MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Proof of a Conjecture of Morrison and Pfirsch

Harold Weitzner and Dieter Pfirsch

IPP 6/294 October 1990

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Proof of a Conjecture of Morrison and Pfirsch

H. Weitzner
Courant Institute of Mathematical Sciences,
New York, USA

D. Pfirsch
Max Planck Institut für Plasmaphysik
Garching bei München, F. R. Germany

Abstract

A necessary and sufficient condition for the existence of negative-energy modes in a plasma described by the Vlasov-Maxwell system is derived from an expression of Morrison and Pfirsch for the second variation of the free-energy of a steady state solution.

In a paper "Free-energy expressions for Vlasov equilibria"[1] Morrison and Pfirsch developed an expression for the second variation of the free-energy of a steady state solution of the Vlasov-Maxwell equations. When the second variation of the free-energy of a system is positive, the system is linearly stable. When the second variation is negative, the linearized system has negative-energy modes. The existence of negative-energy modes of a dynamical system is an important property of that system, as their existence may imply linear instability or indicate the possibility of perturbation of the original system into a neighboring, unstable system, see Refs. 1, 2, and the references cited there for further discussion. Morrison and Pfirsch considered the second variation of the free-energy of Vlasov-Maxwell steady state solutions for modes localized about a point $\mathbf{x}$ and they showed that negative-energy modes exist provided that the steady state distribution function $f_\nu^{(0)}(\mathbf{x},\mathbf{v})$ for the ν^{th} species satisfies the inequality

$$(\mathbf{k}\cdot\mathbf{v})\left(\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right) > 0 \, . \tag{1}$$

for some vector $\mathbf{k}$. They conjectured that if

$$(\mathbf{k}\cdot\mathbf{v})\left(\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right) \le 0 \tag{2}$$

for all $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{k}$, then no negative-energy modes, global or localized, are possible. In this note we prove their conjecture. As observed by them, (2) is equivalent to the conditions that

$$f_\nu^{(0)}(\mathbf{x},\mathbf{v}) = f_\nu^{(0)}(H_\nu^{(0)},\mathbf{x}) \tag{3}$$

and

$$\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}} \le 0 \tag{4}$$

where $H_\nu^{(0)}$ is the Hamiltonian in steady state for the ν^{th} species

$$\begin{aligned} H_\nu^{(0)} &= \frac{1}{2\,m_\nu}\left(\mathbf{p} - \frac{e_\nu}{c}\,\mathbf{A}^{(0)}\right)^2 + e_\nu\,\Phi^{(0)} \\ &= \frac{1}{2}\,m_\nu\,\mathbf{v}^2 + e_\nu\,\Phi^{(0)} \, . \end{aligned} \tag{5}$$

That (4) implies that no negative-energy modes exist is not at all surprising given the closely related stability theorems of Newcomb[3] and Gardner[4] which are based on free-energy methods. The free-energy variations in Ref. 1 based on Eulerian representations and Casimir invariants of the system also suggest strongly that the conjecture should be correct. In this note we obtain a proof of the non-existence of negative-energy modes wholly within the Lagrangian characterization of the variation of the free-energy.

Before we proceed to the proof we make two minor extensions of the preceding work. As was made explicit in (3), the local condition for the non-existence of negative-energy modes (2), or (4) allows the equilibrium distribution function $f_\nu^{(0)}$ to depend on $\mathbf{x}$ explicitly, as well as implicitly through its dependence on $H_\nu^{(0)}$. However, the condition that $f_\nu^{(0)}$ of the form (3) satisfy the time-independent Vlasov equation is

$$\mathbf{v} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}} = 0 \ . \tag{6}$$

With $H_\nu^{(0)}$ fixed one can select $\mathbf{v}$ in the direction of $\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}$, so that (6) requires that

$$\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}} = 0$$

or

$$f_\nu^{(0)} = f_\nu^{(0)}(H_\nu^{(0)}) \ . \tag{7}$$

As was observed in Ref. 1, distribution functions of the form (3) or (7) generate no current, so that steady state solutions to the Vlasov-Maxwell system must have vacuum magnetic fields. We may trivially extend the results by the addition of given external charge and current sources $Q^{ext}(\mathbf{x}, t)$ and $\mathbf{J}^{ext}(\mathbf{x}, t)$, where for energy conservation to hold these sources must, of course, be time independent. We replace the electromagnetic field Lagrangian

$$L_{EM} = \frac{1}{8\pi} \int (\mathbf{E}^2 - \mathbf{B}^2) \, d^3x \tag{8}$$

by

$$L_{EM} = \frac{1}{8\pi} \int (\mathbf{E}^2 - \mathbf{B}^2) \, d^3x - \int (\Phi \, Q^{ext} - \mathbf{A} \cdot \mathbf{J}^{ext} \, / \, c) \, d^3x \tag{9}$$

where

$$\mathbf{E} \;=\; -\,\nabla\,\Phi \;-\; \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t}$$

and

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}\;.$$

Gauge invariance of the theory, or equivalently charge conservation, implies

$$\frac{\partial Q^{ext}}{\partial t} \;+\; \frac{1}{c}\,\nabla\cdot\mathbf{J}^{ext} \;=\; 0\;.$$

