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Complete Solution of the Modified Cherry Oscillator Problem

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Abstract

In 1925, T. M. Cherry [1] presented a simple example demonstrating that linear stability analysis will in general not be sufficient for finding out whether a system is stable or not with respect to small-amplitude perturbations (see also [2]). The example consisted of two nonlinearly coupled oscillators, one possessing positive energy, the other negative energy, with frequencies $\omega_1 = 2\omega_2$ allowing third-order resonance. In a previous paper [3], the present author reformulated Cherry's example and then generalized it to three coupled oscillators corresponding to three-wave interaction in a continuum theory like that of Maxwell-Vlasov [4]. Cherry was able to present a two-parameter solution set for his example which would, however, allow a four-parameter solution set, and in Ref. [3] a three-parameter solution set for the resonant three-oscillator case was obtained which, however, would allow a six-parameter solution set. Nonlinear instability could therefore be proven only for a very small part of the phase space of the oscillators. This paper now gives the complete solution for the three-oscillator case and shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behaviour. This is true of the resonant case. The non-resonant oscillators can sometimes also become explosively unstable, but only if the initial amplitudes are not infinitesimally small.

1 Introduction

In a previous paper [3] Cherry's two nonlinearly coupled oscillators [1] were reformulated and then generalized to three coupled oscillators corresponding to three-wave interaction in a continuum theory like that of Maxwell-Vlasov [4]. Cherry was able to present a two-parameter solution set for his example which would, however, allow a four-parameter solution set, and in Ref. [3] a three-parameter solution set for the resonant three-oscillator case was obtained which, however, would allow a six-parameter solution set. Nonlinear instability could therefore be proven only for a very small part of the phase space of the oscillators. This paper now gives the complete solution for the three-oscillator case and shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behaviour. This is true of the resonant case. The non-resonant oscillators can sometimes also become explosively unstable, but the initial amplitudes must not be infinitesimally small.

2 Reduction of the equations of motion

The Hamiltonian for the three coupled oscillators was

$$H = \frac{1}{2}\sum_{k=1}^{3}\omega_k\xi_k\xi_k^* + \frac{1}{2}\alpha\xi_1\xi_2\xi_3 + \frac{1}{2}\alpha^*\xi_1^*\xi_2^*\xi_3^*. \tag{1}$$

The quantities $\frac{1}{2i}\xi_k$ are coordinates, and the quantities ξ_k^* the canonically conjugated momenta. Invariance to time reversal is guaranteed for purely imaginary α. The equations of motion following from eq. (1) are

$$\dot{\xi}_k = i\omega_k\xi_k + i\alpha^*\xi_1^*\xi_2^*\xi_3^*/\xi_k^* = i\omega_k\xi_k + f^*/\xi_k^*, \tag{2}$$

where

$$f^* = i\alpha^*\xi_1^*\xi_2^*\xi_3^* \tag{3}$$

is independent of the special oscillator. From eq. (2) one finds

$$\frac{d}{dt}\xi_k\xi_k^* = f + f^*. \tag{4}$$

If one defines

$$F = \int_0^t f dt', \tag{5}$$

one obtains from eq. (4)

$$\xi_k \xi_k^* = \lambda_k + F + F^*. \tag{6}$$

λ_k are real positive constants:

$$\lambda_k \geq 0. \tag{7}$$

Equation (2) then has the formal solution

$$\xi_k = \xi_k^0 e^{G_k} \tag{8}$$

with

$$G_k = i\omega_k t + \int_o^t dt' \frac{f^*}{\lambda_k + F + F^*}. \tag{9}$$

When f is decomposed into its real and imaginary parts:

$$f = f_R + i f_I, \tag{10}$$

one can write G_k as

$$G_k = i\omega_k t - i \int_0^t dt' \frac{f_I}{\lambda_k + 2F_R} + \frac{1}{2} \ln(1 + \frac{2F_R}{\lambda_k}). \tag{11}$$

The definition (3) for f then yields the relation

$$f = -i\alpha \prod_{k=1}^{3} \xi_k^0 e^{G_k}. \tag{12}$$

From this relation one finds

$$|f|^2 = |f^0|^2 \prod_{k=1}^{3} (1 + \frac{2F_R}{\lambda_k}) \tag{13}$$

and

$$\dot{f} = \sum_{k=1}^{3} (i\omega_k + \frac{f^*}{\lambda_k + 2F_R}) f. \tag{14}$$

It is helpful to decompose this equation into its real and imaginary parts:

$$\dot{f}_R = \sum_{k=1}^{3} (-\omega_k f_I + \frac{f f^*}{\lambda_k + 2F_R}), \tag{15}$$

$$\dot{f}_I = \sum_{k=1}^{3} \omega_k f_R. \tag{16}$$

3 The resonant case $\Sigma_{k=1}^{3}\omega_k = 0$

In this case one has

$$f_I = const \tag{17}$$

and

$$\begin{aligned}
\dot{f_R} &= |f^0|^2 \sum_{k=1}^{3} \frac{1}{\lambda_k + 2F_R} \prod_{l=1}^{3}(1 + \frac{2F_R}{\lambda_l}) \\
&= |f^0|^2 (\frac{1}{\lambda_3}(1 + \frac{2F_R}{\lambda_1})(1 + \frac{2F_R}{\lambda_2}) \\
&+ \frac{1}{\lambda_1}(1 + \frac{2F_R}{\lambda_2})(1 + \frac{2F_R}{\lambda_3}) + \frac{1}{\lambda_2}(1 + \frac{2F_R}{\lambda_3})(1 + \frac{2F_R}{\lambda_1})) \\
&= |f^0|^2 [(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}) \\
&+ 4F_R(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1}) + (F_R)^2 \frac{12}{\lambda_1\lambda_2\lambda_3}],
\end{aligned} \tag{18}$$

