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On a general Stability Condition in Resistive MHD

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Abstract

The general sufficient condition obtained by the author in a previous work is analysed with respect to its "nearness" to necessity. It is found that for physically reasonable approximations the condition is in some sense necessary and sufficient for stability against all modes. This together with hermiticity makes its analytical and numerical evaluation worth for the optimization of magnetic configurations.

In a previous note [1] the author has derived a general stability condition which is sufficient for stability with respect to purely growing modes. The derivation is made possible by manipulating the linearized equations of resistive MHD in such a way that they can be put in the form(see [1] for the derivation and notations, it is recommended to read [1] together with this letter)

$$N\ddot{\Psi} + P\dot{\Psi} + Q\Psi = 0 \tag{1}$$

with N and P symmetric real and positive operators. Then according to [2] one obtains the condition

$$\delta W = (\Psi, Q_s\Psi) \geq 0, \tag{2}$$

sufficient for stability with respect to purely growing modes. Q_s is the symmetric part of Q which can be split in $Q = Q_s + Q_a$, Q_a being the antisymmetric part. (,) is the usual notation for the scalar product as in [1], where the explicit form of (2) is given. We reproduce it here again

$$\begin{aligned}\delta W &= \int d\tau(\gamma P_0(\nabla\cdot\xi)^2 + (\xi\cdot\nabla P_0)\nabla\cdot\xi) \\ &+ \int d\tau(\nabla\times\mathbf{A})^2 - \int d\tau\xi\times\mathbf{J}\cdot\nabla\times\mathbf{A} + \\ &+ p\cdot p\cdot\int d\tau\mathbf{J}\cdot(\mathbf{A}-\xi\times\mathbf{B})(\mathbf{B}\cdot\nabla)^{-1}(1/\eta_0)(\nabla\eta_0\cdot\nabla\times\mathbf{A}) \\ &- \int d\tau(\mathbf{A}-\xi\times\mathbf{B})\cdot\mathbf{V}\times(\nabla\times\mathbf{A})1/\eta_0 \qquad (3)\end{aligned}$$

Condition (3) is equivalent to (2) if all quantities in it are real but is not hermitean. It is, however, obvious to obtain an hermitean form of (3) by constructing the adjoint (by integrations by parts) of Q or as indicated in [1].

As already known (see [3]) condition (2) becomes necessary and sufficient for all modes if $Q_a = 0$. In the incompressible case with tokamak ordering, Tasso and Virtamo derived in such a limiting case some time ago a necessary and sufficient condition (see [4]) which has been evaluated numerically(see [5]). In the general case it does not seem possible to find a system of dynamic variables for which $Q_a = 0$. One can, however, "upgrade" conditions (2) or (3) for two interesting situations 1) for $Q_a \approx \epsilon$ small, which is related to the tokamak scaling and 2) $N = 0$ or neglecting inertia, which is valid for time scales much larger than the Alfven or acoustic time scales.

1) $Q_a \approx \epsilon$

Let us first show that for $Q_a = 0$ any unstable mode must be purely growing. For that purpose assume

$$\Psi = e^{\omega t}\Psi_0(\mathbf{r}), \qquad (4)$$

$$\omega = i\omega_0 + \gamma_0 \qquad (5)$$

with ω_0 and γ_0 real. Insert (4) and (5) in (1) to obtain

$$(i\omega_0 + \gamma_0)^2 N\Psi_0 + (i\omega_0 + \gamma_0)P\Psi_0 + (Q_s + Q_a)\Psi_0 = 0. \qquad (6)$$

Multiplying by $\Psi_0^\star$, integrating over the plasma volume and using the usual notation for the scalar product, (6) reduces to

$$[(\gamma_0^2 - \omega_0^2) + 2i\gamma_0\omega_0](\Psi_0, N\Psi_0) + (\gamma_0 + i\omega_0)(\Psi_0, P\Psi_0) + (\Psi_0, (Q_s + Q_a)\Psi_0) = 0. \tag{7}$$

Since N, P and Q_s are hermitean the imaginary part of (7) is

$$2\gamma_0\omega_0(\Psi_0, N\Psi_0) + \omega_0(\Psi_0, P\Psi_0) - (\Psi_0, Q_a\Psi_0) = 0. \tag{8}$$

Since N and P are positive and if we assume $Q_a = 0$ and $\gamma_0 \geq 0$ it follows from (8) that $\omega_0 = 0$. This proves that for $Q_a = 0$, exponentially unstable modes must be purely growing.

