$$\psi_o(x) = \cos \nu x; \quad \hat{\psi}_o(x) = a + b x \tag{20}$$

with  $\nu x_f = \frac{\pi}{2}$ ,  $0 < x_f \leq 1$ . Ansatz for the perturbation:

$$\psi(x, y) = \psi_o(x) + \varepsilon \psi_1(x, y), \quad X(y) = x_f + \varepsilon X_1(y). \tag{21}$$

Plugging this into system (19) we get the following system for  $\psi_1$ ,  $X_1$ :

$$\begin{aligned}
 \Delta \psi_1 + \nu^2 \psi_1 &= 0 \text{ in } B_i \\
 \Delta \hat{\psi}_1 &= 0 \text{ in } B_o \\
 \psi_1(x_f, y) + X_1(y) \psi'_o(x_f) &= 0, \\
 \hat{\psi}_1(x_f, y) + X_1(y) \psi'_o(x_f) &= 0 \\
 \frac{\partial \psi_1}{\partial x}(x_f, y) &= \frac{\partial \hat{\psi}_1}{\partial x}(x_f, y) \\
 \frac{\partial \psi_1}{\partial n_x}(0, y) &= \frac{\partial \hat{\psi}_1}{\partial x}(0, y) = 0 \\
 \psi_1(1, y) &= 0
 \end{aligned} \tag{22}$$

where  $\psi'_o(x_f) = -\nu$ . To solve system (22), we make the following ansatz, which is supported by Fig. 8 (remember the transformation in  $x$ ):

$$\begin{aligned}
 \psi_1 &= A \cos(\sqrt{\nu^2 - \mu^2} x) e^{i\mu y} & \hat{\psi}_1 &= (B e^{\mu x} + C e^{-\mu x}) e^{i\mu y} \\
 X_1 &= D e^{i\mu y}
 \end{aligned} \tag{23}$$

Unknown are the wavenumber  $\mu$  and the amplitudes  $A, B, C, D$ . Going with this into system (22) gives us a  $4 \times 4$ -linear system depending on  $\nu$  and  $\mu$ . Bifurcation is possible only for those values of  $(\nu, \mu)$  for which the system is singular. Requesting that the determinant of (22) be zero gives us a *marginal curve*  $\nu^2(\mu)$ . This curve satisfies

$$0 = -\sqrt{\nu^2 - \mu^2} \sin\left(\sqrt{\nu^2 - \mu^2} \frac{\pi}{2\nu}\right) \sinh\left[\mu\left(1 - \frac{\pi}{2\nu}\right)\right] + \mu \cos\left(\sqrt{\nu^2 - \mu^2} \frac{\pi}{2\nu}\right) \cosh\left[\mu\left(1 - \frac{\pi}{2\nu}\right)\right] \quad (24)$$

and is shown in Fig. 9 b. Fig. 9 a shows the transformation of this curve to  $\lambda/4$  versus  $L$ , where  $\lambda$  is the quantity of eq (15) and  $L$  is the length of the domain in  $y$ -direction. The curves have been obtained by numerical evaluation of formula (24) and linear interpolation (10 values inbetween the  $\square$ -marks and inbetween the  $\circ$ -marks).

This analysis applies to domains of arbitrary lengths. How to use this curve for a given domain? Given a domain of length  $L = \pi/\mu$ , we find from eq (24) a countably infinite sequence of bifurcation points on the trivial branch. A necessary condition for bifurcation of a solution of length  $L/n$ ,  $n = 1, 2, 3, 4, \dots$  is that the eigenvalue has the value given by the curve (24). The corresponding  $\alpha$ -values then can be obtained from (9). This was verified with our code: The first four  $\lambda$ -values for which the Jacobian of the numerical approximation to (15) turns singular were compared to the first four values obtained from the analysis in this section, and they agreed quite well. For the fixed length  $L = 2\pi$  Sijbrand [17] computed bifurcation points of (15). Also, the values he gave and the values obtained by the present analysis agreed quite well.

Sijbrand also treated the same problem with the 3-dimensional Laplacian. This does not make sense in the context of plasma physics, a system of PDEs is needed then [11]. We thus don't treat that case here.

