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Reflection and absorption of ordinary waves in an inhomogeneous plasma

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Abstract

This study treats the system of Vlasov and Maxwell equations for the Fourier transform in space and time of a plasma referred to Cartesian coordinates with the coordinate z parallel to the uniform equilibrium magnetic field and with the equilibrium plasma density dependent on ηx, where η is a parameter. The k_y component of the wave vector is taken equal to zero, whereas k_z is different from zero. When the interaction of ordinary and extraordinary waves is neglected, the Fourier transform of the electric field of the ordinary waves obeys a homogeneous integral equation with principal part integrals, which is solved in the case of weak absorption and sufficiently small η (essentially smaller than the vacuum wave vector), but without limitations on the ratio of the wavelength to the Larmor radius (the usual approximation being limited to wavelengths much smaller than the Larmor radius). The reflection and transmission coefficients and the total energy absorption are given in this approximation, whereas the energy conservation theorem for the reflection and transmission coefficients in an absorption–free plasma are derived for every value of η without explicit knowledge of the solutions. Finally, a general and compact equation for the eigenvalues which does not require complex analysis and knowledge of all solutions of the dispersion relation is given.

Introduction

The system of Vlasov and Maxwell equations for the electric field of a plasma referred to Cartesian coordinates with the coordinate z parallel to the uniform equilibrium magnetic field and the equilibrium plasma density depending on ηx, η being a parameter, yields a system of integral equations for the Fourier transform in space and the Laplace transform in time of the electric field. The inhomogeneous term of this equation is given by the initial conditions and possible sources, and the Laplace transform variable ω is the eigenvalue parameter. These equations belong to a class of integral equations which reduce to algebraic equations when a parameter, η in our case, goes to zero because the kernel becomes proportional to a Dirac δ–function. The eigenfunctions (waves in the x space) can be separated, in most physical situations of interest, into two classes of scarcely interacting waves: ordinary and extraordinary ones. The ordinary waves propagate adiabatically into vacuum when the density goes to zero, and can be directly excited by a source in vacuum. The purpose of this paper is to derive the transmission and reflection coefficients for ordinary waves propagating in the plane (x, z) when an electric field source with a given angular frequency is located at $x = -\infty$. In section 1 we relate the required coefficients to the solution of an inhomogeneous integral equation with principal part integrals and show that they verify the energy conservation theorem, which is an obvious requirement, with a less obvious proof, valid for every value of η. In section 2 the homogeneous integral equation corresponding to the inhomogeneous one introduced in section 1 is asymptotically solved when the parameter η is sufficiently small (essentially smaller than the vacuum wavelength of the source). The method we have chosen here consists in deriving asymptotic approximations for the solution in various superposed k–intervals. The intervals are such that one approximation of the solution matches onto another approximation in the common subinterval (a 'matched asymptotic procedure'; see, for example, Murray (1974)). In section 3 the asymptotic solution of the inhomogeneous integral equation is derived by means of two linearly independent, but for general ω values not integrable, solutions of the homogeneous integral equation in the following way. The form of the inhomogeneous term shows that except in the neighbourhood of k_o the solution is proportional to two different solutions of the homogeneous equation in the intervals $k \ll k_o$ and $k \gg k_o$ which are such that they go to zero when $|k|$ goes to infinity. In the neighbourhood of k_o the integral equation is approximated in two steps by a differential equation, which is solved by the usual methods. Then the matching conditions determine the, still free, proportionality factors of the two neighbouring approximate solutions. A by–product of this procedure is the eigenvalue condition, which is obtained in a compact and general form. Finally, the phase changes of the reflected and transmitted waves and the reflection coefficient are explicitly given.

1. Transmission and reflection coefficients

We consider a plasma immersed in a uniform magnetic field in the z direction, described by the linearized Vlasov equation. Let the equilibrium density distribution function be $\hat{h}(\eta x)$, a symmetric function of ηx, typically $\exp(-\eta^2 x^2)$. Let the Larmor radius be smaller than the inhomogeneity length. We want to deduce the transmission and reflection coefficients of the ordinary waves for such a configuration when an electric field source with a given angular frequency ω is located at $x = -\infty$ and the coupling of the z component of the electric field wih the other components (i.e. the coupling of the ordinary and extraordinary waves) can be neglected. Whereas the content of this section is valid for every value of η, the results derived in in sections 2 and 3 are asymptotic in η, i.e. η has to be sufficiently small (essentially smaller than the vacuum wavevector of the source), no assumption being made, however, on the ratio of Larmor radius to wavelength.

As is well known, the space Fourier transform of the z component of the electric field obeys the following integral equation (see appendix):

$$(k^2 - \omega^2/c^2)\mathcal{E}(k,\omega) - \sigma(k,\omega)\int_{-\infty}^{\infty} h(k'-k)\mathcal{E}(k',\omega)\,dk' = 0\,, \tag{1}$$

where, for simplicity, k is used to denote k_x and the dependence on k_z is not indicated; σ is connected with the (z,z) component of the dielectric tensor of a similar configuration, but with the uniform density distribution function $\hat{h}(0)$, by the relation $\sigma = \epsilon_{zz} - 1$; h is the Fourier transform of $\hat{h}$, which in the limit $\eta = 0$ is the Dirac function $\delta(k'-k)$. With the usual approximation that the Larmor radius is small compared with the wavelength eq. (1) reduces to a differential equation, but this approach cannot tackle essential parts of the problem. Our aim is to deduce the solution of eq. (1) whose inverse Fourier transform is such that the amplitude of the ingoing field for x going to minus infinity has the given value S. The modulus squared of the ratio with respect to S of the amplitudes $\mathcal{E}_r$ and $\mathcal{E}_t$ (in general complex) of the outgoing field for x going to minus and plus infinity, respectively, are the required reflection and transmission coefficients.

