**ENERGY OF LINEAR QUASI-NEUTRAL
ELECTROSTATIC DRIFT WAVES**

DIETER PFIRSCH AND DARÍO CORREA-RESTREPO

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Abstract. Linearized drift-wave instabilities are related to the existence of negative-energy perturbations. In this paper, an exact energy expression for linear quasi-neutral electrostatic perturbations is derived within the framework of drift-theory multi-fluid theory that is valid for any geometry. Treating the mass as a tensor with, in general, different masses parallel and perpendicular to a uniform magnetic field allows one to treat the full dynamics and also restrict consideration to parallel dynamics or to the coupled adiabatic case. Application to slab configurations yields the result that the adiabatic approximation does not allow negative-energy perturbations, whereas inclusion of the parallel dynamics does. This is in agreement with a numerical study of drift-wave turbulence within the framework of a fluid two-fluid theory by H. Scott [1].

A dissipationless theory can be formulated in terms of a Lagrangian, from which the energy is immediately obtained. We start with the nonlinear theory; the linearized theory is written in terms of displacement vectors $\xi_{\perp}(x, t)$ such that all constraints are taken into account, as in ideal MHD theory. The procedure used is the same as that developed in a forthcoming paper by Pfirsch and Sudan [2] for ideal MHD theory, which also allows one to obtain the exact nonlinear energy with all the constraints built in [9]. People working in this field [5, 1] have hitherto considered only (positive semi-definite) expressions for the nonlinear energy, which do not immediately allow one to investigate the existence of linear negative-energy perturbations relating to non-linear instability [2, 3, 4].

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Energy of Linear Quasi-neutral Electrostatic Drift Waves

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Abstract

Certain kinds of nonlinear instabilities are related to the existence of negative-energy perturbations. In this paper an exact energy expression for linear quasi-neutral electrostatic perturbations is derived within the framework of dissipationless multi-fluid theory that is valid for any geometry. Taking the mass as a tensor with, in general, different masses parallel and perpendicular to an ambient magnetic field allows one to treat the full dynamics and also restrict consideration to parallel dynamics or to the completely adiabatic case. Application to slab configurations yields the result that the adiabatic approximation does not allow negative-energy perturbations, whereas inclusion of the parallel dynamics does. This is in agreement with a numerical study of drift-wave turbulence within the framework of collisional two-fluid theory by B. Scott [1].

A dissipationless theory can be formulated in terms of a Lagrangian, from which the energy is immediately obtained. We start with the nonlinear theory; the linearized theory is written in terms of displacement vectors $\xi_\nu(\mathbf{x}, t)$ such that all constraints are taken into account, as in ideal MHD theory. The procedure used is the same as that developed in a forthcoming paper by Pfirsch and Sudan [8] for ideal MHD theory, which also allows one to obtain the exact nonlinear energy with all the constraints built in [9]. People working in this field [5, 1] have hitherto considered only (positive semi-definite) expressions for the nonlinear energy, which do not immediately allow one to investigate the existence of linear negative-energy perturbations relating to non-linear instability [2, 3, 4].

1 Introduction

An impressive numerical study of collisional drift-wave turbulence was recently published by B. Scott [1] in which he demonstrated self-sustained turbulence of a linearly stable plasma slab resembling the plasma edge regions of tokamaks. His main results are that all features of nonlinear mode structure are determined by nonlinear processes, divesting linear stability criteria of their relevance to that structure, or its amplitude; contrary to the present common notion in tokamak physics that drift-wave turbulence cannot be the agent behind energy transport in tokamak edge regions, many important features of experimentally observed tokamak edge fluctuations were reproduced, particularly the amplitude ordering $e\tilde{\phi}/T > \tilde{n}/n > \tilde{T}/T$; the transport is found to be gyro-reduced-Bohm like. In Scott's study a certain threshold amplitude is needed. It can, however, even happen that the nonlinear instability occurs with arbitrarily small initial perturbations. This was shown for the first time in 1925 by Cherry [2]. He presented a simple example demonstrating that linear stability analysis will in general not be sufficient for finding out whether a system is stable or not with respect to small-amplitude perturbations. His example consisted of two nonlinearly coupled oscillators, one possessing positive energy, the other negative energy, and the frequency of one oscillator was twice that of the other, which means a third-order resonance. The exact two-parameter solution set he had found exhibited explosive instability after a finite time. D. Pfirsch [3] considered the corresponding three-oscillator case and found the complete solution of this problem. It shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behaviour. This is true of the resonant case. The non-resonant oscillators can sometimes also become explosively unstable, but the initial amplitudes must not be infinitesimally small. (Concerning negative-energy modes and nonlinear instabilities see also [4].)

