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Relaxed plasmas in external magnetic fields

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Abstract

The well-known theory of relaxed plasmas (Taylor states) is extended to external magnetic fields whose field lines intersect the conducting toroidal boundary. Application to an axially symmetric, large-aspect-ratio torus with circular cross section shows that the maximum pinch ratio, and hence the phenomenon of current saturation, is independent of the external field. The relaxed state is explicitly given for an external octupole field. In this case, field reversal is inhibited near parts of the boundary if the octupole generates magnetic x-points within the plasma.

1 Introduction

Various pinch experiments involve a toroidal vacuum vessel in which first a toroidal magnetic field is generated by external coils. Then, after an initial plasma is produced by some ionization process, a toroidal current I is induced. According to a postulate of J. B. Taylor [1], the plasma subsequently relaxes to the state of lowest energy compatible with conservation of the total magnetic helicity H and toroidal magnetic flux ψ (Taylor state). The relaxation process involves turbulent magnetic reconnection, and Taylor's postulate cannot be valid unless the associated relaxation time is much shorter than any other relevant characteristic times, such as those for plasma diffusion, for skin penetration of the boundary, or for changes in external circuits.

Most of the theory of the relaxed state has been developed on the assumption that the boundary is a magnetic surface (an exhaustive list of references is in the review paper by Taylor [2]). The magnetic field is then a solution of the equation [3]

$$\mathrm{curl}\mathbf{B} = \mu \mathbf{B}, \tag{1}$$

and μ is the smallest of all constants which yield the prescribed values of helicity and flux, thus being a function of H and ψ. If the vessel has axial symmetry, large aspect ratio, and a circular cross section of radius a, thus being approximated by a circular cylinder, the leading terms of the Taylor state depend only on one nondimensional parameter, viz. the "pinch ratio"

$$\Theta = 2\pi a I/\psi = \mu a/2. \tag{2}$$

The magnitude of this parameter is bounded by a maximum value $\Theta_0 \approx 1.56$, a phenomenon which is interpreted as "current saturation". While the plasma has the same cylindrical symmetry as the vessel if $|\Theta| < \Theta_0$, it is helically deformed if $|\Theta| = \Theta_0$, and any attempt to increase the current any further leads to an increase of this deformation instead. The cylindrically symmetric relaxed state is described by the "Bessel function model", which predicts reversal of the axial magnetic field near the boundary if the pinch ratio exceeds a critical value $\Theta_c \approx 1.2$.

The theory of relaxed plasmas has not been worked out for the case that the toroidal vessel is not a flux surface (only the flux-core spheromak, whose vessel has the topology of a sphere, has been considered [2]). Since some pinch experiments (e.g. Extrap [5]) feature external magnetic fields whose field lines intersect a toroidal boundary, we extend the theory to such cases. This extension, though it may seem trivial, requires a different definition of the helicity, and a rather involved examination of the boundary terms arising from various integrations by parts. We show that the relaxed state is still governed by Eq. (1), and that μ is stili the smallest constant yielding the prescribed helicity and flux. Only the boundary condition are different: While one has $B_n = 0$ when the wall is a flux surface, B_n may now take prescribed values which are arbitrary within the constraint

$$\int d^2\sigma B_n = 0, \tag{3}$$

which follows from $\mathrm{div}\mathbf{B} = 0$.

Applying the theory to a circular cylinder, we then show that the maximum pinch ratio Θ_0 is independent of the external magnetic field. Further specializing to the external field created by four axial current carrying rods, thus modeling the octupole field of the Extrap experiment, we give explicit expressions for the magnetic field of the Taylor states and show that field reversal is inhibited near parts of the boundary if the rod current is strong enough to generate magnetic x-points within the plasma.

2 General theory

2.1 Fields and potentials

We consider a slightly dissipative magnetized plasma in the toroidal domain T bounded by a highly conducting shell S_T. The exterior of T is a vacuum, and imbedded in this are rigid wires carrying time-independent currents. These currents, together with surface currents in S_T and volume currents in the plasma (both time-dependent), generate a magnetic field $\mathbf{B}(\mathbf{x}, t)$ whose time dependence is accompanied by an electric field $\mathbf{E}(\mathbf{x}, t)$.

We thus have the homogeneous Maxwell equations

$$\mathrm{div}\mathbf{B} = 0, \quad \dot{\mathbf{B}} = -\mathrm{curl}\mathbf{E} \tag{4}$$

(the dot denotes the time-derivative) in all of space. At S_T, the normal component of $\mathbf{B}$ and the tangential components of $\mathbf{E}$ must be continuous. Since the boundary is to leading order a perfect conductor, we thus have the boundary condition

$$\mathbf{n} \times \mathbf{E} = 0. \tag{5}$$

This implies that the normal component of $\mathbf{B}$ is time-independent (since S_T is not assumed to be a magnetic surface, this component need not vanish). As a consequence, the vacuum magnetic field in the exterior of T is time-independent. Therefore, it merely serves to specify the boundary conditions upon the magnetic field in T and need no longer be considered.

