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Negative-energy Modes in Collisionless Kinetic Theories and their Possible Relation to Nonlinear Instabilities

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Abstract

It might be necessary to take nonlinear instabilities into account in order to explain anomalous transport. Such instabilities can exist even for arbitrarily small initial amplitudes if the system possesses linear negative-energy modes, as was shown already in 1925 by Cherry [1]. Since the class of equilibria allowing negative-energy perturbations is much larger than the class of equilibria that are linearly unstable without dissipation, the search for negative-energy modes seems to be rather important. After reformulation and generalization of Cherry's oscillator example, which allows a simple physical interpretation of the nonlinear instabilities and shows the relation to continuum theories such as the Maxwell-Vlasov theory, the question is discussed how to obtain energy expressions for various linearized theories. The known energy expressions are in general rather impractical. Two new methods are described that yield energy expressions which can be used similarly to the potential energy δW in ideal MHD. The simpler one allows the energy of the linearized Maxwell-Vlasov theory to be obtained; the more complicated one can be applied to any Maxwell-collisionless kinetic theory and even yields the whole energy-momentum and angular momentum tensors. The latter method is used to treat the collisionless drift kinetic theory. Only the main points of this treatment can be presented here. Some general results are derived for the Maxwell-Vlasov theory. This is done in the form of examples. For one example a comparison between the Maxwell-Vlasov and the Maxwell-drift kinetic theories is made.

Introduction

Usually the question is asked whether a system is linearly unstable. If it is, an attempt is sometimes made, too, to find out what the nonlinear development of the linearly unstable modes might be. There also exists, however, the possibility of a system being linearly absolutely stable but nonlinearly unstable. (For references on nonlinear explosive instabilities see J. Weiland and H. Wilhelmsson [2] and H. Wilhelmsson [3].) An impressive example, a numerical study of collisional drift-wave turbulence, was recently published by B.Scott [4] in which he demonstrated self-sustained turbulence of a linearly stable plasma slab resembling the plasma edge regions of tokamaks. His main results are that all of the features of nonlinear mode structure are determined by nonlinear processes, divesting linear stability criteria of their relevance to that structure, or its amplitude; contrary to today's common perception in tokomak physics that drift-wave turbulence cannot be the agent behind energy transport in tokamak edge regions, many important features of experimentally observed tokamak edge fluctuations could be reproduced, most particular the amplitude ordering $e\tilde{\phi}/T > \tilde{n}/n > \tilde{T}/T$; the transport is found to be gyro-reduced-Bohm-like. In Scott's study a certain threshold amplitude is needed. It can, however, even happen that the nonlinear instability occurs with arbitrarily small initial perturbations. This was shown for the first time in 1925 by Cherry [1]. He presented a simple example demonstrating that linear stability analysis will in general not be sufficient for finding out whether a system is stable or not with respect to small-amplitude perturbations (see also [5]). His example consisted of two nonlinearly coupled oscillators, one possessing positive energy, the other negative energy, and the frequency of the one oscillator was twice that of the other. The exact two-parameter solution set he had found exhibited explosive instability after a finite time.

What is meant by positive and negative mode energy follows from the definitions below:

- energy of the equilibrium: $\mathcal{E}^{(0)}$;
- frame of reference: $\mathcal{E}^{(0)}$ minimum;

- energy of the perturbed system dynamically accessible from the equilibrium: $\mathcal{E}^{(0)} + \delta\mathcal{E}$;
- $\delta\mathcal{E}$: mode energy;
- $\delta\mathcal{E} < 0$: equilibrium possesses free energy.

A question is then: What does $\delta\mathcal{E}$ tell us? The following list describes situations relating to $\delta\mathcal{E}$:

- $\delta\mathcal{E} > 0$: stable;
- $\delta\mathcal{E} = 0$: necessary for linear instability;
- $\delta\mathcal{E} < 0$: amplitudes of perturbations with $\delta\mathcal{E} < 0$ grow if energy is removed from them by
    1. dissipation,
    2. coupling to positive-energy waves in the same system, which means nonlinear instability.

In order to get some insight into the mechanisms responsible for the nonlinear instabilities described, Cherry's oscillators are discussed in Sec. 1 in some detail; in particular, reformulation and generalization corresponding to three-wave interaction and allowing a simple physical interpretation are presented. In Sec. 2 another simple model is investigated, that of a charged particle on a hill with superimposed magnetic field. This example serves to test a multiple time scale formalism which could also be applied to continuous systems and should generally lead to explosive or non-explosive instability, if there exist negative-energy modes which can resonantly interact with positive-energy modes. Section 3 reviews a number of existing expressions for the energy of linear perturbations and discusses the restrictions and difficulties connected with them. In the past, investigations of nonlinear electrostatic instabilities in homogeneous plasmas were performed on the basis of one of these expressions [6],[7],[8],[9],[10]. Section 4 gives an elegant derivation of the energy of linear perturbations for the Vlasov-Maxwell theory based on the Lie group formalism. Section 5 sketches a more complicated derivation of the energy which, however, allows the whole energy-momentum and angular momentum

tensors for general Maxwell-collisionless kinetic theories to be obtained. For details the reader is referred to Ref. [11]. Sections 4 and 5 also give some first applications.
The class of equilibria allowing negative-energy perturbations is much larger than the class of equilibria that are linearly unstable without dissipation, and anomalous transport might well be caused by nonlinear instabilities relating to such negative-energy perturbations. As regards these perturbations the energy expressions mentioned above can be used similarly to the potential energy in the energy principle of ideal MHD.

1 Cherry Oscillators

Cherry's Hamiltonian is given by

$$\begin{aligned} H \quad &= \quad - \tfrac{1}{2}\,\omega_1\,(p_1^2 + q_1^2) \;+\; \tfrac{1}{2}\,\omega_2\,(p_2^2 + q_2^2) \\ &\quad + \tfrac{\alpha}{2}\,(2q_1p_1p_2 \;-\; q_2(q_1^2 - p_1^2))\;. \end{aligned}$$

$\alpha = 0$ means two uncoupled oscillators of frequencies $\omega_1 > 0$ and $\omega_2 > 0$ which possess negative and positive energy, respectively.
If $\omega_2 = 2\,\omega_1$, one has a third-order resonance. Cherry found for this case the following exact two-parameter solution set:

$$q_1 = \frac{\sqrt{2}}{\epsilon - \alpha\, t}\, sin(\omega_1 t + \gamma)\;,$$

$$p_1 = \frac{-\sqrt{2}}{\epsilon - \alpha\, t}\, cos(\omega_1 t + \gamma)\;,$$

$$q_2 = \frac{-1}{\epsilon - \alpha\, t}\, sin(2\omega_1 t + 2\gamma)\;,$$

$$p_2 = \frac{-1}{\epsilon - \alpha\, t}\, cos(2\omega_1 t + 2\gamma)\;,$$

where ϵ and γ are constants.
These relations show explosive instability for any $\alpha \neq 0$, whereas the linearized theory gives only stable oscillations. There is also no threshold amplitude. Small initial amplitudes only mean that it takes a long time for the explosion to occur. In a continuum theory, such as the Vlasov-Maxwell theory, the assumed resonance corresponds to the conservation law

$$\omega_1 + \omega_2 + \omega_3 = 0$$

for a three-wave interaction. It is therefore of interest to have a formulation and an example which are closer to the structure of a three-wave interaction. To this end, it is advantageous to introduce complex quantities (see [12]) by

$$\xi = p + i\,q \quad , \quad \xi^* = p - i\,q.$$

Canonical variables are

$$\xi^* = P \quad , \quad \xi/2i = Q,$$

and Cherry's Hamiltonian becomes

$$H = -\frac{1}{2}\,\omega_1\,\xi_1^*\,\xi_1 + \frac{1}{2}\,\omega_2\,\xi_2^*\,\xi_2 + \frac{\alpha}{4i}\,(\xi_1^2\,\xi_2 - \xi_1^{*2}\,\xi_2^*).$$

