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On the Nonlinear Stability of Dissipative Fluids

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Abstract

A general sufficient condition for nonlinear stability of steady and unsteady flows in Hydrodynamics and Magnetohydrodynamics is derived. It leads to strong limitations in the Reynolds and magnetic Reynolds numbers. It is applied to the stability of generalized time-dependent planar Couette flows in Magnetohydrodynamics. Reynolds and magnetic Reynolds numbers have to be typically less than $2\pi^2$ for stability.

- 47.20.-k Hydrodynamic Stability
- 47.65.+a Magnetohydrodynamics and Electrohydrodynamics
- 47.10.+g General Theory (Fluid Dynamics)

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For incompressible fluids and in particular in Hydrodynamics (HD) and Magnetohydrodynamics (MHD) the nonlinear terms in the equation of motion are of the quasilinear type and dissipation is present in the form of material viscosity or resistivity. More precisely if $\underline{u}$ is a many components vector field in an L^2 function space representing the frame of the fluid motion, $\underline{u}$ will obey an equation of the form

$$\dot{\underline{u}} = A(\underline{u})\underline{u} + D\underline{u}, \tag{1}$$

where $A(\underline{u})$ is a nonlinear operator depending linearly upon $\underline{u}$ and D is a linear negative definite operator if $\underline{u} = 0$ at the boundary. A simple example is

$$A(\underline{u})\underline{u} = \underline{u} \cdot \nabla \underline{u}\,, \qquad D\underline{u} = \nabla^2. \tag{2}$$

We assume further that

$$(\underline{u}, A(\underline{u})\underline{u}) = 0, \tag{3}$$

where the scalar product is given by

$$(\underline{a}, \underline{b}) = \int \underline{a} \cdot \underline{b}\, dr, \tag{4}$$

the integration being done over the volume occupied by the fluid.

To study the nonlinear stability we split $\underline{u}$ in

$$\underline{u} = \underline{u}_0 + \underline{u}_1, \tag{5}$$

where $\underline{u}_1$ is a finite perturbation zero at the boundary and $\underline{u}_0$ satisfies

$$\dot{\underline{u}}_0 = A(\underline{u}_0)\underline{u}_0 + D\underline{u}_0. \tag{6}$$

The equation for $\underline{u}_1$ is then

$$\dot{\underline{u}}_1 = A(\underline{u}_1)\underline{u}_1 + L\underline{u}_1, \tag{7}$$

with

$$L\underline{u}_1 = A(\underline{u}_0)\underline{u}_1 + A(\underline{u}_1)\underline{u}_0 + D\underline{u}_1. \tag{8}$$

L is a linear operator on $\underline{u}_1$ which in cases like (2) will remain negative definite if $A(\underline{u}_0)$ and $\underline{u}_0$ are small enough. Taking the scalar product of $\underline{u}_1$ with equation (7) we obtain

$$\frac{1}{2}(\underline{u}_1, \underline{u}_1)^{\cdot} = (\underline{u}_1, L\underline{u}_1) \tag{9}$$

by virtue of (3). Since all considered quantities are real we have

$$(\underline{u}_1, L\underline{u}_1) = (\underline{u}_1, L_s\underline{u}_1)\,, \tag{10}$$

where L_s is the symmetric part of L. Nonlinear stability is then warranted by Lyapunov methods if

$$(\underline{u}_1, L_s\underline{u}_1)\ < 0, \tag{11}$$

for all $\underline{u}_1$ satisfying $(\underline{u}_1, \underline{u}_1) =$ finite and $\underline{u}_1 = 0$ at the boundary. Expression (11) is a sufficient condition for nonlinear stability. The stability problem is now reduced to the minimization of the hermetian form $(\underline{u}_1, L_s\underline{u}_1)$. This can always be done for any flow ultimately numerically using standards hermiteans eigenvalues techniques.