The homogeneous Maxwell equations are solved by expressing the fields in terms of an electrostatic potential Φ and a vector potential $\mathbf{A}$:

$$\mathbf{B} = \mathrm{curl}\mathbf{A}, \quad \mathbf{E} = -\dot{\mathbf{A}} - \nabla\Phi. \tag{6}$$

The potentials are unique up to a gauge transformation of the form

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \Phi \to \Phi' = \Phi - \dot{\chi}, \tag{7}$$

with an arbitrary function $\chi(\mathbf{x}, t)$. Though this is not explicitly stated in most of the literature, it is crucial that a gauge be chosen in which $\mathbf{A}$ is single-valued. Since the fields are single-valued, this implies that $\nabla\Phi$ is single-valued. We find it convenient to assume, in addition, that Φ itself is single-valued.

To discuss an implication of these assumptions, we introduce a disk-like surface S_s bounded by a curve C_s which lies in S_T and closes upon itself the short way around the torus, as well as a surface S_l bounded by a curve C_l which also lies in S_T and closes the long way. The surface S_s thus cuts the torus into a singly connected domain, and S_l cuts its exterior into a singly connected domain. We choose the orientations of the unit normals $\mathbf{n}$ at the various surfaces and of the arc length elements $\mathbf{dl}$ along the various curves such that application of Gauss' theorem and Stokes' theorem produces no minus signs. We denote the increase of a function f along C_ν ($\nu = s, l$) by $\delta_\nu f$. Both $\delta_s f$ and $\delta_l f$ are zero if f is single-valued, and f has a singularity in T (or in its exterior) if $\delta_s f \neq 0$ (or $\delta_l f \neq 0$). Both $\delta_s f$ and $\delta_l f$ are path-independent (independent of the choice of the cut surfaces) if ∇f is single-valued. Since Φ is defined throughout T, $\delta_s \Phi$ is automatically zero. Our additional assumption that Φ is single-valued thus means that Φ can be uniquely continued into the exterior of T. Our assumption that $\mathbf{A}$ is single-valued implies that it can also be continued into the exterior of T. It also implies that only gauge functions χ with single-valued gradients must be considered. Hence both $\delta_s\chi$ and $\delta_l\chi$ are path-independent for all admissible gauge transformations.

2.2 Conserved quantities

We will minimize the magnetic energy, keeping fixed quantities which are conserved in an ideal plasma, and still approximately conserved in a slightly dissipative one.

The time derivative of the magnetic flux through a disk-like surface S with a bounding curve C is

$$\frac{d}{dt}\int_S d^2\sigma \mathbf{n}\cdot\mathbf{B} = \int_S d^2\sigma \mathbf{n}\cdot\dot{\mathbf{B}} = -\int_S d^2\sigma \mathbf{n}\cdot \mathrm{curl}\mathbf{E} = \oint_C \mathbf{dl}\cdot\mathbf{E}. \qquad (8)$$

If the curve C is in the boundary S_T, then the boundary condition (5) implies conservation of the flux. In particular, the normal component of $\mathbf{B}$ at S_T, and the fluxes ψ_s and ψ_l through the cut surfaces S_s and S_l are conserved. (note that the latter depend on the choice of the cut surfaces unless the boundary is a magnetic surface). The toroidal flux ψ_s will simply be denoted by ψ. This is because the flux ψ_l through the hole of the torus does not affect the magnetic field in T, and hence need not be considered (it merely serves to normalize the vacuum field in the exterior of T).

While the helicities of infinitesimal flux tubes form an infinite set of conserved quantities in an ideal plasma, only the total helicity remains meaningful in a slightly resistive one and is then approximately conserved. The basic definition of the total helicity,

$$H_0 = \int_T d^3\tau \mathbf{A} \cdot \mathbf{B}, \tag{9}$$

requires modification because H_0 is gauge-invariant only in singly connected domains, and is conserved only if the boundary is a magnetic surface. Following Finn and Antonsen [4], we introduce a vacuum field $\mathbf{B}_0$ in T that has the same normal component as $\mathbf{B}$ at S_T, and the same flux through S_s. This field is unique if $\mathbf{B}$ is given. When continued into the exterior of T, it differs from the externally applied magnetic field only by a vacuum field whose field lines do not intersect the boundary. The total helicity is then defined as

$$H = \int_T d^3\tau (\mathbf{A} + \mathbf{A}_0) \cdot (\mathbf{B} - \mathbf{B}_0), \tag{10}$$

where $\mathbf{A}$ and $\mathbf{A}_0$ are single-valued vector potentials of $\mathbf{B}$ and $\mathbf{B}_0$. In Appendix A we show that H is gauge-invariant, and that it is conserved if the plasma is perfectly conducting.

2.3 Relaxed state

We seek the state of lowest magnetic energy,

$$E_M = \frac{1}{2} \int_T d^3\tau |\mathrm{curl}\mathbf{A}|^2, \tag{11}$$

with fixed helicity H, fixed toroidal magnetic flux ψ, and fixed normal component B_n at the boundary S_T.