In quantum mechanical language $\xi^*/\sqrt{2}$ means creation of quanta and $\xi/\sqrt{2}$ annihilation of quanta. The nonlinear terms then have the property of creating or annihilating two quanta of oscillator 1 and one quantum of oscillator 2 without changing the energy. Thus, the coupling terms do just what was said in the Introduction to be the mechanism responsible for nonlinear instability. With the new formulation we can write down the envisaged generalization to three coupled oscillators in the form of the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{3}\omega_k\,\xi_k^*\,\xi_k + \frac{1}{2}\,\alpha\,\xi_1\xi_2\xi_3 + \frac{1}{2}\,\alpha^*\,\xi_1^*\xi_2^*\xi_3^*.$$

The frequencies ω_k are assumed to satisfy the three-wave conservation law

$$\sum_{k=1}^{3}\omega_k = 0.$$

The equations of motion are

$$\dot{\xi}_k = i\,\omega_k\,\xi_k + i\,\alpha^*\,\xi_1^*\xi_2^*\xi_3^*\,/\,\xi_k^*.$$

For a set of special solutions we make the ansatz

$$\xi_k(t) = a(t)\,e^{i\omega_k t + i\varphi_k} \quad , \quad \sum_{k=1}^{3}\varphi_k = 0.$$

The resulting equation for $a(t)$ is independent of k and is given by

$$\dot{a} = i\,\alpha^*\,a^{*2}.$$

Its solution is

$$a = \gamma\,b(t) \quad , \quad \gamma = \left(\frac{i\alpha^*}{|\alpha|^4}\right)^{1/3} \quad , \quad b^* = b,$$

$$\dot{b} = b^2,$$

$$b = \frac{1}{\hat{\epsilon} - t}.$$

$\hat{\epsilon}$ is a constant of integration. This yields the following three-parameter solution set

$$\xi_k = \left(\frac{i\alpha^*}{|\alpha|}\right)^{1/3} \frac{1}{\epsilon - |\alpha| t}\,e^{i\omega_k t + i\varphi_k} \quad , \quad \sum_{i=1}^{3}\varphi_k = 0$$

$$\epsilon = \hat{\epsilon}\,/\,|\alpha|.$$

These solutions correspond to Cherry's two-parameter solution set .
It is possible to obtain also the complete solution for the three-oscillator case [12]. It shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behaviour. This is true of the resonant case. The non-resonant oscillators can sometimes also become explosively unstable, but the initial amplitudes must not be infinitesimally small.

2 Particle on a Hill

In this section a simple but characteristic physical example is treated which consists of a charged particle on a hill with superimposed magnetic field. To lowest order the hill is axisymmetric and parabolic and is described by the potential

$$V(x,y) = -\frac{1}{2}(x^2+y^2).$$

With a constant magnetic field $\mathbf{B}$ in the z-direction, the equations of motion and their solutions are

$$\ddot{x} = x + B\dot{y}, \qquad \ddot{y} = y - B\dot{x},$$

$$x + iy = \zeta \longrightarrow \ddot{\zeta} + iB\dot{\zeta} = \zeta,$$

$$\zeta \propto e^{-i\omega t} \longrightarrow -\omega^2 + B\omega = 1,$$

$$\omega_\pm = \frac{1}{2}B \pm \sqrt{\frac{B^2}{4} - 1}.$$

Stability exists for $B > 2$.
ω_+ means: mainly gyromotion,
ω_- means: mainly drift motion.
The energy of the motions corresponds to the mode energy and is given by

$$\begin{aligned}
\delta\mathcal{E} &= \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}(x^2+y^2) \\
&= \tfrac{1}{2}|\dot{\zeta}|^2 - \tfrac{1}{2}|\zeta|^2 \\
&= \tfrac{1}{2}|\zeta_0|^2(\omega_\pm^2 - 1) \\
&= |\zeta_0|^2\left(\tfrac{B^2}{4} - 1 \pm \tfrac{B}{2}\sqrt{\tfrac{B^2}{4}-1}\right).
\end{aligned}$$

This shows that

$$\delta\mathcal{E}_+ > 0, \qquad \delta\mathcal{E}_- < 0.$$

When friction is added, one has

$$\ddot{\zeta} + (iB+\gamma)\dot{\zeta} = \zeta.$$

It holds that for all B $\Im\omega > 0$, and hence instability, is possible.
When a potential x^3 is added positive-energy modes and negative-energy modes are coupled. The coupling is, however, only partly of the Cherry

type. Yet numerical results exhibit instability again for very small initial x, if there is resonance given by $\omega_+ = 2\omega_-$. This can also be shown via a multiple time scale formalism for the initial phase until the nonlinear term attains the same order as the linear ones.
Finite Larmor radius stabilization is similar in structure to the particle-on-a-hill problem. Negative δW modes can be stable but become unstable with friction.

3 Expressions Used for $\delta\mathcal{E}$ in the Past

For homogeneous plasmas the best known energy expression is

$$\delta\mathcal{E} = \frac{1}{8\pi}\underline{E}^*(\underline{k},\omega)\cdot\frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2\underline{\underline{\epsilon}}_H\cdot\underline{E}(\underline{k},\omega)),$$

where $\omega = \omega(\underline{k})$,
$\underline{\underline{\epsilon}}_H$: hermitian part of $\underline{\underline{\epsilon}}$,
$\underline{\underline{\epsilon}}_A$: anti-hermitian part of $\underline{\underline{\epsilon}}$, must be negligible.
This kind of energy expression was used in the past for discussing nonlinear instabilities as mentioned in the Introduction. It can also be extended to inhomogeneous systems (see [13]).
Since $\underline{E}(\underline{k},\omega),\omega(\underline{k})$ or $\underline{E}(\underline{x},\omega),\omega$ have to be known, the use of these expressions is in most of the cases of interest rather difficult, if not impossible. A different kind of energy expression is known for 1-d homogeneous plasmas and electrostatic perturbations [14],[15]:

$$\delta\mathcal{E} = \int\frac{E_1^2}{8\pi}dx - \sum_\nu\frac{m_\nu}{2}\int g_\nu^2 v\frac{\partial f_{\nu 0}}{\partial v}dvdx,$$

$$f_{\nu 1} = g_\nu\frac{\partial f_{\nu 0}}{\partial v},$$

$$\frac{\partial g_\nu}{\partial t} + v\frac{\partial g_\nu}{\partial x} = -\frac{e_\nu}{m_\nu}E_1,$$

$$\frac{\partial E_1}{\partial x} = 4\pi\sum_\nu e_\nu\int g_\nu\frac{\partial f_{\nu 0}}{\partial v}dv.$$

A simple application of this expression is the following: For any g_ν^2 one can choose the distribution of the signs in g_ν such that $\int g_\nu\partial f_{\nu 0}/\partial v dv = 0$ and hence $E_1 = 0$. This yields the conclusion

$$\delta\mathcal{E} < 0 \;\text{ possible if }\; v\frac{\partial f_{\nu 0}}{\partial v} > 0$$

for at least one species ν in a small interval of v in a frame of reference with $\mathcal{E}^{(0)}$ = minimum. Therefore

$$f_{\nu 0} \neq \text{monotonous function of } v^2$$

in this frame guarantees the existence of negative-energy perturbations. This result is not easily obtained with the first kind of energy expression. The subject of the following sections is:

- generalization of the energy expression for 1-d homogeneous plasmas with electrostatic perturbations to general 3-d inhomogeneous Maxwell-Vlasov and collisionless drift kinetic equilibria with general electromagnetic perturbations;
- generalization of the $v\frac{\partial f_{\nu 0}}{\partial v} > 0$ criterion.

4 Maxwell-Vlasov Theory

The energy for the nonlinear Maxwell-Vlasov theory is well known and simple. Obtaining from this expression the one for the linearized theory is, however, not a trivial matter. The problem is that $\delta\mathcal{E}$ is of second order in the perturbations, especially

$$\delta\mathcal{E}_{kin} = \int \frac{m}{2} v^2 f^{(2)} d^3v d^3x,$$

and so $f^{(2)}$ has to be expressed in terms of first order quantities. The methods of achieving this are Hamiltonian and Lagrangian ones.
Two methods are described here(see [15],[11]):

- canonical transformation method based on the Lie group formalism;
- modified Hamilton-Jacobi theory,
  also applicable to collisionless kinetic guiding center theories.

