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Localization of Modes in Media with Simple Quasiperiodic Modulation

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Abstract

The spectrum and the solutions of the 1-D Schrödinger equation with a quasiperiodic potential V are investigated numerically. *Quasiperiodicity* in contrast to *periodicity* makes possible the localization of modes. V is taken as the sum of two incommensurate periodic δ-function series. The spectrum consists of bands separated by gaps which correspond to plateaus in the winding number. The spectrum of extended solutions (three-frequency-quasiperiodic) is found from the initial value problem. Localized solutions cannot be found in this way, owing to numerical instability. Instead, localized solutions (exponential decay on both sides) and the concomitant point spectrum are obtained from a boundary value method. An alternative method to find the point spectrum for bounded potentials is proved valid also for δ-function potentials, as used here. Connections with the MHD spectrum of toroidal plasmas and Alfvén wave heating are pointed out.

1. Introduction and Conclusions

The aim of the present investigation is the numerical study of localization of modes in media with a simple quasiperiodic modulation. The type of modulation studied here apparently has not been investigated before in the context of localization. Quasiperiodic "structures" or "modulations" are familiar mainly from solid state physics but they occur in other areas like plasma physics as well. Also, the applicability of a method to find the point spectrum of localized solutions, which is valid for bounded modulations, is tested for unbounded δ-function type modulations. The latter type leads to recursions which are much simpler to handle numerically than differential equations.

The propagation of waves (i.e. existence of modes, after Fourier transformation in time) through a *periodically* modulated medium is one of the key problems in solid state physics and has been solved long ago. It is well known that propagation is possible in continuous energy bands, separated by forbidden gaps. This follows from the Floquet theory of ordinary linear differential operators with periodic coefficients.

In recent years there has been increasing interest in media with "superstructure", namely with an incommensurate modulation superimposed on the primary periodicity. In this case the one-dimensional Schrödinger equation takes on the form

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(k_1 x, k_2 x)\,\psi = E\,\psi \tag{1.1}$$

where V is 2π-periodic in both arguments, and the modulation lengths $L_i = 2\pi/k_i, \quad i = 1,2$, are such that L_1/L_2 is irrational. Such $V(x)$ are called *quasiperiodic.*

In plasma physics equation (1.1), augmented by a first order term, describes the so called "MHD shear Alfvén continuum" in the low pressure limit in general toroidal geometry [1]. It describes the propagation of waves along the magnetic field in a toroidal surface with irrational rotational transform. The two arguments of V correspond, respectively, to the poloidal and toroidal variation of the plasma equilibrium. $E - < V >$ determines the average refractive index and $V - < V >$ is a measure of its variation. The existence of eigenmodes of equ. (1.1) which are single valued over the toroidal surface has been demonstrated for a model potential V in [2]. Such toroidal modes may be destructed by localization if the combined poloidal and toroidal modulation of the plasma equilibrium is too strong [3]. The "shear Alfvén continuum" is important e.g. for the heating of plasmas. (The term "continuum" here originates from consideration of the radial dependence of the modes and should not be confused with the

continuous versus point spectrum of equ. (1.1) proper, to be discussed below. Nevertheless, knowledge of *all* possible types of solutions of equ. (1.1) is instructive or even necessary to understand the full MHD mode equations of general toroidal equilibria.)

The mathematical theory of equ. (1.1) is intricate, owing to problems with small denominators as in KAM theory. For a review, including also more general potentials V, and further pertinent references see [4], and, for short, also [5]. For large E and smooth $V(x)$ the solutions are of generalized Floquet type [6], [7], namely a product of a quasiperiodic function with an exponential function $e^{i\mu x}$ (+ conj. compl.). This holds provided L_1/L_2 satisfies some diophantine inequality. Again, wave propagation is possible, for μ real, and forbidden in an infinity of gaps, for μ imaginary. Although the gaps exist almost everywhere in a Cantor like fashion the measure of the allowed E is finite and the spectrum is still continuous.

For small eigenvalue parameter E, however, different behaviour is possible [4]: solutions $\psi(x)$ may exist which decay to zero exponentially at both sides for $x \to \pm\infty$. Such solutions with $\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi^2(x) < \infty$ are called "localized", in contrast to the "extended" solutions of the (generalized) Floquet type. They belong to the point spectrum of equ. (1.1). Although for almost all quasiperiodic potentials the spectrum has both a continuous component (absolutely continuous and singular continuous) and a point component [4], there exist particular cases where the point component is empty [4], either for arbitrary L_1/L_2, as conjectured for example in [8], or for special classes of L_1/L_2 [9]. Apart from the standard cases mentioned below few explicit analytical investigations of the spectra for quasiperiodic V have been done, e.g. [8], [10], [11], [12].

In order to explain the motivation of the specific potential taken below a short review is necessary. Much of the work on equ. (1.1) has been done with a discretized version of it: in the spirit of Kronig-Penney [13] the medium is replaced by a (one-dimensional) sequence of δ-functions,

$$V(x) \;=\; \sum_{n=-\infty}^{+\infty} F_n\, \delta(\, x \;-\; x_n\,)\,. \tag{1.2}$$

Quasiperiodicity of V is achieved either through the choice of the amplitudes F_n or of the positions x_n. The simplest choice is a periodic set of x_n with periodicity length L_1, say, $x_n = x_0 + n\,L_1$, $n = 0,\ \pm 1,\ \cdots$, and an incommensurate modulation of the amplitudes F_n with periodicity length L_2: $F_n = P(k_2 x_n) = P(2\pi n\alpha)$, $\alpha = L_1/L_2$, and $P(t)$ 2π-periodic in t. In this case equ. (1.1) boils down to the recursion (see below)

$$\psi_{n+1} + \psi_{n-1} + \tilde{P}_n \psi_n = \tilde{E} \psi_n , \tag{1.3}$$

where $\tilde{P}_n = \tilde{P}(2\pi n\alpha, E)$ is proportional to P and is 2π-periodic in its first argument, and $\tilde{E} = \tilde{E}(E)$. Most of the analytic work on the spectrum and on localization has been done with equ. (1.3), notably with $\tilde{P}_n = G \sin(2\pi n\alpha + \theta)$, with G, θ = const, see for example [14] - [18]. Other functions $\tilde{P}_n$ were considered in, for example, [8], [11], [12], [19] and [20].