It is very likely that the results obtained by Scott are understandable in terms of coupling between positive- and negative-energy perturbations and that the threshold amplitude he needed relates to a lack of resonance.

In order to find out whether this conjecture might be correct, it is necessary to determine the sign of the energy of drift-wave perturbations. People working

in this field [5], especially Scott [1], have hitherto considered only (positive semi-definite) expressions for the nonlinear energy, which do not immediately allow one to investigate the existence of linear negative-energy perturbations relating to nonlinear instability [2, 3, 4]. In this paper we derive the exact expression for the energy of linear quasi-neutral electrostatic drift waves and evaluate it for situations corresponding to Scott's calculations. Contrary to Scott, we consider a dissipationless theory, which is the only kind allowing a unique definition of the energy of perturbations. Within such a theory the energy is a constant of the motion for any initial conditions. Whereas the nonlinear energy is just kinetic plus potential plus thermal energy, the energy of perturbations depends on constraints. In a multi-fluid quasi-neutral electrostatic theory, from which we start, such constraints are mass conservation and entropy conservation. The latter is violated if heat conduction, heat sources (e.g. Joule heating) and heat sinks play a role. Hence, the energy expressions obtained in this paper are, strictly speaking, only valid for situations where this is not the case or where these phenomena do not influence the entropy constraint. The latter is the case if the heat conduction is infinitely large such that the equilibrium temperature profiles $T_\nu(\mathbf{x})$ of the various particle species ν are independent of $\mathbf{x}$ and $\delta T_\nu = 0$. A vanishing temperature perturbation results in an entropy-conserving theory if one takes the adiabatic coefficients $\gamma_\nu = 1$. This is only possible, however, for the perturbations; the equilibrium energy would diverge. When we consider this case, we do it by putting the γ 's equal to 1 only after having obtained the perturbed energy for general γ 's.

We should like to mention that the present authors are also investigating the same kind of problems within the framework of Vlasov-Maxwell theory, starting from the energy expression obtained by Morrison and Pfirsch [6]. This theory, of course, automatically takes heat and momentum transport into account. They also intend to use the energy obtained by Pfirsch and Morrison [7] within the framework of Maxwell and drift-kinetic theory.

Scott had found that it is essential to take into account the dynamics parallel to the ambient magnetic field for electrons and ions, whereas the inertial effects connected with the perpendicular electron motion can be neglected. We can treat this in our investigation by expressing the mass of the particles as a tensor with different masses for the parallel and the perpendicular motion

and choose $m_{\nu\perp} = 0$.

We shall explicitly discuss examples in slab geometry. We choose $T_i = v_i = 0$, where v_i is the equilibrium flow velocity of the ions.

2 Quasi-neutral electrostatic multi-fluid theory

A dissipationless theory can be formulated in terms of a Lagrangian from which the energy is immediately obtained. We start from the nonlinear theory; the linearised theory is written in terms of displacement vectors $\xi(\mathbf{x}, t)$ such that all constraints are taken into account as in ideal MHD theory. The procedure used is the same as that developed in a forthcoming paper by Pfirsch and Sudan [8] for ideal MHD theory which also allows one to obtain the exact nonlinear energy with all the constraints incorporated [9].

2.1 The Lagrangian of the nonlinear theory

Quasi-neutrality means that there is no electric field energy term, and, since the magnetic field is even prescribed, there is no magnetic field energy term either; instead we have to use the usual coupling terms for charged particles in electric and magnetic fields. This results in

$$L = \sum_{\nu} \int d^3x \left\{ \frac{1}{2} n_{\nu} \mathbf{v}_{\nu} \cdot \underline{\underline{m}}_{\nu} \cdot \mathbf{v}_{\nu} - \frac{p_{\nu}}{\gamma_{\nu} - 1} - e_{\nu} n_{\nu} \Phi + \frac{e_{\nu}}{c} n_{\nu} \mathbf{v}_{\nu} \cdot \mathbf{A} \right\}, \quad (1)$$

where

$$\underline{\underline{m}}_{\nu} = m_{\nu\perp} + (m_{\nu\parallel} - m_{\nu\perp}) \mathbf{b} \mathbf{b}, \quad \mathbf{b} = \mathbf{B}/B. \quad (2)$$

In the following the index ν will be suppressed if it is not needed.