The previous condition on the dependence of the electric field on x can be translated into the following requirements about the form of $\mathcal{E}(k)$: if k_o is the vacuum wave number $(k_o \equiv \omega/c)$, in the neighbourhood of $k = k_o$ it must hold that

$$\mathcal{E} \sim \frac{k_o}{\pi i}\lim_{\epsilon=0}\left(\frac{S}{k-k_o-i\epsilon} - \frac{\mathcal{E}_t}{k-k_o+i\epsilon}\right), \tag{2}$$

an equation which can also be written as

$$\mathcal{E} \sim k_o(S+\mathcal{E}_t)\,\delta(k-k_o) + \frac{k_o(S-\mathcal{E}_t)}{\pi i}\,\mathcal{P}\frac{1}{k-k_o}\,, \tag{3}$$

where $\mathcal{P}$ denotes the principal value and δ is the Dirac function. Analogously, in the neighbourhood of $k=-k_o$ it must hold that

$$\mathcal{E} \sim k_o\mathcal{E}_r\,\delta(k+k_o) + \frac{k_o\mathcal{E}_r}{\pi i}\,\mathcal{P}\frac{1}{k+k_o}\,. \tag{4}$$

Both conditions are subsumed under the following ansatz for the Fourier transform of the electric field:

$$\mathcal{E} \equiv k_o(S+\mathcal{E}_t)\,\delta(k-k_o) + k_o\mathcal{E}_r\delta(k+k_o) + \mathcal{P}\frac{F(k,\omega)}{k^2-k_o^2}\,. \tag{5}$$

Once $F(k,\omega)$ is known, $\mathcal{E}_r$ and $\mathcal{E}_t$ are given by the equations

$$F(-k_o,\omega) = -\frac{2k_o^2}{\pi i}\,\mathcal{E}_r\,; \qquad F(k_o,\omega) = \frac{2k_o^2}{\pi i}(S-\mathcal{E}_t)\,. \tag{6}$$

By substituting eq. (5) into the homogeneous equation (1) we get an inhomogeneous integral equation with a principal part integral for the function F:

$$F(k,\omega)-\sigma(k,\omega)\,⨍\, h(k'-k)\,\frac{F(k',\omega)}{k'^2-k_o^2}\,dk' = k_o\sigma(k,\omega)\left[(S+\mathcal{E}_t)\,h(k-k_o)+\mathcal{E}_r\,h(k+k_o)\right]. \tag{7}$$

The solution of this equation is easily related to the solution of the simpler equation

$$Q(k,k_o,\omega)-\sigma(k,\omega)\,⨍\, h(k'-k)\,\frac{Q(k',k_o,\omega)}{k'^2-k_o^2}\,dk' = \frac{\pi}{2|k_o|}\sigma(k,\omega)\,h(k-k_o)\,, \tag{8}$$

because one obviously has

$$\frac{\pi}{2k_o^2}F(k,\omega) \equiv (S+\mathcal{E}_t)\,Q(k,k_o,\omega)+\mathcal{E}_r\,Q(k,-k_o,\omega)\,. \tag{9}$$

Notice that only k_o has to be changed in eq. (8), and not the variable ω in σ, although $k_o \equiv \omega/c$. With the help of the function Q eqs. (6) take the explicit form (from now on the ω dependence will be explicitly indicated only when necessary):

$$i\mathcal{E}_r = (S+\mathcal{E}_t)\,Q(-k_o,k_o)+\mathcal{E}_r\,Q(-k_o,-k_o)\,,$$

$$-i(S-\mathcal{E}_t) = (S+\mathcal{E}_t)\,Q(k_o,k_o) + \mathcal{E}_r\,Q(k_o,-k_o)\,. \tag{10}$$

We solve this algebraic system for $\mathcal{E}_r$ and $\mathcal{E}_t$ and obtain

$$(\mathcal{E}_r/S) = \frac{2iQ(-k_o,k_o)}{1+Q(-k_o,k_o)Q(k_o,-k_o)-Q(k_o,k_o)Q(-k_o,-k_o)+i\big(Q(k_o,k_o)+Q(-k_o,-k_o)\big)}\,,$$

$$(\mathcal{E}_t/S) = -1+$$

$$+\frac{2\big(1+iQ(-k_o,-k_o)\big)}{1+Q(-k_o,k_o)Q(k_o,-k_o)-Q(k_o,k_o)Q(-k_o,-k_o)+i\big(Q(k_o,k_o)+Q(-k_o,-k_o)\big)}\,. \tag{11}$$

The reflection and transmission coefficients, which can be derived from these equations as functions of Q, should give energy conservation in a situation corresponding to a stable, absorption–free plasma. Indeed, when σ is real (i.e. $k_z=0$) and hence Q as well, we get with some elementary algebra

$$R+T\;=\;1+$$

$$+\frac{4Q(-k_o,k_o)\big(Q(-k_o,k_o)-Q(k_o,-k_o)\big)}{[Q(-k_o,k_o)Q(k_o,-k_o)-Q(k_o,k_o)Q(-k_o,-k_o)+1]^2+[Q(k_o,k_o)+Q(-k_o,-k_o)]^2}\,. \tag{12}$$

Hence energy is conserved if $Q(-k_o,k_o)=Q(k_o,-k_o)$. This condition is verified when σ is an even function of k, the case of interest in plasma physics also when k_z is different from zero. In this case, however, the function Q is complex and the expression $(R+T-1)$ (which is no longer given by eq. (12)) yields the energy absorbed in plasma. Owing to the symmetry of the function Q eqs. (11) simplify to

$$(\mathcal{E}_r/S) = \frac{2iQ(-k_o,k_o)}{\big(Q(-k_o,k_o)\big)^2-\big(Q(k_o,k_o)\big)^2+1+2iQ(k_o,k_o)}\,,$$

$$(\mathcal{E}_t/S) = \frac{1+\big(Q(k_o,k_o)\big)^2-\big(Q(-k_o,k_o)\big)^2}{\big(Q(-k_o,k_o)\big)^2-\big(Q(k_o,k_o)\big)^2+1+2iQ(k_o,k_o)}\,. \tag{13}$$

It remains to deduce the function $Q(k,k_o)$. Equation (8) will be solved by a modification of the method developed in ref. [1], in particular for discussion of the integral equations which govern the longitudinal component of the electric field in the same plasma configuration we are considering here. The essential difference of eq. (8) from the equations considered in [1] is the presence of the factor describing the vacuum waves, $(k^2-k_o^2)$, which has two zeros on the k axis.