Quasiperiodicity can also occur through geometry alone, at constant amplitude F_n. The one and only case investigated so far is a periodic modulation of the distance $\Delta_n = x_n - x_{n-1}$ of neighbouring pulses, with periodicity length L_2:

$$x_n - x_{n-1} = L_1 + P(2\pi n\alpha) . \tag{1.4}$$

In this case a recursion of the type $A_n(E)\,\psi_{n+1} + B_n(E)\,\psi_{n-1} + C_n(E)\,\psi_n = 0$ results, where $A_n = A(2\pi n\alpha, E)$, etc., are again 2π-periodic functions (see below and refs. [10], [17]). Analytic investigations in references [10], [17] and [21] have shown that localization can occur in this case also.

Since so far only one discrete quasiperiodic geometry has been discussed in the literature it seems desirable to investigate a *qualitatively* different type of geometry also. For this purpose we consider the following case: $V(x) = V(k_1 x, k_2 x)$ is simply the sum of two periodic potentials, each with constant amplitudes $F_n = -F$ and $F_n = -G$, and periodicity length L_1 and L_2, respectively, with L_1/L_2 incommensurate, see Figure 1. (The signs are purely for convention, see below.) According to this simple rule the potential

$$V(k_1 x, k_2 x) = - F \sum_{r=-\infty}^{+\infty} \delta(\, x - r\, L_1\,) - G \sum_{s=-\infty}^{+\infty} \delta(\, x - s\, L_2\,) \tag{1.5}$$

results. The distance $x_n - x_{n-1}$ of neighbouring points is *not* a periodic function of n anymore. This makes the main difference to the previously considered arrangement. Remarkably, in the study of thermodynamic properties of quasicrystals both cases and combinations there-of have already been considered [22].

In Section 2 recursion relations for ψ_n are given for our case. As a novel feature, they involve coefficients which are themselves determined by recursions and cannot be expressed in closed form as functions of n. The (presumably) continuous spectrum of extended solutions is obtained in a straightforward way by running the recursions long enough and keeping only bounded solutions.

The point spectrum of localized solutions, in contrast, is found by consideration of a boundary value problem (see Section 3). Since the eigenvalue parameter E enters nonlinearly it is nonstandard and is handled using the ideas developed in [23]. The localized eigenfunctions obtained decay exponentially on both sides (with superimposed oscillations), with a tendency towards more than one peak if the boundaries are put farther apart. It is shown analytically that the recursion relation is numerically unstable for localized solutions and is therefore unable to (re-)produce them. This holds for the non-discretized equation (1.1) as well, if it is treated as an initial value problem.

Bondeson et al. [24] and Romeiras et al. [25], who considered smooth quasiperiodic potentials in equations related to equ. (1.1), used an alternative method to obtain the point spectrum. Theory [26] predicts that the spectrum of equ. (1.1), for bounded quasiperiodic potentials, consists of those values of E for which the winding number (see below) is not on a plateau. If, in addition, the Ljapunov exponents are non-zero, E belongs to the point spectrum. We compare this method for our potential with the boundary value method and find agreement of the results. This demonstrates the (heuristic) validity of the method just described for δ-function potentials also. The drawback of this method is its inability to produce localized eigenfunctions $\psi(x)$. (It is well suited, however, to demonstate the existence of strange nonchaotic attractors [24], [25] in the "phase" $\phi(x)$ of $\psi(x)$ whenever ψ *is* localized, a topic on which we do not dwell here.)

Localized domains (if they exist) and extended domains are usually found interspersed in the spectrum, up to a critical value E_u which depends on the strength of the modulation of the potential. For $E > E_u$ no localization is observed anymore. In the case $F = G$, considered here, localization is found to set in at $F \gtrsim 0.6$ or $F \lesssim -0.7$ (no Figure given).

Where ever in the spectrum localization occurs, for increasing modulation strength, it is always connected with the "collision" of two plateaus of the winding number w. Since each plateau has its own symmetry (w a specific integer combination of $k_1/2$ and $k_2/2$) localization is seen connected with symmetry breaking.

No indication is found that the δ-function type of the potential or its specific arrangement makes invalid any predictions borrowed from the spectral theory of *bounded* quasiperiodic potentials.

2. Extended solutions

With the δ-function potential (1.2) equation (1.1) goes over into a recursion relation from one puls to the next: ψ itself is continuous across the puls while $\mathrm{d}\psi/\mathrm{d}x$ jumps by an amount F_n. Between the pulses ψ evolves freely and satisfies $\psi(x) = c_1 \sin(\sqrt{E}x + c_2)$ with constants c_1, c_2. If this is put together one obtains for the values $\psi_n = \psi(x = x_n + 0)$ and $\dot{\psi}_n = \mathrm{d}\psi(x = x_n + 0)/\mathrm{d}x$ [10], [17],

$$\underline{\psi}_n \;=\; \mathbf{M}_{n,n-1}\; \underline{\psi}_{n-1}\;, \tag{2.1}$$

where $\underline{\psi}_n = (k\psi_n,\; \dot{\psi}_n)^T$ and

$$\mathbf{M}_{n,n-1} \;=\; \begin{pmatrix} c_{n,n-1} & s_{n,n-1} \\ -\,s_{n,n-1} \;+\; c_{n,n-1}\; F_n \,/\, k & c_{n,n-1} \;+\; s_{n,n-1}\; F_n \,/\, k \end{pmatrix}\;. \tag{2.2}$$