$\mathbf{A} = \mathbf{A}(\mathbf{x})$ is the fixed equilibrium vector potential and is not to be varied, but all other quantities are. As concerns n , p and $\mathbf{v}$, these variations are expressed in terms of virtual displacement vectors $\zeta(\mathbf{x}, t)$:

$$\delta n = -\nabla \cdot (n \zeta), \quad (3)$$

$$\delta \mathbf{v} = \dot{\zeta} + \mathbf{v} \cdot \nabla \zeta - \zeta \cdot \nabla \mathbf{v}, \quad (4)$$

$$\delta p = -\zeta \cdot \nabla p - \gamma p \nabla \cdot \zeta, \quad (5)$$

$$\delta T = -\zeta \cdot \nabla T - (\gamma - 1) T \nabla \cdot \zeta. \quad (6)$$

The first variations, on the one hand, must yield the nonlinear equations and on the other, will serve to yield the second-order variations, from which the linearized equations and the second-order energy follow.

The kinetic energy is denoted by K and the "potential" energy by U . In the following, we write down only the integrands of the variations of these quantities:

$$\delta^{(1)}K : -\frac{1}{2}(\nabla \cdot (n\zeta))\mathbf{v} \cdot \underline{\underline{m}} \cdot \mathbf{v} + n\mathbf{v} \cdot \underline{\underline{m}} \cdot (\dot{\zeta} + \mathbf{v} \cdot \nabla \zeta - \zeta \cdot \nabla \mathbf{v}). \quad (7)$$

After integrations by parts, also with respect to time, as is possible in the action integral, this becomes

$$\begin{aligned} \frac{1}{2}n\zeta \cdot \nabla \mathbf{v} \cdot \underline{\underline{m}} \cdot \mathbf{v} - \zeta \cdot \frac{\partial}{\partial t}(n\underline{\underline{m}} \cdot \mathbf{v}) - \zeta \cdot ((\nabla \cdot n\mathbf{v}\underline{\underline{m}} \cdot \mathbf{v})) - (\zeta \cdot \nabla \mathbf{v}) \cdot \underline{\underline{m}} \cdot \mathbf{v}n \\ = -\zeta \cdot \left(\frac{\partial}{\partial t}(n\underline{\underline{m}} \cdot \mathbf{v}) + (\nabla \cdot \mathbf{v}\underline{\underline{m}} \cdot \mathbf{v}n) \right). \end{aligned} \quad (8)$$

The first variation of the potential energy is

$$\begin{aligned} \delta^{(1)}U : & -\frac{1}{\gamma-1}(\zeta \cdot \nabla p + \gamma p \nabla \cdot \zeta) \\ & -e\Phi \nabla \cdot (n\zeta) + en\delta\Phi \\ & + \frac{e}{c} \mathbf{v} \cdot \mathbf{A} \nabla \cdot (n\zeta) - \frac{e}{c} n(\dot{\zeta} + \mathbf{v} \cdot \nabla \zeta - \zeta \cdot \nabla \mathbf{v}) \cdot \mathbf{A}. \end{aligned} \quad (9)$$

Integrations by parts lead to

$$\begin{aligned} \delta^{(1)}U : & \zeta \cdot \nabla p + en\zeta \cdot \nabla \Phi + en\delta\Phi \\ & - \frac{e}{c} n\zeta \cdot \nabla(\mathbf{v} \cdot \mathbf{A}) + \frac{e}{c} \dot{n}\zeta \cdot \mathbf{A} \\ & + \frac{e}{c} \zeta \cdot (\nabla \cdot (n\mathbf{v}\mathbf{A})) + \frac{e}{c} n\zeta \cdot (\nabla \mathbf{v}) \cdot \mathbf{A}. \end{aligned} \quad (10)$$

With

$$\mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\nabla \times \mathbf{A}) = (\nabla \mathbf{A}) \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{A}$$

and

$$\dot{n} + \nabla \cdot (n\mathbf{v}) = 0$$

this becomes

$$\delta^{(1)}U : \zeta \cdot \nabla p + en \zeta \cdot \nabla \Phi + en \delta \Phi - \frac{en}{c} \zeta \cdot (\mathbf{v} \times \mathbf{B}) . \quad (11)$$

The requirement that the factor of $\delta \Phi$ must vanish leads to the quasi-neutrality condition

$$\sum_{\nu} e_{\nu} n_{\nu} = 0 . \quad (12)$$

Vanishing of the factor of ζ yields the correct nonlinear equations of motion:

$$n \underline{\underline{m}} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + en \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) . \quad (13)$$

2.2 The Lagrangian of the linearized theory

This Lagrangian is the second-order expansion in real displacements of the original Lagrangian for the exact nonlinear theory. It is obtained by varying $\delta^{(1)}K$ and $\delta^{(1)}U$ again and then replacing the virtual displacement ζ by the real displacement ξ . To the envisaged order there is no need to vary the displacement itself, since its factor vanishes because of the equilibrium equations.