where

$$|f^0|^2 = |\alpha|^2 \lambda_1\lambda_2\lambda_3. \tag{19}$$

Since all the λ's are non-negative, the r.h.s. can vanish only for negative values of F_R. The zeros are

$$F_{R\pm} = -\frac{1}{6}\sum_{k=1}^{3}\lambda_k \pm \frac{1}{6\sqrt{2}}\sqrt{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2}\,. \tag{20}$$

According to definition (5) F_R is zero at $t = 0$. Negative values of F_R are therefore obtained only for initially negative $\dot{F}_R = f_R$. But also these initial conditions lead to $F_R > 0$ after some time except in the singular case, leading to

$$F_R(t \to \infty) = F_{R+}. \tag{21}$$

For all other initial conditions the asymptotic behaviour is characterized by the $(F_R)^2$ term on the r.h.s. of eq. (18) becoming the dominant one. First the once-integrated form of eq. (18) is written down:

$$\begin{aligned}\frac{1}{2}(f_R)^2 = & \frac{1}{2}(f_R^0)^2 + \\ & |f^0|^2[(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3})F_R + \\ & 2F_R^2(\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_2\lambda_3}+\frac{1}{\lambda_3\lambda_1}) + 4F_R^3\frac{1}{\lambda_1\lambda_2\lambda_3}].\end{aligned} \tag{22}$$

Its asymptotic form is

$$f_R^2 = |f^0|^2\frac{8}{\lambda_1\lambda_2\lambda_3}F_R^3 = 8|\alpha|^2F_R^3. \tag{23}$$

The solution of this equation with the constant of integration t_0 for $t < t_0$ is

$$F_R = \frac{1}{2|\alpha|^2}\frac{1}{(t-t_0)^2}, \tag{24}$$

from which it follows that

$$f_R = -\frac{1}{|\alpha|^2}\frac{1}{(t-t_0)^3}. \tag{25}$$

Equation (11) then yields for t close to t_0

$$G_k = i\omega_k t + \frac{1}{2}\ln\frac{1}{\lambda_k|\alpha|^2(t-t_0)^2} + const \tag{26}$$

and therefore

$$\xi_k \propto \frac{e^{i\omega_k t}}{t-t_0}. \tag{27}$$

The general solution of eq. (18) or (22) valid for all times except at *reflection* points can be expressed in terms of the Weierstrass $\wp$ function:

$$F_R = \frac{1}{2|\alpha|^2}\wp(t-t_0, g_2, g_3) - \frac{B}{3A}, \tag{28}$$

where g_2 and g_3 are the so-called invariants, which in the present case are given by

$$g_2 = \frac{1}{12}B^2 - \frac{1}{4}CA \tag{29}$$

and

$$g_3 = \frac{1}{48}ABC - \frac{1}{16}DA^2 - \frac{2}{16 \cdot 27}C^3. \tag{30}$$

The quantities A, B, C, D are

$$\begin{aligned} A &= 8|\alpha|^2, \\ B &= 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= 2|\alpha|^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \\ D &= (f_R^0)^2. \end{aligned} \tag{31}$$

These are the coefficients in eq. (22) when this equation is written as

$$(f_R)^2 = A + CF_R + BF_R^2 + AF_R^3. \tag{32}$$

The constant of integration t_0 is obtained from $F_R(t=0) = 0$:

$$\frac{1}{2|\alpha|^2}\wp(-t_0, g_2, g_3) = \frac{B}{3A}. \tag{33}$$

It is necessarily real. Since the Weierstrass $\wp$ function is an even function, t_0 satisfying eq. (33) can have either sign. The sign is determined by

$$\dot{F}_R(t=0) = f_R^0. \tag{34}$$

After a *reflection* at a time t_R the time t runs backward corresponding to the other possible sign of f_R in eq. (32), i.e. the solution is then the one which is obtained by the replacement

$$\wp(t - t_0) \to \wp(2t_R - t_0 - t). \tag{35}$$

Furthermore, $\wp$ is a doubly-periodic meromorphic function in the complex $t - t_0$ plane, and since g_2 and g_3 are real, it is periodic along the real $t - t_0$ axis. If this property is combined with the fact that $\wp$ possesses as the only singularities a double pole at vanishing argument and corresponding periodic points, one always finds the behaviour exhibited by eq. (24). The special solutions corresponding to eq. (21) imply parameters g_2, g_3 which lead to an infinitely long period of $\wp$ along the real axis. Also Cherry's solutions for the two-oscillator case and those for the three-oscillator case presented in Ref. [3] belong to this class.

In order to write down the ξ_k's, one has to do the integral occurring in eq. (11), which can be expressed in terms of Weierstrass's ζ and σ functions. This will, however, not be done here, since the main emphasis is on the time dependence of the amplitudes.