Here $k = \sqrt{E}$ and

$$\begin{aligned} s_{n,m} &= \sin[\,k(x_n \;- x_m)] \\ c_{n,m} &= \cos[\,k(x_n \;- x_m)]\;. \end{aligned} \tag{2.3}$$

If the visually obvious position of the x_n in the potential 1.5 (see Fig. 1) is formalized one obtains the recursion

$$x_n \;= \min\left\{ \left(\left[\frac{x_{n-1}}{L_1}\right] \;+\; 1 \right) L_1\;,\; \left(\left[\frac{x_{n-1}}{L_2}\right] \;+\; 1 \right) L_2 \right\}\;, \tag{2.4}$$

where $[x]$ means the integer part of x. In order to complete the definitions of equ. (2.2) F_n has to be given the value $-F$ or $-G$, depending on whether x_n is a member of the sequence $r\,L_1$ or $s\,L_2$, $r, s = 1,\; 2,\; \cdots$. For simplicity, the relative initial phase of the two sequences is set to zero.

An equivalent scalar form of the recursion relation is obtained by eliminating $\dot{\psi}_n$ from equs. (2.1), (2.2), [10]:

$$s_{n,n-1}\;\psi_{n+1} \;+\; s_{n+1,n}\;\psi_{n-1} \;-\; s_{n+1,n}\;s_{n,n-1}\;\frac{F_n}{k}\;\psi_n \;=\; s_{n+1,n-1}\;\psi_n\;. \tag{2.5}$$

(If F_n is chosen as a periodic function of n and $x_n - x_{n-1} = L_1 = const$, and/or if $x_n - x_{n-1}$ is chosen as a periodic function of n one obtains the types of recursion mentioned in the Introduction.)

The recursion (2.1 has been solved numerically in the interval $E = 0. - 1.$, with the incommensurate values $k_1 = 1$ and $k_2 = (\sqrt{5} - 1)/2$, chosen arbitrarily. 1000 values of E were taken in this interval. Amplitudes $F = G$ of the potential from 0.1 to 1.0 were considered, see Fig. 2. In the Figure dots, blending into lines, mark those values of E for which the sequence of amplitudes $A(n) = \sqrt{k^2\psi_n^2 + \dot{\psi}_n^2}$ stays bounded after $N = 10^4$ iterations, while blank spaces correspond to unbounded cases. Bounded is defined here arbitrarily as staying below $A_\infty = 10^5$, at starting value unity. Changing N to 10^5 or decreasing A_∞ by a factor of 10 does not visibly alter the picture. Fig. 2 shows sequences of bands with bounded solutions, separated by gaps which grow with increasing amplitude. Higher resolution, in agreement with theory, reveals finer and finer gaps. For $F \neq G$ and also for $F < 0$ and/or $G < 0$ the spectra are qualitatively the same. (In the latter case, $E - < V >$ and with it the average refractive index can become negative, for large $|F|$ and/or $|G|$.)

All solutions $\psi(x)$ thus obtained are of the generalized Floquet type, as Fourier analysis reveals. Figure 3, as an example, shows a Poincaré section of amplitudes $A_1(n) = \sqrt{k^2\psi^2(x) + \dot{\psi}^2(x)}$ at $x = nL_1$ at $F = G = 0.8$ and $E = 0.55$, with the two periods $L_2/2$ (short) and $2\pi/(2\mu)$ (long) visible in the figure.

In conclusion, the numerical solution of the recursion relation (2.1) as an initial value problem brings forward the extended type of solutions only. The spectrum thus obtained corresponds to the absolutely continuous spectrum described by Dinaburg and Sinai [6] and Rüssmann [7] although they proved its existence for continuous potentials only. In the next Section the reason for the apparent lack of localized solutions is discussed, and such solutions are constructed by a different method.

3. Localized solutions

Localized solutions go to zero at both $x = \pm\infty$. This suggests treatment as a boundary value problem, with $\psi(x) = 0$ imposed at positions far enough apart.

Let $\psi_0 = \psi_{N+1} = 0$, where N is a suffiently large number. The recursion relation (2.5) then assumes the form

$$\mathbf{N}\,\underline{\hat{\psi}} \;=\; 0\;, \tag{3.1}$$

where $\mathbf{N}$ is a symmetric $N \times N$ tridiagonal matrix with diagonal and off-diagonal elements a_n and b_n, respectively, given by

$$\begin{aligned} a_n \;&=\; -\,\frac{1}{k}\,F_n \;-\; \frac{s_{n+1,n-1}(k)}{s_{n+1,n}(k)\;s_{n,n-1}(k)}\;, \qquad && n = 1,\,2,\,\cdots\,N \\ b_n \;&=\; \frac{1}{s_{n+1,n}(k)}\;, && n = 1,\,2,\,\cdots\,N-1\;, \end{aligned} \tag{3.2}$$

and $\underline{\hat{\psi}} = (\psi_1, \cdots, \psi_N)^T$. Eigenvalues $E = k^2$ are determined by

$$d(E) \;:=\; \det \mathbf{N}(E) \;=\; 0\;. \tag{3.3}$$

Equations (3.1) - (3.3) constitute a nonlinear eigenvalue problem in E. It is handled numerically in the following way: after an LU-factorization of $\mathbf{N}$ the value of $d(E)$ is determined at very closely spaced intervals δE of order 10^{-5}. This allows determination of zero-crossings of $d(E)$ quite accurately. However, in automating the procedure, care has to be taken of the presence of poles of $d(E)$. As pointed out in [23] poles are caused by non-zero solutions of equ. (1.1) which exist between neighbouring supporting points x_n, x_{n-1} of the recursion and have nodes at these points. With equs. (2.1), (2.2) and $\psi_n = \psi_{n-1} = 0$, $\dot{\psi}_{n-1} \neq 0$, the condition for such modes is $\sin[\,k(x_n - x_{n-1})] = 0$. This implies poles at $E = k^2 = (\pi r/(x_n - x_{n-1}))^2$; $r = 0,\ 1,\ 2,\ \cdots$ for all $n = 1,\ 2,\ \cdots,\ N+1$, in agreement with the observed numerical behaviour.