In this section the method of Lie-type canonical transformations is described. (In Appendix A a brief derivation of Lie-type canonical transformations is presented.)
The energy of the nonlinear theory is given by

$$\mathcal{E} = \int (H - e\Phi) f d^3x d^3p + \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{B}^2) d^3x.$$

We now introduce

$$g(\mathbf{x}, \mathbf{p}, t) \ : \ \text{generating function for Lie-type canonical transformations to unperturbed orbits.}$$

This yields the perturbed distribution function

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, t) &= e^{[g,\cdot]} f^0(\mathbf{x}, \mathbf{p}) \\ &= f^0 + [g, f^0] + \frac{1}{2}\left[g, [g, f^0]\right] + \cdots . \end{aligned}$$

g and the Hamiltonian H are needed up to second order:

$$g = g^{(1)} + g^{(2)} + \cdots ,$$

$$H = H_0 + H_1 + H_2 + \cdots .$$

The - not needed - equation for $g^{(1)}$ is

$$\dot{g}^{(1)} + [g^{(1)}, H_0] = H_1 .$$

We first obtain the first-order energy:

$$\begin{aligned} \mathcal{E}^{(1)} = & \\ & \int \left\{ -\frac{e}{c}\mathbf{A}_1 \cdot \frac{\partial H_0}{\partial \mathbf{p}} f^{(0)} + (H_0 - e\Phi_0) f^{(1)} \right\} d^3x d^3p \\ & + \frac{1}{4\pi} \int (\mathbf{E}_0 \cdot \mathbf{E}_1 + \mathbf{B}_0 \cdot \mathbf{B}_1) d^3x . \end{aligned}$$

With

$$\mathbf{E}_0 = -\nabla \Phi_0, \quad \mathbf{B}_1 = \mathrm{curl}\mathbf{A}_1,$$

it follows that

$$\begin{aligned} \frac{1}{4\pi} \int (\mathbf{E}_0 \cdot \mathbf{E}_1 + \mathbf{B}_0 \cdot \mathbf{B}_1) d^3x = & \\ \int (\Phi_0 \rho_1 + \frac{1}{c}\mathbf{j}_0 \cdot \mathbf{A}_1) d^3x . & \end{aligned}$$

Furthermore, one has

$$
\begin{aligned}
&\int H_0 f^{(1)} d^3x d^3p \\
&= \int H_0 [g^{(1)}, f^{(0)}] d^3x d^3p \\
&= -\int g^{(1)} [H_0, f^{(0)}] d^3x d^3p \\
&= 0.
\end{aligned}
$$

Since it holds that

$$
\rho_1 = \int e f^{(1)} d^3p, \quad \mathbf{j}_0 = \int e \frac{\partial H_0}{\partial \mathbf{p}} f^{(0)} d^3p,
$$

all terms cancel and hence it follows that

$$
\mathcal{E}^{(1)} = 0.
$$

We now find the second-order energy:

$$
\begin{aligned}
\mathcal{E}^{(2)} =& \\
&\int d^3x d^3p \Big\{ (H_0 - e\Phi_0) f^{(2)} \\
&+ \frac{\partial H_0}{\partial \mathbf{p}} \cdot (-\frac{e}{c}\mathbf{A}_1) f^{(1)} \\
&+ \frac{\partial H_0}{\partial \mathbf{p}} \cdot (-\frac{e}{c}\mathbf{A}_2) f^{(0)} \\
&+ \frac{1}{2} \frac{\partial^2 H_0}{\partial p_i \partial p_k} \frac{e^2}{c^2} A_{1i} A_{1k} f^{(0)} \Big\} \\
&+ \frac{1}{8\pi} \int (\mathbf{E}_1^2 + \mathbf{B}_1^2) d^3x \\
&+ \frac{1}{4\pi} \int (\mathbf{E}_0 \cdot \mathbf{E}_2 + \mathbf{B}_0 \cdot \mathbf{B}_2) d^3x.
\end{aligned}
$$

As in the first-order case one has

$$
\begin{aligned}
&\int d^3x d^3p \left\{ -e\Phi_0 f^{(2)} + \frac{\partial H_0}{\partial \mathbf{p}} \cdot (-\frac{e}{c}\mathbf{A}_2 f^{(0)} \right\} \\
&+ \frac{1}{4\pi} \int (\mathbf{E}_0 \cdot \mathbf{E}_2 + \mathbf{B}_0 \cdot \mathbf{B}_2) d^3x \\
&= 0.
\end{aligned}
$$

The second-order distribution function is

$$f^{(2)} = [g^{(2)}, f^{(0)}] + \frac{1}{2}\left[g^{(1)}, [g^{(1)}, f^{(0)}]\right].$$

Similar again to the first-order case, one obtains

$$\begin{aligned}
&\int H_0[g^{(2)}, f^{(0)}]d^3x d^3p \\
&= \; -\int g^{(2)}[H_0, f^{(0)}]d^3x d^3p \\
&= \; 0.
\end{aligned}$$

This yields the second-order energy expressed in terms of first-order quantities:

$$\begin{aligned}
\mathcal{E}^{(2)} = \\
&\int d^3x d^3p \left\{\frac{1}{2}H_0\left[g^{(1)}, [g^{(1)}, f^{(0)}]\right]\right. \\
&- \frac{e}{c}\mathbf{A}_1 \cdot \frac{\partial H_0}{\partial \mathbf{p}}[g^{(1)}, f^{(0)}] \\
&+ \left.\frac{1}{2}\frac{\partial^2 H_0}{\partial p_i \partial p_k}\frac{e^2}{c^2}A_{1i}A_{1k}f^{(0)}\right\} \\
&+ \frac{1}{8\pi}\int(\mathbf{E}_1^2 + \mathbf{B}_1^2)d^3x.
\end{aligned}$$

By partial integration one can replace

$$\frac{1}{2}H_0\left[g^{(1)}, [g^{(1)}, f^{(0)}]\right] \text{ by } \frac{1}{2}[H_0, g^{(1)}][g^{(1)}, f^{(0)}].$$

The expression for $\mathcal{E}^{(2)}$ is understood to imply summation over all particle species.

In the sense of initial conditions $g^{(1)}, \mathbf{A}_1, \dot{\mathbf{A}}_1$ can be chosen arbitrarily, whereas Φ_1 is restricted to satisfy

$$\Delta\Phi_1 = -4\pi\rho_1 = -4\pi\sum_\nu e_\nu \int[g_\nu^{(1)}, f_\nu^{(0)}]d^3v,$$

where ν denotes the particle species.

4.1 Examples

The following notation is used below:

$$g^{(1)} = g_\nu^{(1)} \longrightarrow g.$$

4.1.1 Homogeneous Plasma: $\mathbf{B}_0 = \mathbf{E}_0 = 0$, Electrostatic Perturbations

For this case one has

$$f^{(0)} = f^{(0)}(\mathbf{v}),$$

$$\Phi_0 = \mathbf{A}_0 = \mathbf{A}_1 = 0.$$

The minimizing perturbations are

$$\begin{aligned} g &= g(\mathbf{v},\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}, \\ \mathbf{E}_1 &= \mathbf{E}_1(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}. \end{aligned}$$

This leads to the following expression for the wave energy:

$$\begin{aligned} \mathcal{E}^{(2)} = & \\ & \frac{V}{2}\int d^3v \frac{f^{(0)}}{2m}\left\{k^2|g|^2 + (g^*\mathbf{k}\cdot\mathbf{v}\mathbf{k}\cdot\frac{\partial g}{\partial \mathbf{v}} + \text{c.c.})\right\} \\ & + \frac{V}{16\pi}|\mathbf{E}_1|^2, \end{aligned}$$

where V is the volume of a large periodicity box.
Integration by parts yields

$$\begin{aligned} \mathcal{E}^{(2)} = & \\ & -\frac{V}{2}\int d^3v \frac{1}{2m}|g|^2\mathbf{k}\cdot\mathbf{v}\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}} \\ & + \frac{V}{16\pi}|\mathbf{E}_1|^2. \end{aligned}$$

$\mathbf{E}_1 = 0$ is made possible by a proper choice of signs in $g(\mathbf{v},\mathbf{k})$. The frame of reference which has to be used is defined by

$$\sum_\nu \int \mathbf{v} f_\nu^{(0)} d^3v = 0.$$

This means that $\mathcal{E}^{(2)} < 0$ cannot be obtained by changing the center-of-mass velocity. The condition for the existence of negative-energy perturbations is then obviously

$$\mathbf{k} \cdot \mathbf{v} \mathbf{k} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} > 0$$

for at least one $\mathbf{k}, \mathbf{v}$ and one particle species.
(This result also holds with $\mathbf{A}_1$ minimizing $\mathcal{E}^{(2)}$.)