4 The non-resonant case $\Sigma_{k=1}^{3} \omega_k \neq 0$

With

$$\sum_{k=1}^{3} \omega_k = \Omega \tag{36}$$

one obtains from eq. (15)

$$\ddot{f}_R = -\Omega \dot{f}_I + \frac{d}{dt} \sum_{k=1}^{3} \frac{f f^*}{\lambda_k + 2F_R}, \tag{37}$$

which, by means of eq. (16), becomes

$$\ddot{f}_R = -\Omega^2 \dot{F}_R + \frac{d}{dt} \sum_{k=1}^{3} \frac{f f^*}{\lambda_k + 2F_R}. \tag{38}$$

Integration over t with $f_I^0 = f_I(t=0)$ then yields

$$\dot{f}_R = -\Omega^2 F_R + \sum_{k=1}^{3} \frac{f f^*}{\lambda_k + 2F_R} - \Omega f_I^0, \tag{39}$$

which replaces eq. (18). The r.h.s. of this equation is again a polynomial of second order in F_R:

$$\dot{f}_R = c + bF_R + aF_R^2 \tag{40}$$

with

$$\begin{aligned} a &= 12|\alpha|^2, \\ b &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ c &= -\Omega f_I^0 + |\alpha|^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1). \end{aligned} \tag{41}$$

Since a is positive, sufficiently large $F_R > 0$ always leads to runaway of this quantity. If the equation

$$c + bF_R + aF_R^2 = 0 \tag{42}$$

has no real solution, then one has $\dot{f}_R > 0$ for all F_R , which results in explosive behaviour for arbitrary F_R. The same is true if there are two real solutions that are both negative. This was the situation with the resonant case. If at least one solution is positive, a threshold initial amplitude may be needed to obtain explosive behaviour. Of special interest are small amplitudes for which

$$c \ll \frac{b^2}{4a}. \tag{43}$$

The solutions in this case are

$$F_R \approx \begin{cases} -\dfrac{b}{a} & \approx \dfrac{\Omega^2}{12|\alpha|^2}, \\[2ex] \dfrac{c}{b} & \approx \dfrac{f_I^0}{\Omega}. \end{cases} \tag{44}$$

The important solution is the first one. When it is inserted in the *potential* $V(F_R)$ corresponding to the r.h.s. of eq. (43),

$$V(F_R) = -cF_R - \frac{1}{2}bF_R^2 - \frac{1}{3}aF_R^3, \tag{45}$$

one obtains

$$V \approx \frac{1}{3}\frac{\Omega^6}{(24|\alpha|^2)^2}. \tag{46}$$

Since this is non-infinitesimal, there cannot be nonlinear instability with arbitrarily small initial amplitudes.

The general solution of eqs. (40),(41) valid for all times is again given by eqs. (28)-(30),(35), and eq. (32) also holds, but with the following new definitions of the quantities A, B, C, D:

$$\begin{aligned} A &= 8|\alpha|^2, \\ B &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= -2\Omega f_I^0 + 2|\alpha|^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \\ D &= (f_R^0)^2. \end{aligned} \tag{47}$$

With these quantities the *potential* (45) can also be written as

$$V(F_R) = -\frac{1}{2}(CF_R + BF_R^2 + AF_R^3). \tag{48}$$

5 Examples

In this section some examples are given showing explosive behaviour or stability.

Insight into what is to be expected provides a more detailed discussion of the *potential* (48) and its negative derivative, which is the r.h.s. of eq. (40), than the one found in the foregoing section. Figures 1a-1d may be helpful for doing this. They show typical forms of the *potential* $V(F_R)$.

The conditions for runaway to occur at all initial values f_R^0 are obviously that

1. eq. (42) have no real zero, or
2. real zeros of eq. (42) be at negative F_R, or
3. the maximum of $V(F_R)$ be at positive F_R, but that it be negative.

If none of these conditions is fulfilled, stable behaviour is possible if the maximum of V is at positive F_R and is positive. The final condition for runaway not to occur is then

$$\frac{1}{2}(f_R^0)^2 \leq V(F_{Rmax}), \tag{49}$$

where F_{Rmax} is the position of the maximum of $V(F_R)$:

$$F_{Rmax} = -\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}. \tag{50}$$

Since the condition for stable behaviour being possible is a single one, it is easier to discuss this condition instead of the three conditions for runaway. For this discussion the zeros of $V(F_R)$ are of interest. They are given by

$$\begin{aligned} F_{R0} &= 0, \\ F_{R\pm} &= -\frac{B}{2A} \pm \sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}. \end{aligned} \tag{51}$$

One can then distinguish two cases:

1. $c < 0$,

2. $c > 0$.

For the discussion which follows Figs. 1a to 1d may be helpful; they show typical forms of the *potential* $V(F_R)$.

The first case means that the derivative of $V(F_R)$ at $F_R = 0$ is positive, which implies that the condition in question is fulfilled.

If one has $b > 0$ in the second case, then $V(F_R)$ is convex from above at $F_R = 0$, and since the derivative is negative, $V(F_R)$ is negative for $F_R > 0$. Hence the condition is not fulfilled.

If one has $b < 0$ in the second case, the condition is satisfied if F_{R+} is real and positive. The latter condition is automatically fulfilled with the first one. Reality requires that

$$C < \frac{B^2}{4A}. \tag{52}$$

Stable behaviour requires in addition that inequality (49) be satisfied.