The result of this procedure, applied to the case $F = G = 0.8$, for example, is shown in Fig. 4, upper part (short vertical lines). It shows the point spectrum in the region $E = 0. - 0.42$ for increasing extension of the domain, $N = 100 - 600$. Fast convergence towards a structure with well defined bands takes place. (Also visible are isolated eigenvalues which come and go with variation of N. These spurious modes have amplitudes strongly localized at the boundaries of the domains.) For comparison, the spectrum obtained previously from the initial value method is repeated in the middle

trace (dots). It is evident that the boundary value method reveals several eigenvalue bands, — "new" bands —, in the region $E < 0.3$ which are absent in the middle trace. In order to clarify the situation the eigenfunctions $\underline{\hat{\psi}}(x)$ are examined.

For this purpose equation (3.1) together with the normalization $\sum_{n=1}^{N} \psi_n^2 = 1$ was solved with a standard nonlinear-equation solver for the $N+1$ unknowns $\psi_1, \cdots, \psi_n, E$. This also provides an independent check on the eigenvalues E. The method requires initial guesses of the unknowns, for which simple 3-point peaks or flat profiles in $\underline{\hat{\psi}}$ were used, while E was taken close to previously found values. By repeating the procedure with various E_{initial} and various positions of the peaks an overview of the properties of the eigenfunctions is achieved. In all cases considered the eigenvalues so obtained agree to accuracy δE with values from the determinant method.

Figure 5 shows the amplitude $A_1(n)$ of the eigensolution with eigenvalue $E = 0.1060618$ (dots), for example, taken from one of the new bands. Evidently, it is a localized solution with an exponential decay on the average on both sides. Note the logarithmic scale in the Figure. (The Poincaré sections at $x_n = n\,L_1$ and $n\,L_2$ fit smoothely together, so that the full sequence of $A(n)$ looks the same.) All eigenfunctions examined, in the new bands, are found to be of this localized type, with the position of the peaks shifted rather arbitrarily as a function of E, and with occasional "steps" in the decay. (See also below.) In contrast, eigenfunctions taken from the old bands are of the extended type, like in Fig. 3.

If the values of $\psi_0 = 0$, ψ_1 of Fig. 5 are inserted as initial values into the recursion relation (2.5), together with E, the recursion gives the values shown as circles in Fig. 5. For a while the values agree with those of the boundary value method but later on they depart, and the recursion solution switches to an exponential growth. This shows that the initial value method fails in finding localized solutions because the recurrence relation becomes numerically unstable. In addition, localized solutions, for ψ_0 given, require values ψ_1 or $\dot{\psi}_0$ which are not arbitrary and which are not known in advance.

The instability is a consequence of the decay to zero of the localized solution, as the following considerations show. For this and subsequent purposes it is useful to introduce the phase angle ϕ of the vector $\underline{\psi} = (k\psi,\ \dot{\psi})^T$ by the definition $\tan\phi = k\psi/\dot{\psi}$, where $k = \sqrt{E}$. Remarkably, ϕ satisfies a *first order, nonlinear* differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} \;=\; k \;-\; \frac{V(x)}{k}\,\sin^2\phi\,. \tag{3.4}$$

This connection has been observed and exploited for related purposes in, for example, [24], [25]. Linearizing equ. (3.4) one obtains the equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \delta\phi \;=\; -\frac{V(x)}{k} \sin 2\phi \tag{3.5}$$

for a small perturbation $\delta\phi$ of $\phi(x)$. On the other hand, the amplitude $A(x) = \sqrt{k^2\psi^2(x) + \dot{\psi}^2(x)}$ satisfies $k\,\mathrm{d}\ln A^2/\mathrm{d}x = V(x)\sin 2\phi$ so that

$$A^2(x)\;\delta\phi(x) \;=\; const\;. \tag{3.6}$$

The relation $k\,\psi^2 \mathrm{d}(\delta\psi/\psi)/\mathrm{d}x = -A^2\,\delta\phi$ between a perturbation $\delta\psi$ of ψ and $\delta\phi$ finally yields

$$\delta\psi(x) \;=\; \psi(x)\left[\frac{\delta\psi_0}{\psi_0} - \frac{\delta\phi_0 A_0}{k}\int_{x_0}^{x}\frac{\mathrm{d}x}{\psi^2(x)}\right], \tag{3.7}$$

where the index zero refers to the values at an arbitrary $x = x_0$. If $\psi(x)$ goes to zero fast enough, for example exponentially, then $\delta\psi(x)$, owing to the denominator ψ^2, diverges towards infinity. This implies the instability quoted.

For δ-function potentials $V(x) = \sum_n F_n\,\delta(x - x_n)$ equ. (3.4) for ϕ is integrated between the pulses by $\phi_n^- = \phi_{n-1}^+ + k(x_n - x_{n-1})$. If it is rewritten in the form $\mathrm{d}\cot\phi/\mathrm{d}x = [\,V(x) - k^2\,]/k - k\cot^2\phi$, the jump condition across pulses $\cot\phi_n^+ - \cot\phi_n^- = F_n/k$ is obtained. With careful consideration of the branches of the cotangent the sequence of $\phi_n^\pm = \phi(x_n \pm 0)$ results. From $\phi_n^\pm$ the winding number w,

$$w \;=\; \lim_{x\to\infty}\;\phi(x)\;/\;x \;=\; \lim_{n\to\infty}\;\phi_n^\pm\;/\;x_n \tag{3.8}$$

can be calculated.