This procedure is known (see [1, 2, 3]) for steady HD and MHD flows satisfying

$$A(\underline{u}_0)\underline{u}_0 + D\underline{u}_0 = 0, \tag{12}$$

which is equivalent to (6) for time-dependent flows. Though we did not find it in this general form in the literature, it is likely that it has been used (see [4]) for unsteady HD flows also. We are not aware however, of the derivation and application of (11) for MHD unsteady flows. The MHD equations are

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v}\cdot\nabla\mathbf{v} + (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p + \nu\nabla^2\mathbf{v}, && (13)\\
\nabla\cdot\mathbf{v} &= 0, && (14)\\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, && (15)\\
\nabla\cdot\mathbf{B} &= 0. && (16)
\end{aligned}$$

If $\underline{u}$ is defined as

$$\underline{u} = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}, \tag{17}$$

then $A(\underline{u})$ and D are the matrix operators

$$A(\underline{u}) = \begin{pmatrix} -(\mathbf{v}\cdot\nabla) & (\nabla\times\mathbf{B})\times \\ (\mathbf{B}\cdot\nabla) & -(\mathbf{v}\cdot\nabla) \end{pmatrix}, \tag{18}$$

$$D = \begin{pmatrix} \nu\nabla^2 & 0 \\ 0 & \eta\nabla^2 \end{pmatrix}, \tag{19}$$

with $A(\underline{u})$ verifying condition (3), for normal component of $\underline{u}$ zero at the boundary, and D being negative definite if $\underline{u}$ is zero at the boundary.

Let us illustrate the procedure by studying the nonlinear stability of a time-dependent MHD flow generalising the time-dependent planar Couette flow. It consists of a fluid bounded by two horizontal plates, the first plate at $z = 0$ and the second at $z = h$, with velocity parallel to the magnetic field and both depending only on one coordinate (z) and the time (t):

$$\mathbf{v} = v(z,t)\,\hat{e}_y, \tag{20}$$
$$\mathbf{B} = B(z,t)\,\hat{e}_y, \tag{21}$$

satisfying the equations

$$\frac{\partial v}{\partial t} - \nu\frac{\partial^2 v}{\partial z^2} = 0, \tag{22}$$
$$\frac{\partial B}{\partial t} - \eta\frac{\partial^2 B}{\partial z^2} = 0, \tag{23}$$
$$\frac{\partial p}{\partial z} + B\frac{\partial B}{\partial z} = 0. \tag{24}$$

For simplicity special solutions of these equations can be taken as

$$\mathbf{v} = \frac{v_0}{\sin\sqrt{\frac{\alpha}{\nu}}h} e^{-\alpha t}\sin\sqrt{\frac{\alpha}{\nu}}z\;\hat{e}_y, \tag{25}$$
$$\mathbf{B} = \frac{B_0}{\sin\sqrt{\frac{\alpha}{\eta}}h} e^{-\alpha t}\sin\sqrt{\frac{\alpha}{\eta}}z\;\hat{e}_y, \tag{26}$$
$$p = -\frac{B^2}{2} + f(t), \tag{27}$$

with the following boundary conditions:

$$v(0,t) = B(0,t) = 0, \tag{28}$$
$$v(h,t) = v_0\,e^{-\alpha t}, \tag{29}$$
$$B(h,t) = B_0\,e^{-\alpha t}. \tag{30}$$

and $f(t)$ fixed by the boundary conditions on p. In the limit $\alpha \to 0$ this system reduces to a stationary MHD flow. For $B_0 \to 0$ we have the time dependent Couette flow and when both α and $B_0 \to 0$, we obtain the stationary Couette flow. After calculating L (see equation (8)) for this case, we obtain its symmetric part L_s

$$L_s = \frac{1}{2}\begin{pmatrix} 2\nu\nabla^2 & 0 & 0 & 0 & -B\frac{\partial}{\partial x} & 0 \\ 0 & 2\nu\nabla^2 & 0 & 0 & -B\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 2\nu\nabla^2 & 0 & -B\frac{\partial}{\partial z} - \frac{\partial B}{\partial z} & 0 \\ 0 & 0 & 0 & 2\eta\nabla^2 & 0 & 0 \\ B\frac{\partial}{\partial x} & B\frac{\partial}{\partial y} & B\frac{\partial}{\partial z} & 0 & 2\eta\nabla^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\eta\nabla^2 \end{pmatrix} +$$
$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial v}{\partial z} & 0 & 0 & \frac{\partial B}{\partial z} \\ 0 & -\frac{\partial v}{\partial z} & 0 & 0 & -\frac{\partial B}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial B}{\partial z} & 0 & 0 & \frac{\partial v}{\partial z} \\ 0 & \frac{\partial B}{\partial z} & 0 & 0 & \frac{\partial v}{\partial z} & 0 \end{pmatrix}. \qquad (31)$$