Equivalent to this condition is that $f^{(0)}$ ought not to be a monotonous function of v^2 in the center-of-mass frame, in which the equilibrium energy is minimum.

4.1.2 General Maxwell-Vlasov Equilibria - Localized Perturbations

We now assume localization of g to intervals

$$\triangle x,\ \triangle y,\ \triangle z\ \ll r_g.$$

Furthermore, perturbations inside these intervals are chosen to be $\propto e^{i\mathbf{k}\cdot\mathbf{x}}$ with

$$k_x \triangle x,\ k_y \triangle y,\ k_z \triangle z\ \gg 1.$$

This has the consequence that $\mathcal{E}^{(2)}$ is dominated by $\dfrac{\partial g}{\partial \mathbf{x}}$ and $\dfrac{\partial}{\partial \mathbf{x}} \cdot \dfrac{\partial g}{\partial \mathbf{v}}$. In this way we arrive at the same expression for $\mathcal{E}^{(2)}$ as in the first example.

4.1.3 General Maxwell-Vlasov Equilibria - General Perturbations

A conjecture by Morrison and Pfirsch [15] which is proved by Weitzner and Pfirsch [16] says that necessary and sufficient for the non-existence of negative-energy modes is that

$$f_\nu^{(0)}(\mathbf{x}, \mathbf{v}) = f_\nu^{(0)}(H_\nu^{(0)})$$

and

$$\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}} \leq 0.$$

This condition is closely related to the stability theorems of Newcomb [17] and Gardner[18].

4.1.4 Conclusions from the Examples

The problem is not whether an equilibrium possesses negative-energy modes or not, but:
What is the necessary localization for $\mathcal{E}^{(2)} < 0$ *?*
It is therefore of special interest to have an energy expression for the linearized Maxwell-drift kinetic theory. The next section describes how to obtain such an expression.

5 Maxwell-Drift Kinetic Theory

5.1 Modified Hamilton-Jacobi Approach

We begin with the following definitions:

- $H(p_i, q_i, t)$: Hamiltonian of particles;
- $p_1, \ldots, p_n, q_1, \ldots, q_n$: phase space;
- $(q_1, q_2, q_3) = (x_1, x_2, x_3) = \mathbf{x}$;
- $(p_1, p_2, p_3) = \mathbf{p}$;
- $n = 4$: needed for guiding center motion;
- $H^{(0)}(P_i, Q_i, t)$: reference Hamiltonian;
- $S(P_i, q_i, t)$: mixed variable generating function for canonical transformation

$$p_i, q_i \longrightarrow P_i, Q_i$$

$$p_i = \frac{\partial S}{\partial q_i} \quad , \quad Q_i = \frac{\partial S}{\partial P_i}.$$

The modified Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q_i}, q_i, t) = H_0(P_i, \frac{\partial S}{\partial P_i}, t),$$

and

$$S_0 = \sum P_i q_i$$

is the time-independent solution to $H = H_0$.
The original Hamilton-Jacobi theory is the special case

$$H_0 \equiv 0.$$

We shall later choose H_0 as the time-independent equilibrium Hamiltonian. As will be shown below a Lagrangian for the whole theory, irrespective of the special choice of H_0, is

$$\begin{aligned} L = & -\int dq\, dP\, \phi(P_i, q_i, t) \\ & \left\{ \frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q_i}, q_i, t) - H_0(P_i, \frac{\partial S}{\partial P_i}, t) \right\} \\ & + \frac{1}{8\pi} \int d^3x (\mathbf{E}^2 - \mathbf{B}^2) \end{aligned}$$

with

$$dq\, dP \equiv dq_1 \cdots dq_n\, dP_1 \cdots dP_n, \qquad d\hat{q} \equiv dq_4 \cdots dq_n.$$

Quantities to be varied are

$$\phi,\ S,\ \mathbf{A},\ \Phi.$$

The variational principle is

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

with

$$\delta\phi = \delta S = \delta \mathbf{A} = \delta \Phi = 0$$

at t_1, t_2 and some boundaries in q,P space. Gauge invariance requires

$$\begin{aligned} & H(p_i, q_i, t) = \\ & \quad \hat{H}(\mathbf{p} - \frac{e}{c}\mathbf{A}, \mathbf{E}, \mathbf{B}, \text{derivatives of}\, \mathbf{E}, \mathbf{B}, p_4, \ldots, p_n, q_4, \ldots, q_n) \\ & \quad + e\Phi. \end{aligned}$$

Lowest-order guiding center motion does not involve derivatives of $\mathbf{E}, \mathbf{B}$. In this case the Euler-Lagrange equations are as follows
variation with respect to ϕ:

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q_i}, q_i, t) = H_0(P_i, \frac{\partial S}{\partial P_i}, t),$$

variation with respect to S:

$$\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\phi\right)-\frac{\partial}{\partial P_i}\left(\frac{\partial H_0}{\partial Q_i}\phi\right)=0,$$

variation with respect to Φ:

$$\frac{1}{4\pi}\frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{E}=e\int\phi d\hat{q}dP+\frac{\partial}{\partial \mathbf{x}}\cdot\int\frac{\partial H}{\partial \mathbf{E}}\phi d\hat{q}dP,$$

variation with respect to $\mathbf{A}$:

$$\begin{aligned}&\frac{1}{4\pi}\left(\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}+curl\,\mathbf{B}\right)=\\&\quad\frac{e}{c}\int\frac{\partial H}{\partial \mathbf{p}}\phi d\hat{q}dP\\&\quad-\frac{1}{c}\frac{\partial}{\partial t}\int\frac{\partial H}{\partial \mathbf{E}}\phi d\hat{q}dP-curl\int\frac{\partial H}{\partial \mathbf{B}}\phi d\hat{q}dP\end{aligned}$$

with

$$\frac{\partial H}{\partial p_i}\equiv\left.\frac{\partial H(p_i,q_i,t)}{\partial p_i}\right|_{p_i=\frac{\partial S}{\partial q_i}}$$

and

$$\frac{\partial H_0}{\partial Q_i}\equiv\left.\frac{\partial H_0(P_i,Q_i,t)}{\partial Q_i}\right|_{Q_i=\frac{\partial S}{\partial P_i}}.$$

Proof of the correctness of the above Lagrangian follows from the properties of the density functions ϕ. In order to obtain these properties, we introduce a modified Van Vleck determinant [19],[20] defined by

$$\hat{\phi}=det\left\|\frac{\partial^2 S}{\partial q_i\partial P_k}\right\|,$$

which solves the mixed-variable continuity equation for ϕ. The general solution of this equation can then be written as

$$\phi(P_i,q_i,t)=\hat{\phi}\hat{f}(P_i,q_i,t)$$

with

$$\hat{f}(P_i,q_i,t)=f\left(\frac{\partial S}{\partial q_i},q_i,t\right)$$

or

$$\hat{f}(P_i, q_i, t) = f^{(0)}\left(P_i, \frac{\partial S}{\partial P_i}, t\right).$$

The function $f(p_i, q_i, t)$ solves the "Vlasov" equation

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{\partial H(p_i, q_i, t)}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \\
&= \frac{\partial f}{\partial t} - [H, f] = 0.
\end{aligned}$$

The other function $f^{(0)}(P_i, Q_i, t)$ solves the "Vlasov" equation for the reference system

$$\begin{aligned}
&\frac{\partial f^{(0)}}{\partial t} + \frac{\partial H_0(P_i, Q_i, t)}{\partial P_i}\frac{\partial f^{(0)}}{\partial Q_i} - \frac{\partial H_0}{\partial Q_i}\frac{\partial f^{(0)}}{\partial P_i} \\
&= \frac{\partial f^{(0)}}{\partial t} - [H_0, f^{(0)}] = 0.
\end{aligned}$$

These "Vlasov" equations show that the density function ϕ has the right properties, which proves the correctness of the above Lagrangian.