2. Homogeneous integral equation

In order to obtain the solution of eq. (8), we need two solutions of the corresponding homogeneous equation which do not go to zero for k going to both minus and plus infinity since they are not eigenfunctions. Hence we derive (without many details, because the method is already described in ref. [1]) two solutions of the following equation:

$$\Psi(k,\omega) - \sigma(k,\omega) \fint h(k'-k)\,\frac{\Psi(k',\omega)}{k'^2 - k_o^2}\,dk' = 0\,. \tag{14}$$

We first assume that k_z is zero and that σ is positive on the whole k axis. We then set

$$\Psi(k) \equiv \exp\left[\int^{k} (g(k')/\eta)\,dk'\right]$$

and expand the exponent in powers of $(k'-k)$ because the function h is peaked around $k'=k$; since it holds that

$$h(k'-k) \equiv (1/2\eta\sqrt{\pi})\exp\left[-(k'-k)^2/4\eta^2\right]\,,$$

from eq. (14) we get

$$1 - \frac{\sigma}{4\sqrt{\pi}\,k_o\eta}\cdot$$

$$\cdot \fint \mathrm{e}^{-y^2(1-2\eta g'(k))}\mathrm{e}^{2g(k)y}\left[\frac{1}{y+(k-k_o)/2\eta} - \frac{1}{y+(k+k_o)/2\eta}\right] dy \;=\; 0\,. \tag{15}$$

To the first order we neglect $\eta g'$ with respect to unity and use the representation of the principal part of $(1/x)$ as $i\pi\delta(x) + \lim_{\epsilon=0} 1/(x+i\epsilon)$ to derive, by means of a contour integral, an alternative expression of eq. (15), valid when the imaginary part of g is not negative:

$$1 - \frac{\sigma}{4\sqrt{\pi}\,k_o\eta}\cdot\left[i\pi\mathrm{e}^{g^2}\mathrm{e}^{-\left(g+(k-k_o)/\eta\right)^2} - i\pi\mathrm{e}^{g^2}\mathrm{e}^{-\left(g+(k+k_o)/\eta\right)^2} + \right.$$

$$\left.\mathrm{e}^{g^2}\int \mathrm{e}^{-t^2}\left(\frac{1}{t+g+(k-k_o)/2\eta+i\epsilon} - \frac{1}{t+g+(k+k_o)/2\eta+i\epsilon}\right)\right] dt \;=\; 0\,. \tag{16}$$

The integrals in this equation are a representation of the plasma dispersion function, and so eq. (16) can be written as

$$1 - \frac{\sigma}{4\,k_o\eta}\mathrm{e}^{g^2}\cdot$$

$$\cdot\Big[i\sqrt{\pi}\,\mathrm{e}^{-\left(g+(k-k_o)/\eta\right)^2} - i\sqrt{\pi}\,\mathrm{e}^{-\left(g+(k+k_o)/\eta\right)^2} -$$

$$-Z\big(g+(k-k_o)/2\eta\big) + Z\big(g+(k+k_o)/2\eta\big)\Big] = 0\,. \tag{17}$$

Since the plasma dispersion function is analytic on the complex plane $\big(g+(k-k_o)/2\eta\big)$, eq. (17) is valid also when the imaginary part of g is negative. An alternative derivation of eq. (17) which is valid from the beginning for both signs of g_I is obtained by writing

$$y + i\epsilon + (k \pm k_o)/2\eta = -i\int_0^\infty \exp\left[it(y + i\epsilon + (k \pm k_o)/2\eta)\right]\,dt\,.$$

It will be seen that the solution g verifies the inequality $|g + (k \pm k_o)/2\eta| \gg 1$ everywhere; then the plasma dispersion functions can be replaced by their asymptotic expressions, so that eq. (17) becomes

$$1 - \frac{\sigma}{4\,k_o\eta}\mathrm{e}^{g^2}\cdot$$

$$\cdot\left[i\sqrt{\pi}\,\mathrm{e}^{-\left(g+(k-k_o)/\eta\right)^2} - i\sqrt{\pi}\,\mathrm{e}^{-\left(g+(k+k_o)/\eta\right)^2} + \frac{4k_o\eta}{\left(2\eta g+k\right)^2 - k_o^2}\right] = 0\,, \tag{18}$$

an equation which is valid when the imaginary part of g is positive; the corresponding equation valid for $g_I < 0$ is obtained by replacing i by $-i$; for $g_I = 0$ the terms proportional to $\pm i$, the poles' contributions, are not present. Let us introduce the notation $\tau \equiv \sigma(k,\omega)/(k^2 - k_o^2)$. In the intervals of amplitude of the order $\eta^{2/3}|\partial\tau/\partial k|$ about $\tau = 1$, i.e. in the neighbourhood of the real values $k_j(\omega)$ $(j = 1, 2, ..., 2N)$, which are solutions of the dispersion relation $\sigma(k,\omega) = k^2 - k_o^2$, the condition $|\eta g'| \ll 1$ is not fulfilled; these intervals will be considered later. When τ is sufficiently larger than unity, the real part of g is negligible, and one gets $g_I \approx \pm\ln\tau$. When τ is sufficiently smaller than unity, but positive, the imaginary part of g is negligible, yielding $g_R \approx \pm\ln\tau$. When τ is negative, i.e. when $k^2 < k_o^2$, then g^2 becomes complex; also the corresponding complex conjugate quantity is a solution of eq. (15). The solution for $k^2 \approx k_o^2$ will be derived later, with a different approach (see eq. (41)). An approximation to the solution of the original eq. (15) which takes the presence of $\eta g'$ into account is obtained by adding a correction g_1 to the solution of eq. (17). By denoting for brevity the integral in eq. (15) by $I(g,g')$, one obviously has

$$g_1 = -g'\left[\frac{\partial}{\partial g'}I(g,g')\Big/\frac{\partial}{\partial g}I(g,g')\right]_{g'=0}\,. \tag{19}$$