One finds

$$\begin{aligned} \delta^{(2)}K : & \frac{1}{2} (\nabla \cdot (\xi \nabla \cdot (n \xi))) \mathbf{v} \cdot \underline{\underline{m}} \cdot \mathbf{v} \\ & - 2 (\nabla \cdot (n \xi)) \mathbf{v} \cdot \underline{\underline{m}} \cdot (\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \\ & + n \mathbf{v} \cdot \underline{\underline{m}} \cdot ((\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \cdot \nabla \xi - \xi \cdot \nabla (\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v})) \\ & + n (\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \cdot \underline{\underline{m}} \cdot (\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) . \end{aligned} \quad (14)$$

The contribution to the wave energy $\mathcal{E}$ is given by the term bilinear in $\dot{\xi}$ and the negative of the terms not containing $\dot{\xi}$, whereas the terms linear in $\dot{\xi}$ do not contribute:

$$\begin{aligned} 2\mathcal{E}_K : & n \dot{\xi} \cdot \underline{\underline{m}} \cdot \dot{\xi} - n (\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \cdot \underline{\underline{m}} \cdot (\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \\ & - \frac{1}{2} (\nabla \cdot (\xi \nabla \cdot (n \xi))) \mathbf{v} \cdot \underline{\underline{m}} \cdot \mathbf{v} + 2 (\nabla \cdot (n \xi)) \mathbf{v} \cdot \underline{\underline{m}} \cdot (\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \\ & - n \mathbf{v} \cdot \underline{\underline{m}} \cdot ((\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \cdot \nabla \xi - \xi \cdot \nabla (\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v})) . \end{aligned} \quad (15)$$

For the second variation of the potential energy we start with $\delta^{(1)}U$ as given by eq. (12) and obtain after some integrations by parts

$$\begin{aligned} \delta^{(2)}U : & -\xi \cdot \nabla(\xi \cdot \nabla p + \gamma p \nabla \cdot \xi) - e \nabla \cdot (n\xi) \xi \cdot \nabla \Phi \\ & + en \xi \cdot \nabla \delta \Phi - e \nabla \cdot (n\xi) \delta \Phi \\ & + \frac{e}{c} \nabla \cdot (n\xi)(\mathbf{v} \times \mathbf{B}) \cdot \xi - \frac{e}{c} n[(\dot{\xi} + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \times \mathbf{B}] \cdot \xi . \end{aligned} \quad (16)$$

The variation of $\delta \Phi$ yields the linearized quasi-neutrality condition

$$\sum_{\nu} e_{\nu} \nabla \cdot (n_{\nu} \xi_{\nu}) = 0 . \quad (17)$$

The equilibrium relations

$$e(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}) = \frac{1}{n} \nabla p$$

and integrations by parts lead to the following contribution to $\mathcal{E}$:

$$\begin{aligned} 2\mathcal{E}_U : & (\nabla \cdot \xi)(\xi \cdot \nabla p + \gamma p \nabla \cdot \xi) + \frac{1}{n} \nabla \cdot (n\xi) \nabla p \cdot \xi \\ & - \frac{e}{c} n[(\mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}) \times \mathbf{B}] \cdot \xi . \end{aligned} \quad (18)$$

The sum of this expression and expression (15) is twice the total wave energy.

2.3 Adiabatic approximation

This approximation means

$$m_e = m_i = 0 . \quad (19)$$

Like Scott [1], we choose in addition

$$p_i = \mathbf{v}_i = 0 \implies \mathbf{E} = 0 . \quad (20)$$

Then we obtain from the second-order variations of the preceding section the following second-order Lagrangian for the ions:

$$L_i^{(2)} = \int n_i (-e_i \xi_i \cdot \nabla \delta \Phi + \frac{1}{2} \frac{e_i}{c} (\dot{\xi} \times \mathbf{B}) \cdot \xi) d^3x ; \quad (21)$$

the second-order Lagrangian for the electrons is

$$L_e^{(2)} = \int \left\{ \frac{1}{2} \xi_e \cdot \nabla (\xi_e \cdot \nabla p_e + \gamma_e p_e \nabla \cdot \xi_e) - \frac{1}{2} \frac{1}{n_e} \nabla \cdot (n_e \xi_e) \xi_e \cdot \nabla p_e - e_e n_e \xi_e \cdot \nabla \delta \Phi + \frac{1}{2} \frac{e_e}{c} n_e \xi_e \cdot [(\dot{\xi}_e + \mathbf{v}_e \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{v}_e) \times \mathbf{B}] \right\} d^3x. \quad (22)$$