The constraint of fixed H is observed by introducing a Lagrangian multiplier μ. One thus minimizes the functional

$$W = E_M - \frac{1}{2}\mu H = \frac{1}{2} \int_T d^3\tau \left[|\mathrm{curl}\mathbf{A}|^2 - \mu(\mathbf{A} + \mathbf{A}_0) \cdot \mathrm{curl}(\mathbf{A} - \mathbf{A}_0) \right]. \tag{12}$$

Forming its first variation, integrating by parts, and applying Gauss' theorem, one obtains

$$\delta W = \int_T d^3\tau [\delta \mathbf{A} \cdot (\mathrm{curl}\mathbf{B} - \mu \mathbf{B}) + \int_{S_T} d^2\sigma \mathbf{n} \cdot (\delta \mathbf{A} \times \mathbf{C}), \tag{13}$$

where

$$\mathbf{C} = \mathrm{curl}\mathbf{A} - \frac{1}{2}\mu(\mathbf{A} + \mathbf{A}_0). \tag{14}$$

Putting $\delta W = 0$ for all admissible variations $\delta\mathbf{A}$, one obtains the Euler equation (1). The boundary term in the variation δW must vanish, too, in order that a solution of the Euler equation correspond to a state of stationary energy. We show in Appendix B that this is automatically the case as a consequence of keeping B_n and ψ fixed.

We thus have the result that the states of stationary energy are solutions of the Euler equation satisfying the boundary condition of prescribed B_n, and with a constant μ to be determined from the prescribed values of H and ψ. Since there may be several such states, it is not a priori clear which one has the lowest energy. We show in Appendix C that among all states of stationary energy the state of lowest energy has the smallest value of $|\mu|$.

3 Circular cylinder

3.1 Fourier decomposition

If the toroidal vessel is axially symmetric, if its poloidal cross section is circular, and if its aspect ratio is large, then it is approximately a circular cylinder, and physical quantities are to leading order periodic in the axial coordinate z, with a period L much larger than the radius a. In cylindrical coordinates (r, θ, z), the magnetic field is then Fourier-decomposed according to

$$\mathbf{B} = \sum_{k,m} e^{i(kz+m\theta)}\mathbf{B}^{km}(r), \qquad \mathbf{B}^{km} = B_r^{km}\nabla r + rB_\theta^{km}\nabla\theta + B_z^{km}\nabla z, \tag{15}$$

where m runs through all integers, and k runs through all multiples of $2\pi/L$. Since $L \gg a$, the values of ka are closely spaced.

The components of the Euler equation (1) are

$$\frac{im}{r}B_z^{km} - ikB_\theta^{km} - \mu B_r^{km} = 0, \tag{16}$$

$$ikB_r^{km} - \frac{dB_z^{km}}{dr} - \mu B_\theta^{km} = 0, \tag{17}$$

$$\frac{1}{r}\frac{d(rB_\theta^{km})}{dr} - \frac{im}{r}B_r^{km} - \mu B_z^{km} = 0. \tag{18}$$

If $\mu \neq 0$, the Euler equation implies div$\mathbf{B} = 0$ or, when written in components,

$$\frac{1}{r}\frac{d(rB_r^{km})}{dr} + \frac{im}{r}B_\theta^{km} + ikB_z^{km} = 0. \tag{19}$$

This equation thus may be ignored if $\mu \neq 0$, but must be adjoined if $\mu = 0$.

The relaxed state is now determined by first solving the Euler equation for given B_n, ψ, and μ, then computing H as a function of μ, and finally inverting the latter to obtain μ as function of H, choosing among the branches $\mu(H)$ the one with the smallest $|\mu|$.

3.2 Solution of the Euler equation

Since the Euler equation is a second-order system, it has two fundamental solutions. It turns out that only one of them is regular at the origin. The solution of interest is thus unique within an arbitrary constant factor, and we can write

$$B_r^{km} = a_{km}\beta_r^{km}(r), \qquad B_\theta^{km} = a_{km}\beta_\theta^{km}(r), \qquad B_z^{km} = a_{km}\beta_z^{km}(r), \tag{20}$$

where β_r^{km}, β_θ^{km}, β_z^{km} is some particular non-trivial solution regular at the origin, and the complex coefficients a_{km} are arbitrary within the constraint that $\mathbf{B}$ be real or, equivalently, that $\mathbf{B}^{-k-m}$ be the complex conjugate of $\mathbf{B}^{km}$. If the particularsolution is chosen swith this property, then the constraint upon the coefficients is $a_{-k-m} = a_{km}^*$.

For given m and k, case distinctions are necessary, depending on the sign of $\mu^2 - k^2$. These distinctions were previously ignored [2], but this omission did not affect the result in the cases which were considered (the boundary was assumed to be a flux surface).

If $\mu^2 - k^2 \neq 0$,then β_r^{km} and β_θ^{km} can be expressed in terms of β_z^{km},

$$\beta_r^{km} = \frac{i}{\mu^2 - k^2}\left(k\frac{d\beta_z^{km}}{dr} + \mu\frac{m}{r}\beta_z^{km}\right), \tag{21}$$

$$\beta_\theta^{km} = \frac{-1}{\mu^2 - k^2}\left(\mu\frac{d\beta_z^{km}}{dr} + k\frac{m}{r}\beta_z^{km}\right), \tag{22}$$

and β_z^{km} satisfies the equation

$$\frac{d^2\beta_z^{km}}{dr^2} + \frac{1}{r}\frac{d\beta_z^{km}}{dr} + \left(\mu^2 - k^2 - \frac{m^2}{r^2}\right)\beta_z^{km} = 0. \tag{23}$$

A regular solution is

$$\beta_z^{km} = J_m(\sqrt{\mu^2 - k^2}r) \tag{24}$$

if $\mu^2 - k^2 > 0$, but

$$\beta_z^{km} = I_m(\sqrt{k^2 - \mu^2}r) \tag{25}$$

if $\mu^2 - k^2 < 0$ (J_m is a Bessel function, and I_m a modified one).