When f_R^0 and f_I^0, referring to eq. (19), are written in the forms

$$f_I^0 = \sin\beta|\alpha|\sqrt{\lambda_1\lambda_2\lambda_3}\,,\ f_R^0 = \cos\beta|\alpha|\sqrt{\lambda_1\lambda_2\lambda_3}\,,\ 0 \leq \beta \leq 2\pi\,, \tag{53}$$

the quantities involved in the foregoing conditions become

$$\begin{aligned} a &= 12|\alpha|^2, \\ b &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ c &= -\Omega\sin\beta|\alpha|\sqrt{\lambda_1\lambda_2\lambda_3} + |\alpha|^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \\ A &= 8|\alpha|^2, \\ B &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= -2\Omega\mu|\alpha|\sqrt{\lambda_1\lambda_2\lambda_3} + 2|\alpha|^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \\ D &= \cos^2\beta|\alpha|^2\lambda_1\lambda_2\lambda_3. \end{aligned} \tag{54}$$

Figures 2 present typical examples; shown are the *potential* $V(F_R)$ together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ together with $\dot{F}_R = f_R$ (dashed line):

2a: $\Omega = 0; \lambda_1 = 0.01; \lambda_2 = 0.01; \lambda_3 = 0.01; \beta = 0$; *potential* like the one shown in Fig. 1a; runaway; the solution belongs to the special class of Cherry-type solutions found for the resonant case;

2b: $\Omega = 0; \lambda_1 = 0.01; \lambda_2 = 0.01; \lambda_3 = 0.01; \beta = \pi$; *potential* like the one shown in Fig. 1a; $F_R(t)$ comes to a stop at the maximum of $V(F_R)$; the solution again belongs to the special class of Cherry-type solutions found for the resonant case;

2c: $\Omega = .46; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = .5$; *potential* like the one shown in Fig. 1b; stable behaviour;

2d: $\Omega = .5; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = 0$; *potential* like the one shown in Fig. 1b; runaway;

2e: $\Omega = .4; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = 2$; *potential* like the one shown in Fig. 1c; stable behaviour;

2f: $\Omega = .542; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = -.565$; *potential* like the one shown in Fig. 1d; runaway.

6 Conclusions

The discussion of the complete solution of the three-oscillator case has shown that for almost all initial conditions resonance leads to an explosive behaviour. The nonlinear coupling of the three oscillators, however, allows runaway to occur in the nonresonant case as well, but the initial amplitudes must not be infinitesimally small. In a continuum theory the three-wave coupling expression usually contains terms additional to the ones considered here. They are generally of a kind which introduces nonresonant behaviour even in the otherwise resonant case. One can speculate that their effect averages out so as to make the resonant terms dominant. This would mean that one can expect nonlinear instability rather generally, when a continuum theory allows negative energy perturbations. In Ref. [2] the same conclusion was obtained especially by referring to the original two-oscillator case of Cherry and his class of solutions. The present paper can be considered as further support for this conclusion.

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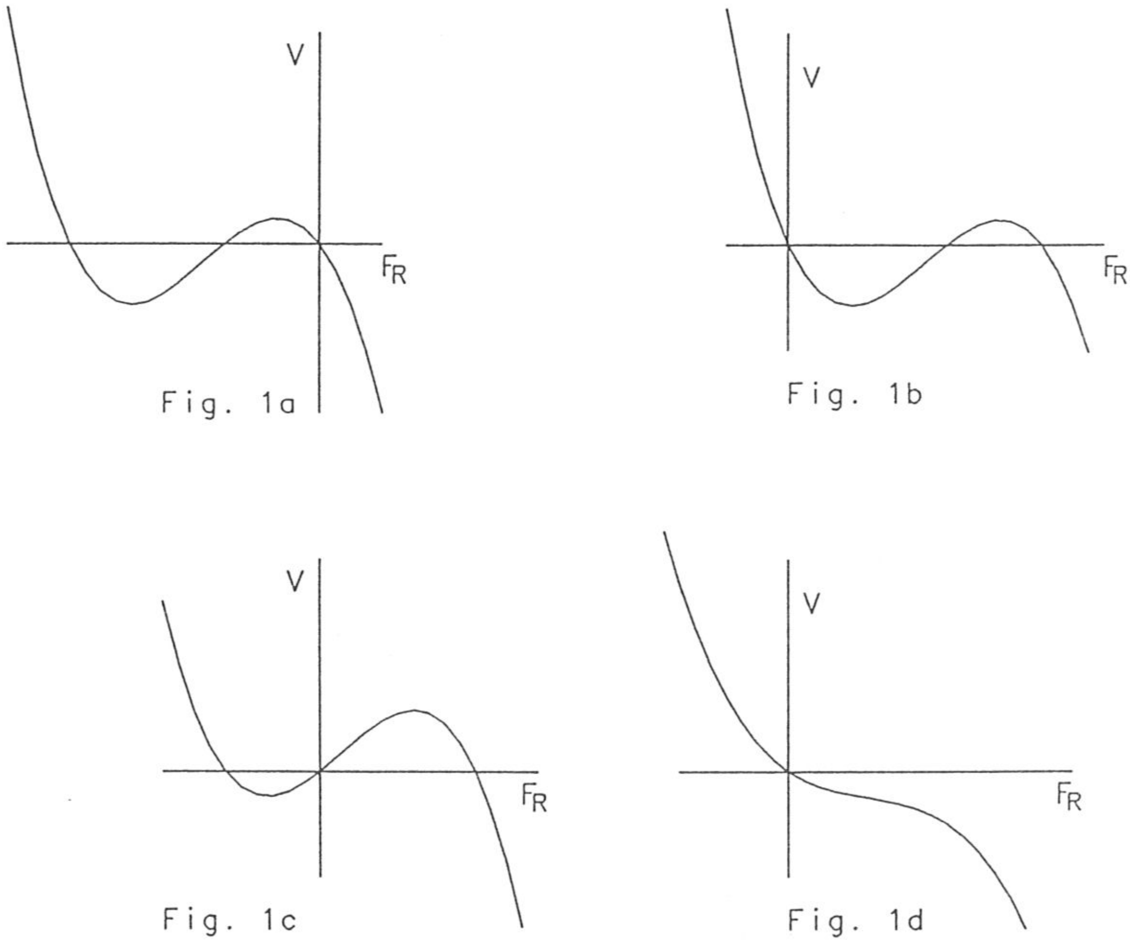