Variation of $\delta \Phi$ yields

$$\nabla \cdot (e_i n_i \xi_i) + \nabla \cdot (e_e n_e \xi_e) = 0 \quad (23)$$

and variation of ξ_i

$$0 = e_i (\delta \mathbf{E} + \frac{1}{c} \dot{\xi}_i \times \mathbf{B}). \quad (24)$$

The last equation has the solution

$$\xi_{i\parallel} : \text{arbitrary}$$

$$\dot{\xi}_{i\perp} = c \frac{\delta \mathbf{E} \times \mathbf{B}}{B^2}. \quad (25)$$

Variation with respect to ξ_e yields

$$\begin{aligned} & \frac{1}{2} \nabla (\xi_e \cdot \nabla p_e + \gamma_e p_e \nabla \cdot \xi_e) - \frac{1}{2} \nabla \cdot \xi_e \nabla p_e + \frac{1}{2} \gamma_e \nabla (p_e \nabla \cdot \xi_e) \\ & - \frac{1}{2} \frac{1}{n_e} \nabla \cdot (n_e \xi_e) \nabla p_e + \frac{1}{2} n_e \nabla \left(\frac{1}{n_e} \xi_e \cdot \nabla p_e \right) + e_e n_e (\delta \mathbf{E} + \frac{1}{c} \dot{\xi}_e \times \mathbf{B}) \\ & + \frac{1}{2} \frac{e_e}{c} n_e (\mathbf{v}_e \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{v}_e) \times \mathbf{B} + \frac{1}{2} \frac{e_e}{c} \nabla \cdot (\mathbf{v}_e n_e \xi_e \times \mathbf{B}) \\ & + \frac{1}{2} \frac{e_e n_e}{c} (\nabla \mathbf{v}_e) \cdot (\xi_e \times \mathbf{B}) \\ & = 0. \end{aligned} \quad (26)$$

From the parallel component of this equation one derives

$$\delta \Phi = \frac{1}{e_e n_e} (\xi_e \cdot \nabla p_e + \gamma_e p_e \nabla \cdot \xi_e) = -\frac{1}{e_e n_e} \delta p_e. \quad (27)$$

When this "adiabatic" relation is used in eq. (26), this equation becomes

$$\begin{aligned}
& \left(\frac{1}{2n_e} \xi_e \cdot \nabla p_e + \gamma_e \frac{p_e}{n_e} \nabla \cdot \xi_e \right) \nabla n_e \\
& - \left(\frac{1}{2} \nabla \cdot \xi_e + \frac{1}{2n_e} \nabla \cdot (n_e \xi_e) \right) \nabla p_e \\
& + e_e n_e \frac{1}{c} \dot{\xi}_e \times \mathbf{B} \\
& + \frac{1}{2} \frac{e_e}{c} n_e (\mathbf{v}_e \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{v}_e) \times \mathbf{B} + \frac{1}{2} \frac{e_e}{c} \nabla \cdot (\mathbf{v}_e n_e \xi_e \times \mathbf{B}) \\
& + \frac{1}{2} \frac{e_e n_e}{c} (\nabla \mathbf{v}_e) \cdot (\xi_e \times \mathbf{B}) \\
& = 0 .
\end{aligned} \tag{28}$$

A first expression for the wave energy is then

$$\begin{aligned}
\mathcal{E} = \int \left\{ \frac{1}{2} (\nabla \cdot \xi_e) (\xi_e \cdot \nabla p_e + \gamma_e p_e \nabla \cdot \xi_e) + \frac{1}{2n_e} \nabla \cdot (n_e \xi_e) \xi_e \cdot \nabla p_e \right. \\
\left. - \frac{e_e n_e}{2c} \xi_e \cdot ((\mathbf{v}_e \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{v}_e) \times \mathbf{B}) \right\} d^3 x .
\end{aligned} \tag{29}$$