If $\mu^2 - k^2 = 0$, then different relative signs of k, m, and μ must be distinguished. Let σ be the sign of m. Then a regular solution is

$$\beta_r^{km} = r^{|m|-1}, \tag{26}$$

$$\beta_\theta^{km} = i\sigma r^{|m|-1}, \tag{27}$$

$$\beta_z^{km} = 0 \tag{28}$$

if $\sigma \neq 0$ and $\mu = \sigma k$, but

$$\beta_r^{km} = r^{|m|-1} + \frac{k^2}{m^2 + |m|} r^{|m|+1}, \tag{29}$$

$$\beta_\theta^{km} = i\sigma \left(r^{|m|-1} - \frac{k^2}{m^2 + |m|} r^{|m|+1} \right), \tag{30}$$

$$\beta_z^{km} = \frac{2ik}{|m|} r^{|m|} \tag{31}$$

if $\sigma \neq 0$ and $\mu = -\sigma k$, and

$$\beta_r^{k0} = -\frac{1}{2} ikr, \tag{32}$$

$$\beta_\theta^{k0} = \frac{1}{2} \mu r, \tag{33}$$

$$\beta_z^{k0} = 1 \tag{34}$$

if $\sigma = 0$ and $\mu = k$ or $\mu = -k$.

3.3 Determination of coefficients

The boundary condition is

$$a_{km}\beta_r^{km}(a) = f_{km}, \tag{35}$$

where f_{km} is a Fourier coefficient of the given function $B_n(\theta, z)$. The coefficient a_{km} is thus determined as long as $\beta_r^{km}(a) \neq 0$. Since the function $\beta_r^{km}(a)$ depends on μ, there are thus certain values of μ for which the coefficient a_{km} is not uniquely determined.

The case $k = m = 0$ is exceptional because $\beta_r^{00}(a) \equiv 0$ for all μ. This is consistent with $f_{00} = 0$, which is equivalent to the constraint (3) upon the boundary condition. The coefficient a_{00} is thus never determined from the boundary condition. It is instead determined from the toroidal magnetic flux because this is given by

$$\mu\psi = 2\pi a J_1(|\mu|a)a_{00}, \tag{36}$$

thus depending only on a_{00}, not on any other coefficients. We will see that $J_1(|\mu|a) \neq 0$ for all possible values of μ. This implies that the relation (36) can indeed be inverted to yield a_{00}.

If k and m are not both zero, then $\beta_r^{km}(a)$ vanishes only on a discrete set values of μ (the eigenvalues of the Euler equation with the boundary condition $B_n = 0$ and the additional constraint $\psi = 0$). For each pair (k, m), there is an infinite sequence of eigenvalues which we label with an integer n, thus denoting them by μ_{kmn}. There are two possibilities: either $f_{km} \neq 0$, in which case a_{km}, as a function of μ, has a pole at each μ_{kmn}, or $f_{km} = 0$, in which case a_{km} is arbitrary for $\mu = \mu_{kmn}$. We conclude that the magnetic field is uniquely determined as long as μ is not an eigenvalue. For $\mu = \mu_{kmn}$, it does not exist if $f_{km} \neq 0$ ($\mathbf{B}_{km}$ diverges as $\mu \to \mu_{kmn}$), and it exists, but is unique only up to an arbitrary multiple of $\mathbf{B}_{km}$, if $f_{km} = 0$.

3.4 Determination of pinch ratio

This implies that the helicity $H(\mu)$ exists and is uniquely determined as long as μ is not an eigenvalue, but has a pole at $\mu = \mu_{kmn}$ if $f_{km} \neq 0$, and is arbitrary at $\mu = \mu_{kmn}$ if $f_{km} = 0$. Hence the line $\mu = \mu_{kmn}$ belongs to the graph of the function $H(\mu)$ if $f_{km} = 0$, but is an asymptote if $f_{km} \neq 0$. As a consequence, the inverse $\mu(H)$ has infinitely many branches, one in each interval between adjacent eigenvalues.

The branch corresponding to the relaxed state is that with the smallest $|\mu|$. This branch is obtained by observing the following two facts: First, the spectrum is symmetric to the origin because the Euler equation is invariant upon changing the signs of μ, m, and B_θ, or of μ, k, and B_z. Second, the function $H(\mu)$ is odd. We conclude that the branch $\mu(H)$ with the smallest $|\mu|$ is an odd function with $|\mu(H)| \leq \mu_0$, where μ_0 is the smallest positive eigenvalue. This function is monotonic even if the function $H(\mu)$ is nonmonotonic for $|\mu| < \mu_0$; in the latter case it has jumps, thus being only piecewise continuous. Since the relationship between H and μ is never needed, we do not give it here. However, we remark that the function $H(\mu)$ depends on the boundary conditions, each Fourier coefficient f_{km} giving rise to an additive contribution. The contribution of f_{00} was given by Taylor [1]; it is a monotonic function of μ.