Since the derivative of I with respect to g' is equal to $(\eta/2)$ times the second derivative with respect to g, we can use eq. (17) to get $g_1 \approx (-\eta g'/2g) - \eta g g'$; it thus follows that $\int (g_1(k')/\eta)\, dk' \approx -\ln\sqrt{g} - g^2/2$. This correction is the usual one in a WKB approximation. It shows that the approximate solution $g^2 = \ln\tau$ is valid if $2\eta g' \ll \ln\tau$, i.e. if $|k - k_i| \gg \left(\eta|\partial\tau/\partial k|\right)^{2/3}$. The intervals where this condition is not fulfilled separate on the k axis the intervals where g is approximately real, which will be denoted by s_j $(j = 1, 3, .., 2N+1)$ from the other intervals, which will be denoted by S_j $(j = 2, 4, .., 2N)$. In each interval of these two classes Ψ is asymptotically represented by a linear superposition of the functions $\exp\left[(1/\eta)\int^k g_{1,2}\, dk'\right]$ ($g_{1,2}$ are the two independent solutions of eq. (15) previously derived), the coefficients of the superposition being different in the different intervals. For practical reasons we choose different superpositions of $\exp\left[(1/\eta)\int^k g\, dk'\right]$ as 'elementary solutions' in the two classes of intervals. In the intervals s_j (where $0 < \tau < 1$) we choose

$$f_1 \equiv \frac{\sqrt{\tau}}{2(-\ln\tau)^{1/4}} \exp\left[\frac{1}{\eta}\int_{k_1}^{k} \sqrt{-\ln\tau}\, dk'\right] + c.c.\,, \tag{20}$$

$$f_2 \equiv \frac{\sqrt{\tau}}{2(-\ln\tau)^{1/4}} \exp\left[\frac{-1}{\eta}\int_{k_1}^{k} \sqrt{-\ln\tau}\, dk'\right] + c.c. \tag{21}$$

The function f_1 goes to zero for k going to minus infinity and diverges for k going to plus infinity; the opposite is true of f_2. In the intervals S_j (where $\tau > 1$) the 'elementary solutions' we choose are

$$F_1 \equiv \frac{\sqrt{\tau}}{(\ln\tau)^{1/4}} \cos\left(G(k) - (\pi/4)\right)\,, \quad F_2 \equiv \frac{\sqrt{\tau}}{(\ln\tau)^{1/4}} \sin\left(G(k) - (\pi/4)\right)\,, \tag{22}$$

where

$$G(k,\omega) \equiv (1/\eta)\int_{k_1}^{k} \sqrt{\ln\tau}\, dk'\,. \tag{23}$$

The intervals where eq. (15) is not valid have amplitudes of the order of $\left(\eta|\partial\tau/\partial k|\right)^{2/3}$, as we have already remarked; they are contained in intervals whose amplitudes are of the order of one, which will be denoted by d_j $(j = 1, 2, .., 2N)$, where the solutions were shown in ref. [1] to be proportional to the Airy functions. This can be seen by expanding $\Psi(k')$ in eq. (14) in powers of $(k'-k)$; one obtains the differential equation

$$(1-\sigma(k)F(k))\,\Psi(k)-\frac{\sigma(k)\eta^2}{2}F(k)'\Psi(k)'-\frac{\sigma(k)\eta^2}{2}\big(\frac{\eta^2}{2}F(k)''+F(k)\big)\Psi(k)''=0\,, \quad (24)$$

where $F(k)\equiv\fint h(k'-k)/(k'^2-k_o^2)\,dk'$, a function which is easily seen to be given by the real part of the plasma dispersion function:

$$F=(1/2k_o\eta)\left[Z_R\big((k+k_o)/\eta\big)-Z_R\big((k-k_o)/\eta\big)\right]\,.$$

In an interval d the function $\sigma(k)$ can be considered as constant, and since $|k\pm k_o|\gg\eta$, one has $F\approx 1/(k^2-k_o^2)$. Hence with the ansatz $\Psi(k)\equiv\left(k^2-k_o^2\right)^{1/2}\Phi^{(j)}(k)$ we obtain for $\Phi^{(j)}$ the following Airy equation:

$$\frac{\eta^2}{2}\Phi^{(j)}(k)''+\left(\frac{\sigma}{k^2-k_o^2}-1\right)\Phi^{(j)}(k)=0\,. \quad (25)$$

As 'elementary solutions' we choose the functions

$$\Phi_1^{(j)}\equiv\frac{\sqrt{y}}{\cos(\pi/6)}\left(J_{1/3}\left(\frac{2\pi y^{3/2}}{\left(a^{(j)}\right)^2}\right)+J_{-1/3}\left(\frac{2\pi y^{3/2}}{\left(a^{(j)}\right)^2}\right)\right), \quad (26)$$

$$\Phi_2^{(j)}\equiv\frac{\sqrt{y}}{\sin(\pi/6)}\left(J_{1/3}\left(\frac{2\pi y^{3/2}}{\left(a^{(j)}\right)^2}\right)-J_{-1/3}\left(\frac{2\pi y^{3/2}}{\left(a^{(j)}\right)^2}\right)\right), \quad (27)$$

where $a^{(j)}$ is equal to the quantity $\sqrt{(3\eta|\partial\tau/\partial k|/\pi)}$ taken at $k=k_j$, and $y\equiv(k-k_j)\left[\partial\tau\partial k\right]_{k=k_j}$. For more details see ref. [1]. A schematic representation of the various intervals is given in fig. 1.

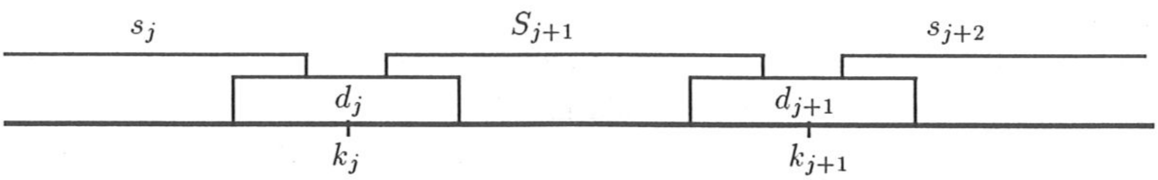

Fig. 1: *Overlapping intervals for the matched asymptotic expansion.*

The asymptotic solution of eq. (14) can now be derived by matching the approximations valid in the different intervals. At the end one gets a couple of linearly independent solutions which have the same form as the solutions of the homogeneous integral equation of ref. [1], but for a different definition of the function g, $\sigma(k,\omega)$ now being replaced by $\tau(k,\omega)$. In particular, if $\Psi_p = \alpha_{pq}^{(i)}\Phi_q^{(i)}$ holds in the interval d_i, in the intervals d_{i+n} $(n = 1, 2, ..., 2N - i)$ one has