Fig. 1a Fig. 1b Fig. 1c Fig. 1d

Figures 1a-d:
Typical forms of the *potential* $V(F_R)$.

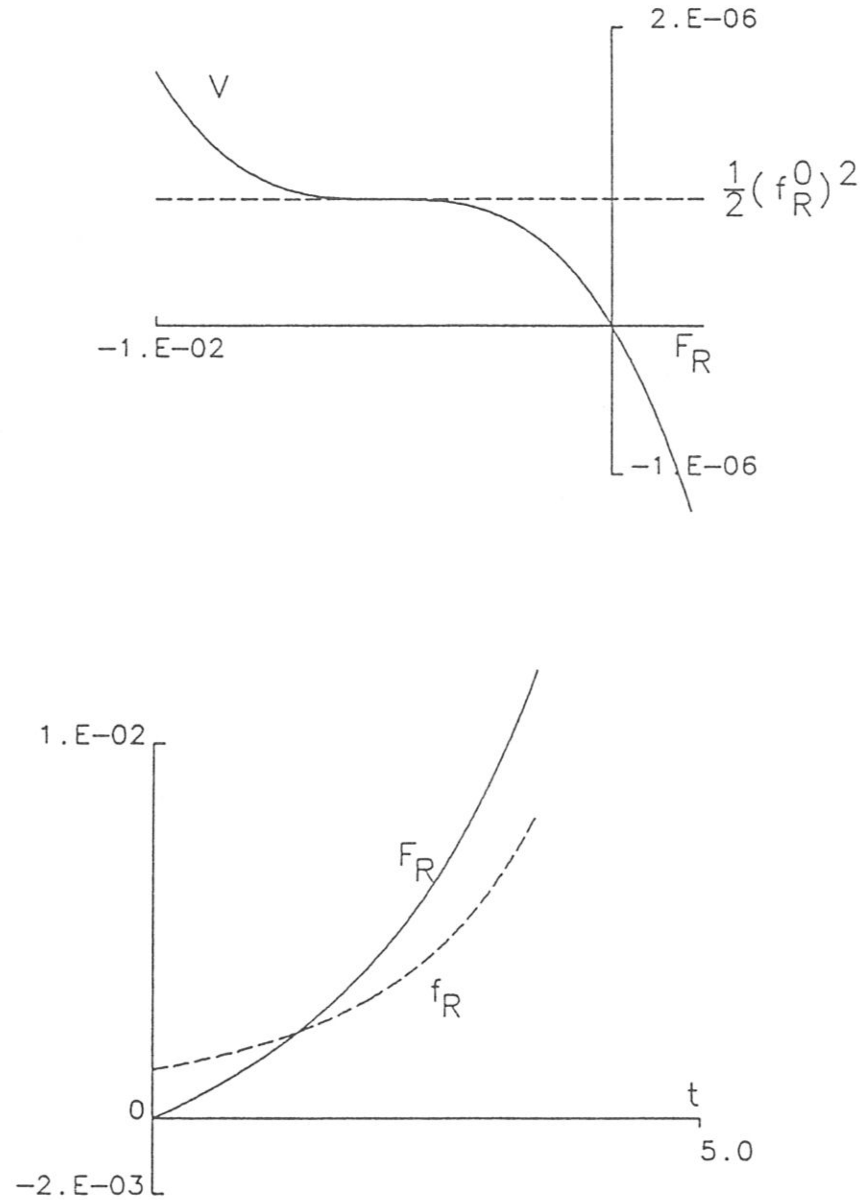


Figure 2a:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = 0; \lambda_1 = 0.01; \lambda_2 = 0.01; \lambda_3 = 0.01; \beta = 0$; *potential* like the one shown in Fig. 1a; runaway; the solution belongs to the special class of Cherry-type solutions found for the resonant case.