5.2 Energy Expressions

With a Lagrangian, it is straightforward to find an expression for the energy.

5.2.1 Nonlinear Theory

We first treat the nonlinear theory, which is simpler than the linearized one. With the help of Noether's theorem even the energy momentum tensor and the angular-momentum tensor can be obtained. A sketch of the derivation is found below. We present first the expression for the energy [11],[15] that follows from the energy momentum tensor:

$$\begin{aligned}
\mathcal{E} &= \int d^3x d\hat{q} dp\, f\left(H - e\Phi - \mathbf{E}\cdot\frac{\partial H}{\partial \mathbf{E}}\right) \\
&+ \frac{1}{8\pi}\int d^3x(\mathbf{E}^2 + \mathbf{B}^2).
\end{aligned}$$

For the derivation we introduce a notation which allows a simpler procedure than the one used up to now:

$$\tilde{q}_0, \cdots, \tilde{q}_n = ct, \mathbf{x}, q_4, \cdots, q_n,$$

$$\tilde{Q}_0, \cdots, \tilde{Q}_n = ct, \mathbf{x}, Q_4, \cdots, Q_n,$$

$$\tilde{p}_0, \cdots, \tilde{p}_n = p_0, \mathbf{p}, p_4, \cdots, p_n,$$

$$\tilde{P}_0, \cdots, \tilde{P}_n = P_0, \mathbf{P}, P_4, \cdots, P_n,$$

$$\mathcal{H}(\tilde{p}_i, \tilde{q}_i) = cp_0 + H(p_1, \cdots, p_n, q_1, \cdots, q_n, t),$$

$$\mathcal{H}^{(0)}(\tilde{P}_i, \tilde{Q}_i) = H_0(P_1, \cdots, P_n, Q_1, \cdots, Q_n, t),$$

$$\frac{\partial \phi}{\partial \tilde{P}_0} = \frac{\partial S}{\partial \tilde{P}_0} \equiv 0,$$

$$A_i \equiv 0 \quad \text{for } i > 3,$$

$$\mathcal{H} = \mathcal{H}(\tilde{p}_i - A_i e/c, F_{\mu\lambda}),$$

$$d\tilde{q}d\tilde{P} = d\tilde{q}_1 \cdots d\tilde{q}_n d\tilde{P}_1 \cdots d\tilde{P}_n = dqdP,$$

$$p_0 : \text{can be replaced by } \frac{\partial S}{\partial t}.$$

The Lagrangian is then

$$L = \\ -\int d\tilde{q}d\tilde{P}\phi\left(\mathcal{H}(\frac{\partial S}{\partial \tilde{q}_i}, \tilde{q}_i) - \mathcal{H}^{(0)}(\tilde{P}_i, \frac{\partial S}{\partial \tilde{P}_i})\right) \\ -\frac{1}{16\pi}\int d^3x F_{\mu\lambda}F^{\mu\lambda}$$

and the corresponding Lagrange density

$$\mathcal{L} = \\ -\int d\hat{\tilde{q}}d\tilde{P}\phi\left(\mathcal{H}(\frac{\partial S}{\partial \tilde{q}_i}, \tilde{q}_i) - \mathcal{H}^{(0)}(\tilde{P}_i, \frac{\partial S}{\partial \tilde{P}_i})\right) \\ -\frac{1}{16\pi}F_{\mu\lambda}F^{\mu\lambda}.$$

The variation of the Lagrange density is given by

$$\begin{aligned}\delta\mathcal{L} &= \int d\hat{\tilde{q}}d\tilde{P}\left[\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{q}_i)}\delta\frac{\partial S}{\partial\tilde{q}_i} + \frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{P}_i)}\delta\frac{\partial S}{\partial\tilde{P}_i}\right] \\ &\quad + \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\delta F_{\mu\lambda}.\end{aligned}$$

From this relation one finds the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0,$$

$$\frac{\partial}{\partial\tilde{q}_i}\left(\frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{q}_i)}\right) + \frac{\partial}{\partial\tilde{P}_i}\left(\frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{P}_i)}\right) = 0,$$

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - 2\frac{\partial}{\partial x^\lambda}\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}} = 0.$$

When these relations are used, $\delta\mathcal{L}$ becomes

$$\delta\mathcal{L} =$$

$$\int d\hat{\tilde{q}}d\tilde{P}\left[\frac{\partial}{\partial\tilde{q}_i}\left(\delta S\frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{q}_i)}\right) + \frac{\partial}{\partial\tilde{P}_i}\left(\delta S\frac{\delta\mathcal{L}}{\delta(\partial S/\partial\tilde{P}_i)}\right)\right]$$
$$+2\frac{\partial}{\partial x^\lambda}\left(\delta A_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\right).$$

Integration over $\tilde{P}$ makes the $\frac{\partial}{\partial\tilde{P}_i}$ term vanish. Integration over $\hat{\tilde{q}}$ reduces the sum over i in the $\frac{\partial}{\partial\tilde{q}_i}$ term to 0,1,2,3. Hence

$$\delta\mathcal{L} =$$

$$\frac{\partial}{\partial x^\lambda}\left[\int d\hat{\tilde{q}}d\tilde{P}\delta S\frac{\delta\mathcal{L}}{\delta(\partial S/\partial x^\lambda)} + 2\delta A_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\right].$$

In the sense of Noether we define variations first by translating the whole system in space and time by

$$\delta x^\mu = const \qquad \mu = 0,1,2,3.$$

For any function $F(x^\mu)$ in the original system and the corresponding function $\hat{F}(x^\mu)$ in the new system it holds

$$\hat{F}(x^\mu + \delta x^\mu) = F(x^\mu)$$

and therefore

$$\delta F = \hat{F}(x^\rho) - F(x^\rho) = -\delta x^\rho \frac{\partial F}{\partial x^\rho}.$$

Application to $\mathcal{L}, S, A_\mu$ yields, since the δx^ρ are arbitrary,

$$\frac{\partial}{\partial x^\lambda} \Theta^\lambda_\rho = 0,$$

where

$$\Theta^\lambda_\rho = \int d\hat{\tilde{q}} d\tilde{P} \frac{\partial S}{\partial x^\rho} \frac{\delta \mathcal{L}}{\delta(\partial S/\partial x^\lambda)} + 2 \frac{\partial A_\mu}{\partial x^\rho} \frac{\partial \mathcal{L}}{\partial F_{\mu\lambda}} - \delta^\lambda_\rho \mathcal{L}$$

is the canonical tensor which is not gauge invariant. The energy-momentum tensor T^λ_ρ is the corresponding gauge invariant expression:

$$T^\lambda_\rho =$$

$$\int d\hat{\tilde{q}} d\tilde{P} \left(\frac{\partial S}{\partial x^\rho} - \frac{e}{c} A_\rho \right) \frac{\delta \mathcal{L}}{\delta(\partial S/\partial x^\lambda)} + 2 F_{\mu\rho} \frac{\partial \mathcal{L}}{\partial F_{\mu\lambda}} - \delta^\lambda_\rho \mathcal{L}.$$

Since $\mathcal{L}$ depends only on $\frac{\partial S}{\partial x^\rho} - \frac{e}{c} A_\rho$, one has

$$\int d\hat{\tilde{q}} d\tilde{P} \frac{e}{c} \frac{\delta \mathcal{L}}{\delta(\partial S/\partial x^\lambda)} = -\frac{\partial \mathcal{L}}{\partial A_\lambda}.$$