One can put this expression into a form which is similar to δW in ideal MHD. To this end we introduce

$$e_e n_e \mathbf{v}_e = \mathbf{j} . \tag{30}$$

The magnetic field term then becomes

$$\int \left\{ -\frac{1}{2} \frac{1}{c} \xi_e \cdot ((\mathbf{j} \cdot \nabla \xi_e - n_e \xi_e \cdot \nabla (\frac{1}{n_e} \mathbf{j})) \times \mathbf{B}) \right\} d^3 x .$$

We write

$$\begin{aligned}
n_e \xi_e \cdot \nabla (\frac{1}{n_e} \mathbf{j}) &= -\frac{1}{n_e} (\xi_e \cdot \nabla n_e) \mathbf{j} + \xi_e \cdot \nabla \mathbf{j} , \\
\mathbf{j} \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{j} &= \nabla \times (\xi_e \times \mathbf{j}) + \mathbf{j} \nabla \cdot \xi_e ,
\end{aligned}$$

use

$$\frac{1}{c} \mathbf{j} \times \mathbf{B} = \nabla p_e ,$$

and obtain for the magnetic field term

$$\int \left\{ \frac{1}{2c} (\boldsymbol{\xi}_e \times \mathbf{j}) \cdot \nabla \times (\boldsymbol{\xi}_e \times \mathbf{B}) - \frac{1}{2n_e} \nabla \cdot (n_e \boldsymbol{\xi}_e) \boldsymbol{\xi}_e \cdot \nabla p_e \right\} d^3x .$$

Integration by parts with the first term and combination with the other contributions to $\mathcal{E}$ yields for this quantity the somewhat familiar-looking expression

$$\mathcal{E} = \int \left\{ \frac{1}{2} (\nabla \cdot \boldsymbol{\xi}_e) (\boldsymbol{\xi}_e \cdot \nabla p_e + \gamma_e p_e \nabla \cdot \boldsymbol{\xi}_e) + \frac{1}{2} \boldsymbol{\xi}_e \cdot \frac{1}{c} (\mathbf{j} \times \nabla \times (\boldsymbol{\xi}_e \times \mathbf{B})) \right\} d^3x . \quad (31)$$

Note that the last term enters the ideal MHD potential energy with the opposite sign. The reason for this difference is that there the magnetic field is not a fixed quantity but is determined self-consistently, which also leads to an additional term $(\delta \mathbf{B})^2 / 8\pi$.

Example

As an example, we consider a plasma slab dependent on x only; we choose

$$\gamma_e = 1 , \quad T_e = T = \text{const.} , \quad \mathbf{v}_e = (0, 0, v_e) = \text{const.}$$

The minimum of $\mathcal{E}$ for this configuration is zero:
The equilibrium magnetic field follows from

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} e_e n_e \mathbf{v}_e .$$

There is no x -component of the magnetic field:

$$\mathbf{B} = (0; B_y(x); B_z(x)) .$$

For the two non-vanishing components one finds

$$B_z = B_z^0 , \quad B_y = B_y^0 + \int_0^x dx' \frac{4\pi}{c} e_e n_e(x') v_e .$$

From

$$\frac{1}{c} e_e n_e \mathbf{v}_e \times \mathbf{B} = \nabla p_e$$

one obtains

$$\frac{\partial p_e}{\partial x} = -\frac{1}{c} e_e n_e(X) v_e \left(B_y^0 + \int_0^x dx' \frac{4\pi}{c} e_e n_e(x') v_e \right).$$

We take $x = 0$ as being a resonant surface and consider perturbations which are localized there and have $k_{\parallel} = 0$:

$$k_y B_y^0 + k_z B_z^0 = 0.$$

We treat first a simple case given by the initial conditions for ξ_e such that

$$\nabla \cdot (n_e \xi_e) = 0.$$

This leads to

$$\delta\Phi = 0, \quad \xi_{i\perp} = 0$$

and

$$\mathcal{E} = \frac{1}{2} \int \left(\frac{T}{n} (n'_e)^2 + \frac{1}{4\pi} (B'_y)^2 \right) (\xi_{ex})^2 d^3x.$$

The minimum of this expression is zero, which is obtained for the possible choice $\xi_{ex} = 0$. This need not, however, be the absolute minimum of $\mathcal{E}$ under some normalization constraint, which would mean solving a Hermitian eigenvalue problem. It is, easier, however, to find the solution of the linearized equations (28). For the present example these equations are

$$-i\omega \frac{e_e n_e}{c} \xi_e \times \mathbf{B} + ik_z \frac{e_e n_e v_e}{c} \xi_e \times \mathbf{B} = 0.$$

Furthermore, we have

$$-i\omega \xi_{i\perp} = -\frac{cT}{e_e B^2} \nabla \left(\frac{1}{n_e} \nabla \cdot (n_e \xi_e) \right) \times \mathbf{B}$$

and

$$\xi_{i\parallel} : \text{arbitrary}.$$

The quasi-neutrality conditions are

$$\nabla \cdot (e_e n_e \xi_e) = -\nabla \cdot (e_i n_i \xi_i), \quad e_e n_e = -e_i n_i.$$

The ξ_e equation is satisfied either with

$$\omega = k_z v_e$$

leaving ξ_e arbitrary, or by

$$\xi_e = \lambda(x) B$$

with ω still open.