Assuming that condition (2) is violated for some test function, it follows that for $\epsilon = 0$ a purely growing mode with $\omega_0 = 0$ exists and satisfies (6) for $Q_a = 0$. Now supposing that Q_a is small and of order ϵ, we expand (1) up to first order in ϵ

$$\Psi = \Psi_0 + \epsilon\Psi_1, \tag{9}$$

$$\omega = \gamma_0 + \epsilon\gamma_1, \tag{10}$$

$$\gamma_0^2 N\Psi_0 + \gamma_0 P\Psi_0 + Q_s\Psi_0 = 0, \tag{11}$$

$$2\gamma_0\omega_1 N\Psi_0 + \gamma_0^2 N\Psi_1 + \omega_1 P\Psi_0 + \gamma_0 P\Psi_1 + Q_s\Psi_1 + Q_a\Psi_0 = 0. \tag{12}$$

Multiplying (11) and (12) by $\Psi_0^\star$ and integrating over the plasma volume we obtain

$$\gamma_0^2(\Psi_0, N\Psi_0) + \gamma_0(\Psi_0, P\Psi_0) + (\Psi_0, Q_s\Psi_0) = 0, \tag{13}$$

$$\begin{aligned} &2\gamma_0\omega_1(\Psi_0, N\Psi_0) + \omega_1(\Psi_0, P\Psi_0) + (\Psi_0, Q_a\Psi_0) + \\ &\gamma_0^2(\Psi_0, N\Psi_1) + \gamma_0(\Psi_0, P\Psi_1) + (\Psi_0, Q_s\Psi_1) = 0. \end{aligned} \tag{14}$$

Using (11) and the fact that N, P and Q_s are hermitean (14) reduces to

$$\omega_1[2\gamma_0(\psi_0, N\Psi_0) + (\Psi_0, P\Psi_0)] + (\Psi_0, Q_a\Psi_0) = 0. \tag{15}$$

Since the original system of equations is real and the mode is purely growing ($\omega_0 = 0$), Ψ_0 can be chosen real without loss of generality. It follows then from (15) together with the positivity of γ_0, N and P and the antisymmetry of Q_a that

$$\omega_1 = 0. \tag{16}$$

This means that if the purely growing eigenmode had to acquire some real frequency through $Q_a \approx \epsilon$, this frequency is of order ϵ^2 or higher. A small Q_a affects the unstable spectrum very weakly.

2) $N = 0$, or neglecting inertia

Equation (1) becomes

$$P\dot{\Psi} + (Q_s + Q_a)\Psi = 0. \tag{17}$$

Multiplying by Ψ real and integrating over the volume we obtain

$$\frac{\partial(\Psi, P\Psi)}{\partial t} = -2(\Psi, Q_s\Psi). \tag{18}$$

We see that the positive form $(\Psi, P\Psi)$ is a Liapunov functional if conditions (2) or (3) are verified. Now these conditions are sufficient for stability against all modes not the purely growing only.

The analysis brougth in this note cannot make condition (2) necessary and sufficient for all modes but gives more weigth to it.One could say that the condition is "nearly" necessary so that its analytical and numerical evaluations may be worth doing. As mentioned in [1] condition (2) reduces to the ideal MHD energy prínciple and to the resistive energy principle of Tasso and Virtamo (see [4]) in the appropriate limits. Extensive numerical calculations in those particular limits for ideal MHD (see [6]) and for resistive MHD (see [5]) show that both ideal and resistive modes can be stabilized if β is small enough, the safety factor large enough and the current distribution well chosen.

Condition (2), however, may be violated in general by test functions reminiscent of resistive ballooning or resistive drift modes or other residual modes. Nevertheless, its degree of violation can be taken as a "measure" for the optimization of magnetic configurations. Similarly to previous work (see [5]) the numerical evaluation of the condition is made possible due to its hermitean form.

References

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