$$\Psi_p = \alpha_{ps}^{(i)} T_{sq}(i, i+n)\Phi_q^{(i+n)}\,, \tag{28}$$

where the matrices $T_{sq}(i, i+n)$ are recursively defined as

$$T_{sq}(i,i) \equiv \delta_{sq}\,, \qquad T_{sq}(i,j+1) \equiv T_{sr}(i,j)\frac{a^{(j)}}{a^{(j+1)}}B_{rq}(j+1)$$

$$T_{sq}(i,j+2) \equiv T_{sr}(i,j+1)\frac{a^{(j+1)}}{a^{(j+2)}}C_{rq}(j+2)\,, \qquad (i+1) \leq (j = 1,3,5,...) \leq (2N-1)\,, \tag{29}$$

with the following definitions:

$$B_{rq}(j+1) \equiv \begin{pmatrix} \sin\left(G^{(j+1)} - G^{(j)}\right) & -\cos\left(G^{(j+1)} - G^{(j)}\right) \\ -\cos\left(G^{(j+1)} - G^{(j)}\right) & -\sin\left(G^{(j+1)} - G^{(j)}\right) \end{pmatrix}, \quad G^{(j)} \equiv G(k_j)\,, \tag{30}$$

$$C_{rq}(j+2) \equiv$$

$$\begin{pmatrix} 0 & -2\sin^2(\pi/6)\exp\left[G^{(j+2)} - G^{(j+1)}\right] \\ -(1/2\sin^2(\pi/6))\exp\left[-G^{(j+2)} + G^{(j+1)}\right] & 0 \end{pmatrix}. \tag{31}$$

The matrix $\alpha_{pq}^{(i)}$ can also be obtained by means of eq. (28) when $\alpha_{pq}^{(1)}$ is given, because one obviously has

$$\alpha_{pq}^{(i)} = \alpha_{ps}^{(1)} T_{sq}(1,i)\,. \tag{32}$$

By choosing the free constants of the functions Ψ_s such that for k going to minus infinity one has $\Psi_1 = f_1$ and $\Psi_2 = f_2$, the function $\alpha_{ps}^{(1)}$ is defined by $\alpha_{ps}^{(1)} = \delta_{ps}\left(b_p/a^{(1)}\right)$, where the quantities $b_1 = \left(1/2\sin(\pi/6)\right)$ and $b_2 = -\sin(\pi/6)$ are due to the matching of the approximations valid in s_1 and in d_1, respectively. All the quantities we have introduced depend on ω directly and indirectly through the solutions $k_i(\omega)$ of the dispersion relation $k^2 - k_o^2 - \sigma(k,\omega) = 0$, although the dependence is not indicated, for the sake of handiness.

We now have to consider the effect of possible zeros of the function $\sigma(k,\omega)$. Let $\sigma \equiv (k^2 - k_s^2)\sigma_p(k,\omega)$, where σ_p is positive definite (it is easy to deduce the effect of the presence of more couple of zeros from this case); for the sake of definiteness let us assume $k_s^2 > k_o^2$. We introduce the function $\Phi \equiv \Psi/(k^2 - k_s^2)$, which verifies the following equation, derived from eq. (14):

$$\Phi(k,\omega) - \sigma_p(k,\omega) \fint h(k'-k)\frac{k'^2 - k_s^2}{k'^2 - k_o^2}\,\Phi(k',\omega)\,dk' = 0\,. \tag{33}$$

When $|k^2 - k_s^2| \gg \eta$, i.e. far from the zeros of σ, we can consider the factor $(k'^2 - k_s^2)$ as being taken at $k' = k$, owing to the presence of the function $h(k'-k)$, and take it out of the integral; we thereby get the same equation as for Ψ. In general, eq. (33) can be written as

$$\Phi(k,\omega) - \sigma_p(k,\omega) \int h(k'-k)\,\Phi(k',\omega)\,dk' - (k_o^2 - k_s^2)\,\sigma_p(k,\omega) \fint h(k'-k)\frac{\Phi(k',\omega)}{k'^2 - k_o^2}\,dk' = 0\,. \tag{34}$$

By proceeding as was done for eq. (15), we derive the following equation, which replaces eq. (18) in the neighbourhood of a zero of σ :

$$1 - \sigma_p\,\mathrm{e}^{-g^2}\left[1 + \frac{k_o^2 - k_s^2}{(\eta g - k)^2 - k_o^2}\right] = 0\,, \tag{35}$$

from which one obtains

$$1 - \sigma_p\,\mathrm{e}^{-g^2}\,\frac{k^2 - k_s^2 - 2\eta k g}{k^2 - k_o^2} = 0\,. \tag{36}$$

At variance with eq. (18), which in the neighbourhood of a zero of σ yields $g^2 \approx \ln\tau \approx \ln(k - k_s)$, which diverges, eq. (36) gives for g^2 the finite value $g^2 \approx -\ln\eta$ for $|k - k_s| < \eta$. We conclude that Φ is finite at the zeros of σ, and therefore Ψ is zero there.

When k_z is different from zero, i.e. when σ is complex and the plasma absorbs energy, one can repeat all previous arguments if $|\eta g'|$ becomes much larger than $|\ln\tau$ in the intervals d_i, i.e. if at k_i the quantity $|\sigma_I/\sigma_R|$ is much smaller than $(\eta|\partial\tau/\partial k|)^{2/3}$. Otherwise the absorption is too strong to allow the formation of waves.

3. Inhomogeneous equation and reflection coefficient

We now come back to the inhomogeneous equation (8). Since the function h on the RHS of eq. (8) becomes a δ–function when $\eta = 0$, the solution is approximately a linear superposition of the real functions Ψ_1 and Ψ_2, except in the neighbourhood of $k = k_o$. The coefficients of the superposition can be different in $k < k_o$ and $k > k_o$; in particular, for $k \leq k_o^- < k_o$ the solution must have the form $\mathcal{C}_1\Psi_1$ since Ψ_2 diverges for k going to minus infinity. For $k \geq k_o^+ > k_o$ the solution Q must have the form $\mathcal{C}_2(\Psi_1 + c\Psi_2)$, where c is determined by the condition that Q go to zero for k going to plus infinity; in practice, we do not need to evaluate c, because the symmetry of $\sigma(k)$ plainly makes $\Psi_1(k) + c\Psi_2(k)$ proportional to $\Psi_1(-k)$. In order to determine the solution in the neighbourhood of k_o integral equation (8) will be approximated by a differential equation. As first step we deduce from eq. (8) an integro–differential equation without principal part integrals. We derive eq. (8) with respect to k, and by taking eq. (8) into account we get (with the notation $\Phi \equiv Q(k)/\sigma(k) - \frac{\pi}{2k_o}h(k-k_o)$)

$$\frac{\partial}{\partial k}\Phi(k) + \frac{k-k_o}{2\eta^2}\Phi(k) - \frac{1}{2\eta^2}⨍ h(k'-k)\frac{Q(k')}{k'+k_o}\,dk' = 0\,. \tag{37}$$