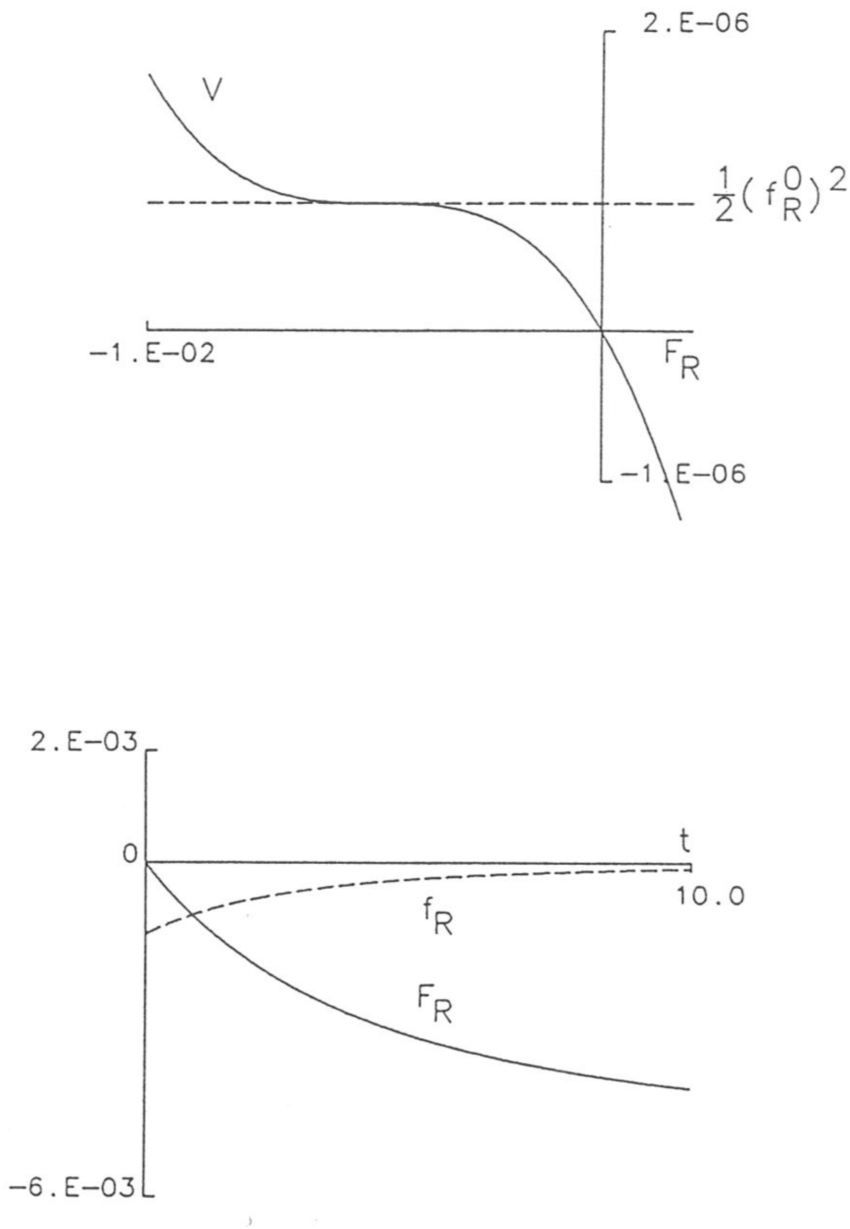
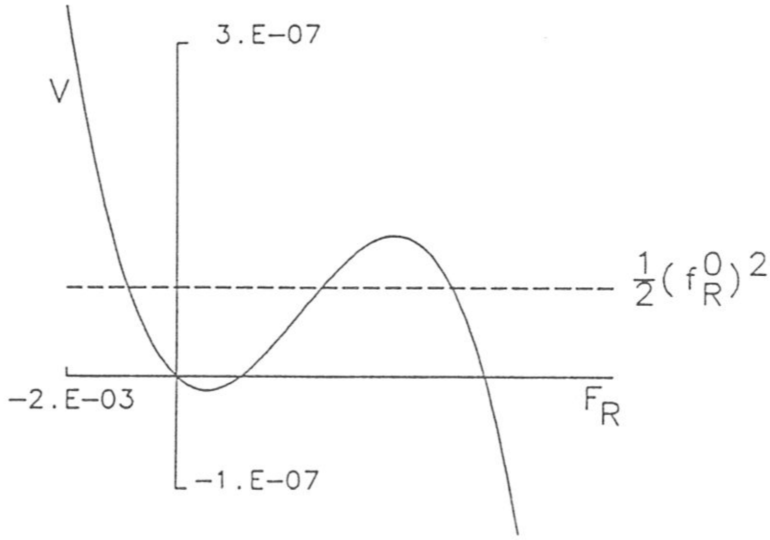


Figure 2b:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = 0; \lambda_1 = 0.01; \lambda_2 = 0.01; \lambda_3 = 0.01; \beta = \pi$; *potential* like the one shown in Fig. 1a; $F_R(t)$ comes to a stop at the maximum of $V(F_R)$; the solution again belongs to the special class of Cherry-type solutions found for the resonant case.