With this and the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\lambda} (\Theta^\lambda_\rho - T^\lambda_\rho) = 0$$

follows and hence

$$\frac{\partial}{\partial x^\lambda} T^\lambda_\rho = 0.$$

We prove now that T^λ_ρ possesses the non-relativistically reqired symmetry. To this end we consider variations resulting from an infinitsimal rotation of our whole system:

$$\delta x^\lambda = \epsilon^\lambda_\mu x^\mu,$$

$$\epsilon^\lambda_\mu = -\epsilon^\mu_\lambda,$$
$$\epsilon^\lambda_\mu = 0 \quad \text{for } \lambda \text{ and/or } \mu = 0.$$

The components of the vector potential $A_\mu, \mu = 1,2,3$ transform like the components of a gradient, $\frac{\partial F}{\partial x^\mu}$. A similar procedure as before leads to

$$\frac{\partial}{\partial x^\lambda}\left(T^\lambda_\mu x^k - T^\lambda_k x^\mu\right) = T^k_\mu - T^\mu_k = 0, \quad k,\mu = 1,2,3,$$

which proves the non-relativistically required symmetry.
We introduce now p_i instead of P_i. Because of the meaning of ϕ this is done by the substitutions

$$dP\,\phi \longrightarrow dp\,f(p_i,q_i),$$
$$\frac{\partial S}{\partial x^\rho} \rightarrow p_\rho \quad \rho \neq 0,$$
$$\frac{\partial S}{\partial x_0} - \frac{e}{c}A_0 \rightarrow \frac{1}{c}\left(H^{(0)} - (H(p_i,q_i,t) - e\phi(q_i,t))\right).$$

$H^{(0)}$ can be dropped since in $\int T^0_0 d^3x$ one has, before p_i is introduced,

$$\int dqdP\,\phi H^{(0)}(P_i, \frac{\partial S}{\partial P_i}) =$$
$$= \int dQdP\, f^{(0)}(P_i,Q_i)H^{(0)}(P_i,Q_i) = \text{const.}$$

Hence $H^{(0)}$ leads to vanishing four-divergence of T^λ_0. This leaves

$$T^\lambda_\rho =$$
$$-\int d\hat{q}dpf\left[\left(p_\rho - \frac{e}{c}A_\rho\right)\frac{\partial \mathcal{H}}{\partial p_\lambda} + 2F_{\mu\rho}\frac{\partial \mathcal{H}}{\partial F_{\mu\lambda}}\right]$$
$$-\frac{1}{4\pi}F_{\mu\rho}F^{\mu\lambda} + \delta^\lambda_\rho \frac{1}{16\pi}F_{\mu\sigma}F^{\mu\sigma}$$

with

$$p_0 - \frac{e}{c}A_0 \rightarrow$$
$$-\left(H(p_i,q_i,t) - e\Phi(q_i,t)\right).$$

The energy expression presented in the beginning of this subsection follows from

$$\int d^3x T^0_0 = \mathcal{E}.$$

5.2.2 Linearized Theory

For the linearized theory we first find the corresponding Lagrangian.
Let H_0 be the time-independent unperturbed Hamiltonian and $\phi_0, \ldots$ the unperturbed quantities such that

$$\phi = \phi_0 + \delta\phi, \quad \ldots .$$

We then expand the Lagrangian up to second order in $\delta\phi, \ldots$:

$$L = L_0 + L_1 + L_2 + \ldots .$$

The unperturbed quantities make the L_1 contribution in the variational principle vanish. Hence L_2 is the Lagrangian for the first-order perturbations $\delta\phi = \phi^{(1)}, \delta S = S_1, \cdots$. It serves to find the second-order energy. The general expression is again obtained via Noether's theorem [11]. In it one can express

$$\hat{f}^{(1)} = \frac{\partial f^{(0)}}{\partial q_i} \frac{\partial S_1}{\partial P_i},$$

$$\phi^{(1)} = \frac{\partial}{\partial q_i} \left(f^{(0)} \frac{\partial S_1}{\partial P_i} \right) .$$

The resulting expression is then a functional of

$$S_1, \ \mathbf{\Phi}_1, \ \mathbf{A}_1$$

similar to the situation in the Vlasov case, with S_1 replacing g.
For the usual particle Hamiltonian and $n = 3$ the second-order Maxwell-Vlasov energy is regained. In order to treat the kinetic guiding center theory, we first have to find the Hamiltonian for the guiding center motion, which is done in the following subsection.

5.3 Hamiltonian for the Guiding Center Motion

5.3.1 Lagrangian of Littlejohn [21] / Wimmel [22]

The Lagrangian is defined in terms of the variables

$$t, \qquad \mathbf{x} = (q_1, q_2, q_3), \qquad q_4,$$

where q_4 is an additional variable needed in guiding center theory.
The (non-regularized) Lagrangian is

$$L = \frac{e}{c}\mathbf{A}^* \cdot \dot{\mathbf{x}} - e\Phi^*,$$

$$\mathbf{A}^* = \mathbf{A} + \frac{mc}{e}(q_4\mathbf{b} + \mathbf{v}_E),$$

$$e\Phi^* = e\Phi + \mu B + \frac{m}{2}(q_4^2 + \mathbf{v}_E^2),$$

$$\mathbf{v}_E = c(\mathbf{E} \times \mathbf{B})/B^2,$$

$$\mathbf{b} = \mathbf{B}/B.$$

μ being the magnetic moment, is treated as a constant of motion. The corresponding equations of motion are

$$\mathbf{E}^* + \frac{1}{c}\mathbf{v} \times \mathbf{B}^* - \frac{m}{c}\dot{q}_4\mathbf{b} = 0,$$

$$\mathbf{b} \cdot \dot{\mathbf{x}} = v_{\|} = q_4$$

with

$$\mathbf{E}^* = -\frac{1}{c}\frac{\partial \mathbf{A}^*}{\partial t} - \frac{\partial \Phi^*}{\partial \mathbf{x}},$$

$$\mathbf{B}^* = \operatorname{curl} \mathbf{A}^*,$$

$$\mathbf{v} = \dot{\mathbf{x}}.$$

From these equations one obtains

$$\mathbf{v} = \mathbf{v}_g = q_4\frac{\mathbf{B}^*}{B_{\|}^*} + c\frac{\mathbf{E}^* \times \mathbf{b}}{B_{\|}^*},$$

$$\dot{q}_4 = \dot{v}_{\|} \equiv V_4 = \frac{e}{m}\mathbf{E}^* \cdot \frac{\mathbf{B}^*}{B_{\|}^*}.$$

For the Hamiltonian one needs

$$\mathbf{p} = \partial L/\partial \dot{\mathbf{x}} = \frac{e}{c}\mathbf{A}^*,$$

$$p_4 = \partial L/\partial \dot{q}_4 = 0.$$

These relations are *constraints* between the momenta and the coordinates. The existence of such constraints has the consequence that Hamilton's equations based on the usual Hamiltonian corresponding to the above - non-standard - Lagrangian are *not* the equations of motion.
What can be done about this?
An elegant way of obtaining a Hamiltonian for such Lagrangians was once developed by Dirac in the form of his constraint theory [23], which will be adopted here.
The usual or *primary* Hamiltonian is

$$\begin{aligned} H_p &= \dot{\mathbf{x}} \cdot \partial L/\partial \dot{\mathbf{x}} + \dot{q}_4 \partial L/\partial \dot{q}_4 - L \\ &= e\Phi^*. \end{aligned}$$

Dirac's Hamiltonian for the present problem is the following expression:

$$H = e\Phi^* + \mathbf{v}_g \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}^*) + V_4 p_4.$$

This Hamiltonian correctly yields

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}_g,$$

$$\dot{q}_4 = \frac{\partial H}{\partial p_4} = V_4.$$

In addition, however, one has

$$\begin{aligned} \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} &= -e\frac{\partial \Phi^*}{\partial \mathbf{x}} \\ &- \left(\frac{\partial}{\partial \mathbf{x}}\mathbf{v}_g\right) \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right) \\ &+ \frac{e}{c}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^*\right) \cdot \mathbf{v}_g - \frac{\partial V_4}{\partial \mathbf{x}} p_4 \end{aligned}$$

and

$$\dot{p}_4 = -\frac{\partial H}{\partial q_4} = -\frac{\partial \mathbf{v}_g}{\partial q_4} \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right).$$