It can be shown that $\beta_r^{km}(r) \neq 0$ for $r > 0$ unless $\mu^2 > k^2$. This implies that all eigenvalues are obtained from Eqs. (21) and (24). The smallest positive eigenvalue was calculated by Taylor [1], k being treated as a continuous variable. It is given by $\mu_0 a \approx 3.11$ (or equivalently, $\Theta_0 = 1.56$), and it arises for $|m| = 1$, and for $|k| = k_0$ with $k_0 a \approx 1.25$. If the discrete nature of the variable k is taken into account, both μ_0 and k_0 depend on the aspect ratio, but are near the values given by Taylor because the aspect ratio is large.

The maximum pinch ratio Θ_0 is thus independent of the external magnetic field. It is attained at some finite H or only approached for large $|H|$, depending on whether f_{1k_0} vanishes or not. The latter case (the external magnetic field resonates with the eigenvalue μ_0) is exceptional because it is excluded if the length of the cylinder is not a multiple of $2\pi/k_0$ (it is also excluded if the boundary is a magnetic surface or if the external magnetic field has plane symmetry).

3.5 Summary

To summarize our results for a circular cylinder, we state that the magnetic field of a relaxed state is a solution of the Euler equation with some $|\mu| \leq \mu_0$ depending on H and ψ. If $\mu < \mu_0$, only those Fourier components a_{km} are present which are imposed by the external field. If $f_{1k_0} = 0$ (which is the general case), then $\mu = \mu_0$ is possible, and implies $a_{\pm 1 k_0} \neq 0$; if $f_{1k_0} \neq 0$, then $\mu = \mu_0$ is excluded, but $a_{\pm 1 k_0} \neq 0$ for all $\mu < \mu_0$.

The helicity has been used to derive these results, but is not needed to determine relaxed states. Using the pinch ratio instead is more convenient for two reasons: Firstly, it appears more explicitly in the Euler equation; secondly, it is easier to measure. Accordingly, showing that a given magnetic field represents a relaxed state merely requires verifying that the field is force-free with a constant ratio of the magnetic field and current density, and that the magnitude of this ratio does not exceed μ_0. It should be noted, however, that the pinch ratio, unlike the helicity, is not conserved during relaxation. Hence the latter is needed for predicting what relaxed state arises from a given initial state.

4 Extrap

If the external field has plane symmetry (i.e. is independent of z), then $f_{km} = 0$ for $k \neq 0$. The relaxed state then has the same symmetry if $|\Theta| < \Theta_0$, but a term proportional to $\cos(\theta + k_0 z)$ is superimposed if $|\Theta| = \Theta_0$.

To determine relaxed states with plane symmetry, one can, instead of Fourier-decomposing, determine the axial field B_z as that solution of the linear elliptic equation

$$\nabla^2 B_z + \mu^2 B_z = 0 \tag{37}$$

which satisfies the boundary condition

$$\partial B_z / \partial \theta = \mu a B_n, \tag{38}$$

and which is normalized so as to yield the prescribed axial flux ψ. The magnetic field vector is then given by

$$\mathbf{B} = \frac{1}{\mu} \nabla B_z \times \nabla z + B_z \nabla z. \tag{39}$$

Its axial component is constant on the magnetic surfaces in this case (it is proportional to the flux function, and Eq. (37) is the familiar Grad-Shafranov equation).

We now discuss in detail the even more special case of an external octupole field, i.e. the field produced by four axial currents which are symmetrically placed at a radius much larger than the wall radius. We thus put $f_{km} = 0$ unless $k = 0$ and $m = \pm 4$, and assume that f_{04} is real (this implies $f_{04} = f_{0-4}$). The normal component of the magnetic field at the boundary is then

$$B_r(a, \theta) = \delta \frac{\psi}{\pi a^2} \cos(4\theta), \tag{40}$$

where the nondimensional parameter δ has been chosen as the ratio of the amplitude of the modulation and the average axial field (this modulation parameter remains defined, and the relations of this section are still approximately valid, if the modulation is not exactly sinusoidal).

If $|\Theta| < \Theta_0$, the relaxed state has plane symmetry. Its axial magnetic field is then given by

$$B_z = \frac{\mu\psi}{2\pi a}\left[\frac{J_0(|\mu|r)}{J_1(|\mu|a)} - \delta\frac{J_4(|\mu|r)}{J_4(|\mu|a)}\cos(4\theta)\right]. \tag{41}$$

If the rod current vanishes, we have $\delta = 0$, the boundary is a magnetic surface, and the Bessel function model of the reversed field pinch (RFP) obtains. In this case, field reversal obtains (the axial field at the surface is opposite to that at the center) if the quantity $|\mu|a$ exceeds the smallest positive zero of the Bessel function J_0 or, equivalently, if $|\Theta| > \Theta_c \approx 1.2$.