Instead of solving equ. (3.5) direct it is easier to take the derivative of the implicit recursion relation $\phi_n^+ = \phi_n^+(\phi_{n-1}^+)$ from above. For $\delta\phi_n = \delta\phi_n^+$ this yields

$$\frac{\delta\phi_n}{\delta\phi_0} \;=\; \prod_{i=0}^{n}\;\frac{1\;+\;\cot^2\phi_i^-}{1\;+\;\cot^2\phi_i^+}\;. \tag{3.9}$$

This allows evaluation of the Ljapunov exponent $\Lambda_\phi = \lim_{x\to\infty}(\ln|\delta\phi/\delta\phi_0|)/x$. Λ_ϕ is a measure of divergence of neighbouring ϕ-trajectories and, via equ. (3.6), also a measure of the speed with which $\psi(x)$ goes to infinity — or to zero, if $\Lambda_\phi < 0$. In Figure 6 Λ_ϕ is plotted for $F = G = 0.8$ and $E = 0.\,-1$. As expected from the previous discussion it is either zero, in the region of extended solutions, or negative, both in the gaps and the regions of localized solutions. (A different functional relation $\Lambda_\phi(E)$, however, is evident in the latter case.)

Also shown in Fig. 6 is the winding number $w(E)$. For E fixed, w is independent of the starting value $\phi = \phi_0$, i.e. of $\psi_0/\dot{\psi}_0$. This implies that a solution with $\psi(x) \to \infty$ has the same value w as a solution with $\psi(x) \to 0$, if it exists. In this sense w is a "robust" quantity. In agreement with theory [4], [26] (although developed for bounded potentials only, apart from special cases) $w(E)$ is a continuous nondecreasing function with plateaus at $w = (m\,\omega_1 + n\,\omega_2)/2$, m, n integer. Theory further states that the spectrum of equ. (1.1) consists of those E which are in the complement of the plateaus. For the spectrum of extended states this is confirmed by comparison of Fig. 2, $F = 0.8$, and Fig. 6. For localized states it is made evident in Fig. 4. In the bottom trace those values of E are shown for which w is not in a plateau and $\Lambda_\phi < 0$ (taking $\Delta w \geq \epsilon_2 = 2 \cdot 10^{-3}$ for neighbouring values of E and $|\Lambda_\phi| \geq \epsilon_1 = 2 \times 10^{-3}$ proved to be numerically reasonable criteria). The bands thus found (long vertical lines) indeed agree with those of the boundary value method (short vertical lines). This, therefore provides an independent method to locate the positions of localized solutions in the spectrum, as applied in [24], [25] for smooth potentials. Of course, neither method allows one to check whether the spectrum of localized solutions is *dense*, as suggested from theory for bounded potentials or potentials leading to equ. (1.3), since the limit $N \to \infty$ is non practicable.

With the method just explained Figure 7 is obtained. It shows both the extended spectrum of Fig. 2 and the spectrum of localized states, in the region $E = 0. - 1.0$ for various $F = G$ between 0.1 and 1.0. Inspection also at higher values of E indicates that there is an upper bound $E_u(F)$ for the existence of localized states. $E_u - E_0 = const \cdot F^2$ seems compatible with the numerical results, for $E_0 = < V(x) > = -F/L_1 - G/L_2$. This fits together with the result proved in [7] for analytic potentials $\tilde{V}(x)$, $|\tilde{V}| \leq \tilde{F}$, that there is a constant c such that for $E - E_0 > c\,\tilde{F}^2$ all modes in the spectrum are extended. For $E < E_u$ bands with localized solutions are interspersed in general with bands of extended solutions, see for example $F = 0.9$ and $F = 1.0$ in Fig. 7.

Fig. 7 shows that localization sets in where gaps begin to overlap, in particular the big ones like $(m, n) = (1, 0)$ and $(0, 2)$, or $(0, 3)$ and $(2, 0)$. With m or n being zero, these gaps correspond to the symmetry of purely periodic potentials, the periodicity length L being equal either to L_1 or L_2. This suggests the following simple picture for localized solutions: If gaps with different symmetries overlap, $\psi(x)$ clings to one symmetry for the ascent and to the other symmetry for the descent. (More generally: to different combinations of the primary symmetries.) In the purely periodic case, in contrast, according to the Floquet theory, an ascent or a descent must go on forever.

In line with this picture it is conceivable that in the quasiperiodic case the switching between ascent and descent occurs repeatedly. This is indeed observed, see for example Figure 8. If the extension of the intervall $x_N - x_0$ in the boundary value problem is increased the number of such secondary etc. peaks often goes up. When starting at very low amplitudes, they seem to be triggered by numerical round-off effects.