These relations yield

$$\frac{d}{dt}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right) =$$

$$-\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{v}_g\right)\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right)-\frac{\partial V_4}{\partial \mathbf{x}}p_4,$$

$$\dot{p}_4 = -\frac{\partial \mathbf{v}_g}{\partial q_4}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right).$$

These equations are solved by the constraints. It is important for what follows, however, to be aware of the fact that the constraints do not represent special values of some constants of the motion. Therefore δ-functions of the constraints are not constants of motion either. The distribution function f must, however, be proportional to such δ-functions in order to ensure that the constraints are fulfilled, but it must also be a constant of motion. Both conditions are satified by

$$f =$$

$$\delta(p_4)\delta(\mathbf{p}-\frac{e}{c}\mathbf{A}^*)B_{\|}^* f_g(\mathbf{x}, v_{\|}, \mu, t),$$

where f_g is a solution of the *drift kinetic equation*

$$\frac{\partial f_g}{\partial t}+\mathbf{v}_g\cdot\frac{\partial f_g}{\partial \mathbf{x}}+V_4\frac{\partial f_g}{\partial q_4}=0.$$

5.3.2 Linearized Theory

Since the constraints are to hold along the perturbed orbits, it is natural that a displacement vector $(\boldsymbol{\xi}, \xi_4)$ from the unperturbed to the perturbed orbit must play a role. This means that, to the order needed, one has [11]

$$S_1 = \hat{S}_1(\mathbf{x}, q_4) - \boldsymbol{\xi}\cdot\left(\mathbf{P}-\frac{e}{c}\mathbf{A}_0^*(\mathbf{x}, q_4)\right)-\xi_4 P_4$$

so that

$$\frac{\partial S_1}{\partial \mathbf{P}} = -\boldsymbol{\xi}\,,\ \frac{\partial S_1}{\partial P_4} = -\xi_4.$$

$\boldsymbol{\xi}, \xi_4$ are *displacements* in $\mathbf{x}, q_4$ space similar to the displacement vector in macroscopic theory. The constraints yield for them

$$\xi_4 = \frac{1}{mB_0^*}\mathbf{B}_0^* \cdot \left(\frac{\partial \hat{S}_1}{\partial \mathbf{x}} - \frac{e}{c}\mathbf{A}_1^*\right),$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{\perp *} + \lambda(\mathbf{x}, q_4)\mathbf{B}_0^*,$$

with

$$\begin{aligned} \boldsymbol{\xi}_{\perp *} = \\ & \frac{c}{eB_0^{*2}}\left[\mathbf{b}_0^* \cdot \left(\frac{\partial \hat{S}_1}{\partial \mathbf{x}} - \frac{e}{c}\mathbf{A}_1^*\right)\mathbf{B}_0^* \times \mathbf{b}_0 \right. \\ & \left. -\mathbf{B}_0^* \times \left(\frac{\partial \hat{S}_1}{\partial \mathbf{x}} - \frac{e}{c}\mathbf{A}_1^*\right)\right], \end{aligned}$$

$$\lambda = -\frac{1}{mB_{0\|}^*}\left[\frac{\partial \hat{S}_1}{\partial q_4} + m\mathbf{b}_0 \cdot \boldsymbol{\xi}_{\perp *}\right].$$

With these relations the second-order energy is a functional (see Ref. [11]) of

$$\mathbf{A}_1 \,,\ \dot{\mathbf{A}}_1 \,,\ \Phi_1 \,,\ \hat{S}_1(\mathbf{x}, q_4, \mu).$$

Except for Φ_1, which is constrained to

$$\mathrm{div}\mathbf{E}_1 = 4\pi\rho_1,$$

these quantities can be freely chosen in the sense of initial conditions.

5.3.3 Extremization of the Second-order Energy

Introducing any convenient norm, one can try to find the minimum of the second-order energy by varying the quantities

$$\mathbf{A}_1 \,,\ \dot{\mathbf{A}}_1 \,,\ \Phi_1 \,,\ \hat{S}_1(\mathbf{x}, q_4, \mu)$$

with the constraint

$$\mathrm{div}\mathbf{E}_1 = 4\pi\rho_1.$$

This yields a *Hermitian eigenvalue problem* as in MHD.

5.4 Example

As an example we consider a homogeneous plasma with

$$\mathbf{B}_0 = \text{const} \neq 0 \ , \ \mathbf{E}_0 = 0,$$

$$\mathbf{A}_1 = \dot{\mathbf{A}}_1 = \Phi_1 = \rho_1 = 0.$$

As in the Maxwell-Vlasov theory one can try to obtain, with the help of the above formalism, conditions for the existence of negative-energy modes (see [11]). The results for both the Maxwell-Vlasov and the Maxwell- drift kinetic theory, are:
Maxwell-Vlasov theory:
Localized modes: $k_\perp r_g \gg 1$

$$\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}} > 0 \qquad \text{for some } \mathbf{k},\mathbf{v};$$

Maxwell-drift kinetic theory:
Extended modes: $\mathbf{k}$ arbitrary

$$v_{\|}\frac{\partial f_{g0}}{\partial v_{\|}} > 0 \qquad \text{for some } v_{\|}.$$

These conditions must hold, in the center-of-mass system, for at least one particle species.
The last condition is only a sufficient one. In the kinetic guiding center theory initially non-vanishing field perturbations might be important.

Summary and Additional Remarks

The nonlinear instabilities of the kind discussed here relate to the existence of linear negative-energy perturbations. They were found to be explainable in terms of *creation and annihilation operators.* A discussion of the *complete* solution of the *three-oscillator case with Cherry-like nonlinear coupling* shows that for almost all initial conditions resonance leads to explosive behaviour. In addition, the nonlinear coupling of the three oscillators allows runaway to occur in the nonresonant case as well, but the initial amplitudes ought not to be infinitesimally small.

In a *continuum theory* the three-wave coupling expression usually contains terms additional to those considered here. They are generally of a kind which introduces nonresonant behaviour even in the otherwise resonant case. One can speculate that their effect averages out so as to make the resonant terms dominant. Among these resonant terms can be those which shuffle the energy between positive-energy modes only or between negative-energy modes only. But this can be expected to only modify the exchange process between positive- and negative-energy modes. If this is so, *one can expect nonlinear instability rather generally when a continuum theory allows negative-energy perturbations.* An example of this behaviour is the particle on a hill with a superimposed magnetic field in vertical direction and a small nonlinear term destroying the axisymmetry of the linear system: numerical and some analytical results indeed show instability under arbitrarily small initial conditions in the resonant case. This example has only exchange of energy between positive- and negative-energy modes. A four-oscillator system with one negative-energy mode ($\omega = -3$) and three positive-energy modes ($\omega = 1; 2; 3$) modes can be shown, within a multiple time scale formalism, to be nonlinearly unstable, although not always explosively, where the nonlinear terms include those shuffling energy between the positive-energy modes only.

For the *Maxwell-Vlasov theory* a formally simple derivation of the second-order energy based on the *Lie group formalism* was presented. A *modified Hamilton-Jacobi formalism* allows one to derive the full energy momentum tensor and angular-momentum tensor for *general Maxwell- kinetic theories*, especially the *Maxwell-collisionless drift kinetic theory.* In the latter case, as in MHD, a *displacement vector in the* $\mathbf{x}, q_4 = v_{||}$ *space* plays an essential role.

The second-order energy can be considered as being expressed in terms of initial conditions which can be chosen freely exept for the constraint given by Poisson's equation. *The minimum of this energy is therefore obtainable via variations of these quantities leading to a Hermitian eigenvalue problem similar to the situation in ideal MHD.*
Special results are:

- *Maxwell-Vlasov theory*:
  Necessary and sufficient for the existence of negative-energy modes is the deviation of the distribution function of at least one particle species

in the frame of minimum equilibrium energy from being a monotonous function of v^2. This implies in general strongly localized perturbations.