While the RFP depends only on the single parameter Θ ($\mu a = 2\Theta$), our model of Extrap also depends on δ. To study the phenomenon of field reversal for $\delta \neq 0$, we assume $\mu \geq 0$ and $\delta \geq 0$ (due to obvious symmetries there is no loss of generality). There are now two critical values $\Theta_\pm(\delta)$, one for field reversal somewhere (partial reversal) and one for field reversal along the entire boundary (total reversal). These two values are obtained by inverting the two relations

$$F_\pm(\Theta,\delta) = 2J_0(2\Theta) \pm \delta J_1(2\Theta) = 0; \tag{42}$$

they coalesce at Θ_c for $\delta = 0$. No field reversal obtains if $F_- > 0$, partial reversal obtains if $F_- < 0$ and $F_+ > 0$, and total reversal obtains if $F_+ < 0$. Figure 1 shows the two curves $F_\pm = 0$ in the (Θ,δ)-plane. It also shows the curve on which the four x-points of the poloidal magnetic field ($\nabla B_z = 0$) are at the boundary. Since this curve does not intersect the curve $F_+ = 0$ for $|\Theta| \leq \Theta_0$, total field reversal is impossible if the x-points are in the plasma. In other words, field reversal is suppressed at least near parts of the boundary if the octupole field is sufficiently strong to generate x-points within the vessel.

5 Summary

We have extended the theory of relaxed toroidal plasmas to external magnetic fields whose field lines intersect the wall ($B_n \not\equiv 0$). As in the case $B_n \equiv 0$, the magnetic field is force-free, with a constant ratio of the current density and magnetic field whose magnitude has the smallest possible value yielding the given flux and helicity. Applying the general theory to a large-aspect-ratio torus with circular cross section, we have shown that the maximum pinch ratio (or, in other words, the phenomenon of current saturation) is independent of the external field. We have explicitly determined the relaxed state in an external octupole field modeling the Extrap experiment. In this case, field reversal is inhibited near parts of the boundary if the octupole is strong enough to generate magnetic x-points within the vessel.

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Appendix A: Properties of helicity

First, we show that the helicity H is invariant under gauge transformations, provided the vector potentials are single-valued. If $\mathbf{A}$ is replaced by $\mathbf{A}' = \mathbf{A} + \nabla\chi$, and $\mathbf{A}_0$ is replaced by $\mathbf{A}_0' = \mathbf{A}_0 + \nabla\chi_0$, then H is replaced by $H' = H + \Delta H$ and the change of the helicity is

$$\Delta H = \int_T d^3\tau(\mathbf{B} - \mathbf{B}_0) \cdot \nabla(\chi + \chi_0) = \int_T d^3\tau \mathrm{div}[(\chi + \chi_0)(\mathbf{B} - \mathbf{B}_0)]. \quad \text{(A1)}$$

Since χ and χ_0 may be multi-valued, Gauss' theorem can be applied only if the torus is made singly connected by cutting it with the surface S_s. The result is

$$\Delta H = \int_{S_T} d^2\sigma(\chi + \chi_0)\mathbf{n} \cdot (\mathbf{B} - \mathbf{B}_0) + \delta_l(\chi + \chi_0) \int_{S_s} d^2\sigma \mathbf{n} \cdot (\mathbf{B} - \mathbf{B}_0). \quad \text{(A2)}$$

The integrand of the first term vanishes because $\mathbf{B}$ and $\mathbf{B}_0$ have the same normal component, and the second term integrates to zero because $\mathbf{B}$ and $\mathbf{B}_0$ have the same flux.

Second, we show that H is conserved if the plasma is perfectly conducting. Since B_n and ψ are conserved, $\mathbf{B}_0$ is conserved, too, and $\mathbf{A}_0$ can be chosen time-independent. The time derivative of H is then

$$\dot{H} = \int_T d^3\tau[\dot{\mathbf{A}} \cdot (\mathbf{B} - \mathbf{B}_0) + (\mathbf{A} + \mathbf{A}_0) \cdot \dot{\mathbf{B}}]. \tag{A3}$$

Using the equations for the various time-derivatives, we write this as

$$\dot{H} = \int_T d^3\tau[-(\mathbf{B} - \mathbf{B}_0) \cdot (\mathbf{E} + \nabla\Phi) - (\mathbf{A} + \mathbf{A}_0) \cdot \mathrm{curl}\mathbf{E}] \tag{A4}$$

or, equivalently,

$$\dot{H} = \int_T d^3\tau\{-2\mathbf{B} \cdot \mathbf{E} - \mathrm{div}[\Phi(\mathbf{B} - \mathbf{B}_0) + \mathbf{E} \times (\mathbf{A} + \mathbf{A}_0)]\}. \tag{A5}$$

Applying Gauss' theorem now yields

$$\dot{H} = -2\int_T d^3\tau\mathbf{B} \cdot \mathbf{E} - \int_{S_T} d^2\sigma\mathbf{n} \cdot [\Phi(\mathbf{B} - \mathbf{B}_0) + (\mathbf{A} + \mathbf{A}_0) \times \mathbf{E}]. \tag{A6}$$

The integrand in the volume term vanishes in an ideal plasma because the fields and the mass velocity $\mathbf{U}$ are related by $\mathbf{E} + \mathbf{U} \times \mathbf{B} = 0$, and the integrand in the surface term vanishes because of our boundary conditions.