In order to examine stability, we calculate I given by

$$\begin{aligned} I &= (\underline{u}_1, L_s\, \underline{u}_1) = \\ & \int d\mathbf{r} \left(\nu \left(v_{1x}\nabla^2 v_{1x} + v_{1y}\nabla^2 v_{1y} + v_{1z}\nabla^2 v_{1z}\right) + \right. \\ & \eta\left(B_{1x}\nabla^2 B_{1x} + B_{1y}\nabla^2 B_{1y} + B_{1z}\nabla^2 B_{1z}\right) + \\ & \left.\left(\frac{\partial v}{\partial z}(-v_{1y}v_{1z} + B_{1y}B_{1z}) + \frac{\partial B}{\partial z}(v_{1y}B_{1z} - v_{1z}B_{1y})\right)\right). \qquad (32) \end{aligned}$$

We suppose for convenience that $v_0, B_0 > 0$, $0 < h < \sqrt{\nu/\alpha}\ \pi/2$ and $0 < h < \sqrt{\eta/\alpha}\ \pi/2$, which guarantees that $(\partial v/\partial z) > 0$ and $(\partial B/\partial z) > 0$. To satisfy condition (11) for all $\underline{u}_1$ we make a first estimate of I using

$$-\left(\frac{\partial v}{\partial z}\right) v_{1y}v_{1z} < \left(\frac{\partial v}{\partial z}\right)\frac{1}{2}\left(v_{1y}^2 + v_{1z}^2\right) < \left(\frac{\partial v}{\partial z}\right)_m \frac{1}{2}\left(v_{1y}^2 + v_{1z}^2\right). \qquad (33)$$

where $(\partial v/\partial z)_m$ is the maximum of $(\partial v/\partial z)$ with respect to z and t, which occurs at $t = 0$, $z = 0$. Similar estimates can be done for the other cross terms, so that

$$
\begin{aligned}
I < I_m &= \int d\mathbf{r} \left(\nu \left(v_{1x} \nabla^2 v_{1x} + v_{1y} \nabla^2 v_{1y} + v_{1z} \nabla^2 v_{1z} \right) + \right. \\
&\quad \eta \left(B_{1x} \nabla^2 B_{1x} + B_{1y} \nabla^2 B_{1y} + B_{1z} \nabla^2 B_{1z} \right) + \\
&\quad \left. C_m \left(v_{1y}^2 + v_{1z}^2 + B_{1y}^2 + B_{1z}^2 \right) \right), \qquad (34)
\end{aligned}
$$

with

$$
\begin{aligned}
C_m &= \left(\frac{\partial v}{\partial z} \right)_m + \left(\frac{\partial B}{\partial z} \right)_m, \qquad (35) \\
\left(\frac{\partial v}{\partial z} \right)_m &= \frac{v_0}{\sin \sqrt{\frac{\alpha}{\nu}} h} \sqrt{\frac{\alpha}{\nu}}, \qquad (36) \\
\left(\frac{\partial B}{\partial z} \right)_m &= \frac{B_0}{\sin \sqrt{\frac{\alpha}{\eta}} h} \sqrt{\frac{\alpha}{\eta}}. \qquad (37)
\end{aligned}
$$

Now we look for the extremum of I_m subject to the condition $(\underline{u}_1, \underline{u}_1) =$ finite.

$$\delta I' = \delta \left(I_m - \lambda \left(\underline{u}_1, \underline{u}_1 \right) \right) = 0, \qquad (38)$$

where λ is the Lagrange multiplier. This leads to the following system of equations

$$
\begin{aligned}
&\nabla^2 v_{1x} - \frac{\lambda}{\nu} v_{1x} = 0, \qquad (39) \\
&\nabla^2 v_{1y} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} \right) v_{1y} = 0, \qquad (40) \\
&\nabla^2 v_{1z} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} \right) v_{1z} = 0, \qquad (41) \\
&\nabla^2 B_{1x} - \frac{\lambda}{\eta} B_{1x} = 0, \qquad (42) \\
&\nabla^2 B_{1y} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} \right) B_{1y} = 0, \qquad (43) \\
&\nabla^2 B_{1z} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} \right) B_{1z} = 0. \qquad (44)
\end{aligned}
$$