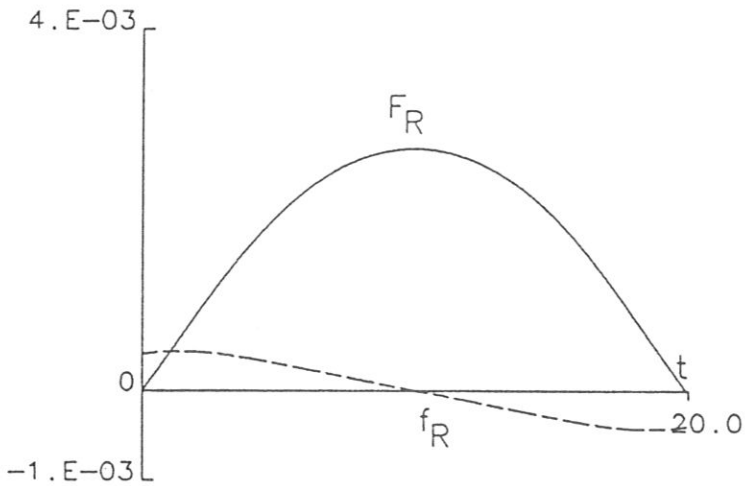
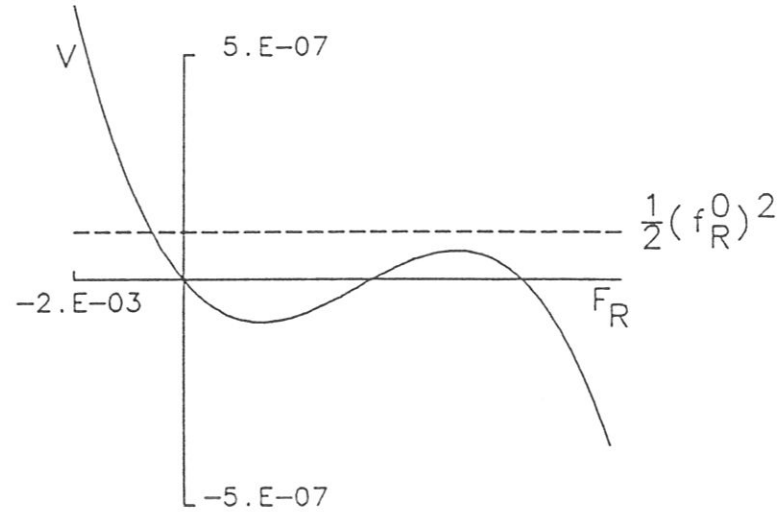


Figure 2c:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = .46$; $\lambda_1 = 0.01$; $\lambda_2 = 0.0049$; $\lambda_3 = 0.0025$; $\beta = .5$; *potential* like the one shown in Fig. 1b; stable behaviour.



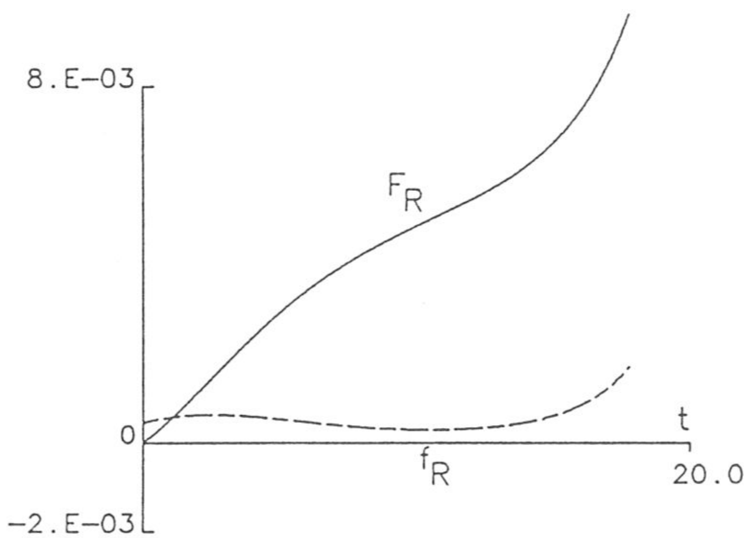
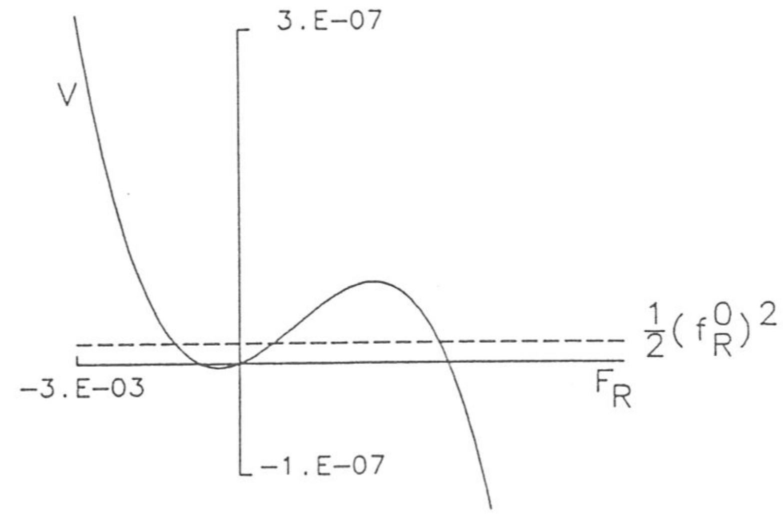


Figure 2d:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = .5; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = 0$; *potential* like the one shown in Fig. 1b; runaway.