We again derive with respect to k and by taking eq. (8) into account we get

$$\frac{\partial^2}{\partial k^2}\Phi(k) + \frac{1}{2\eta^2}\frac{\partial}{\partial k}\Big((k-k_o)\Phi(k)\Big) +$$

$$+\frac{4}{\eta^4}(k+k_o)⨍ h(k'-k)\frac{Q(k')}{k'+k_o}\,dk' - \frac{4}{\eta^4}\int_{-\infty}^{\infty} h(k'-k)Q(k')\,dk' = 0\,. \tag{38}$$

Finally, by using eq. (37) we obtain

$$\frac{\partial^2}{\partial k^2}\Phi(k) + \frac{k}{\eta^2}\frac{\partial}{\partial k}\Phi(k) + \frac{2}{\eta^2}\left[\frac{2(k^2-k_o^2)}{\eta^2} + 1\right]\Phi(k) - \frac{4}{\eta^4}\int_{-\infty}^{\infty} h(k'-k)Q(k')\,dk' = 0\,. \tag{39}$$

The last term of this equation can also be written as

$$-\frac{4}{\eta^4}\int_{-\infty}^{\infty} h(k'-k)\sigma(k')\Phi(k')\,dk' + \frac{2\pi}{\eta^4 k_o}\int_{-\infty}^{\infty} h(k'-k)\,\sigma(k')h(k'-k_o)\,dk'$$

or, approximately,

$$-\frac{4\sigma(k)}{\eta^4}\int_{-\infty}^{\infty} h(k'-k)\Phi(k')\,dk' + \frac{2\pi\sigma(k)}{\eta^4 k_o}\int h(k'-k)\,h(k'-k_o)\,dk'\,. \tag{40}$$

Thus eq. (39) becomes an inhomogeneous integro–differential equation for the function $\Phi(k)$. As announced in the previous section we now derive an approximation for the function g valid in the neighbourhood of k_o, in the approximation (which will be seen to be correct) that $\eta g' \ll 1$. From eq. (39) we get the following equation for g :

$$4\eta^2 g^2 + 4k\eta g + 2\eta^2 + (k^2 - k_o^2) = \sigma \exp\left[g^2\right] . \tag{41}$$

In the neighbourhood of k_o the term $(k^2 - k_o^2)$ can be neglected, and the approximate solution is $g^2 \approx \ln \eta + \ln f$, where for f^2 one gets $f^2 \approx (4k_o/\sigma)^2(\ln \eta + \ln f) \approx (4k_o/\sigma)^2 \ln \eta$.

We now approximate eq. (39) in the interval $k \approx k_o$ with a differential equation, by expanding $\Phi(k')$ in powers of $k'-k$, similarly to the procedure in the neighbourhood of the zeros of the 'dispersion relation' of eq. (14). Since the solutions of the homogeneous equation corresponding to eq. (39) are known, being the functions $\Psi(k)$ derived in the preceding section, we are in a position to write the solution of the inhomogeneous equation for the interval considered by means of the method of the variation of the arbitrary constants. As one has

$$\int_{-\infty}^{\infty} h(k'-k)\Phi(k')\,dk' \approx \Phi(k)\int_{-\infty}^{\infty} h(k'-k)\,dk' - \frac{1}{2}\frac{\partial^2}{\partial k^2}\Phi(k)\int_{-\infty}^{\infty}(k'-k)^2 h(k'-k)\,dk'$$

$$= \Phi(k) - \frac{\eta^2}{4}\frac{\partial^2}{\partial k^2}\Phi(k) , \tag{42}$$

equation (39) becomes

$$\frac{\partial^2}{\partial k^2}\Phi(k) + \frac{4k}{\eta^2}\frac{\partial}{\partial k}\Phi(k) + \frac{2}{\eta^2}\left[\frac{2(k^2-k_o^2)}{\eta^2} + 1\right]\Phi(k) -$$

$$-\frac{4\sigma}{\eta^4}\Phi(k) + \frac{\eta^2\sigma}{4}\frac{\partial^2}{\partial k^2}\Phi(k) = \frac{2\pi\sigma}{\eta^4 k_o}\int_{-\infty}^{\infty} h(k'-k)\,h(k'-k_o)\,dk' . \tag{43}$$

The solution of eq. (42) can be written as

$$\Phi(k) = \Psi_1(k)\int^{k} \frac{H(k')}{W(k')}\,\Psi_2(k')\,dk' \; - \Psi_2(k)\int^{k}\frac{H(k')}{W(k')}\,\Psi_1(k')\,dk' , \tag{44}$$

where $W(k)$ is the Wronskian of $\Psi_{1,2}(k)$:

$$W(k) \equiv \Psi_1(k)(\partial\Psi_2(k)/\partial k) - (\partial\Psi_1(k)/\partial k)\Psi_2(k) ,$$

and $H(k) \equiv \sqrt{\pi/2}\,(\sigma/k_o\eta^3)\exp\left(-(k-k_o)^2/8\eta^2\right)$ is the RHS of eq. (43). As lower limits of the integrals in eq. (44) we choose plus infinity for the first integral and minus infinity for the second one. With this choice the solution is almost independent of $\Psi_2(k)$ at the beginning of the interval we are considering, as it is of $\Psi_1(k)$ at the end of the interval. The form of Q in the neighbourhood of k_o follows from the definition of Φ :

$$Q(k) = \frac{\pi\sigma}{2k_o} h(k-k_o) + \Psi_1(k) \int_{\infty}^{k} \frac{H(k')}{W(k')} \Psi_2(k')\, dk' \; - \Psi_2(k) \int_{-\infty}^{k} \frac{H(k')}{W(k')} \Psi_1(k')\, dk' \, . \quad (45)$$