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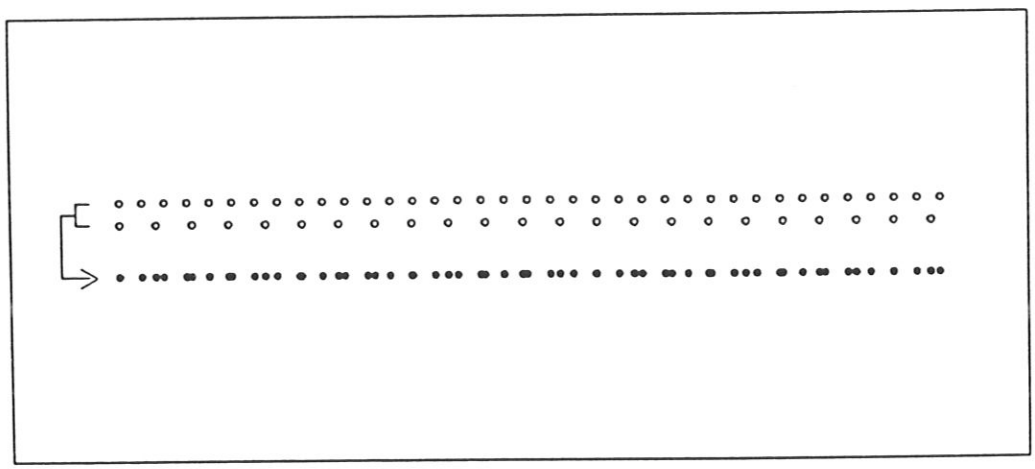

Figure 1: Quasiperiodic set (dots) from addition of two periodic sets (circles).

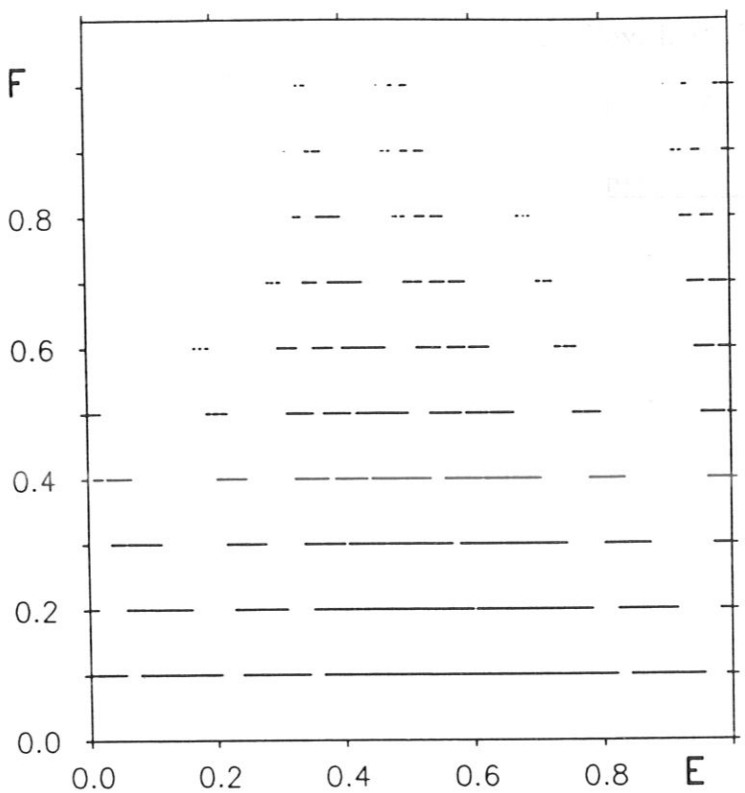

Figure 2: Spectrum of extended states, for various amplitudes $F = G$.

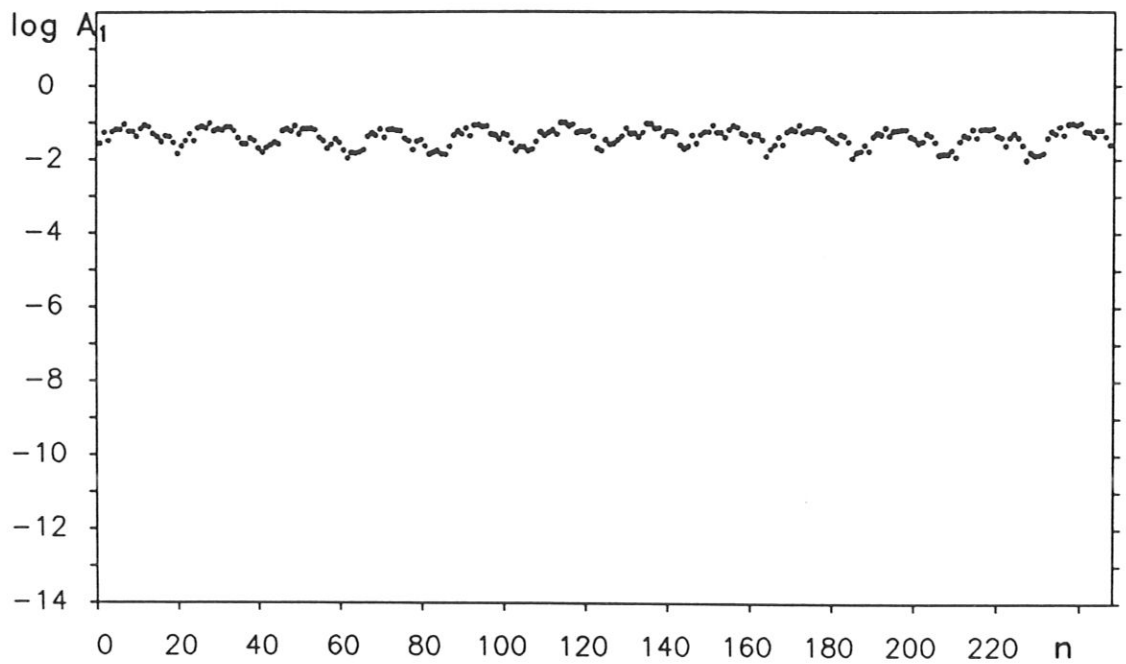

Figure 3: Amplitude $A_1(n)$ of extended solution (Poincaré cut at $x = nL_1$).
$F = G = 0.8,\ E = 0.55$.