Fourier analysing in x and y

$$\underline{\mathrm{u}}_1 = \sum_{\mathbf{k}} A_{\mathbf{k}}\left(z\right)\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{45}$$

where

$$\begin{aligned} \mathbf{k} &= k_x\, \hat{e}_x + k_y\, \hat{e}_y & (46) \\ \mathbf{r} &= x\, \hat{e}_x + y\, \hat{e}_y & (47) \end{aligned}$$

and calling each component of the vector $A_{\mathbf{k}}\left(z\right)$ as

$$\left(A_{\mathbf{k}}\left(z\right)\right)_j = A_j \tag{48}$$

the system of equations (38)-(43) can be reduced to

$$\begin{aligned} &\frac{d^2 A_1}{dz^2} - \left(\frac{\lambda}{\nu} + k^2\right) A_1 = 0, & (49) \\ &\frac{d^2 A_2}{dz^2} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} - k^2\right) A_2 = 0, & (50) \\ &\frac{d^2 A_3}{dz^2} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} - k^2\right) A_3 = 0, & (51) \\ &\frac{d^2 A_4}{dz^2} - \left(\frac{\lambda}{\eta} + k^2\right) A_4 = 0, & (52) \\ &\frac{d^2 A_5}{dz^2} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} - k^2\right) A_5 = 0, & (53) \\ &\frac{d^2 A_6}{dz^2} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} - k^2\right) A_6 = 0. & (54) \end{aligned}$$

Since the boundary conditions for the perturbations are

$$\begin{aligned} \mathbf{v}_1\left(x, y, 0, t\right) &= \mathbf{B}_1\left(x, y, 0, t\right) = 0, & (55) \\ \mathbf{v}_1\left(x, y, h, t\right) &= \mathbf{B}_1\left(x, y, h, t\right) = 0, & (56) \end{aligned}$$

the non-trivial solutions of this system, which satisfy the boundary conditions (54) are sine solutions. To satisfy also the other boundary conditions (55) we obtain some restrictions on λ. When the maximum value of λ is negative, the system is stable, this can occur in two ways.

- For $\nu < \eta$ the system is stable if

$$Re\frac{\sqrt{\frac{\alpha}{\nu}}\,h}{\sin\sqrt{\frac{\alpha}{\nu}}\,h} + S\frac{\sqrt{\frac{\alpha}{\eta}}\,h}{\sin\sqrt{\frac{\alpha}{\eta}}\,h} < 2\pi^2, \tag{57}$$

where

$$Re = \frac{v_0 h}{\nu}, \tag{58}$$

$$S = \frac{B_0 h}{\nu}. \tag{59}$$

- For $\eta < \nu$ the system is stable if

$$Re_m\frac{\sqrt{\frac{\alpha}{\eta}}\,h}{\sin\sqrt{\frac{\alpha}{\eta}}\,h} + S_m\frac{\sqrt{\frac{\alpha}{\eta}}\,h}{\sin\sqrt{\frac{\alpha}{\eta}}\,h} < 2\pi^2, \tag{60}$$

where

$$Re_m = \frac{v_0 h}{\eta}, \tag{61}$$

$$S_m = \frac{B_0 h}{\eta}. \tag{62}$$

In the limit $\alpha \to 0$ (steady MHD flow) we obtain

$$Re + S < 2\pi^2 \quad for \quad \nu < \eta \tag{63}$$

and

$$Re_m + S_m < 2\pi^2 \quad for \quad \eta < \nu. \tag{64}$$

For the time-dependent Couette flow ($B_0 \to 0$), we have

$$Re\frac{\sqrt{\frac{\alpha}{\nu}}\,h}{\sin\sqrt{\frac{\alpha}{\nu}}\,h} < 2\pi^2, \tag{65}$$

and for the stationary Couette flow ($\alpha, B_0 \to 0$)

$$Re < 2\pi^2. \tag{66}$$

It should be mentioned that for the stationary Couette flow the last condition is also obtained without introducing the estimate $I < I_m$.

The sufficient condition (11) is general and robust, but also too stringent. It is fulfilled in HD and MHD only if the Reynolds and magnetic Reynolds numbers are small enough. Since viscosity and resistivity especially for hot plasmas are small, condition (11) would allow only a very low level of electrical currents and flows. Linear stability analysis and experimental evidence, however, seem to show that in some cases, values for currents and flows far beyond those allowed by condition (11) occur without any sign of gross instabilities. It will be, however, much more difficult to do the nonlinear stability theory for such situations and in contrast with the present method, it is likely that it may have to be done differently for each case.

References

[1] J. Serrin, "Handbuch der Physik" *vol VIII/1* p.125- Springer Verlag, 1959.

[2] D. Joseph, "Stability of Fluid Motions", Springer Verlag, 1976 and references therein.

[3] S. Rionero, Ann. Mat. Pura Appl. *LXXVI*, 75, 1967 and *LXXVIII*, 339, 1968.

[4] J.G. Heywood, "Open Problems in the Theory of the Navier-Stokes Equations for Viscous Incompressible Flows", to be published, and references therein.