The modified field Lagrangian (9) merely adds the explicit external charge Q^{ext} and current $\mathbf{J}^{ext}$ to the plasma charge and current in Maxwell's equations. Since the modification to the Lagrangian is linear in the electromagnetic fields it does not affect the expression for the second variation of the free-energy. Thus, for our proof we take over the formalism of Refs. 1 and 2 and we allow time independent electric and magnetic fields restricted only by

$$\nabla\times\mathbf{E}^{(0)} \;=\; 0 \qquad \text{and} \qquad \nabla\cdot\mathbf{B}^{(0)} \;=\; 0\;.$$

Further we assume that the equilibrium distribution functions are of the form (7) and satisfy (4). Our proof starts from eq. (13) in Ref. 2, an expression for the second variation of the free-energy of the system. When one restricts the equilibrium distribution functions by (7), one finds

$$\begin{aligned}\delta^2 F \;=\; \sum_\nu \int d^3x\, d^3p\, \Big\{ -\frac{1}{2}\,\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}} \Big(\Big\{ [g_\nu,\, H_\nu^{(0)}] + \frac{e_\nu}{m_\nu c}\,\delta\mathbf{A}\cdot(\mathbf{p} - \frac{e_\nu}{c}\,\mathbf{A}^{(0)}) \Big\}^2 \\ - \frac{e_\nu^2}{m_\nu^2 c^2}\,[\delta\mathbf{A}\cdot(\mathbf{p} - \frac{e_\nu}{c}\,\mathbf{A}^{(0)})]^2 \Big) \;+\; \frac{e_\nu^2}{2\,m_\nu c^2}\, f_\nu^{(0)}\,\delta\mathbf{A}\cdot\delta\mathbf{A} \Big\} \qquad (10) \\ +\; \frac{1}{8\pi}\int d^3x\,[(\delta E)^2 \;+\; (\delta B)^2]\;.\end{aligned}$$

Let us consider

$$I \;=\; \frac{1}{m_\nu}\,\delta A_i\,\delta A_j\,(p_i - \frac{e_\nu}{c}\,A_i^{(0)})(p_j - \frac{e_\nu}{c}\,A_j^{(0)})\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}} \;+\; f_\nu^{(0)}\,\delta A_i\,\delta A_i\;. \quad (11)$$

Since

$$\frac{\partial}{\partial p_i} f_\nu^{(0)} \;=\; \frac{1}{m_\nu}\,(p_i - \frac{e_\nu}{c}\,A_i^{(0)})\,\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}}\;,$$

$$I \;=\; \frac{\partial}{\partial p_i}\Big((p_j \;-\frac{e_\nu}{c}\,A_j^{(0)})\;f_\nu^{(0)}\Big)\;\delta A_i\;\delta A_j\;, \tag{12}$$

and

$$\delta^2 F \;=\; \sum_\nu \int d^3x\; d^3p \,\Big\{\, -\,\frac{1}{2}\,\frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}}\Big([g_\nu,\; H_\nu^{(0)}] + \frac{e_\nu}{m_\nu c}\,\delta\mathbf{A}\cdot(\mathbf{p} - \frac{e_\nu}{c}\,\mathbf{A}^{(0)})\Big)^2\Big\}$$

$$+\;\frac{1}{8\pi}\int d^3x\;[(\delta E)^2 \;+\; (\delta B)^2]\;. \tag{13}$$

The expression (13) shows that (4) is clearly sufficient to guarrantee that no negative-energy modes exist. Combined with the local condition for the existence of negative-energy modes (1) as given in Ref. 1, it follows that (4) is necessary and sufficient for the non-existence of negative-energy modes. We note that expression (13) for the second variation of the free-energy is similar to the Eulerian expression (24) in Ref. 1, but the two are not the same.

1 References

[1] P. J. Morrison and D. Pfirsch, Phys. Rev. A **40**, 3898 (1989)

[2] P. J. Morrison and D. Pfirsch, Phys. Fluids B **33**, 1105 (1990)

[3] A result of W. Newcomb quoted in I. B. Bernstein, Phys. Rev. **109**, 10 (1958)

[4] A result of C. S. Gardner quoted in T. K. Fowler, J. Math. Phys. **4**, 559 (1963)