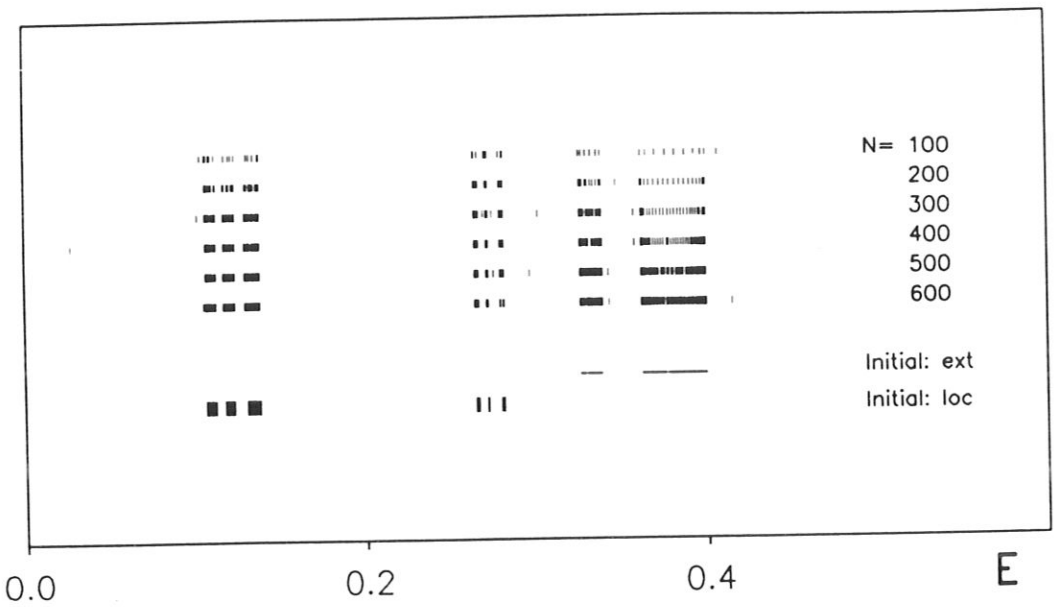

Figure 4: Upper traces: spectrum of eigenvalues, for different domain length N. Middle trace: spectrum of extended states from initial value problem. Lower trace: spectrum of localized states from initial value problem. $F = G = 0.8$.

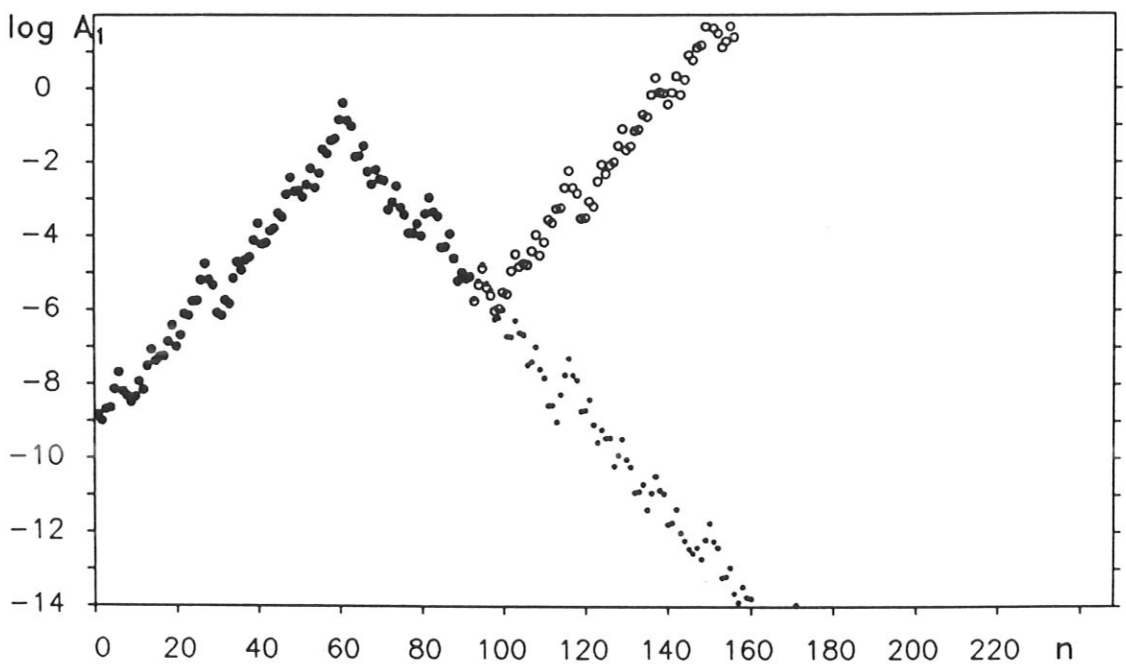

Figure 5: Amplitude $A_1(n)$ of localized solution (dots). Unstable solution from initial value problem (circles). $F = G = 0.8,\ E = 0.1060618$.