For modes localized around $x = 0$ with $k_{\parallel} = 0$ the component $\xi_{i\parallel}$ does not enter the linearized quasi-neutrality condition, in which

$$\nabla \cdot (n_i \xi_i) = \nabla \cdot (n_i \xi_{i\perp}) = \xi_{ix} n'_i + n_i \nabla \cdot \xi_{i\perp}.$$

In this expression we can insert ξ_{ix} from the relation for $\xi_{i\perp}$:

$$\begin{aligned} \xi_{ix} &= \frac{cT}{\omega B^2 e_e n_e} \nabla \cdot (n_e \xi_e) (k_y B_z - k_z B_y^0) \\ &= - \frac{k_z cT}{\omega B_y^0 e_e n_e} \nabla \cdot (n_e \xi_e), \end{aligned}$$

where use has been made of $k_y B_y^0 + k_z B_z = 0$.

With the help of the equilibrium relations one finds

$$\nabla \cdot \xi_{i\perp} = - 3 \frac{4\pi T}{B^2} \xi_{ix} n'_i$$

and therefore

$$\nabla \cdot (n_i \xi_i) = \xi_{ix} n'_i + 0(\beta).$$

This means that the quasi-neutrality condition is met with the above ω only if

$$\nabla \cdot (n_i \xi_e) = 0,$$

which corresponds to the "simple" case above.

For the other solution,

$$\xi_e = \lambda(x) B,$$

the frequency is given by

$$\omega = k_z v_e + 0(\beta).$$

In order to evaluate the energy we can use complex quantities according to the scheme

$$\dots \xi \dots \xi \dots = \frac{1}{4} [\dots (\xi + \xi^*) \dots (\xi + \xi^*) \dots]$$

$$\rightarrow \text{Re} \frac{1}{2} [\dots \xi^* \dots \xi \dots] .$$

There is no contribution to $\mathcal{E}$ from the term containing $\mathbf{B}$ and ∇p_e ; but the last term still to be considered,

$$\frac{1}{2} \gamma_e p_e (\nabla \cdot \xi_e)^2 ,$$

also vanishes because of

$$\nabla \cdot \xi_e = \mathbf{B} \cdot \nabla \lambda = \mathbf{k} \cdot \mathbf{B} \lambda = k_{\parallel} B \lambda = 0 .$$

The minimum of $\mathcal{E}$ is therefore indeed zero.

2.4 Parallel dynamics in a force-free configuration

The last subsection has shown that the thermal energy alone does not lead to negative-energy perturbations. We therefore discuss here a one-dimensional system in which the thermal energy is altogether irrelevant, i.e. a force-free configuration, but take the parallel dynamics into account.

The configuration again depends only on x . The densities and pressures are constant:

$$n_e, p_e = \text{const.} , \quad n_i, p_i = \text{const.}$$

The ion equilibrium velocity is assumed to be zero: $\mathbf{v}_i = 0$.

The magnetic field follows from

$$\nabla \times \mathbf{B} = \lambda \mathbf{B} , \quad \lambda = \text{const.}$$

We choose as solutions

$$B_y = B \sin \lambda x , \quad B_z = B \cos \lambda x .$$

The electron velocity $\mathbf{v}_e$ determines the current density and is therefore related to the magnetic field by

$$\frac{4\pi}{c} e n_e \mathbf{v}_e = \lambda \mathbf{B} \implies \nabla \times \mathbf{v}_e = \lambda \mathbf{v}_e \implies \nabla \cdot \mathbf{v}_e = 0 .$$

A consequence of this relation is that the center-of-mass velocity vanishes, and this means that the frame chosen is the frame of minimum equilibrium

energy. Any negative-energy perturbation is therefore dynamically relevant. A simple discussion is possible in terms of initial conditions. We choose

$$\nabla \cdot \xi_e = 0, \quad \xi_{ex} = 0.$$

It is convenient to introduce a new quantity $\mathbf{w}$:

$$\mathbf{w} = \mathbf{v}_e \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{v}_e = \nabla \times (\xi_e \times \mathbf{v}_e) \quad \nabla \cdot \mathbf{w} = 0.$$