It shows the separate effects of the source and the plasma. The function h decreases from the value $1/\eta$ for $k = k_o$ to a quantity of the order of unity at the boundary of the interval we are considering, and is exponentially small outside it. The coefficient of $\Psi_1(k)$ is exponentially small for k sufficiently larger than k_o and is a constant otherwise. Analogously, the coefficient of $\Psi_2(k)$ is a constant when k is sufficiently larger than k_o, and is exponentially small otherwise. Since these properties imply that $Q(k) \approx \mathcal{C}_{1,2}\Psi_{1,2}(k)$ far enough from $k = k_o$, eq. (45) can be considered valid on the whole k axis if the constants $\mathcal{C}$ are defined as follows:

$$\mathcal{C}_1 \equiv -\frac{\sqrt{\pi/2}\sigma(k_o)}{k_o\eta^3} \int_{-\infty}^{\infty} \exp\left[-(k'-k_o)^2/8\eta^2\right] \frac{\Psi_2(k')}{W(k')}\, dk' \, ,$$

$$\mathcal{C}_2 \equiv \frac{\sqrt{\pi/2}\sigma(k_o)}{k_o\eta^3} \int_{-\infty}^{\infty} \exp\left[-(k'-k_o)^2/8\eta^2\right] \frac{\Psi_1(k')}{W(k')}\, dk' \, . \quad (46)$$

Approximations of $\Psi(k)$ and $W(k)$ for $k \approx k_o$ necessary for the evaluation of the constants $\mathcal{C}$ are now derived. Since one has $g_R \ll g_I$ one can write

$$\Psi_1(k) = a \exp\left[\int_{k_o}^{k} (g(k')/\eta)\, dk'\right] + c.c.\, ,$$

where a is a complex constant whose explicit form can be obtained from eq. (28) with $i = 1$. In the case that there are no zeros of the dispersion relation in the interval $(-k_o, k_o)$ (this particular case has been chosen only to simplify notations), for Ψ_2 one has

$$\Psi_2(k) \equiv \Psi_1(-k) = a \exp\left[\int_{k_o}^{-k} (g(k')/\eta)\, dk'\right] + c.c.$$

$$= a\exp\left[-\int_{-k_o}^{k}(g(k')/\eta)\,dk'\right] + c.c.$$

$$= a^*\exp\left[\int_{-k_o}^{k_o}(g(k')/\eta)\,dk'\right]\exp\left[\int_{k_o}^{k}(g(k')/\eta)\,dk'\right] + c.c. \tag{47}$$

With the notation $b \equiv a\exp\left[-\int_{-k_o}^{k_o}(g(k')/\eta)\,dk'\right]$ one then obtains $W(k) \approx 4|g(k)|(ab)_I/\eta$, which depends only slightly on k. Hence

$$\mathcal{C}_1 \approx -\frac{\sqrt{\pi/2}\sigma(k_o)}{k_o\eta^3 W(k_o)}\int_{-\infty}^{\infty}\exp\left[-(k'-k_o)^2/8\eta^2\right]\Psi_2(k')\,dk' ,$$

$$\mathcal{C}_2 \approx \frac{\sqrt{\pi/2}\sigma(k_o)}{k_o\eta^3 W(k_o)}\int_{-\infty}^{\infty}\exp\left[-(k'-k_o)^2/8\eta^2\right]\Psi_1(k')\,dk' . \tag{48}$$

The integrals can be evaluated as was done for the derivation of g in the neighbourhood of k_o, the final result being

$$\mathcal{C}_1 \approx -\frac{2\pi\sigma(k_o)}{k_o\eta^2 W(k_o)}\left(b^*\exp\left[2g^2\right] + c.c.\right) \approx -\frac{4\pi k_o}{\sigma(k_o)W(k_o)}g^2 b_R ,$$

$$\mathcal{C}_2 \approx \frac{2\pi\sigma(k_o)}{k_o\eta^2 W(k_o)}\left(a\exp\left[2g^2\right] + c.c.\right) \approx \frac{4\pi k_o}{\sigma(k_o)W(k_o)}g^2 a_R \tag{49}$$

because g^2 is approximately real. For a general value of ω and k_z these expressions are of the order of η, owing to the presence of W, so that $Q(k_o) \approx (\pi\sigma/2k_o)h(0)$, the source term in the equation for Q. However, as the necessary and sufficient condition for the eigenvalues is $W(k,\omega) = 0$ for every value of k, as is shown afterwards, $Q(k)$ is singular at the eigenvalues, as it ougth to. In order to show that $W = 0$ is the eigenvalue condition, let us note that an eigenfunction must be proportional to $\Psi_1(k)$ because this function goes to zero for k going to minus infinity for every value of ω, in particular for the eigenvalues. At the same time it must be proportional to $\Psi_2(k) \equiv \Psi_1(-k)$, for an analogous reason. Hence at every k one must have $\Psi_1(k) = A\Psi_2(k)$ and $\Psi_1(k)' = A\Psi_2(k)'$, where A is some constant; the necessary and sufficient condition for the solution of these two equations is precisely $W = 0$. The previous considerations show that the reflection coefficient, which is the modulus squared of $\mathcal{E}_r/S$ given by eq. (13),

varies from zero (when $Q(-k_o, k_o) = 0$, i.e. at the eigenvalues or when $\Psi_1(-k_o) = 0$), to unity. The last case is given when $1 + \left(Q(k_o, k_o)\right)^2 - \left(Q(-k_o, k_o)\right)^2 = 0$, a condition which can be verified only in the neighbourhood of the eigenvalues because the quantity $h(0)$, which appears in $Q(k_o, k_o)$, is of the order $1/\eta$.
The phase angles of $(\mathcal{E}_r/S)$ and $(\mathcal{E}_t/S)$ are easily deduced from eq. (13); when $k_z = 0$ one obtains

$$\operatorname{tg}\phi_r = \frac{1 - \left(Q(k_o, k_o)\right)^2 + \left(Q(-k_o, k_o)\right)^2}{2Q(k_o, k_o)}, \qquad \operatorname{tg}\phi_t = -1/(\operatorname{tg}\phi_r). \tag{50}$$