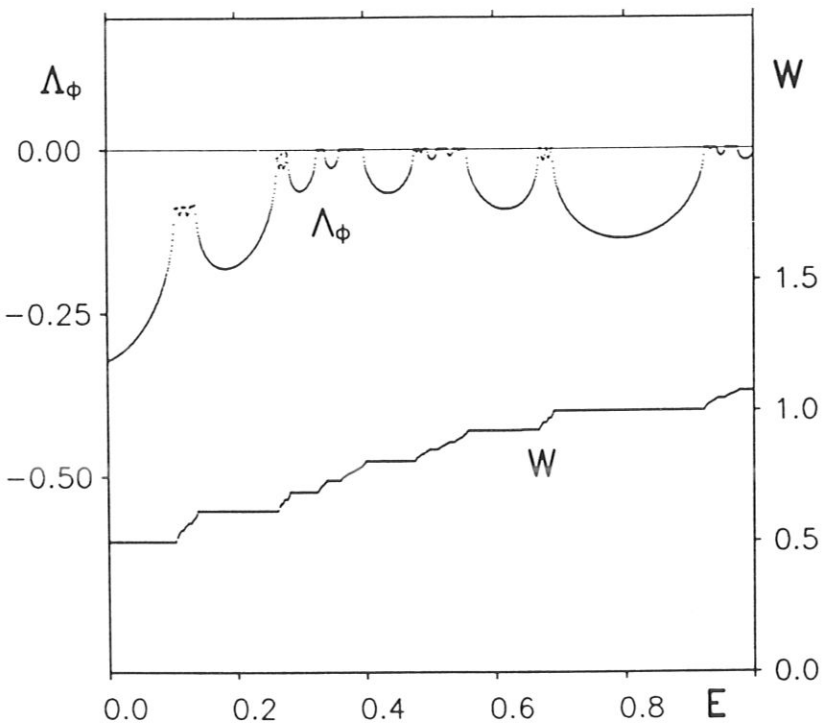

Figure 6: Ljapunov exponent Λ_ϕ and winding number w for $F = G = 0.8$.

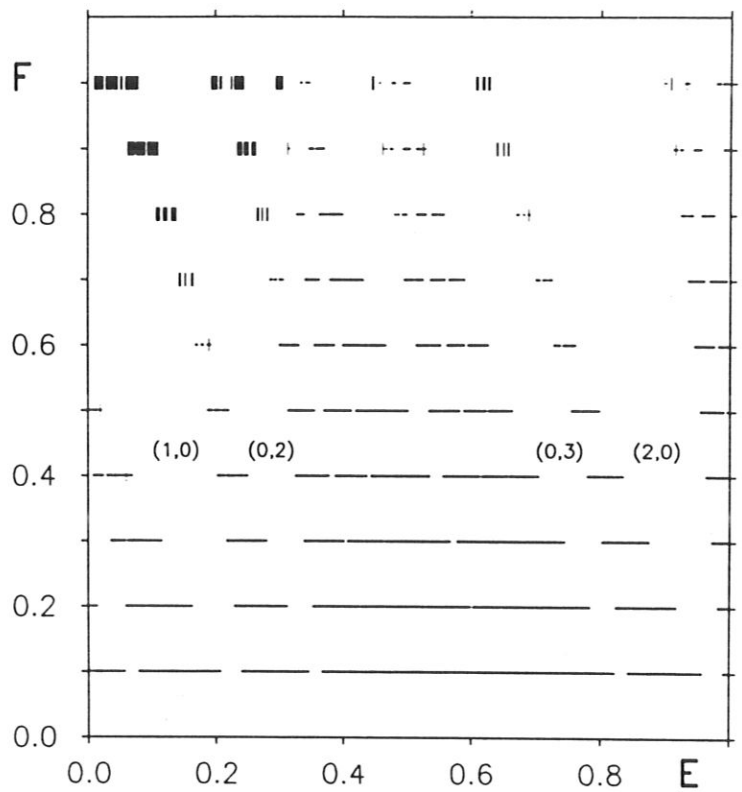

Figure 7: Spectrum of extended states (points), and localized states (bars), for various amplitudes $F = G$.

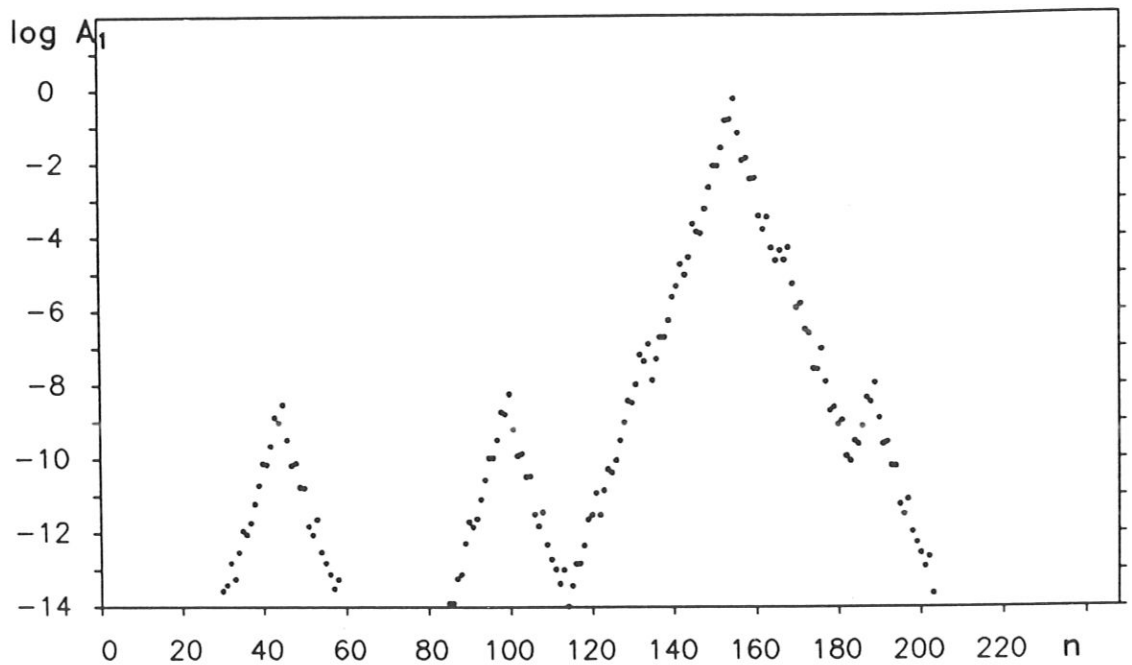

Figure 8: Amplitude $A_1(n)$ of localized solution with several peaks. $F = G = 1.0,\ E = 0.0751363$.