- *Maxwell-collisionless drift kinetic theory*:
  Sufficient for the existence of negative-energy modes in a magnetized homogeneous plasma is that, in the center-of-mass system,

  $$v_{\|}\frac{\partial f_{g0}}{\partial v_{\|}} > 0 \quad \text{for some } v_{\|}$$

  is valid for at least one particle species.

Much remains to be done in linear as well as in nonlinear theory.

Appendix A

Derivation of Lie-type canonical transformations

Infinitesimal transformations

A generating function for an infinitesimal transformation is defined by

$$\frac{1}{N}L(\mathbf{x}, \mathbf{p}, t)$$

$$N : \text{integer}$$

$$N \longrightarrow \infty.$$

Let the Poisson bracket between a and b be

$$[a, b] = \sum_i \left(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial x_i}\frac{\partial a}{\partial p_i}\right).$$

The infinitesimal canonical transformations from $\mathbf{x}$ to $\mathbf{X}_1$ and from $\mathbf{p}$ to $\mathbf{P}_1$ are then

$$\mathbf{X}_1 = (1 + \frac{1}{N}[L, \cdot])\mathbf{x}$$

$$
\begin{aligned}
&= \mathbf{x} - \frac{1}{N}\frac{\partial L}{\partial \mathbf{p}} \\
\mathbf{P}_1 &= (1 + \frac{1}{N}[L, \cdot])\mathbf{p} \\
&= \mathbf{p} + \frac{1}{N}\frac{\partial L}{\partial \mathbf{x}}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
[X_{1i}, P_{1k}] &= \delta_{ik} - \frac{1}{N}\frac{\partial^2 L}{\partial p_i \partial x_k} + \frac{1}{N}\frac{\partial^2 L}{\partial x_k \partial p_i} \\
&\quad -\frac{1}{N^2}\left[\frac{\partial L}{\partial p_i}, \frac{\partial L}{\partial x_k}\right] \\
&= \delta_{ik} + O(\frac{1}{N^2}).
\end{aligned}
$$

Finite transformation

N iterations of the infinitesimal transformations described above yield the new variables

$$\mathbf{X_N} = \left(1 + \frac{1}{N}[L, \cdot]\right)^N \mathbf{x},$$

$$\mathbf{P_N} = \left(1 + \frac{1}{N}[L, \cdot]\right)^N \mathbf{p},$$

from whichit follows that

$$[X_{N_i}, P_{N_k}] = \delta_{ik} + N \cdot O(\frac{1}{N^2}).$$

For the limit

$$N \to \infty : \mathbf{X_N} \to \mathbf{X}, \quad \mathbf{P_N} \to \mathbf{P}$$

with

$$\mathbf{X} = e^{[L,\cdot]}\mathbf{x}, \quad \mathbf{P} = e^{[L,\cdot]}\mathbf{p}$$

one finds

$$[X_i, P_k] = \delta_{ik},$$

which shows that the new variables are again canonical.

Proof of the theorem $e^{[L,\cdot]}F(\mathbf{x},\mathbf{p}) = F(e^{[L,\cdot]}\mathbf{x}, e^{[L,\cdot]}\mathbf{p})$

We start again with the infinitesimal transformations:

$$\left(1 + \frac{1}{N}[L,\cdot]\right) F(\mathbf{x},\mathbf{p}) =$$
$$F(\mathbf{x},\mathbf{p}) + \frac{1}{N}\left(\frac{\partial L}{\partial x_i}\frac{\partial F}{\partial p_i} - \frac{\partial L}{\partial p_i}\frac{\partial F}{\partial x_i}\right),$$

$$\begin{aligned}
&F\left((1 + \frac{1}{N}[L,\cdot])\mathbf{x}, (1 + \frac{1}{N}[L,\cdot])\mathbf{p}\right) \\
&= F(\mathbf{x} - \frac{1}{N}\frac{\partial L}{\partial \mathbf{p}}, \mathbf{p} + \frac{1}{N}\frac{\partial L}{\partial \mathbf{x}}) \\
&= F(\mathbf{x},\mathbf{p}) + \frac{1}{N}(\frac{\partial L}{\partial x_i}\frac{\partial F}{\partial p_i} - \frac{\partial L}{\partial p_i}\frac{\partial F}{\partial x_i}) \\
&\quad + O(\frac{1}{N^2}).
\end{aligned}$$

N iterations lead to

$$O(\frac{1}{N^2}) \to N \cdot O(\frac{1}{N^2}) = O(\frac{1}{N})$$

$$\lim_{N\to\infty}\left(1 + \frac{1}{N}[L,\cdot]\right)^N = e^{[L,\cdot]}.$$

Hence $N \to \infty$ yields the statement.

References

[1] T. M. Cherry, Trans.Cambridge Philos.Soc. **23**, 199,(1925); see E.T. Whittaker, Analytical Dynamics (Cambridge, London, 1937), Sec. 182, p. 142; and A. Wintner, The Analytical Foundations of Celestial Mechanics, (Princeton University, Princeton, 1947), Sec. 136, p. 101

[2] J. Weiland and H. Wilhelmsson, Coherent Non-linear Interaction of Waves in Plasmas (Pergamon Press, 1977)

[3] H. Wilhelmsson, Nuclear Physics, **A518**, 84 (1990)

[4] B. D. Scott, Phys. Rev. Lett. **65**, 3289 (1990)

[5] P. J. Morrison and M. Kotschenreuther, The Free Energy Principle, Negative Energy Modes, and Stability, Report DOE/ET-53088-280, IFSR # 280, October 1989, Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

[6] V. M. Dikasov, L. I. Rudakov, and D. D. Ryutov, Zh. Eksperim. i Teor. Fiz. **48**, 913 (1965). [English trans.: Soviet Phys.-JETP **21**, 608 (1966)]

[7] R. E. Aamodt and M. L. Sloan, Phys. Rev. Lett. **19**, 1227 (1967)

[8] R. Z. Sagdeev and A. A. Galeev, in *Nonlinear Plasma Theory*, edited by T. M. O'Neil and D. L. Book (Benjamin, New York, 1969)

[9] B. Coppi, M. N. Rosenbluth, and R. Sudan, Ann. Phys. **55**, 207 (1969)

[10] Further references can be found in [2],[3]

[11] D. Pfirsch and P. J. Morrison, Phys.Fluids B **3**, 271 (1991)

[12] D. Pfirsch, Z. Naturforsch. **45a**, 839 (1990)

[13] H. L. Berk and D. Pfirsch, Phys. Fluids **31**, 1532 (1988)

[14] C. S. Gardner, Phys. Fluids **6**, 839 (1963)

[15] P. J. Morrison and D. Pfirsch, Phys.Fluids B **2**,1105 (1990)

[16] H. Weitzner and D. Pfirsch, Phys. Rev. A **43**, 4532 (1991)

[17] A result of W. Newcomb quoted in I. B. Bernstein, Phys. Rev. **109**, 10 (1958)

[18] A result of C. S. Gardner quoted in T. K. Fowler, J. Math. Phys. **4**, 559 (1963)

[19] J. H. Van Vleck, Proc. Nat. Acad. Sci. **14**, 178 (1928)

[20] D. Pfirsch, Arch. Rational Mech. Anal. **29**, 323 (1968)

[21] R. G. Littlejohn, J. Plasma Physics **29**, 111 (1983)

[22] H. K. Wimmel, Physica Scripta 29, 141 (1984)

[23] P. A. M. Dirac, Canad. J. Math **2**, 129 (1950); Proc. Roy. Soc. **A 246**, 326 (1958); *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series Number 2, Yeshiva University, New York, (1964)
K. Sundermeyer, *Constraint Dynamics*, Lecture Notes in Physics No. 169, Springer Verlag (1962)
D. Pfirsch, New Formulation of Dirac's Hamiltonian Method, Report IPP 6/246, Max-Planck-Institut für Plasmaphysik, D-8046 Garching (1985)