Hence, for values of ω sufficiently far from the eigenvalues the phase change of the reflected wave with respect to the source is approximately $\pi/2$, whereas the phase change of the transmitted wave is of the order η. In the neighbourhood of $R = 0$ it is Φ_r which goes to zero, whereas Φ_t is approximately $\pi/2$.
We now derive the explicit form of the function $F(k, \omega)$ from eq. (9). By taking eqs. (13) into account we get

$$\left[\left(Q(-k_o, k_o)\right)^2 - \left(Q(k_o, k_o)\right)^2 + 1 + 2iQ(k_o, k_o)\right] F(k, \omega) =$$

$$\frac{4k_o^2}{\pi} S\Big(\left(1 + iQ(k_o, k_o)\right) Q(k, k_o) + iQ(-k_o, k_o) Q(k, -k_o)\Big) \tag{51}$$

or

$$\left[\left(Q(-k_o, k_o)\right)^2 - \left(Q(k_o, k_o)\right)^2 + 1 + 2iQ(k_o, k_o)\right] F(k, \omega) =$$

$$\frac{4k_o^2}{\pi} S\Big(\left(1 + iQ(k_o, k_o)\right) Q(k, k_o) + iQ(-k_o, k_o) Q(-k, k_o)\Big). \tag{52}$$

Although the function Q^2 appears in both the numerator and the denominator, the function F is singular at the eigenvalues because $\left(Q(k_o, k_o) + Q(-k_o, k_o)\right)W$ is zero at the eigenvalues, as is easy to deduce from eqs. (49). For $k < -k_o$ eq. (52) can be written in the form

$$\left[\left(Q(-k_o, k_o)\right)^2 - \left(Q(k_o, k_o)\right)^2 + 1 + 2iQ(k_o, k_o)\right] F(k, \omega) =$$

$$\frac{4k_o^2}{\pi} S\Big(\left(1 + iQ(k_o, k_o)\right)\mathcal{C}_1 + iQ(-k_o, k_o)\mathcal{C}_2\Big) \Psi_1(k). \tag{55}$$

A general remark can now be made about the relative amplitude of the electric field in the intervals S_2 and S_4 (we assume that there are only three S intervals, with

S_6 symmetric with respect to S_2), i.e. about the relative energy content of short–wavelength and long–wavelength waves. In the interval s_3 the function Ψ_1 is a superposition of the functions $f_{1,2}$; for general values of ω and k_z the coefficients of $f_{1,2}$ are comparable and therefore $|\Psi_1|$ is much larger in S_4 than in S_2. However, when ω and k_z are such that the coefficient of f_1 is zero (this would be the eigenvalue condition if S_2 were the only S interval; see also ref. [1]), the function Ψ_1 is proportional only to the function f_2 in s_3, and consequently $|\Psi_1|$ is much larger in S_2 than in S_4. The values of ω and k_z with the required property are easily deduced from eq. (29); one obtains the equation

$$\beta_{1p} b_{p2} = 0\,, \qquad i.e. \qquad \cos\left(\int_{k_1}^{k_2} (g/\eta)\, dk'\right) = 0\,. \tag{54}$$

Finally, some remarks should be made about $(R + T)$ in the presence of absorption, i.e. when $k_z \neq 0$. The first consequence of absorption is that one has $|\Psi_1(-k_o)| \ll |\Psi_1(k_o)|$, from which it follows that $|Q(-k_o, k_o)|^2 \ll |Q(k_o, k_o)|^2$. With this approximation, and some algebra, one obtains

$$R + T - 1 \approx Q_I(k_o, k_o)/\big(1 + (Q_R(k_o, k_o))^2\big)\,. \tag{55}$$

At the eigenvalues the plasma reaction dominates over the source; then R is zero, T is equal to one and there is no energy absorption. However, in the immediate neighbourhood of the eigenvalues the real part of the plasma reaction can cancel the source term, thereby producing the largest absorption, given by

$$R + T - 1 \approx \frac{2k_o}{\sigma_R}\frac{\sigma_I}{\sigma_R}\eta\,, \tag{56}$$

where σ_I is negative. The same result is obtained when eq. (54) is verified, which also happens in the neighbourhood of the eigenvalues.

Summary of the results

We consider the system of Vlasov and Maxwell equations for a plasma referred to Cartesian coordinates with the coordinate z parallel to the uniform equilibrium magnetic field and with the equilibrium plasma density dependent on ηx, where η is a parameter; we assume that $k_y = 0$, but that k_z is different from zero. The interaction of ordinary and extraordinary waves is neglected. The (space) Fourier transform of the electric field of the ordinary waves obeys a homogeneous integral equation with principal part integrals, which is solved in the case of weak absorption and sufficiently small η (essentially smaller than the vacuum wave vector), but without limitations on the ratio of the wavelength to the Larmor radius (the usual approximation being limited to wavelengths much smaller than the Larmor radius). We show for every value of η that the reflection and transmission coefficients verify the energy conservation theorem for an absorption–free plasma, with a proof which does not require explicit knowledge of the solutions. A physically interesting property of the distribution of energy over the different k intervals is also derived, as well as the total energy absorbed, again for weak absorption and asymptotically in η. A by–product of our derivation is a general and compact equation for the eigenvalues which does not require complex analysis and knowledge of all solutions of the dispersion relation.

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Appendix

Let the equilibrium distribution function for the particles of species a be $f_a \equiv h(\eta x + \eta v_y/\Omega_a) f_M$, where Ω_a is the gyration frequency of a particle and f_M is the Maxwell distribution function. Since for the ordinary waves the term $(\vec{E} + \vec{v} \times \vec{B})$ has no y component, one has

$$\sum_i (\vec{E} + \vec{v} \times \vec{B})_i \frac{\partial}{\partial v_i} f_a = h(\eta x + \eta v_y \Omega_a) \sum_i (\vec{E} + \vec{v} \times \vec{B})_i \frac{\partial}{\partial v_i} f_a \, .$$

When $\eta v_{ta} \Omega_a \ll 1$, where v_{ta} is the thermal velocity of the particles, one gets the final equation by substituting the space Fourier transform of the product $h(\eta x) E_z(x)$ for the Fourier transform of $h(0) E_z(x)$ in the equation valid for $\eta = 0$. The same is obtained by first integrating along the characteristics (on which the function h is constant) and then making the approximation on the argument of the function h before integrating over velocity.

References

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