In terms of this quantity one finds

$$2\mathcal{E}_K = \int \left\{ m_e n_e \dot{\xi}_{e\parallel}^2 - m_e n_e w_{\parallel}^2 - m_e n_e w_{\parallel} (\mathbf{w} \cdot \nabla \xi_e - \xi_e \cdot \nabla \mathbf{w})_{\parallel} \right\} d^3x$$

$$2\mathcal{E}_U = - \int \frac{e_e}{c} n_e (\mathbf{w} \times \mathbf{B}) \cdot \xi_e d^3x = \int m_e n_e \frac{4\pi e_e^2 n_e}{m_e c^2 \lambda} (\xi_e \times \mathbf{v}_e) \cdot \mathbf{w} d^3x.$$

Complex displacement vectors allow one to write the following relations:

$$\mathbf{w} = i\mathbf{k} \cdot \mathbf{v}_e \xi_e, \quad w_{\parallel} = \frac{i}{v_e} \mathbf{k} \cdot \mathbf{v}_e \xi_e \cdot \mathbf{v}_e,$$

$$\xi_e^* \times \mathbf{v}_e = (\xi_{ey}^* v_{ex} - \xi_{ez}^* v_{ey}; 0; 0) \rightarrow (\xi_e^* \times \mathbf{v}_e) \cdot \mathbf{w} = 0,$$

$$(w^* \cdot \nabla \xi_e - \xi_e^* \cdot \nabla \mathbf{w})_{\parallel} = (w^* \cdot \nabla \xi_e)_{\parallel} = -i\mathbf{k} \cdot \mathbf{v}_e i\mathbf{k} \cdot \xi_e^* \xi_{e\parallel} = 0.$$

With these relations one obtains

$$2\mathcal{E} = \frac{1}{2} \int m n \left\{ |\dot{\xi}_{e\parallel}|^2 - (\mathbf{k} \cdot \mathbf{v}_e)^2 |\xi_{e\parallel}|^2 \right\} d^3x, \quad \mathbf{k} = (0, k_y, k_z). \quad (32)$$

In order to satisfy $\nabla \cdot \xi_e = 0$ with $\xi_{ex} = 0$, we represent ξ_e as

$$\xi_e = \boldsymbol{\eta} \times \mathbf{k}, \quad \xi_{ex} = \eta_y k_x - \eta_z k_y = 0,$$

from which it follows that

$$\xi_{e\parallel} = (\boldsymbol{\eta} \times \mathbf{k}) \cdot \frac{\mathbf{v}_e}{v_e} = \eta_x (k_y \cos \lambda x - k_z \sin \lambda x).$$

η_x can be chosen arbitrarily and the expression inside the braces vanishes only at points where $\mathbf{k} \parallel \mathbf{B}$. Since $\xi_{e\parallel}$ can be described as initially independent of $\xi_{e\parallel}$, one can have both positive- and negative-energy perturbations.

This situation is similar to the Cherry oscillator case. The configuration considered should therefore allow nonlinear instabilities. A possibly necessary threshold amplitude will depend on the degree of resonance present in the initial conditions.

As concerns negative-energy modes which are necessarily stable, their frequencies must obviously satisfy

$$\left(\frac{\omega}{kv_e}\right)^2 < \frac{\int \left(\frac{k_y}{k} \sin \lambda x + \frac{k_y}{k} \cos \lambda x\right)^2 |\xi_{e\parallel}|^2 d^3x}{\int |\xi_{e\parallel}|^2 d^3x}.$$

Summary

An exact energy expression for linear quasi-neutral electrostatic perturbations has been derived within the framework of dissipationless multi-fluid theory that is valid for any geometry. Taking the mass as a tensor with, in general, different masses parallel and perpendicular to an ambient magnetic field allowed us to treat the full dynamics, but also to restrict consideration to parallel dynamics only or to the completely adiabatic case. Application to slab configurations yielded results in agreement with Scott's [1] numerical study within the framework of collisional two-fluid theory. Arguments were given that a comparison with such a theory is reasonable. The result is that in plane geometry the adiabatic approximation does not allow negative-energy perturbations, whereas inclusion of the parallel dynamics does. This might, of course, be different with other kinds of configurations, but parallel dynamics, as pointed out by Scott, should always play an essential role.

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