Appendix B: Boundary terms in the energy variation

First, the constraint of fixed normal component B_n,

$$\mathbf{n} \cdot (\mathrm{curl}\delta\mathbf{A}) = 0, \tag{B1}$$

is equivalent to the existence of a scalar function a such that

$$\mathbf{n} \times \delta\mathbf{A} = \mathbf{n} \times \nabla a \tag{B2}$$

at the boundary. Second, the constraint of fixed flux,

$$\int_{S_s} d^2\sigma\mathbf{n} \cdot \mathrm{curl}\delta\mathbf{A} = \int_{C_s} \mathbf{dl} \cdot \delta\mathbf{A} = 0, \tag{B3}$$

implies $\delta_s a = 0$. Third, the single-valuedness of $\delta\mathbf{A}$ implies that a has a single-valued surface gradient. Hence $\delta_l a$ is path-independent.

The boundary term is

$$\delta W_B = \int_{S_T} d^2\sigma \mathbf{n} \cdot (\delta \mathbf{A} \times \mathbf{C}) \tag{B4}$$

or, with $\delta \mathbf{A}$ expressed in terms of a,

$$\delta W_B = \int_{S_T} d^2\sigma \mathbf{n} \cdot (\nabla a \times \mathbf{C}) = \int_{S_T} d^2\sigma \mathbf{n} \cdot [\mathrm{curl}(a\mathbf{C}) - a\mathrm{curl}\mathbf{C}]. \tag{B5}$$

Here the second term vanishes because the Euler equation implies

$$\mathrm{curl}\mathbf{C} = \frac{1}{2}\mu(\mathbf{B} - \mathbf{B}_0), \tag{B6}$$

and because $\mathbf{B}$ and $\mathbf{B}_0$ have the same normal components. To apply Stokes' theorem to the first term, S_T is cut along both C_s and C_l to become a singly connected surface. The result is

$$\delta W_B = \oint_{C_s} \mathbf{dl} \cdot (\delta_l a\mathbf{C}) + \oint_{C_l} \mathbf{dl} \cdot (\delta_s a\mathbf{C}). \tag{B7}$$

Here the second term vanishes because $\delta_s a = 0$. Since $\delta_l a$ is path-independent, the first term is

$$\delta W_B = \delta_l a \int_{C_s} \mathbf{dl} \cdot \mathbf{C}. \tag{B8}$$

Again applying Stokes' theorem yields

$$\delta W_B = \delta_l a \int_{S_s} d^2\sigma \mathbf{n} \cdot \mathrm{curl}\mathbf{C}. \tag{B9}$$

The above equation for $\mathbf{C}$ now implies

$$\delta W_B = \frac{1}{2}\mu\delta_l a \int_{S_s} d^2\sigma \mathbf{n} \cdot (\mathbf{B} - \mathbf{B}_0). \tag{B10}$$

This vanishes because $\mathbf{B}$ and $\mathbf{B}_0$ have the same toroidal flux.

Appendix C: Energies of different stationary states

We wish to show the following: If two magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfy the Euler equations

$$\mathrm{curl}\mathbf{B}_1 = \mu_1 \mathbf{B}_1, \quad \mathrm{curl}\mathbf{B}_2 = \mu_2 \mathbf{B}_2 \tag{C1}$$

in T, have the same total helicities H_1 and H_2, the same fluxes ψ_1 and ψ_2 through the cut surface S_s, and the same normal components at the boundary S_T, then it follows that

$$(\mu_2^2 - \mu_1^2)(E_2 - E_1) > 0, \tag{C2}$$

and hence that $E_1 < E_2$ if $|\mu_1| < |\mu_2|$.

Equation (C2) follows from the identity

$$\int_T d^3\tau f = 0, \quad f = \frac{1}{2}(\mu_1 + \mu_2)(|\mathbf{B}|_2^2 - |\mathbf{B}|_1^2) + \frac{1}{2}(\mu_1 - \mu_2)|\mathbf{B}_2 - \mathbf{B}_1|^2 \tag{C3}$$

upon multiplication by $\mu_1 - \mu_2$. To prove it, we first write the condition $H_2 - H_1 = 0$ as

$$\int_T d^3\tau[(\mathbf{A}_1 + \mathbf{A}_0) \cdot (\mathbf{B}_1 - \mathbf{B}_0) - (\mathbf{A}_2 + \mathbf{A}_0) \cdot (\mathbf{B}_2 - \mathbf{B}_0)] = 0 \tag{C4}$$

or, equivalently,

$$\int_T d^3\tau\{\mathbf{A}_1 \cdot \mathbf{B}_1 - \mathbf{A}_2 \cdot \mathbf{B}_2 + \mathrm{div}[\mathbf{A}_0 \times (\mathbf{A}_2 - \mathbf{A}_1)]\} = 0. \tag{C5}$$