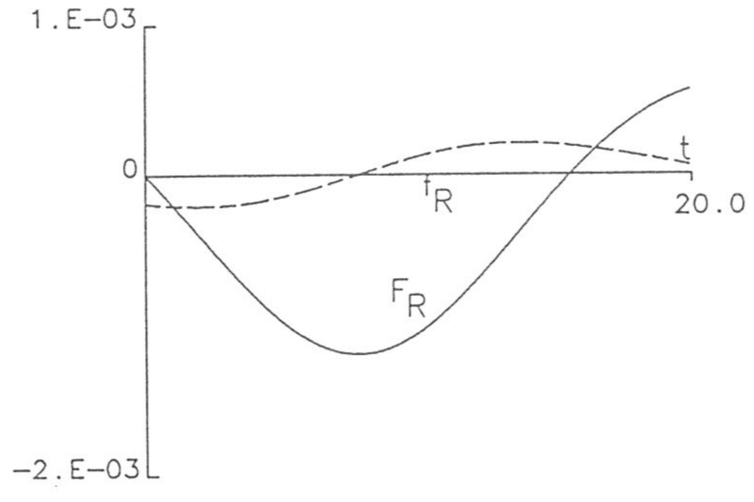


Figure 2e:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = .4; \lambda_1 = 0.01; \lambda_2 = 0.0049; \lambda_3 = 0.0025; \beta = 2$; *potential* like the one shown in Fig. 1c; stable behaviour.

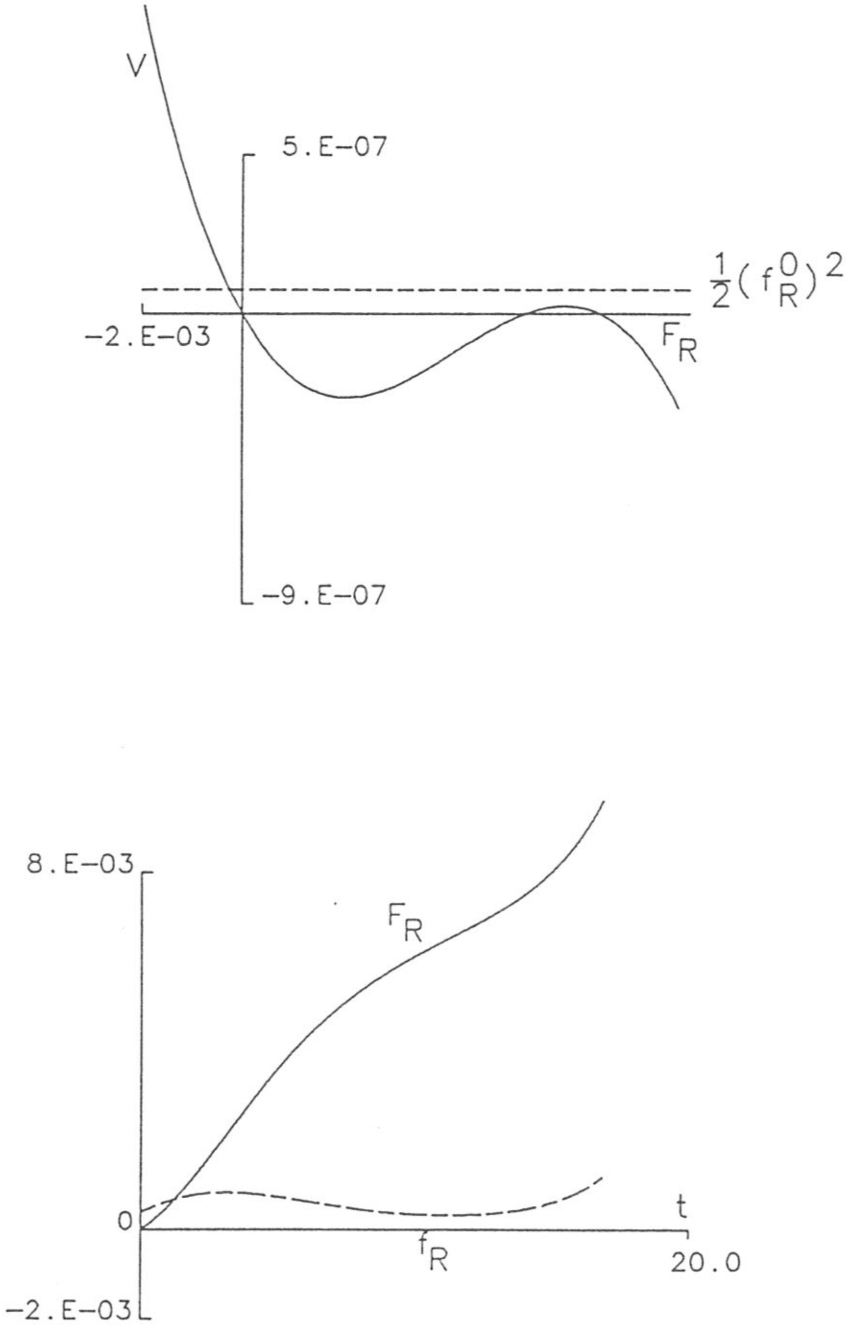


Figure 2f:
Example showing the *potential* $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line); $\Omega = .542$; $\lambda_1 = 0.01$; $\lambda_2 = 0.0049$; $\lambda_3 = 0.0025$; $\beta = -.565$; *potential* like the one shown in Fig. 1d; runaway.