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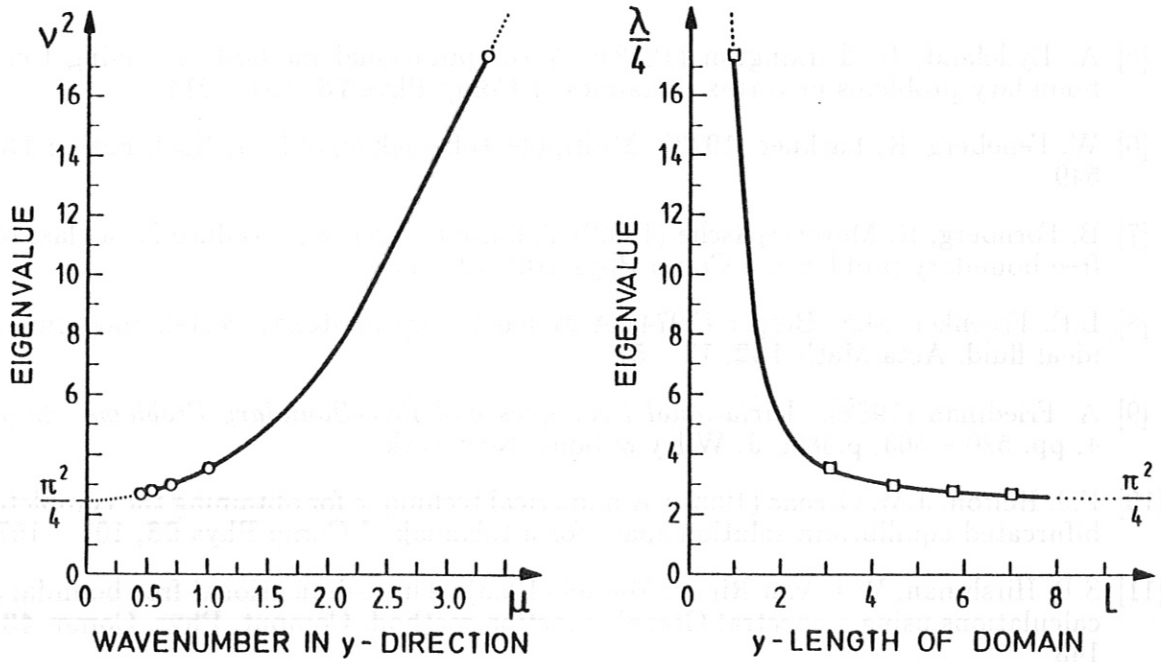


Figure 9: Marginal curves. a) Eigenvalue  $\nu^2$  versus wavenumber  $\mu$ , as obtained by numerical evaluation of the formula for the marginal curve. b) Transformation of the curve of part a) to the  $\lambda/4$  versus  $L$  plane, where  $\lambda$  is the eigenvalue used earlier and  $L$  is the length of the domain in  $y$ -direction.

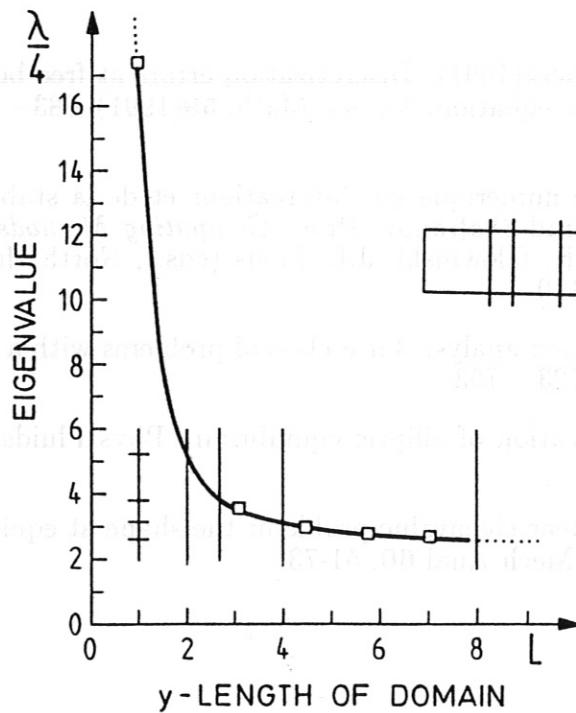


Figure 10: Sequence of eigenvalues for a domain of fixed given length  $L = 8$ .

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