Since $\mathbf{B}_1$ and $\mathbf{B}_2$ have the same normal components at S_T, a gauge can be chosen such that $\mathbf{n} \times (\mathbf{A}_2 - \mathbf{A}_1) = 0$ at S_T. Then the second term in the integrand integrates to zero after applying Gauss' theorem. Hence

$$\int_T d^3\tau(\mathbf{A}_1 \cdot \mathbf{B}_1 - \mathbf{A}_2 \cdot \mathbf{B}_2) = 0. \tag{C6}$$

Second, we use some simple algebra to write

$$f = (\mathbf{B}_2 - \mathbf{B}_1) \cdot (\mu_2\mathbf{B}_1 + \mu_1\mathbf{B}_2) \tag{C7}$$

for the integrand in the identity which we want to prove. The Euler equations now imply

$$f = \mu_1\mu_2(\mathbf{A}_2 - \mathbf{A}_1) \cdot (\mathbf{B}_1 + \mathbf{B}_2) + \mathrm{div}[(\mathbf{A}_2 - \mathbf{A}_1) \times (\mu_2\mathbf{B}_1 + \mu_1\mathbf{B}_2)]. \tag{C8}$$

Upon applying Gauss' theorem, the second term integrates to zero in our present gauge. Hence

$$\int_T d^3\tau f = \mu_1\mu_2 \int_T d^3\tau(\mathbf{A}_2 - \mathbf{A}_1) \cdot (\mathbf{B}_1 + \mathbf{B}_2) \tag{C9}$$

or, equivalently,

$$\int_T d^3\tau f = \mu_1\mu_2 \int_T d^3\tau[(\mathbf{A}_1 \cdot \mathbf{B}_1 - \mathbf{A}_2 \cdot \mathbf{B}_2) + \mathrm{div}(\mathbf{A}_1 \times \mathbf{A}_2)]. \tag{C10}$$

Here the first term vanishes because $H_1 = H_2$, and the second term vanishes in our present gauge. This proves our identity.

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Figure caption

Figure 1: Regions of different field reversal properties in the upper right quadrant of the parameter plane. The two solid curves separate the following three regions: No reversal obtains for small pinch ratios, total reversal obtains for large pinch ratios and small modulation parameters, and partial reversal obtains in the intermediate regime. In the region below the dashed curve there are no x-points within the vessel. The vertical dashed line marks the maximum value of the pinch ratio. The figure is extended into the entire parameter plane by using the symmetry with respect to both coordinate axes.

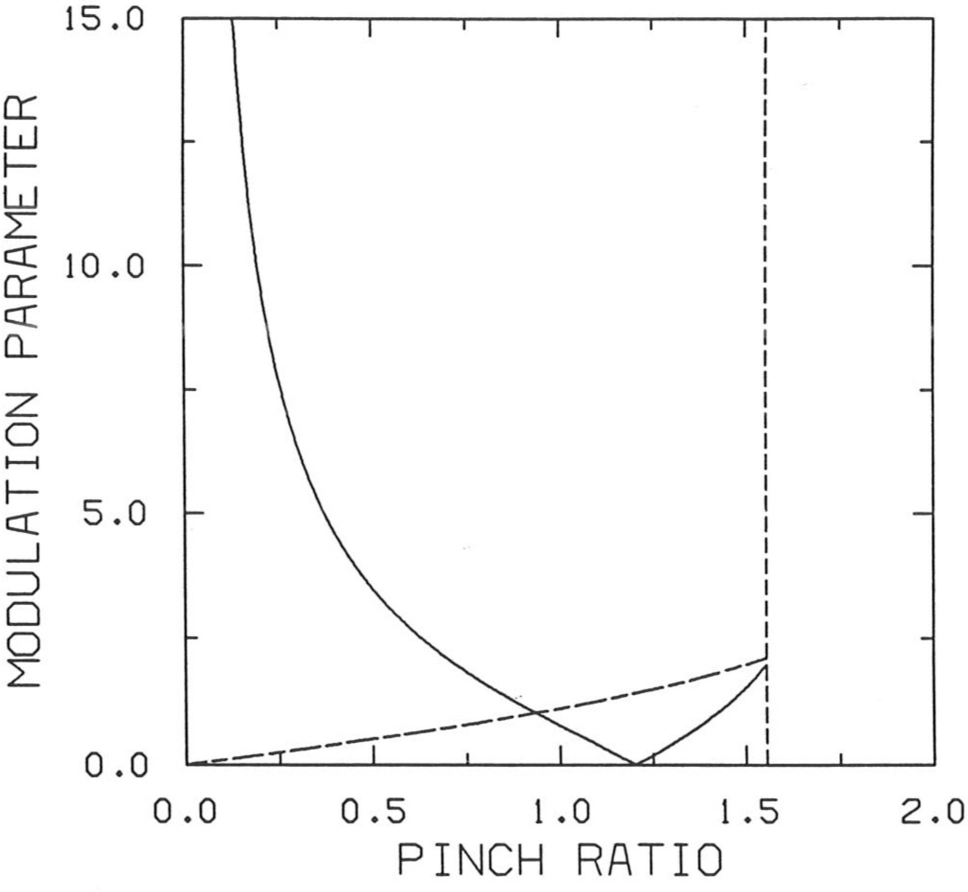

Fig. 1